StatPhys Seminar 2023/5/26

Duality, criticality, topology and integrability in quantum spin-1 chains

Hosho Katsura (UTokyo)

Collaborators:
Hong Yang, Linhao Li (UTokyo)
Koichi Okunishi (Niigata Univ.)

➢ Phys. Rev. B 107, 125158 (2023) [arXiv:2203.15791]

2D classical Ising model

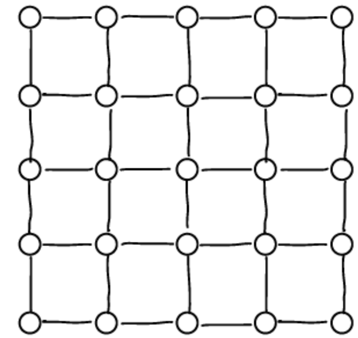
■ Model

- Spin configuration $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}, \quad \sigma_i = \pm 1$
- Hamiltonian $H(\boldsymbol{\sigma}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad (J > 0)$
- Partition function

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}$$

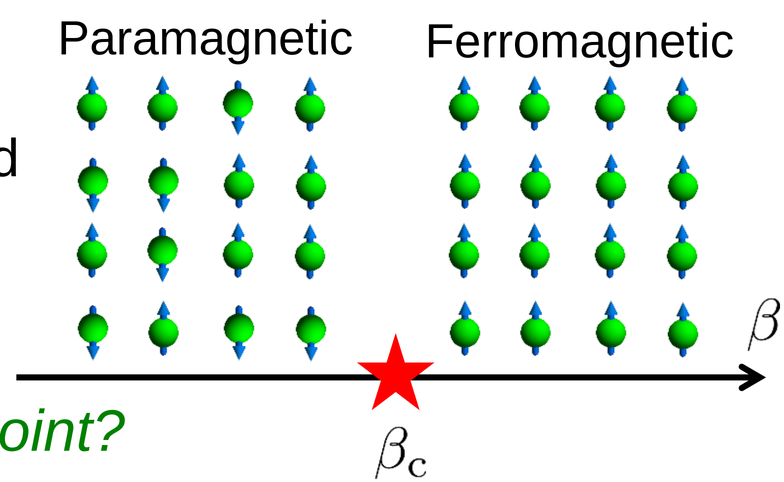
Solved by Onsager, Phys. Rev. **65**, 117 (1944)
Majorana-fermion trick by Kauffmann (1949)

Square lattice



■ Phases

- Zero temperature ($\beta = \infty$)
  All-up and all-down states are realized
- Infinite temperature ($\beta = 0$)
  All states occur with equal probability



Where is the transition (critical) point?

Kramers-Wannier duality (1)

Phys. Rev. **60**, 252 (1941)

- "High-temperature" expansion

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \exp\left(\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \right)$$

$\beta J \to K$

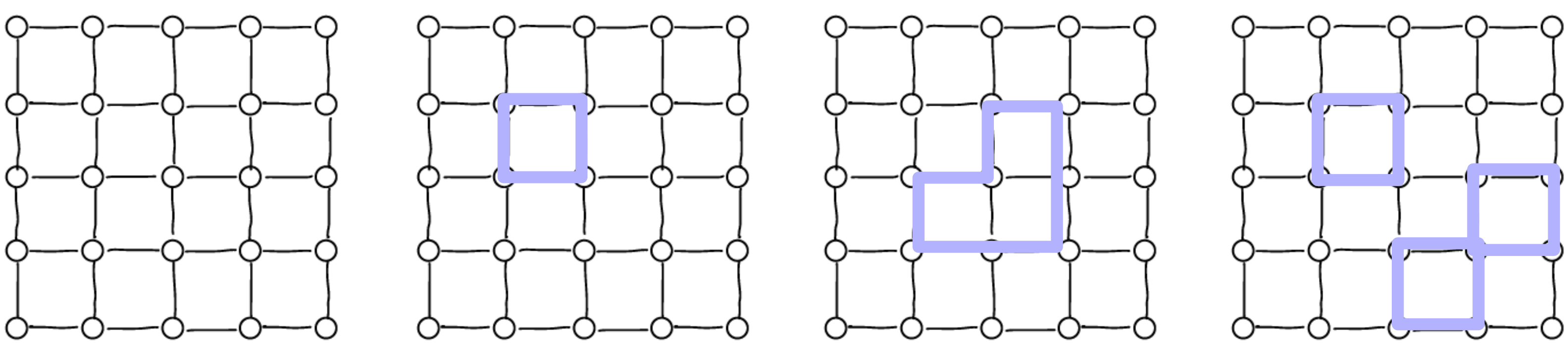
- Useful identity: $e^{K\sigma_i\sigma_j} = \cosh K (1 + \sigma_i \sigma_j \tanh K)$

$$Z_N(K) = (\cosh K)^{2N} \sum_{\boldsymbol{\sigma}} \prod_{\langle i,j \rangle} (1 + \sigma_i \sigma_j \tanh K)$$

$$\sum_{\sigma = \pm 1} \sigma^n = 1 + (-1)^n$$

$$= 2^N (\cosh K)^{2N} \sum_P (\tanh K)^{\ell(P)}$$

P: Sum over loop configurations

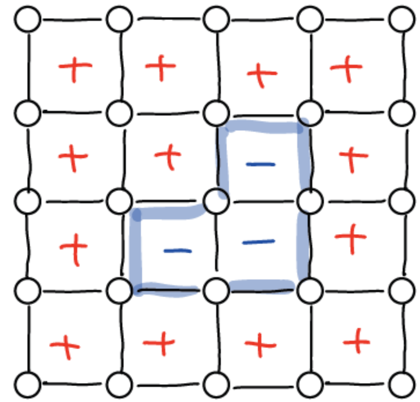


$\ell(P) = 0$ $\ell(P) = 4$ $\ell(P) = 8$ $\ell(P) = 12$

Kramers-Wannier duality (2)

■ Z in terms of dual variables

- Label dual lattice sites by $a = 1, 2, \ldots$
- Dual spin config. $\boldsymbol{\mu} = \{\mu_1, \mu_2, \ldots\}$
  μ=-1 (+1) inside (outside) a loop



$$\ell(P) = \sum_{\langle a,b \rangle} \frac{1 - \mu_a \mu_b}{2}$$

$$Z_N(K) = (\sinh K)^N \sum_{\boldsymbol{\mu}} \prod_{\langle a,b \rangle} \exp(\tilde{K} \mu_a \mu_b)$$

- Dual coupling $\tilde{K} = \frac{1}{2} \ln \coth K$

■ Critical temperature

- Assumption: a single critical temperature
- Then $K = \tilde{K} = \beta_c J$ must hold
  Solving $K = \frac{1}{2} \ln \coth K$ leads to $\beta_c J = \frac{1}{2} \ln(1 + \sqrt{2}) \sim 0.44$

