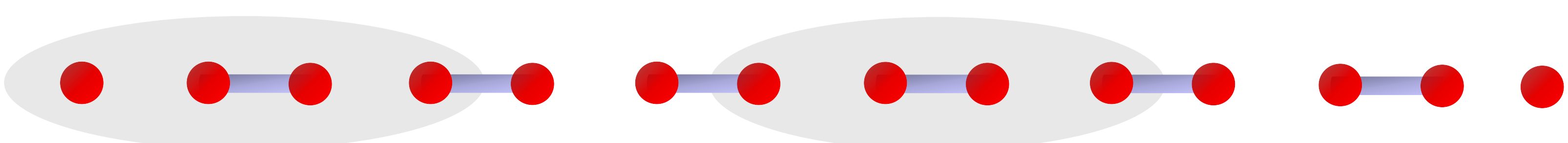
統計力学セミナー@本郷（2015/6/23）

Exact ground states and topological order in interacting Kitaev chains

Hosho Katsura (U. Tokyo)

東京大学
THE UNIVERSITY OF TOKYO

Collaborators:
Masahiro Takahashi (Gakushuin Univ.)
Dirk Schuricht (Utrecht Univ.)



Outline

1. **Introduction & Motivation**

- Majorana fermions in particle and cond-mat.
- Majorana fermions in stat. mech.

2. **Kitaev/Majorana chain**

- Trivial and topological phases, edge zero modes
- Mapping to quantum Ising chain

3. **Interacting Kitaev/Majorana chain**

- Operator inequalities, frustration-free Hamiltonian
- Exact ground states, solvable line in the phase diagram

4. **Energy gap, edge zero modes & topo. order**

- Poor man's definition of topo. order
- Proof of the gap, exact edge zero modes

Much ado about Majorana fermions

- What are Majorana fermions?

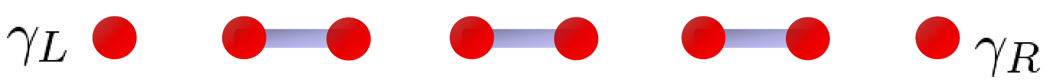
  "*Majorana returns*", F. Wilczek, *Nat. Phys*. **5** (2009).
  Particles that are own antiparticles $\gamma^\dagger = \gamma$
  Real solutions of Dirac eq. → Electrically neutral
  Neutrinos might be Majoranas (seems unlikely…)

- Emergent Majoranas in condensed matter

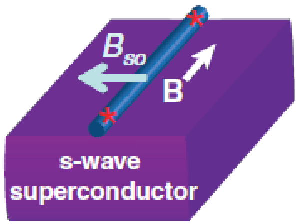
  "*Unpaired Majorana fermions in quantum wires*",
  A. Kitaev, *Phys. Usp*., cond-mat/0010440 (2000).
  Majorana fermions = edge zero modes

$$[H, \gamma_{L/R}] = 0 \qquad \{|0\rangle,\ \gamma_L|0\rangle,\ \cancel{\gamma_R|0\rangle},\ \cancel{\gamma_L\gamma_R|0\rangle}\}$$

qubit?



Experimental Signatures
Mourik *et al*., *Science* **336**, 1003 (2012).
Nadj-Perge *et al*., Science **346**, 602 (2014).



Majorana fermions in Stat. Mech.

- Kitaev chain = Quantum Ising chain

P. Fendley, *J. Stat. Mech.* P11020 (2012).

> Surprisingly, the word "**Ising**" does not appear in Kitaev's paper (except inside "Surprisingly") nor in the excellent Alicea's review article; often this chain is now referred to as the "Kitaev chain".

- History of the 2d Ising model

"*Romance of the Ising model*", B. M. McCoy, arXiv:1111.7006.

1941: Kramers-Wannier, duality → critical temperature
1941-1944: Onsager, exact free energy, Onsager algebra
1949: Kaufman, Majorana-fermion technology, Pffafian
1949-1952: Onsager, Yang, exact magnetization
1970~: McCoy-Wu, Jimbo-Miwa-Sato, …, still ongoing!

Spontaneous magnetization

$$\langle \sigma_j^z \rangle = (1-k^2)^{1/8} \qquad 1/k = \sinh(2\beta J)\sinh(2\beta J')$$

Tower of Babel

Exotic "particles" appear in several seemingly-unrelated fields.
People in these fields use different languages. ***Dictionary*** reads

Topological cond-mat.	2d stat-mech.	Math-phys.
Majorana fermions $\psi^\dagger = \psi,\ \psi^2 = 1$	**2d Ising model**	**Onsager algebra**
Parafermions $\psi^\dagger = \psi^{m-1},\ \psi^m = 1$	**Chiral Potts model**	**Yang-Baxter w/o difference property**
	superintegrable case	PLUS Onsager alg.
Fibonacci anyons $\mathbf{1} \times \tau = \tau$ $\tau \times \tau = \mathbf{1} + \tau$	**RSOS (ABF) model**	**Temperley-Lieb alg.**

Their critical points are described by 2d CFTs,
correlators of which give FQHE wavefunctions.

What people are doing these days are just a rephrasing of what have been known for many decades…?

