Onsager's scars in non-integrable spin chains

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Acknowledgment:
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- N. Shibata, H.K., and N. Yoshioka, *Phys. Rev. Lett*. **124**, 180604 (2020)

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Outline

1. Introduction and Motivation

- Eigenstate thermalization hypothesis (ETH)
- Violation of ETH
- Experiment on Rydberg-atom array
- Quantum many-body scars (QMBS)

2. Models with exact QMBS

3. Results and generalizations

4. Summary

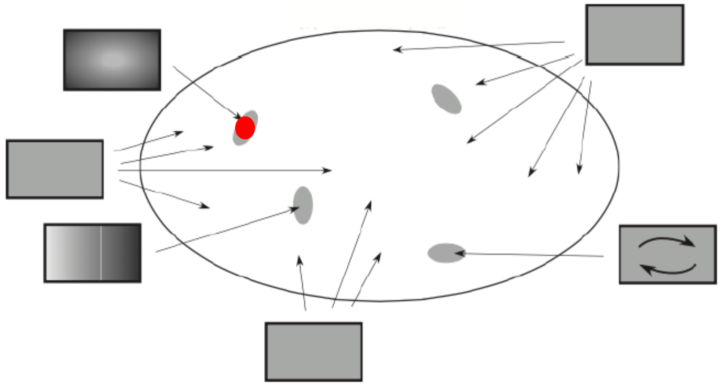
Foundation of equilibrium stat-mech

An isolated macro classical/quantum system relaxes towards a steady state at late times.

- Typicality

A great majority of states with the same energy are indistinguishable by macroscopic observables!

H. Tasaki, J. Stat. Phys. **163** (2016) and his book

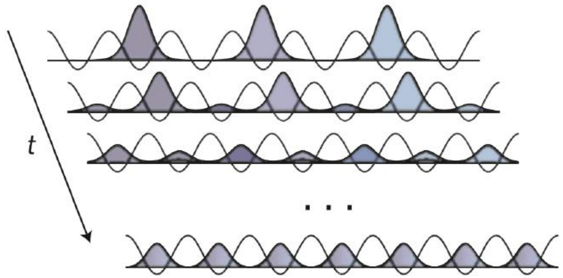


"thermal equilibrium"
= common properties shared by the majority of states

→ Microcanonical(MC) ensemble works!

- Thermalization
  The approach to these typical states

1d Bose-Hubbard, ^{87}Rb



- Experiments and numerics
  S. Trotzky *et al*., Nat. Phys. **8** (2012)
  M. Rigol *et al*., Nature **452** (2008), …

Eigenstate thermalization hypothesis (ETH)

- Setup

  H: Hamiltonian, $|E_n>$: (normalized) energy eigenstate,
  O: macroscopic observable, ρ_{mc}: MC ensemble,
  Energy shell: $\mathrm{span}\{|E_n\rangle : H|E_n\rangle = E_n|E_n\rangle, E_n \in [E-\Delta E, E)\}$

- Thermal states

  A state $|E_n>$ is said to be thermal if $\langle E_n|O|E_n\rangle \simeq \mathrm{Tr}[\rho_{\mathrm{mc}}O]$.

- Strong ETH: All $|E_n>$ in the energy shell are thermal.

  Believed to be true for a large class of non-integrable systems

  Concept: von Neumann, Deutsch, Srednicki, Tasaki, …
  Numerical evidence: D'Alessio et al., Adv. Phys. **65** (2016).

- Weak ETH: Almost all $|E_n>$ in the energy shell are thermal.

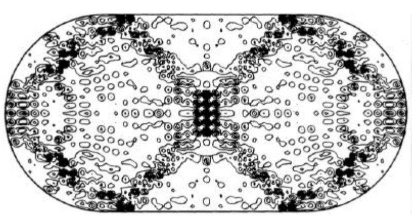
  Proved under certain conditions: translational sym., local interaction
  Biroli, Kollath, Lauchli, PRL **105** (2010), Iyoda, Kaneko, Sagawa, PRL **119** (2017)

Violation of ETH

- Exceptions of strong ETH
  1. Integrable systems
     Many conserved charges
     Strong ETH , Weak ETH
  2. Many-body localized (MBL) systems
     Emergent local integrals of motion
     Strong ETH , Weak ETH
  3. Quantum many-body scarred (QMBS) systems
     Strong ETH , Weak ETH
     Non-integrable but have *scarred* states which
     do not thermalize for an anomalously long time!