Outline

1. Quantum Ising and Potts chains
   - Duality and Majorana translation
   - Kitaev chain: topology matters
   - Duality and parafermion translation

2. Kennedy-Tasaki Duality in spin-1 chains
3. Criticalities in interpolating models
4. Summary

Quantum Ising chain

- **Spin operators**
  - Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - Spin op. at site *j* : $\sigma_j^\alpha = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes \sigma^\alpha \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j},$

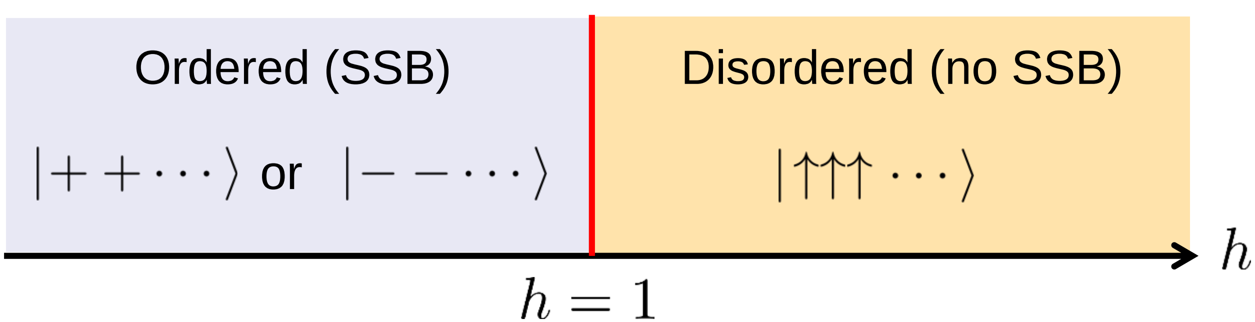
- **Hamiltonian (J, h >0)**

$$H_{\text{Ising}} = -J \sum_j \sigma_j^x \sigma_{j+1}^x - h \sum_j \sigma_j^z$$

Local basis

$$\sigma^z |\uparrow\rangle = +|\uparrow\rangle$$
$$\sigma^z |\downarrow\rangle = -|\downarrow\rangle$$
$$\sigma^x |\pm\rangle = \pm|\pm\rangle$$

- **Phase diagram (J =1)**



h =1 is a critical point described by Ising CFT (c=1/2)

Kramers-Wannier Duality

- Unitary transformation — Huang & Chen, *PRB* **91**, 195143 (2015)

$$U = \prod_j \frac{1 + \mathrm{i}\,\sigma_j^z}{\sqrt{2}}, \quad V = \prod_j \frac{\sigma_j^x \sigma_{j+1}^x + \sigma_{j+1}^z}{\sqrt{2}}$$

$$(UV)\,\sigma_j^z\,(UV)^\dagger = \sigma_j^x \sigma_{j+1}^x$$

$$(UV)\,\sigma_j^x \sigma_{j+1}^x\,(UV)^\dagger = \sigma_{j+1}^z$$

- Dual Hamiltonian

$$\widetilde{H}_{\mathrm{Ising}} = (UV) H_{\mathrm{Ising}} (UV)^\dagger = -h \sum_j \sigma_j^x \sigma_{j+1}^x - J \sum_j \sigma_j^z$$

- The roles of J and h are interchanged
- Must have the same spectrum
  NOTE) Ignore the boundary terms
- If there exists a single gapless point, it should be at $h = J$

Anything to do with topology?

Quantum Ising chain = Kitaev chain

■ Jordan-Wigner tr.

- Majorana fermions

$$\gamma_{2j-1} = \left(\prod_{k<j} \sigma_k^z\right) \sigma_j^x, \quad \gamma_{2j} = \left(\prod_{k<j} \sigma_k^z\right) \sigma_j^y$$

$$(\gamma_m)^\dagger = \gamma_m, \quad \{\gamma_m, \gamma_n\} = 2\delta_{m,n}$$

■ Fermionic Hamiltonian

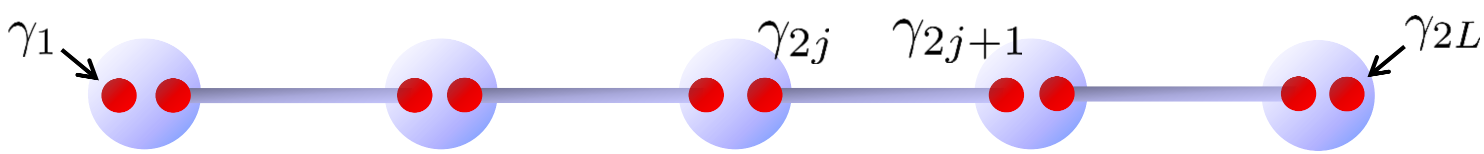
A. Kitaev, *Phys. Usp*. (2001)

$$H_{\mathrm{Ising}} = \mathrm{i}\, J \sum_j \gamma_{2j}\gamma_{2j+1} - \mathrm{i}\, h \sum_j \gamma_{2j-1}\gamma_{2j}$$

Duality = Translation by 1 Majorana site

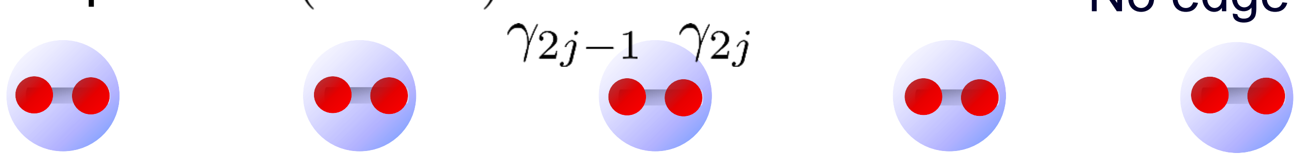
- Topological phase $(J \gg h)$

Edge zero modes!