Interacting Majorana fermions

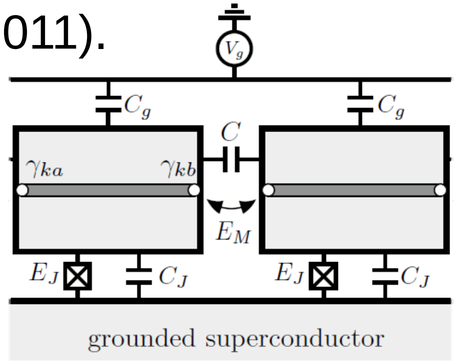
■ Motivation

Do what no one else can do! Let's *solve interacting Majorana fermions exactly*. What is topo. order in interacting systems?

If that's not motivation enough…

1. Interactions narrow or expand the topo. phase?
   Narrow) S. Gangadharaiah *et al*., *PRL* **107**, 036801 (2011).
   Expand) E.M. Stoudenmire *et al*., *PRB* **84**, 014503 (2011).

2. Physical realization of coupled Majorana wires
   Array of Josephson junctions
   F. Hassler & D. Schuricht, *New. J. Phys*. **14** (2012).



■ Dictionary again…

Via Jordan-Wigner transformation,

Kitaev chain	**Quantum Ising chain**
Interacting Kitaev chain	**XYZ chain in a field**

E-mail discussion

On May 6th, 2014:

H.K.: I found that the interacting Kitaev/Majorana chain is exactly solvable if the chemical potential is tuned to a particular function of the other parameters (t, Δ, U).

D.S.: That sounds very interesting. But did you know the **Peschel-Emery** line? [*Z. Phys. B*, **43**, 241, (1981).]

H.K.: No, I didn't. But I will look into their paper...

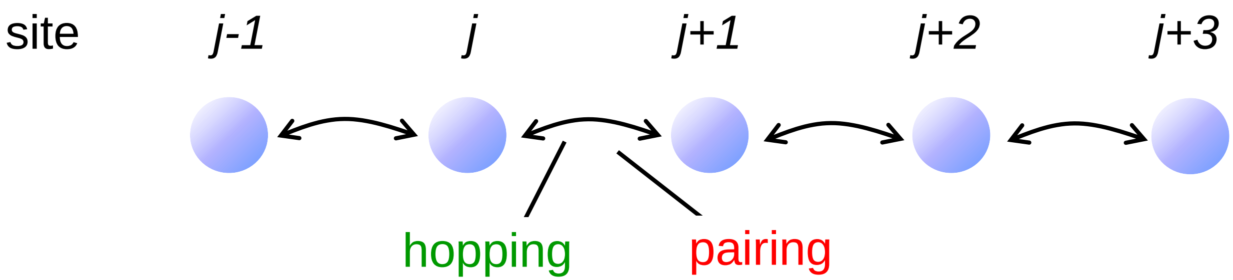
It turned out that I'm also the guy who is just rephrasing the known results in the literature…

Outline

1. **Introduction & Motivation**
   - Majorana fermions in particle and cond-mat.
   - Majorana fermions in stat. mech.
2. **Kitaev/Majorana chain**
   - Trivial and topological phases, edge zero modes
   - Mapping to quantum Ising chain
3. **Interacting Kitaev/Majorana chain**
   - Operator inequalities, frustration-free Hamiltonian
   - Exact ground states, solvable line in the phase diagram
4. **Energy gap, edge zero modes & topo. order**
   - Poor man’s definition of topo. order
   - Proof of the gap, exact edge zero modes

Kitaev/Majorana chain (non-interacting)

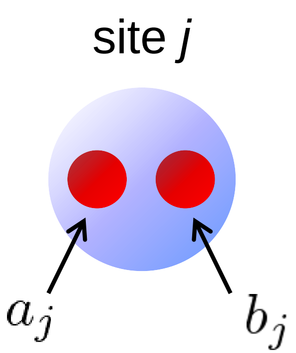
- Hamiltonian (complex fermions, with OBC)   A. Kitaev, *Phys. Usp*. (2001).



Chemical potential

$$H_0 = -w \sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + w \sum_{j=1}^{L-1} (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) - \mu \sum_{j=1}^{L-1} (c_j^\dagger c_j - 1/2)$$

- Complex fermion = pair of real (Majorana) fermions



$$\boxed{c_j = \frac{1}{2}(a_j + i b_j), \quad c_j^\dagger = \frac{1}{2}(a_j - i b_j)}$$

$$a_j = c_j + c_j^\dagger, \quad b_j = (c_j - c_j^\dagger)/i$$

$$a_j^\dagger = a_j, \quad b_j^\dagger = b_j$$

$$\{a_j,\, a_k\} = \{b_j,\, b_k\} = 2\delta_{jk}, \quad \{a_j,\, b_k\} = 0$$

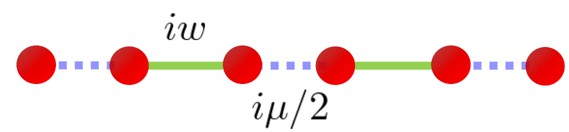
Defining relations

Which particles are fundamental and which are secondary may be a matter of interpretation. ("Nuclear democracy", G.F. Chew)

Phases in the Kitaev/Majorana chain

- Hamiltonian (Majorana fermions, with OBC)