- A very nice blog article

  "**Quantum Machine Appears to Defy Universe's Push for Disorder**",
  Marcus Woo, Quanta magazine, March 2019.



1-particle wavefunction in a Bunimovich stadium
E. Heller, PRL **53** (1984)

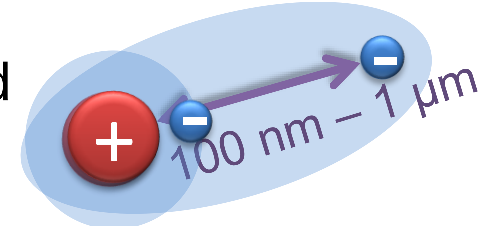
Experiment on a Rydberg atom chain

Bernien *et al.*, Nature **551** (2017)

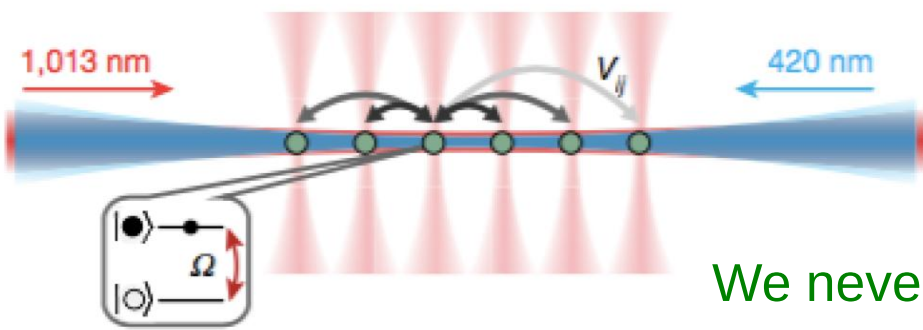
- Rydberg atoms

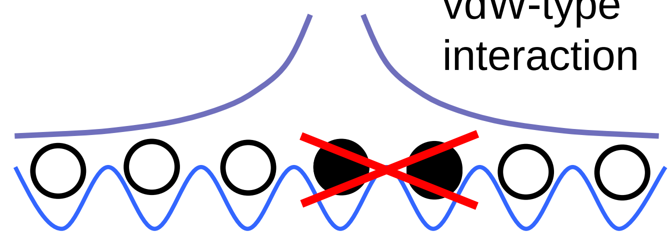
  Atoms in which one of the electrons is in an excited state with a very high principal quantum number.

^{87}Rb: el. in 5s → 70s



- Rydberg blockade





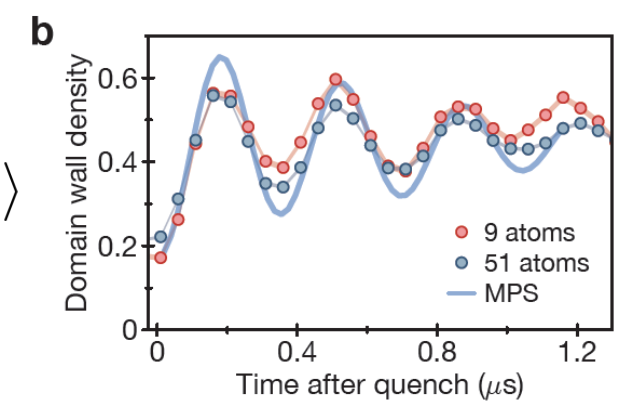
We never have adjacent excited states

- A surprising finding!

  Special initial states

$$|\mathbf{Z}_2\rangle = |\bullet\circ\bullet\circ\cdots\rangle,\ \ |\mathbf{Z}_2'\rangle = |\circ\bullet\circ\bullet\cdots\rangle$$

  Exhibit robust oscillations. Other initial states thermalize much more rapidly.