- Trivial phase $(J \ll h)$

No edge modes…



Self-duality = Translation symmetry

■ Topology matters

- "Bloch vector" of *k*-space Hamiltonian

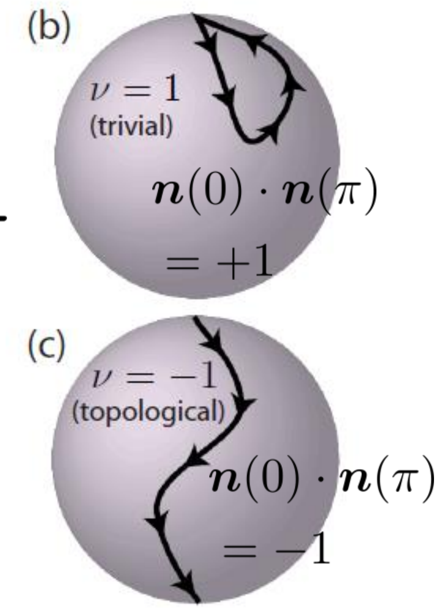
$$H_{\text{Ising}} = \frac{1}{2} \sum_{0 \le k \le \pi} \Psi^\dagger(k) \mathcal{H}(k) \Psi(k), \quad \mathcal{H}(k) = \boldsymbol{d}(k) \cdot \boldsymbol{\sigma}$$

Symmetry

$$d_{x,y}(k) = -d_{x,y}(-k), \quad d_z(k) = d_z(-k)$$

n-vector $\boldsymbol{n}(k) := \dfrac{\boldsymbol{d}(k)}{|\boldsymbol{d}(k)|}$

Nonzero gap $\Leftrightarrow \ |\boldsymbol{d}(k)| > 0$



J. Alicea, *Rep. Prog. Phys*. **75** (2012)

- There must be a critical point b/w (b) & (c)

■ At the critical point $J = h$

$$H_{\text{Ising}} = \mathrm{i}J \sum_m \gamma_m \gamma_{m+1}$$

- New symmetry: Translation by 1 Majorana site!

Quantum Potts chains

■ Shift & clock matrices

$$\sigma|i\rangle = |i-1\rangle, \quad \tau|i\rangle = \omega^i|i\rangle, \quad i = 1, \ldots, N, \ \omega = e^{2\pi \mathrm{i}/N}$$

$$\sigma^N = \tau^N = 1, \quad \sigma^\dagger = \sigma^{N-1}, \quad \tau^\dagger = \tau^{N-1} \qquad \sigma\tau = \omega\,\tau\sigma$$

■ Hamiltonian (J, h >0)

$$H_{\mathrm{Potts}} = -J\sum_j (\sigma_j^\dagger \sigma_{j+1} + \mathrm{h.c.}) - h\sum_j (\tau_j + \tau_j^\dagger)$$

- Duality: $\tau_j \to \sigma_j^\dagger \sigma_{j+1}, \quad \sigma_j^\dagger \sigma_{j+1} \to \tau_{j+1}$
- Parafermions Fradkin & Kadanoff, *NPB* **170** (1980)

$$\chi_{2j-1} = \sigma_j \prod_{k<j} \tau_k, \quad \chi_{2j} = -\omega^{1/2}\tau_j\sigma_j \prod_{k<j}\tau_k$$

$$H_{\mathrm{Potts}} = J\sum_j (\omega^{1/2}\chi_{2j-1}^\dagger \chi_{2j} + \mathrm{h.c.}) + h\sum_j (\omega^{1/2}\chi_{2j}^\dagger \chi_{2j+1} + \mathrm{h.c.})$$

- Translation invariant at the self-dual point $h = J$
- Gap closing (2nd order for N=2, 3, 4, 1st order for N>4)

Self-duality → Gap closing??

- 3-state U(1)-invariant clock model
  - Raising & lowering ops. $S^+ = \frac{1}{3}(2 - \omega\tau - \omega^2\tau^\dagger)\sigma^\dagger, S^- = (S^+)^\dagger$
  - Hamiltonian

$$H_0 = \sum_j \left[3((S_j^+)^2(S_{j+1}^-)^2 - S_j^+ S_{j+1}^- + \mathrm{h.c.}) - \tau_j - \tau_j^\dagger \right]$$

Fateev & Zamolodchikov (1980), Vernier, O'Brien & Fendley, *J. Stat. Mech.* 043107 (2019); *PRB* **101**, 235108 (2020)

✓ U(1) symmetric, integrable, Onsager symmetries, …
✓ Self-dual and gapless (*c*=1 CFT)
✓ Criticality persists in the interpolating model

$$\alpha H_{\mathrm{Potts}}(J = h) + (1 - \alpha) H_0$$

- Today's subject
  - Is this the end of the story?
  - No! One can consider another duality transformation (KT tr.)

Outline

1. Quantum Ising and Potts chains

2. Kennedy-Tasaki Duality in spin-1 chains

- Spin-1 bilinear-biquadratic chain
- Haldane phase & AKLT model
- SPT & Hidden symmetry breaking

3. Criticalities in interpolating models

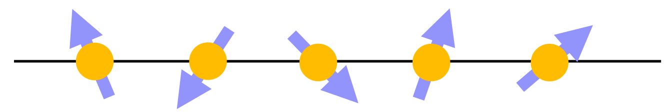
4. Summary

Haldane "conjecture" (early 80s)

■ Spin-S Heisenberg antiferromagnetic chain

- Hamiltonian ($J > 0$)

$$H_{\mathrm{Heis}} = J \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



- S=1/2, 3/2, 5/2, …
  Gapless, power-law decay of spin correlations
  NOTE) S=1/2 case is solvable (Bethe 1931)

- S=1, 2, 3, …
  a. Unique ground state
  b. Non-zero gap Δ (Haldane gap)
  c. Exponential decay of spin correlation

Established in many different ways!

$AgVP_2S_6$, NENP, …; ED, QMC, …

$$\Delta(S) = \begin{cases} 0.41048(6) & \text{for } S = 1 \\ 0.08917(4) & \text{for } S = 2 \\ 0.01002(3) & \text{for } S = 3 \end{cases}$$

Todo & Kato, *PRL* **87** (2001)

Quantum spin-1 chain with SU(2)

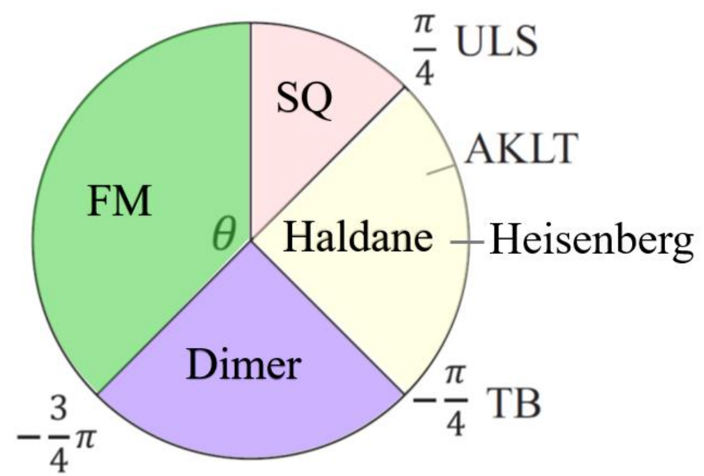
- Bilinear-biquadratic (BLBQ) model

$$H_{\mathrm{BLBQ}}(\theta) = \sum_{j=1}^{L} \left[\cos\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \sin\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\right]$$

Spin-1 op. at site *j*

$$\boldsymbol{S}_j = (S_j^x, S_j^y, S_j^z)$$

- Phase diagram Lauchli, Schmid & Trebst, *PRB* **74**, 144426 (2006)
  - Spin-quadrupolar (SQ): gapless, dominant nematic corr.
  - Ferromagnetic (FM)
  - Dimer: gapped, 2-fold degenerate g.s.
  - Haldane phase
    - ✓ Gapped unique g.s.
    - ✓ Edge states
    - ✓ Hidden AFM order (string order)



Prototype of Symmetry-protected topological (SPT) phase!