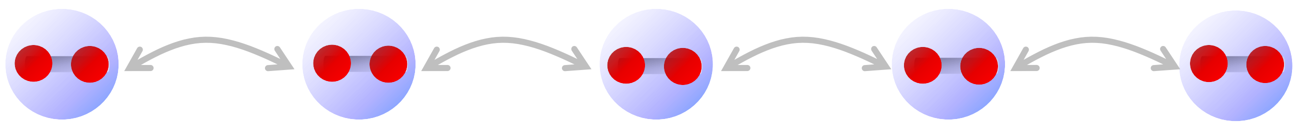
$$H_0 = iw\sum_{j=1}^{L-1} b_j a_{j+1} - \frac{i}{2}\mu\sum_{j=1}^{L} a_j b_j$$



- Trivial phase $(\, w \ll \mu \,)$

$$H_0 \sim -\frac{i}{2}\mu\sum_{j=1}^{L} a_j b_j = -\mu\sum_{j=1}^{L}(c_j^\dagger c_j - 1/2)$$

The fully filled state is the unique g.s. (μ>0)

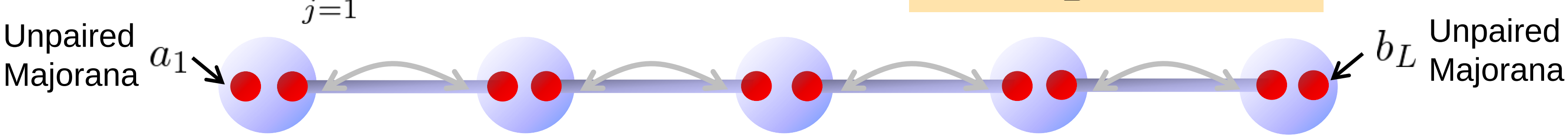


- Topological phase $(\, w \gg \mu \,)$

$$H_0 \sim iw\sum_{j=1}^{L-1} b_j a_{j+1}$$

New, non-local fermion:

$$f = \frac{1}{2}(a_1 + ib_L)$$



presence/absence of *f* ⇔ two-fold degenerate g.s. $|0\rangle, \quad |1\rangle = f^\dagger|0\rangle$

Non-local zero mode commuting with H_0 exists as long as $w>\mu$.
Topological order! Quantum phase transition occurs at $w=\mu$.

Kitaev chain = Quantum Ising chain

- Hamiltonian in terms of spins

  Jordan-Wigner transformation

$$a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^x, \qquad b_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^y$$

Ising model! $J_x = w, \ h = -\frac{\mu}{2}$

$$H_0 = iw \sum_{j=1}^{L-1} b_j a_{j+1} - \frac{i}{2}\mu \sum_{j=1}^{L} a_j b_j = -J_x \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^{L} \sigma_j^z$$

- Ground states of the spin model ($J_x >> |h|$)

$$\sigma^x |\pm\rangle = \pm|\pm\rangle \qquad \sigma^z |\pm\rangle = |\mp\rangle$$

2-fold degenerate g.s.: $|+\rangle_1 |+\rangle_2 \cdots |+\rangle_L$

$|-\rangle_1 |-\rangle_2 \cdots |-\rangle_L$

Ferromagnetically ordered in *x* direction

Order parameter

$$\mathcal{O} = \sum_{j=1}^{L} \sigma_j^x = \sum_{j=1}^{L} \exp\left[i\pi \sum_{k=1}^{j-1} n_k\right] (c_j + c_j^\dagger)$$

Local in spin variables, but non-local in fermions!

Topological order and Majorana edge zero modes

- Fermionic ($\boldsymbol{Z}_2$) Parity

  H_0 conserves fermion number mod 2, which means H_0 commutes with

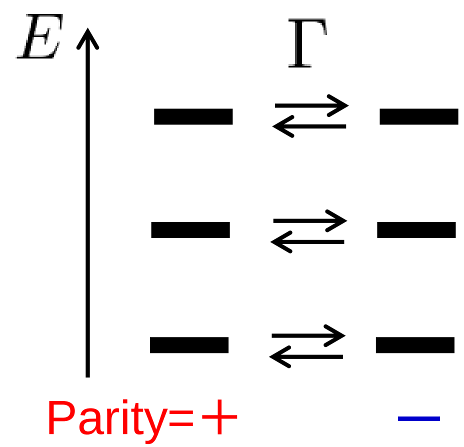
$$(-1)^F = \prod_{j=1}^{L}(-ia_j b_j) = \prod_{j=1}^{L}\sigma_j^z$$

- Characteristic of topological order

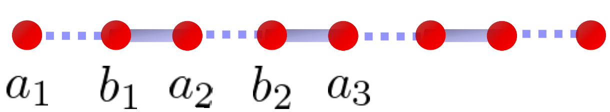
  i) Nonvanishing energy gap, ii) g.s. degeneracy (OBC), and
  iii) locally indistinguishable g.s. (in the fermionic basis)

  PLUS iv) existence of Majorana edge zero modes such that

  - $\Gamma^\dagger = \Gamma$
  - $[H_0, \Gamma] = 0$
  - $\{(-1)^F, \Gamma\} = 0$
  - localized near the edge, and normalizable as $\Gamma^2 = 1$ even in the infinite-size limit.



NOTE) i), …, iv) are not totally independent.



$$\Gamma_L \propto a_1 - \frac{\mu}{2w}a_2 + \left(\frac{\mu}{2w}\right)^2 a_3 + \cdots$$

Outline

1. Introduction & Motivation

- Majorana fermions in particle and cond-mat.
- Majorana fermions in stat. mech.