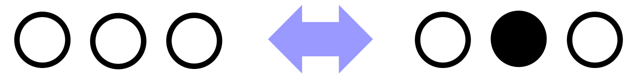
PXP model

- Hamiltonian

Turner *et al*., Nat. Phys. **14**, 745 (2018)

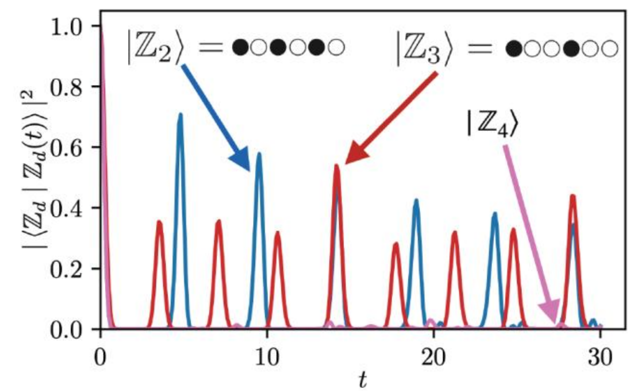
$$H_{\rm PXP} = \sum_j P_{j-1} X_j P_{j+1},$$

$$P = |\circ\rangle\langle\circ|,\ X = |\circ\rangle\langle\bullet| + |\bullet\rangle\langle\circ|$$



- Properties
  1. Level statistics
     → Wigner-Dyson, non-integrable
  2. Long-time oscillations are observed
  3. Energy (E) v.s. entanglement entropy (S)
     → Anomalously low S at high E

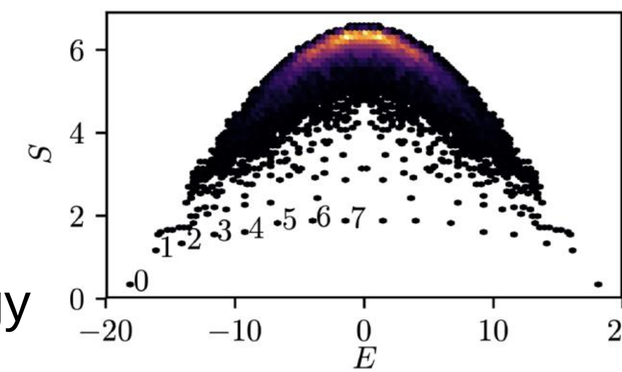
Revivals of fidelity



- Exact QMBS

Lin and Motrunich, PRL **122**, 173401 (2019)

Exact eigenstates of $H_{\rm PXP}$ in the form of matrix product states (MPS)
→ Low entanglement states at high energy



Exact QMBS

- Embedding method

  Shiraishi, Mori, PRL **119**, 030601 (2017)

- AKLT models

  Moudgalya, Regnault, Bernevig, PRB **98**, 235156 (2018)
  Mark, Lin, Motrunich, PRB **101**, 195131 (2020)

- Ising and XY-like models

  Iadecola, Schecter, PRB **101**, 024306 (2020)
  Chattopadhyay, Pichler, Lukin, Ho, PRB **101**, 174308 (2020)

- Floquet scars

  Driven PXP: Sugiura, Kuwahara, Saito, arXiv:1911.06092

■ Today's subject

- A new class of exact QMBS via Onsager algebra
- Spin-S and the interaction range can be arbitrary
- Models allow for spatially varying couplings (disorder)

Outline

1. Introduction and Motivation

2. Models with exact QMBS

- Exactly solvable models
- Shiraishi-Mori method
- More algebraic approach
- Example: perturbed S=1/2 XY chain

3. Results and generalizations

4. Summary

Exactly solvable models

■ (Crude) Classification

- Integrable systems

  Free fermions/bosons, Bethe ansatz

  Many conserved charges

Not exclusive!

- Frustration-free systems

  Ground state (g.s.) minimizes each local Hamiltonian

  Explicit g.s., but hard to obtain excited states

  (A few exact excited states in AKLT chains)

■ Heisenberg Hamiltonian

$$\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin op. on *j*-th site: $S_j^\alpha = \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{j-1} \otimes S_j^\alpha \otimes \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{L-j}$

$$H = -\sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z)$$

Eigenstates take the Bethe-ansatz form (1931)

A crash course in inequalities

■ Positive semidefinite operators

Appendix in H.Tasaki, *Prog. Theor. Phys.* **99**, 489 (1998).

$\mathcal{H}$: finite-dimensional Hilbert space.

A, *B*: Hermitian operators on $\mathcal{H}$

- **Definition 1.** We write $A \geq 0$ and say *A* is positive semidefinite (p.s.d.) if $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}.$
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0.$

■ Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0.$
  Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0.$

Frustration-free systems

■ Anderson's bound

Phys. Rev. **83**, 1260 (1951)

- Total Hamiltonian: $H = \sum_j h_j$
- Sub-Hamiltonian: h_j that satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
  ($E_j^{(0)}$ is the lowest eigenvalue of h_j)

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Used to obtain a lower bound on the g.s. energy of AFM Heisenberg model

■ Frustration-free Hamiltonian

The case where the *equality* holds.