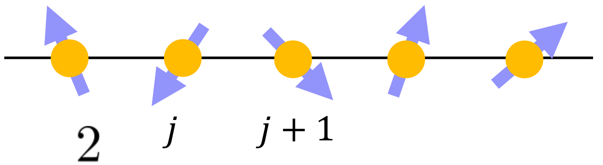
AKLT model/state

■ S=1 AKLT model

Affleck, Kennedy, Lieb & Tasaki, *PRL* **59** (1987), *CMP* **115** (1987)

- Hamiltonian $(\tan\theta = 1/3)$

$$H_{\mathrm{AKLT}} = \sum_j \left[\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 \right]$$



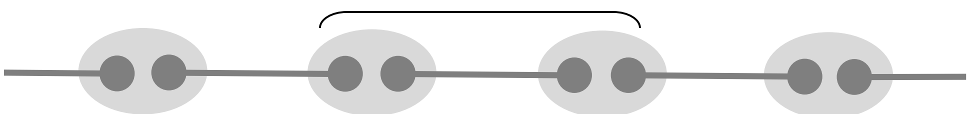
$$= 2P^{S=2}_{j,j+1} - \frac{2}{3}$$

- Exact ground state


: Spin singlet $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

: Projection to spin-1 space

$S_{\mathrm{tot}} < 2$



Frustration-free! The g.s. minimizes each local Hamiltonian

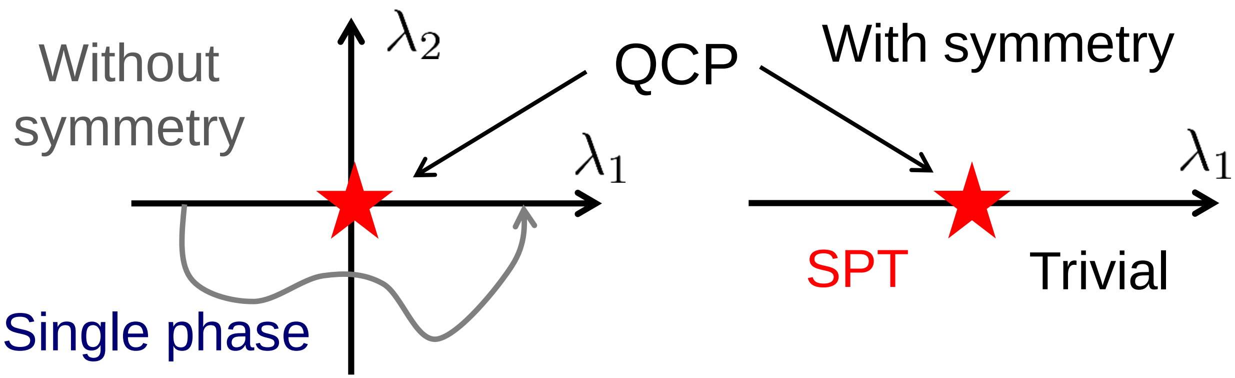
(i) Non-zero gap above the g.s, (ii) exponential decay of correlations, supporting Haldane conjecture

Edge states = nearly free edge spin-1/2s in open chain

Haldane phase as SPT phase

- **What is SPT?**

Gu & Wen, *PRB* **80** (2009). Pollmann, Berg, Turner & Oshikawa, *PRB* **81** (2010); **85** (2012)

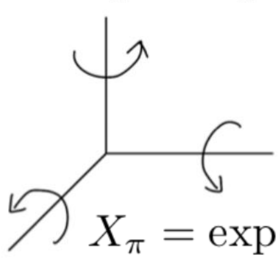


Two distinct phases!

- **Symmetry protection**

S=1 Haldane phase is protected by ANY one of three symmetries:
(i) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (ii) time-reversal, (iii) bond centered inversion

$$Z_\pi = X_\pi Y_\pi = \exp(-i\pi \sum_j S_j^z)$$



$$Y_\pi = \exp(-i\pi \sum_j S_j^y)$$

$$X_\pi = \exp(-i\pi \sum_j S_j^x)$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, X_\pi, Y_\pi, Z_\pi\}$$

BLBQ has symmetries (i), (ii) & (iii)

Symmetry-breaking interpretation? (1)

■ Kennedy-Tasaki transformation

- Non-local unitary

Kennedy & Tasaki, *PRB 4*5 (1992), Oshikawa, *JPC* **4** (1992)

$$U_{\mathrm{KT}} = \prod_{j<k} \exp(\mathrm{i}\pi S_j^z S_k^x), \quad U_{\mathrm{KT}}^\dagger = U_{\mathrm{KT}}, \ (U_{\mathrm{KT}})^2 = 1$$

- Transformation rules

Any local operator with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is transformed into a local operator in the new basis

$$S_j^x S_{j+1}^x \quad \overset{U_{\mathrm{KT}}}{\Longleftrightarrow} \quad U_{\mathrm{KT}} S_j^x S_{j+1}^x U_{\mathrm{KT}} = -S_j^x S_{j+1}^x$$

$$S_j^y S_{j+1}^y \quad \overset{U_{\mathrm{KT}}}{\Longleftrightarrow} \quad S_j^y e^{\mathrm{i}\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y$$

$$S_j^z S_{j+1}^z \quad \overset{U_{\mathrm{KT}}}{\Longleftrightarrow} \quad -S_j^z S_{j+1}^z$$

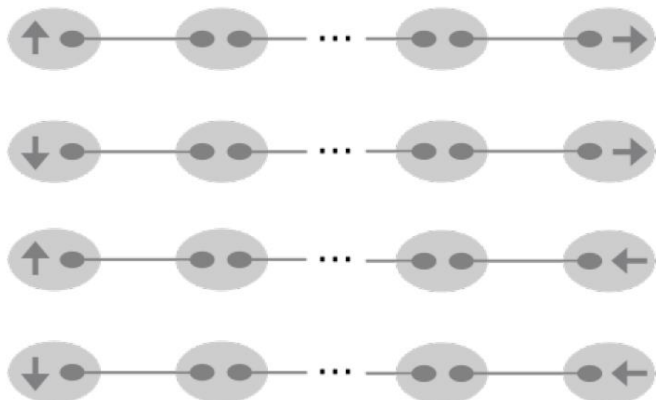
- “Hidden order” ⇔ Ferromagnetic order $(\alpha = x, z)$

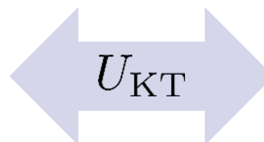
$$-\lim_{r\to\infty} \langle S_j^\alpha \exp[\mathrm{i}\pi \sum_{k=j+1}^{j+r-1} S_k^\alpha] S_{j+r}^\alpha \rangle \quad \overset{U_{\mathrm{KT}}}{\Longleftrightarrow} \quad \lim_{r\to\infty} \langle S_j^\alpha S_{j+r}^\alpha \rangle$$

Symmetry-breaking interpretation? (2)

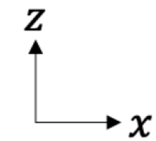
- What AKLT states look like after KT tr.?