2. Kitaev/Majorana chain

- Trivial and topological phases, edge zero modes
- Mapping to quantum Ising chain

3. Interacting Kitaev/Majorana chain

- Operator inequalities, frustration-free Hamiltonian
- Exact ground states, solvable line in the phase diagram

4. Energy gap, edge zero modes & topo. order

- Poor man's definition of topo. order
- Proof of the gap, exact edge zero modes

Interacting Kitaev/Majorana chain

- Hamiltonian (complex fermions, with OBC)

$$H = -t\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta\sum_{j=1}^{L-1}(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$

4-Majorana int.

$$-\sum_{j=1}^{L}\mu_j(c_j^\dagger c_j - 1/2) + U\sum_{j=1}^{L-1}\underbrace{(2c_j^\dagger c_j - 1)(2c_{j+1}^\dagger c_{j+1} - 1)}_{-a_j b_j a_{j+1} b_{j+1}}$$

$H \rightarrow H_0$ when $t=\Delta=w$, $\mu_j=\mu$, and $U=0$. μ_j may depend on j.

- Spin Hamiltonian = XYZ chain in a magnetic field

$$H = \sum_{j=1}^{L-1}(-J_x\sigma_j^x\sigma_{j+1}^x - J_y\sigma_j^y\sigma_{j+1}^y + \underline{J_z\sigma_j^z\sigma_{j+1}^z}) + \frac{1}{2}\sum_{j=1}^{L}\mu_j\sigma_j^z$$

$$J_x = (t+\Delta)/2,\ \ J_y = (t-\Delta)/2,\ \ J_z = U$$

Symmetries: H commutes with fermionic parity $(-1)^F = \prod_{j=1}^{L}\sigma_j^z$

Integrable when $\mu_j=0$ for all j (Baxter, 1971).

Easily solvable for a particular set of μ_j. (Frustration free!!)

A crash course in *inequalities*

- Positive semidefinite operators (H. Tasaki, *PTP* **99**, 489 (1998).)

  Let $\mathcal{H}$ be a finite dimensional Hilbert space.

  Definition 1. For a hermitian matrix on $\mathcal{H}$, we write $A \geq 0$ and say *A* is positive semidefinite if we have $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}.$

  Definition 2. For two hermitian matrices *A* and *B* on $\mathcal{H}$, we write $A \geq B$ if $A - B \geq 0.$

  Lemma 1. $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.

  Lemma 2. Let *C* be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0.$

  Lemma 3. If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0.$

- Min-max theorem (Courant-Fischer-Weyl)

  Theorem. Let *A* and *B* be two hermitian matrices on $\mathcal{H}$, and let a_i and b_i be the *i*-th eigenvalues of *A* and *B*, respectively. a_i and b_i are arranged so that $a_1 \leq a_2 \leq \cdots, \ b_1 \leq b_2 \leq \cdots.$ If $A \geq B$, then we have $a_i \geq b_i, \ \forall i.$

  Ex.) For *i*=1, Theorem simply implies the variational principle.

Frustration-free Hamiltonian

- Anderson's bound (*Phys. Rev*. **83** , 1260 (1951).

Suppose Hamiltonian takes the form $H = \sum_j h_j$,
where each local h_j satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
($E_j^{(0)}$ is the lowest eigenvalue of h_j .) Then we have

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Gives a lower bound on the g.s. energy of AFM Heisenberg model.

- Frustration-free Hamiltonian

The case where the *equality* holds.

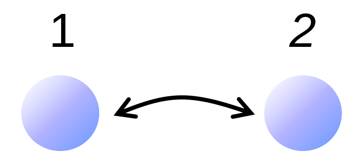
(Pseudo-)Definition. A Hamiltonian $H = \sum_j h_j$ is said to be *frustration-free* when the ground state is obtained by minimizing each term independently.

Examples include *Majumdar-Ghosh*, *AKLT*, *Kitaev's toric code*, *Shastry-Sutherland*, *Rokhsar-Kivelson* (quantum dimer), …
A similar concept, *Bogomolnyi's bound* appears in field theories.

From two to many (1)

Cook up a (frustration free) toy model from 2 sites

$$h_1 = -t(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta(c_1 c_2 + c_2^\dagger c_1^\dagger) - \frac{\mu}{2}(n_1 + n_2 - 1) + U(2n_1 - 1)(2n_2 - 1)$$



- Even and odd Hamiltonians

  h_1 commutes with fermion parity. → Even and odd sectors.

  Even subspace: $|\circ\circ\rangle := |\mathrm{vac}\rangle, \ |\bullet\bullet\rangle := c_1^\dagger c_2^\dagger |\mathrm{vac}\rangle$ $\qquad \tan\theta = 2\Delta/\mu$

$$h_{1,\mathrm{even}} = \begin{pmatrix} U + \mu/2 & -\Delta \\ -\Delta & U - \mu/2 \end{pmatrix}$$

g.s.1: $\sin\frac{\theta}{2}|\circ\circ\rangle + \cos\frac{\theta}{2}|\bullet\bullet\rangle$

$$E_{0,\mathrm{even}} = U - \sqrt{\Delta^2 + (\mu/2)^2}$$

  Odd subspace: $|\bullet\circ\rangle := c_1^\dagger|\mathrm{vac}\rangle, \ |\circ\bullet\rangle := c_2^\dagger|\mathrm{vac}\rangle$

$$h_{1,\mathrm{odd}} = \begin{pmatrix} -U & -t \\ -t & -U \end{pmatrix}$$

g.s.2: $|\bullet\circ\rangle + |\circ\bullet\rangle$

$$E_{0,\mathrm{odd}} = -U - t$$

g.s.1 and g.s.2 become degenerate if $\mu = \mu^* = 4\sqrt{U^2 + tU + \frac{t^2 - \Delta^2}{4}}$.