Definition. $H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j |\psi\rangle = E_j^{(0)} |\psi\rangle$ for all *j*.

Ex.) *S*=1 Affleck-Kennedy-Lieb-Tasaki (AKLT), toric code

$$H = \sum_j h_j, \quad h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2$$

(Generalized) Shiraishi-Mori

■ Universal form of frustration-free system

- Hamiltonian $$H = \sum_j A_j^\dagger A_j \geq 0$$
- Zero-energy g.s.

$$|\psi\rangle \text{ s.t. } A_j|\psi\rangle = 0, \forall j.$$

Can we cook up a model with exact/explicit excited states? YES!

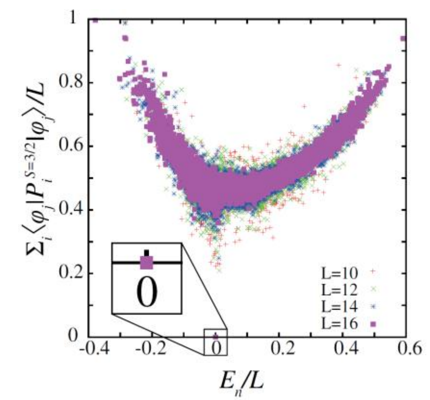
■ Embedding method

- New Hamiltonian $$H_{\text{new}} = \sum_j A_j^\dagger C_j A_j, \quad (C_j : \text{Hermitian})$$

  The g.s. of H remain the g.s. of H_{new} if $C_j \geq 0.$
  They become higher energy states when C_j are not p.s.d.

- Shiraishi-Mori PRL **119**, 030601 (2017)
  Particular case where $A_j = A_j^\dagger = P_j$ (projection).

- Further generalization
  Witten's conjugation: $A_j \to \tilde{A}_j = M A_j M^{-1}$



More algebraic approach

- Strategy

1. Starting point:
   Integrable model with conserved charges $Q_1, Q_2, \ldots$
   They commute with the Hamiltonian H_{int}

2. Take a subalgebra $\{Q_1, Q_2, \ldots\}$

3. Find a reference eigenstate $H_{\mathrm{int}}|\psi_0\rangle = E_0|\psi_0\rangle$
   ψ_0: simple state, e.g., product state or MPS

4. Find a tower of eigenstates generated by acting with the subalgebra on the reference state:

   $(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle$ ← QMBS in non-integrable *H*

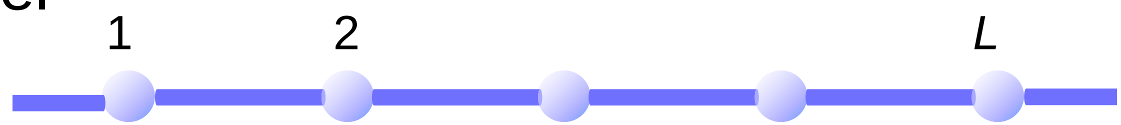
   They have the same energy with ψ_0

5. Add to H_{int} perturbations that break the integrability but leave the tower of states unchanged

   $H = H_{\mathrm{int}} + H_{\mathrm{pert}},$ e.g., $H_{\mathrm{pert}}\,(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle = 0$

Example: *S*=1/2 XY chain

■ Model



S=1/2 at each site
L: even
PBC imposed

- Hamiltonian

$$H_{\mathrm{int}} = \sum_{j=1}^{L} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \qquad S_j^{\pm} := \frac{S_j^x \pm \mathrm{i} S_j^y}{2}$$

Can be mapped to free fermions via Jordan-Wigner tr.

Lieb-Schultz-Mattis (1961), Katsura (1962)

- Conserved charges

Total S^z: $Q = \sum_{j=1}^{L} S_j^z$

Other charges are not very obvious in the spin basis...