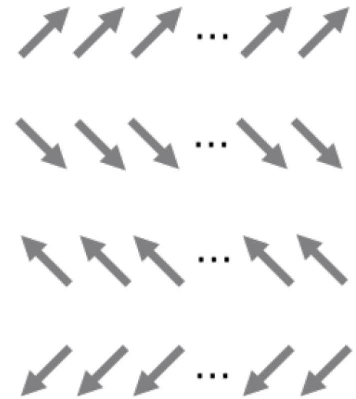
4 ground states on open chain





4 SSB states





$\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|+\rangle$

$\sqrt{\frac{1}{3}}|0\rangle - \sqrt{\frac{2}{3}}|+\rangle$

$\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|-\rangle$

$\sqrt{\frac{1}{3}}|0\rangle - \sqrt{\frac{2}{3}}|-\rangle$

Break $\mathbf{Z}_2$ x $\mathbf{Z}_2$ symmetry!

- Trivial state remains trivial
  - Trivial model

$$H_{\mathrm{triv}} = \sum_j (S_j^z)^2$$

has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Invariant under KT transformation.
Ground state $|0, 0, \cdots, 0\rangle$ is also invariant under KT.
No FM order. It does not break $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

Outline

1. Quantum Ising and Potts chains

2. Kennedy-Tasaki Duality in spin-1 chains

3. Criticalities in interpolating models

- Model & phase diagram
- Self-duality and Gaussian criticality
- SPT & trivial Ising criticalities

4. Summary

Model

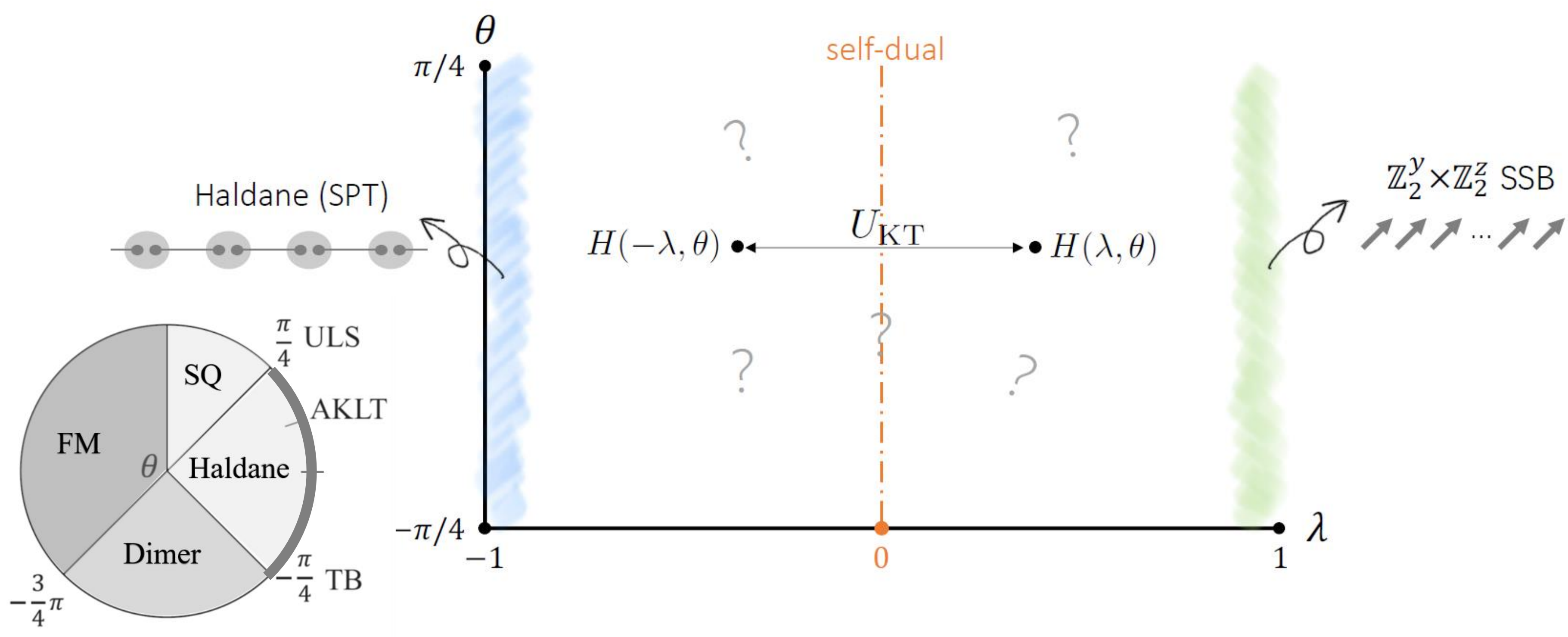
- Hamiltonian

$$H(\lambda,\theta) = (1-\lambda)H_{\mathrm{BLBQ}}(\theta) + (1+\lambda)\widetilde{H}_{\mathrm{BLBQ}}(\theta) \qquad (-1 \le \theta \le 1)$$

$$\widetilde{H}_{\mathrm{BLBQ}}(\theta) = U_{\mathrm{KT}} H_{\mathrm{BLBQ}}(\theta) U_{\mathrm{KT}}$$

$$H_{\mathrm{BLBQ}}(\theta) = \sum_j \left[\cos\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \sin\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\right]$$

- Phase diagram?