From two to many (2)

- Product ground states

When μ=μ*, the g.s. of h_1 can be expressed as

$$\left(\alpha^2|\circ\circ\rangle+|\bullet\bullet\rangle\right)\pm\alpha(|\bullet\circ\rangle+|\circ\bullet\rangle)=(\alpha|\circ\rangle_1\pm|\bullet\rangle_1)(\alpha|\circ\rangle_2\pm|\bullet\rangle_2)$$

g.s.1: g.s.2:

$$=(\alpha\pm c_1^\dagger)(\alpha\pm c_2^\dagger)|\mathrm{vac}\rangle. \quad \alpha^2=\tan\frac{\theta}{2}$$

Disentangled (product) states are the ground states!

L-site Hamiltonian

$$H=\sum_{j=1}^{L-1}h_j, \quad \begin{aligned} h_j=&-t(c_j^\dagger c_{j+1}+c_{j+1}^\dagger c_j)+\Delta(c_jc_{j+1}+c_{j+1}^\dagger c_j^\dagger)\\ &-\frac{\mu}{2}(n_j+n_{j+1}-1)+U(2n_j-1)(2n_{j+1}-1)\end{aligned}$$

is *frustration free* if μ=μ*, in which case the *unique* g. s. are

$$|\Psi_\pm\rangle=(\alpha\pm c_1^\dagger)(\alpha\pm c_2^\dagger)(\alpha\pm c_2^\dagger)\cdots(\alpha\pm c_L^\dagger)|\mathrm{vac}\rangle.$$

$$|\Psi_+\rangle+|\Psi_-\rangle\in\mathcal{H}_{\mathrm{even}}$$
$$|\Psi_+\rangle-|\Psi_-\rangle\in\mathcal{H}_{\mathrm{odd}}$$

NOTE) The boundary chemical potential is half the bulk one.

A fermionic rephrasing of the known results in the XYZ spin chain.

Barouch-McCoy(1971), Peschel-Emery (1981), Mueller-Schrock (1985), …

Solvable line in the phase diagram

- Phase diagram (t=Δ)

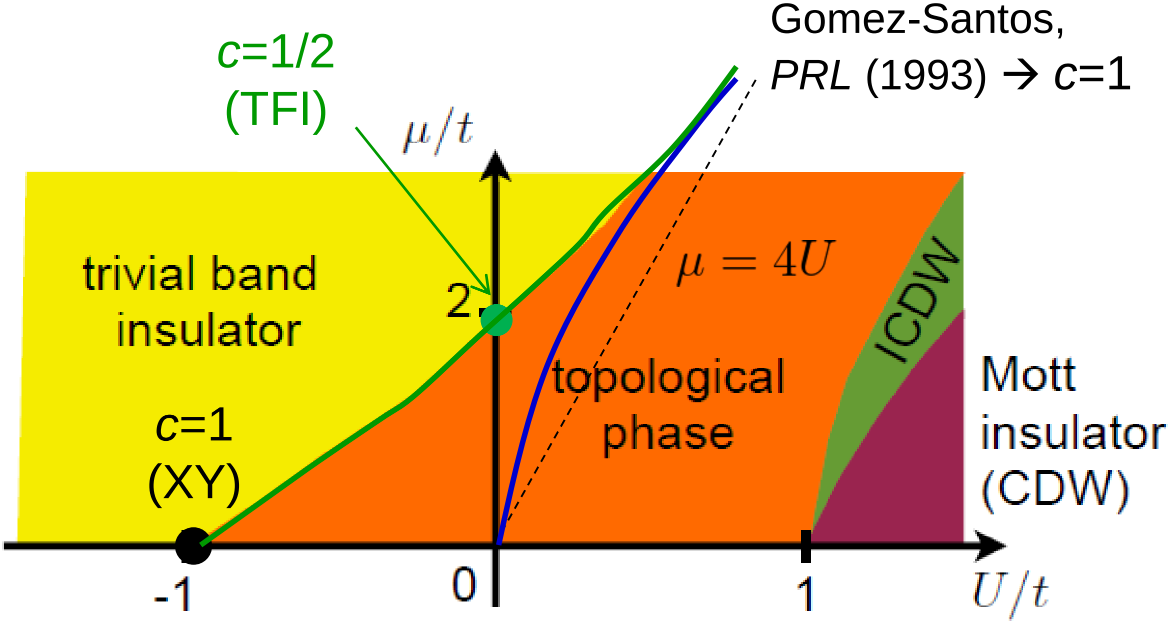
  → Quantum ANNNI model
  Beccaria *et al.*, *PRB* (2007); Sela & Pereira, *PRB* (2011); Hassler and Schuricht, *New. J. Phys*. **14** (2012).