"bi-magnon" operator: $Q^{\pm} = \sum_{j=1}^{L} (-1)^{j+1} S_j^{\pm} S_{j+1}^{\pm}, \quad [H_{\mathrm{int}}, Q^{\pm}] = 0$

- Reference eigenstate

All down state: $|\Downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \quad H_{\mathrm{int}} |\Downarrow\rangle = 0$

E=0 is in the middle of the spectrum

Tower of exact eigenstates

- Eigenstates with fixed total S^z

$$|\Downarrow\rangle,\ Q^+|\Downarrow\rangle, \dots, (Q^+)^k|\Downarrow\rangle, \dots, (Q^+)^{L/2}|\Downarrow\rangle \qquad (\,(Q^+)^{L/2+1} = 0\,)$$

- "Coherent state" $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle = \sum_{k=0}^{L/2} \frac{\beta^{2k}}{k!}(Q^+)^k|\Downarrow\rangle$

- Matrix-product operator (MPO)

$$\begin{aligned}\exp(\beta^2 Q^+) &= \exp(\beta^2 S_1^+ S_2^+)\exp(-\beta^2 S_2^+ S_3^+)\cdots\exp(-\beta^2 S_L^+ S_1^+)\\ &= (1+\beta^2 S_1^+ S_2^+)(1-\beta^2 S_2^+ S_3^+)\cdots(1-\beta^2 S_L^+ S_1^+)\\ &= (1, \beta S_1^+)\begin{pmatrix}1\\ \beta S_2^+\end{pmatrix}(1,-\beta S_2^+)\begin{pmatrix}1\\ \beta S_3^+\end{pmatrix}\cdots(1,-\beta S_L^+)\begin{pmatrix}1\\ \beta S_1^+\end{pmatrix}\\ &= \mathrm{Tr}\left[\begin{pmatrix}1 & \beta S_1^+\\ \beta S_1^+ & 0\end{pmatrix}\begin{pmatrix}1 & -\beta S_2^+\\ \beta S_2^+ & 0\end{pmatrix}\cdots\begin{pmatrix}1 & -\beta S_L^+\\ \beta S_L^+ & 0\end{pmatrix}\right]\end{aligned}$$

- Coherent state = MPS with bond dimension 2

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[\begin{pmatrix}|\downarrow\rangle_1 & \beta|\uparrow\rangle_1\\ \beta|\uparrow\rangle_1 & 0\end{pmatrix}\begin{pmatrix}|\downarrow\rangle_2 & -\beta|\uparrow\rangle_2\\ \beta|\uparrow\rangle_2 & 0\end{pmatrix}\cdots\begin{pmatrix}|\downarrow\rangle_L & -\beta|\uparrow\rangle_L\\ \beta|\uparrow\rangle_L & 0\end{pmatrix}\right]$$

Telescoping trick

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[M_1 \cdots M_j M_{j+1} \cdots M_L\right], \qquad M_j = \begin{pmatrix} |{\downarrow}\rangle_j & (-1)^{j+1}\beta|{\uparrow}\rangle_j \\ \beta|{\uparrow}\rangle_j & 0 \end{pmatrix}$$

$$(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) M_j M_{j+1} = L_j M_{j+1} - M_j L_{j+1}, \qquad L_j = \begin{pmatrix} 0 & 0 \\ 0 & |{\downarrow}\rangle_j \end{pmatrix}$$

- Proves $H_{\mathrm{int}}|\psi(\beta)\rangle = 0$. $[H_{\mathrm{int}}, Q^+] = 0$ isn't so important (?)
- A similar idea was used in Baxter's solution of XYZ chain
- Also in stochastic integrable models ("hat relation")

Possible perturbations

$$M_{2k-1}M_{2k}M_{2k+1} = \begin{pmatrix} |{\downarrow\downarrow\downarrow}\rangle - \beta^2(|{\downarrow\uparrow\uparrow}\rangle - |{\uparrow\uparrow\downarrow}\rangle) & \beta|{\downarrow\downarrow\uparrow}\rangle + \beta^3|{\uparrow\uparrow\uparrow}\rangle \\ \beta|{\uparrow\downarrow\downarrow}\rangle - \beta^3|{\uparrow\uparrow\uparrow}\rangle & \beta^2|{\uparrow\downarrow\uparrow}\rangle \end{pmatrix}_{2k-1,2k,2k+1}$$

- We never have $|{\downarrow\uparrow\downarrow}\rangle$ or $(|{\downarrow\uparrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow}\rangle)/\sqrt{2}$ in any three consecutive sites
- Identify Hermitian operators that annihilate $|\psi(\beta)\rangle$

$$H_{\mathrm{pert}}|\psi(\beta)\rangle = 0$$

Couplings can be random!