Self-duality and criticality (1)

■ Self-dual model

- Hamiltonian $H(0,\theta) = H_{\mathrm{BLBQ}}(\theta) + \widetilde{H}_{\mathrm{BLBQ}}(\theta)$
- Commutes with U_{KT} $[H(0,\theta), U_{\mathrm{KT}}] = 0$
- Change of basis

S^z basis : $\{|+\rangle, |0\rangle, |-\rangle\}$

$$\left\{\begin{array}{l} |\uparrow\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ |\downarrow\rangle = |0\rangle \\ |\mathrm{h}\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \end{array}\right.$$

Effective spin-1/2 subspace

$$\left\{\begin{array}{l} \sigma^x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \\ \sigma^y = -\mathrm{i}|\uparrow\rangle\langle\downarrow| + \mathrm{i}|\downarrow\rangle\langle\uparrow| \\ \sigma^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \end{array}\right.$$

$$n = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$h = |\mathrm{h}\rangle\langle\mathrm{h}|$$

■ Connection to integrable XXZ

$$H(0,\theta) = H_{\mathrm{XXZ}} + \sin\theta \sum_j (2h_j h_{j+1} + n_j n_{j+1} + 2)$$

$$H_{\mathrm{XXZ}} = (\cos\theta - \sin\theta) \sum_j (-\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sin\theta \sum_j \sigma_j^z \sigma_{j+1}^z$$

✓ Holes are not dynamical (fragmented model)

✓ Heisenberg point (θ=0) → XY (free-fermion) model!

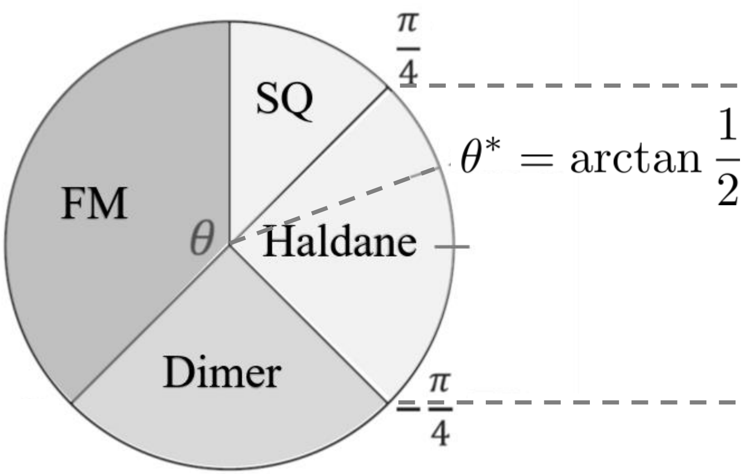
Self-duality and criticality (2)

■ Phase diagram at λ=0

- Anisotropy

$$\Delta(\theta) = \frac{\sin\theta}{|\cos\theta - \sin\theta|}$$

$$\begin{cases} -1 \le \Delta(\theta) < 1 & c = 1 \text{ Gaussian CFT} \\ \Delta(\theta) = 1 & z = 2 \text{ FM Heisenberg} \\ |\Delta(\theta)| > 1 & \text{gapped degenerate} \end{cases}$$



$\theta^* < \theta < \pi/4$	Gapped degenerate
$\theta = \theta^*$	*z*=2 FM Heisenberg
$-\pi/4 < \theta < \theta^*$	Gapless, *c*=1 Gaussian CFT

- Presence or absence of holes in g.s.

(green)	$\theta^* \le \theta < \pi/4$	Holes are present
(lavender)	$-\pi/4 < \theta < \theta^*$	Holes are absent

Can be proved using *classic* Bethe ansatz results

Yang-Yang, *PR* (1966), Hamer, Quispel & Batchelor, *JPA* (1987)

Around self-dual line

■ Perturbation theory around λ =0

Hole motion

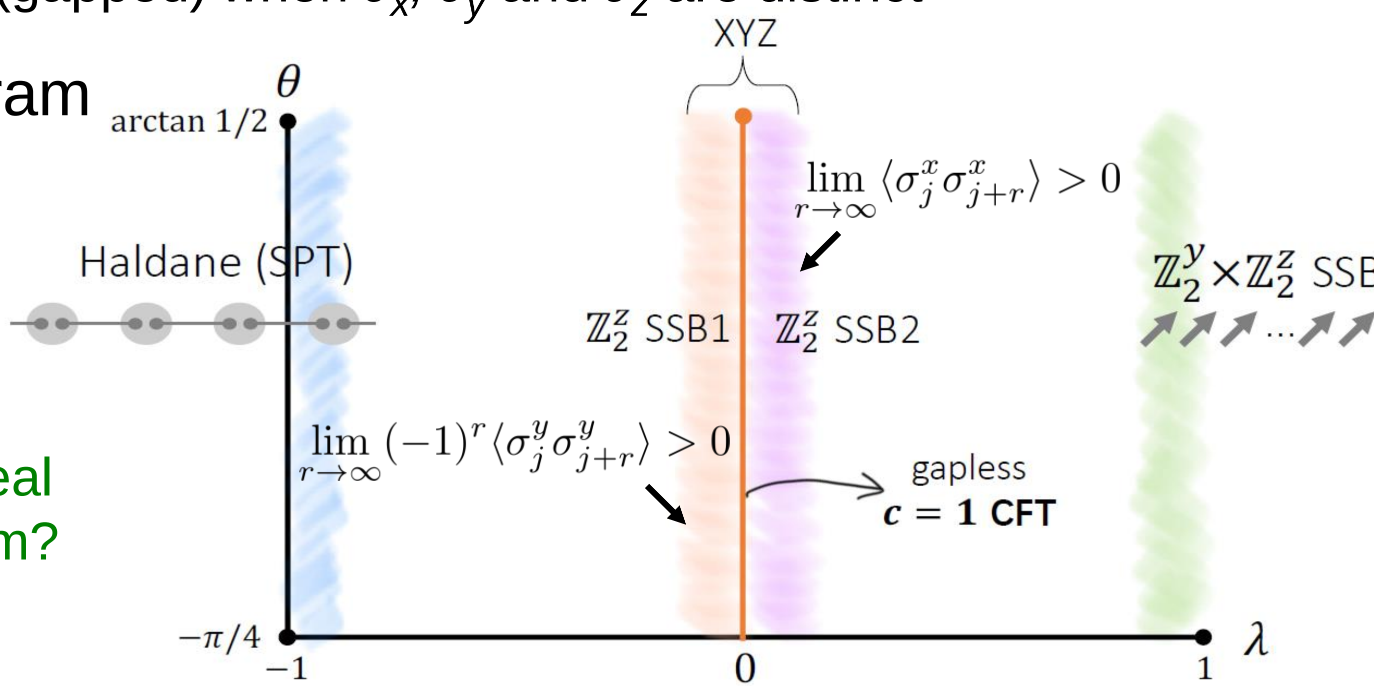
$$H(\lambda,\theta) \sim H_{\mathrm{XYZ}} + \sin\theta \sum_j (n_j n_{j+1} + 2) + 2\lambda\cos\theta \sum_{j,\sigma} (|\mathrm{h}\sigma\rangle_{j,j+1}\langle\sigma\mathrm{h}| + \mathrm{h.c.})$$

$$H_{\mathrm{XYZ}} = \sum_j [-J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z]$$

$$J_x = [(1+\lambda)\cos\theta - \sin\theta],\ \ J_y = [(1-\lambda)\cos\theta - \sin\theta],\ \ J_z = \sin\theta$$