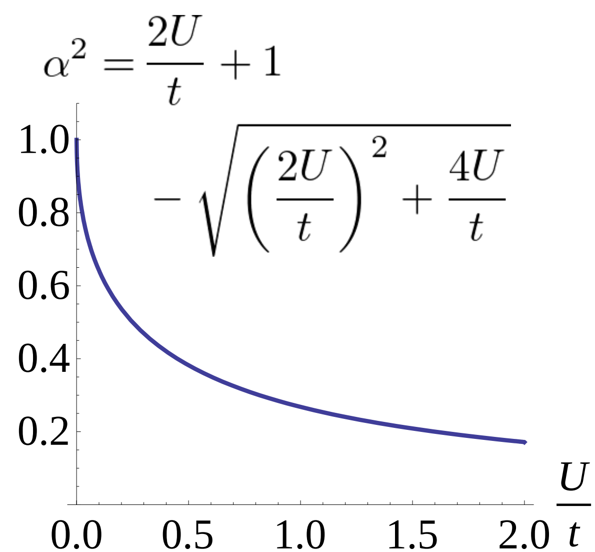
Solvable line: $\mu = \mu^* = 4\sqrt{U^2 + tU}$

Exact g.s.: $|\Psi_\pm\rangle = (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)\cdots(\alpha \pm c_L^\dagger)|\mathrm{vac}\rangle.$

$\mu = 4U \to \infty$

Gomez-Santos, *PRL* (1993) → c=1





Outline

1. **Introduction & Motivation**

- Majorana fermions in particle and cond-mat.
- Majorana fermions in stat. mech.

2. **Kitaev/Majorana chain**

- Trivial and topological phases, edge zero modes
- Mapping to quantum Ising chain

3. **Interacting Kitaev/Majorana chain**

- Operator inequalities, frustration-free Hamiltonian
- Exact ground states, solvable line in the phase diagram

4. **Energy gap, edge zero modes & topo. order**

- Poor man's definition of topo. order
- Proof of the gap, exact edge zero modes

A poor man’s definition of topological order

- Topo. order in 1d interacting Majorana fermions

Fidkowski and Kitaev, *PRB* **81**, 134509 (2010); *PRB* **83**, 075103 (2011).
$\boldsymbol{Z}$ classification reduces to $\boldsymbol{Z}_8$ one.

> **Definition (My ver.)** The interacting Kitaev chain is said to be in a **topological phase** if it can be adiabatically transformed into a non-interacting Kitaev chain in a topological phase.

We need to check …

i) Nonvanishing energy gap

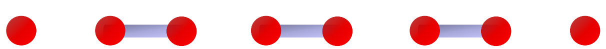
Interacting model	Non-interacting model

$$H(s) \to H(0) = H_0$$

The energy gap must be nonzero along the entire path.

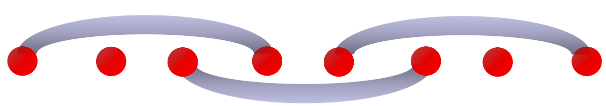
ii) topo. phase of H_0

I. Δ>0, $|\mu/2t|$<1



II. Δ<0, $|\mu/2t|$<1

$$\boxed{a_j \leftrightarrow b_j}$$



Proof of the energy gap –outline- (1)

- Hamiltonian

For the *frustration-free* case, re-parametrization yields

$$H(s,\theta)=\sum_{j=1}^{L-1} h_j(s,\theta) \qquad (0\le\theta\le\pi,\ -\theta \text{ is achieved by } c_j\to ic_j.)$$

$$h_j(s,\theta)=\left(1+\frac{s}{2}\right)-(c_j^\dagger c_{j+1}+\mathrm{h.c.})+(1+s)\sin\theta(c_jc_{j+1}+\mathrm{h.c.})$$
$$-(1+s)\cos\theta(n_j+n_{j+1}-1)+\frac{s}{2}(2n_j-1)(2n_{j+1}-1)$$

where we set *t* =1. $H(2U,\theta)=H+\mathrm{const.},\ H(0,\theta)=H_0+\mathrm{const.}$

- Positive semi-definiteness

Each local Hamiltonian h_j can be expressed as

$$h_j(s,\theta)=Q_jQ_j^\dagger+(1+s)Q_j^\dagger Q_j,$$

$$Q_j(\theta)=\frac{1}{2}\cos\frac{\theta}{2}(c_j+c_{j+1})(c_j^\dagger-c_{j+1}^\dagger)(c_j^\dagger+c_{j+1}^\dagger)$$
$$+\frac{1}{2}\sin\frac{\theta}{2}(c_j^\dagger-c_{j+1}^\dagger)(c_j+c_{j+1})(c_j-c_{j+1})$$

Local supercharge?

$$(Q_j)^2=(Q_j^\dagger)^2=0$$

Clearly, $H(s,\theta)\ge 0$ for *s*≧0. (Remember Lemma 2 & 3.)

Proof of the energy gap –outline- (2)

- Ground states

$$H(s,\theta)=\sum_{j=1}^{L-1}[Q_jQ_j^\dagger+(1+s)Q_j^\dagger Q_j] \qquad Q_j|\Psi_\pm\rangle=Q_j^\dagger|\Psi_\pm\rangle=0,\ \forall j.$$

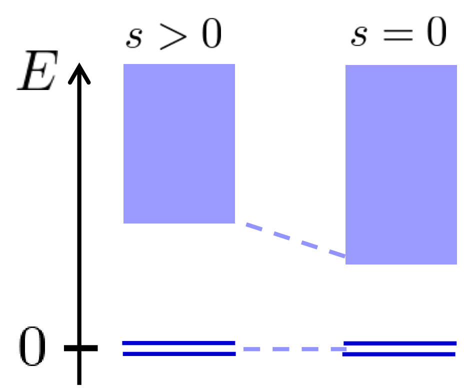
$$|\Psi_\pm\rangle=(\alpha\pm c_1^\dagger)(\alpha\pm c_2^\dagger)\cdots(\alpha\pm c_L^\dagger)|\mathrm{vac}\rangle. \qquad \alpha^2=\tan\frac{\theta}{2}$$

H(2*U*, *θ*) (interacting) and *H* (0, *θ*) (non-interacting) share the same ground states! *H* (0, *θ*) is like a *Kohn-Sham* system.