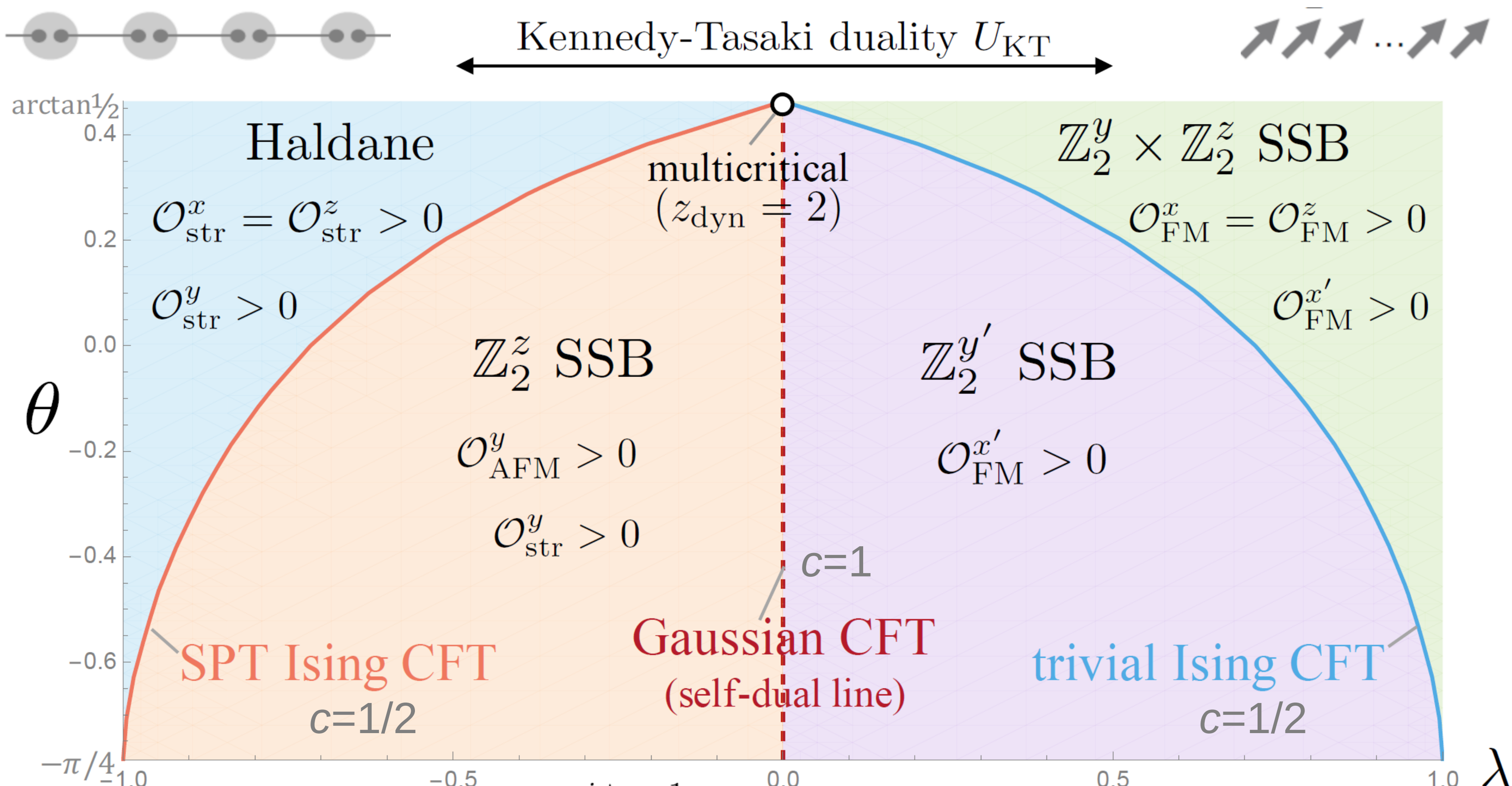
- S=1/2 XYZ chain is integrable
- Massive (gapped) when J_x, J_y and J_z are distinct

■ Phase diagram (updated)



What is the real phase diagram?