- Uniqueness of g.s. & existence of an energy gap

For *s*>0, we have

$$H(s,\theta)-H(0,\theta)=s\sum_{j=1}^{L-1}Q_j^\dagger Q_j\geq 0.$$



From the min-max theorem, (i) the g.s. of *H*(2*U*, *θ*) are unique if those of *H* (0, *θ*) are unique, (ii) *gap of H*(2*U*, *θ*) ≧ gap of *H* (0, *θ*).

We still need to solve the non-interacting model H (0, θ) …

Spectrum of the non-interacting model: *H* (0, *θ*)

- Single-particle spectrum

$$H(0,\theta)+\text{const.}=\frac{i}{2}\sum_{j,k=1}^{L}B_{j,k}\,a_j b_k\;=\sum_{k=1}^{L}\epsilon_k\left(f_k^\dagger f_k-\frac{1}{2}\right)$$

A real matrix *B*, may not be diagonalizable, but can be written in the SVD form: $B=U\Lambda V^{\mathrm{T}},\ \Lambda=\mathrm{diag}(\epsilon_1,\dots,\epsilon_L)$, where $U,\,V\in\mathrm{O}(L)$.
Λ can be obtained by diagonalizing BB^{T} or $B^{\mathrm{T}}B$.

- Miraculous properties of *H*(0, *θ*)

Special factorization

$$BB^{\mathrm{T}}=C^2$$

$$C=\begin{pmatrix}\boxed{1-\mathfrak{s}} & \mathfrak{c} & & & \\ \mathfrak{c} & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & \mathfrak{c} \\ & & & \mathfrak{c} & \boxed{1+\mathfrak{s}}\end{pmatrix}$$

$\mathfrak{s}=\sin\theta$

$\mathfrak{c}=\cos\theta$

BB^T is pentadiagonal, but
C is tridiagonal & symmetric!

Exact eigenvalues of *C*

$$0\quad\text{and}\quad 2+2\mathfrak{c}\cos\left(\frac{n\pi}{L}\right),\qquad n=1,2,\dots,L-1$$

Zero-energy state! Topological order!!

Existence of many-body gap

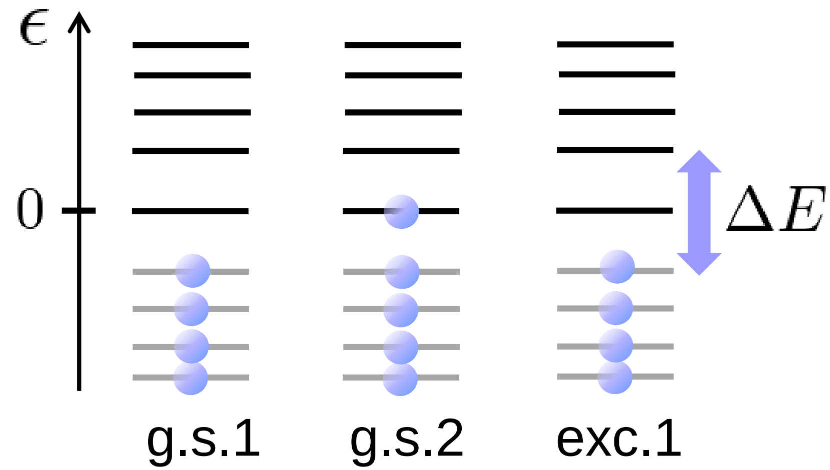
- Many-body eigenstates of $H(0, \theta)$

  I. two g.s. with opposite parities.
  (Unique in each parity sector.)

  II. Many-body gap ($\theta \neq 0, \pi$)

  $$\Delta E \geq 2(1 - |\cos\theta|)$$

  is nonzero in the infinite-L limit.

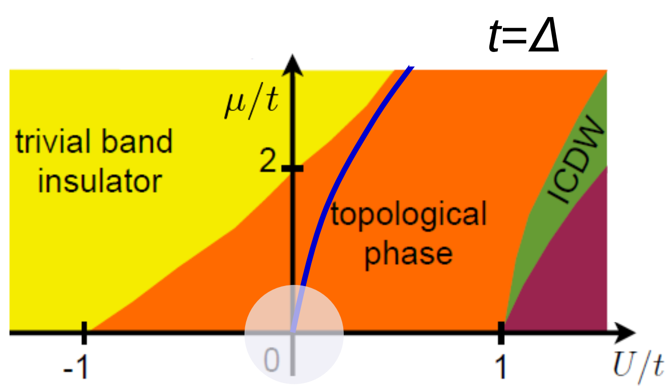


Properties I & II also hold for $H(2U, \theta)$.
(From $H(2U, \theta) \geqq H(0, \theta)$ & min-max theorem.)

> $H(2U, \theta)$ is adiabatically connected to $H(0, \theta)$ which is in a topological phase! The gap does not close along the path.

- Stability away from frustration-free line?
  1. Kato's theorem ($||V|| < \Delta E/2$)
  2. Cluster expansion
  3. Lieb-Robinson bound

  Main difficulty: open boundaries …



Majorana edge zero modes

- Edge zero mode of $H(0,\theta)$ ⇔ *Left* or *Right* null vector of B

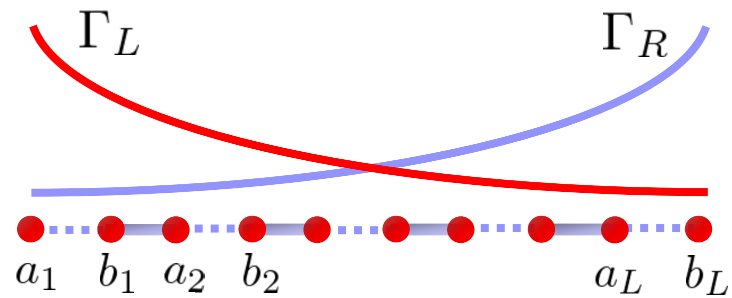
$$B = -\begin{pmatrix} \mathfrak{c} & 1-\mathfrak{s} & & & \\ 1+\mathfrak{s} & 2\mathfrak{c} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2\mathfrak{c} & 1-\mathfrak{s} \\ & & & 1+\mathfrak{s} & \mathfrak{c} \end{pmatrix}$$

$$(1,\, r,\, r^2,\, \cdots,\, r^{L-1})B = 0$$

$$r = -\frac{\mathfrak{c}}{1+\mathfrak{s}}, \quad \mathfrak{s}^2 + \mathfrak{c}^2 = 1$$

$$\Gamma_L = \mathcal{N}(\, a_1 + r a_2 + \cdots r^{L-1} a_L\,)$$
$$\Gamma_R = \mathcal{N}(\, b_L + r b_{L-1} + \cdots r^{L-1} b_1\,)$$

Exact zero modes even for finite L.



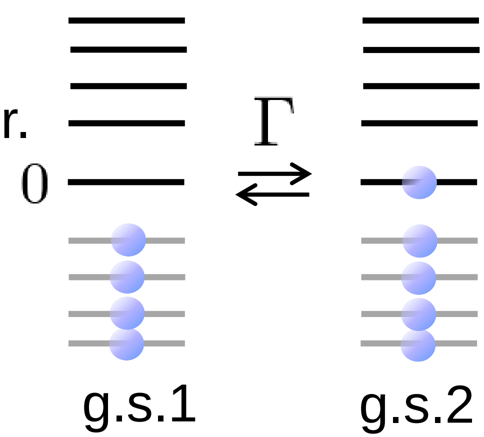
- Role of Γ_L *and* Γ_R in $H(2U,\theta)$

They do not exactly commute with $H(2U,\theta)$.
Nevertheless, they map one of the g.s. to the other.

A significant overlap with many-body Majorana?

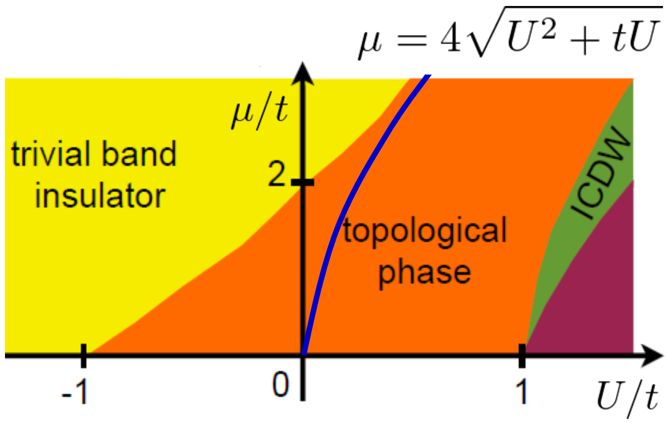
$$\Gamma^{(\text{adiab})}_{L/R} := W(s)\Gamma_{L/R}W(s)^\dagger, \quad W(s) = \sum_n |\psi_n(s)\rangle\langle\psi_n(0)|$$

M.B. Hastings & X-G. Wen, *PRB* **72**, 045141 ('05).



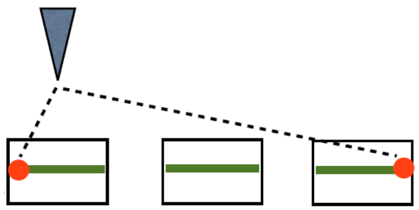
Conclusions

- Studied effect of interactions on Kitaev/Majorana chains
- Solvable (frustration-free) line
- Exact ground states and proof of the gap
- Exact solution of the BdG equation
- Topological order and edge zero modes



Future directions

- Spectral function & tunneling conductance
- Many-body edge zero modes
  1. Superoperator technique: $[H, v_\mu] = \mathcal{H}_{\mu\nu} v_\nu$
     Find "zero-energy state" in the space of operators
  2. Continuum limit: Sine-Gordon, boundary bound state, …
     Ghoshal-Zamolodchikov (1994); Schuricht *et al.*, *PRB* **83** (2011).



- Parafermionic generalizations $\psi^m = 1 \qquad \psi_i\psi_j = \omega\psi_j\psi_i \quad (i < j)$
  Frustration-free Hamiltonian? parafermionic zero-modes?