DMRG results

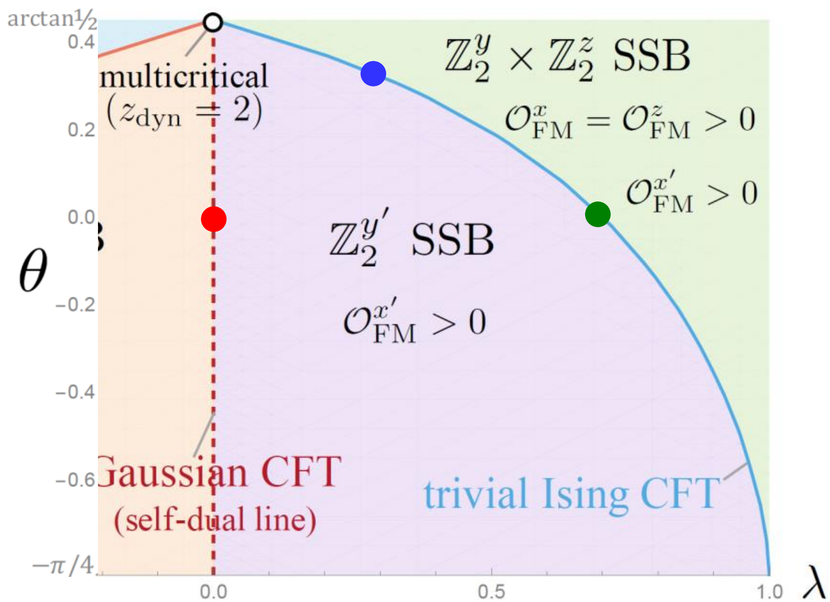


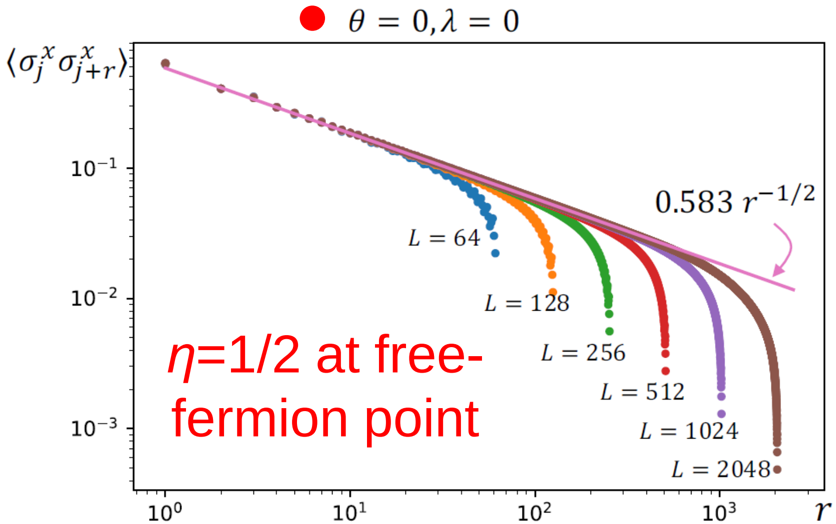
$$\mathcal{O}^\alpha_{\mathrm{str}} = -\lim_{r\to\infty} \langle S^\alpha_j \exp[i\pi \sum_{k=j+1}^{j+r-1} S^\alpha_k] S^\alpha_{j+r} \rangle \qquad \mathcal{O}^{x'}_{\mathrm{FM}} = \lim_{r\to\infty} \langle \sigma^x_j \sigma^x_{j+r} \rangle$$

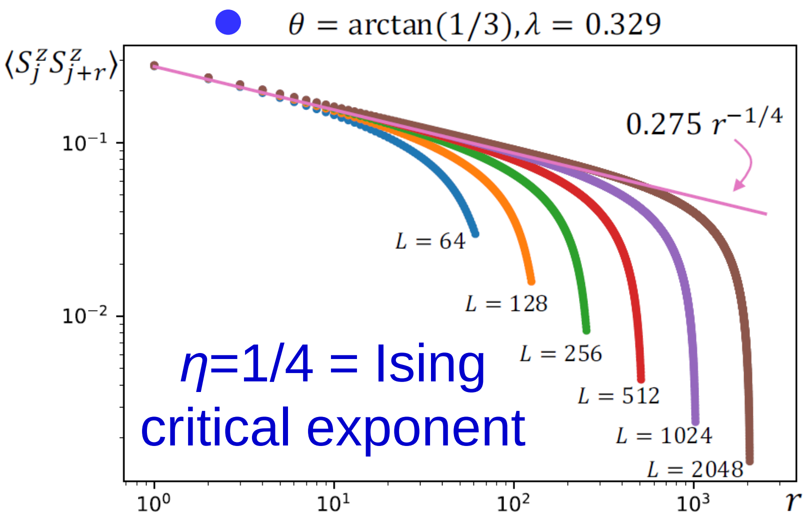
$$\mathcal{O}^y_{\mathrm{AFM}} = \lim_{r\to\infty} (-1)^r \langle S^y_j S^y_{j+r} \rangle \qquad \mathcal{O}^\alpha_{\mathrm{FM}} = \lim_{r\to\infty} \langle S^\alpha_j S^\alpha_{j+r} \rangle$$

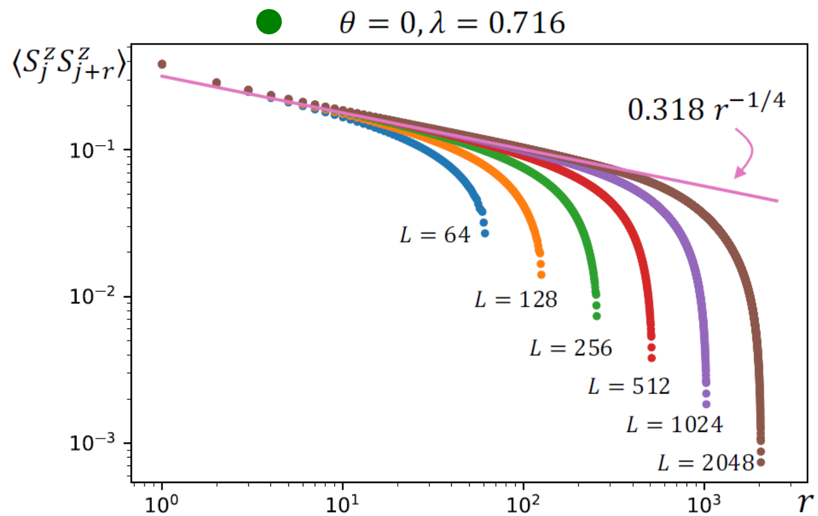
- $\mathcal{O}^y_{\mathrm{str}} > 0$ on the SPT Ising, while $\mathcal{O}^{x'}_{\mathrm{FM}} > 0$ on the trivial Ising

Critical properties



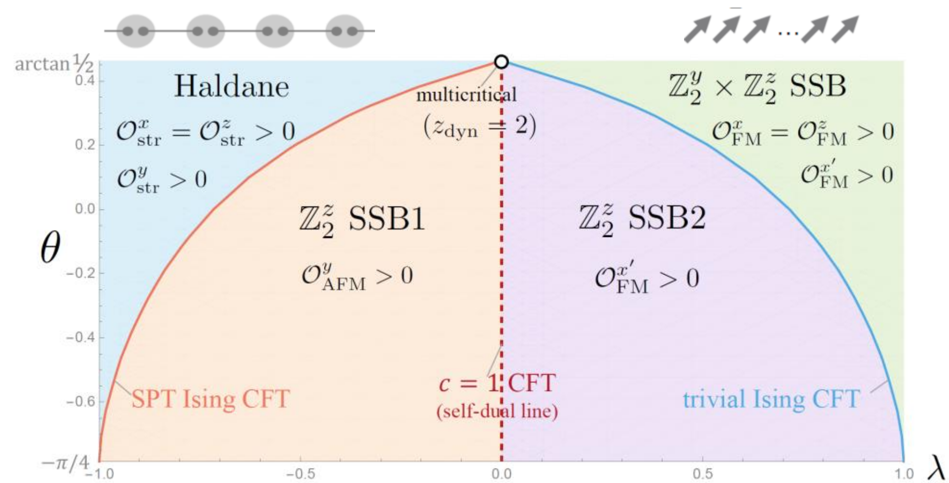






Summary

- Studied interpolating models between BLBQ chain in SPT and its Kennedy-Tasaki dual
- Rich phase diagram
- Gaussian, SPT Ising & trivial Ising criticalities



Future directions

- What about higher integer spins? $$H^{(S)}_{\rm SD} = H^{(S)}_{\rm Heis} + U_{\rm KT} H^{(S)}_{\rm Heis} U_{\rm KT}$$
  Conjecture: *S* = 1, 3, 5, … → Critical
  S = 2, 4, 6, … → Gapped & trivial

- SO(2*n*+1) generalizations?
  KT tr.: H. Tu, G-M. Zhang & T. Xiang, *PRB* **78** (2008)

- More general duality transformations?
  Lootens, Delcamp, Ortiz & Verstraete, arXiv:2112.09091