$$H_{\mathrm{pert}} = \sum_{j=1}^{L} (c_j^{(1)} |{\downarrow\uparrow\downarrow}\rangle\langle{\downarrow\uparrow\downarrow}| + \frac{c_j^{(2)}}{2}(|{\downarrow\uparrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow}\rangle)(\langle{\downarrow\uparrow\uparrow}| + \langle{\uparrow\uparrow\downarrow}|) + c_j^{(3)}[|{\downarrow\uparrow\downarrow}\rangle(\langle{\downarrow\uparrow\uparrow}| + \langle{\uparrow\uparrow\downarrow}|) + \mathrm{h.c.}])$$

Outline

1. Introduction and Motivation

2. Models with exact QMBS

3. Results and generalizations

- Level statistics → Non-integrability
- Entanglement and dynamics
- Longer-range extensions
- Higher-spin generalizations

4. Summary

Is the perturbed model non-integrable?

■ Level-spacing statistics

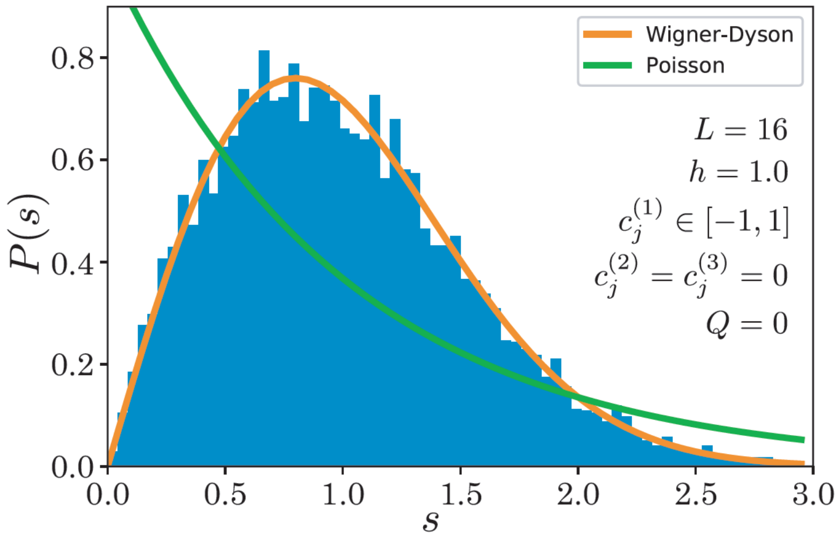
- Perturbed Hamiltonian $H = H_{\text{int}} + H_{\text{pert}} + hQ,$
- Energy levels $E_1 \leq E_2 \leq E_3 \leq \cdots \qquad \Delta E_i = E_{i+1} - E_i$
- Level spacing $s_i := \frac{\Delta E_i}{\langle \Delta E_i \rangle} \qquad \langle \Delta E_i \rangle : \text{ average}$
- *H* is integrable
  → Poisson distribution, PDF: $P(s) = \exp(-s)$
- *H* is non-integrable (GOE)
  → Wigner-Dyson distribution, $P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi s^2}{4}\right)$

Casati *et al*, PRL **54** (1985),
Pal, Huse, PRB **82** (2010)

■ Numerical result

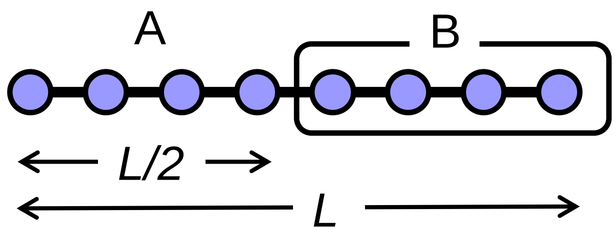
- System size: *L*=16
- Only diagonal perturbation
- Zero magnetization sector

Clearly Wigner-Dyson!
H is non-integrable!



Entanglement diagnosis

■ Half-chain entanglement



- Reduced density matrix

$$\rho = |\psi\rangle\langle\psi|, \quad \rho_A = \mathrm{Tr}_B[\rho]$$

- Entanglement entropy (EE) $\mathcal{S}_A = -\mathrm{Tr}_A[\rho_A \ln \rho_A]$

- Thermodynamic entropy ~ EE Mori *et al*., J. Phys. B **51**, 112001 (2018)

  Volume law $\mathcal{S}_A \propto L$ → Thermal

  Sub-volume law → non-thermal (including area law $\mathcal{S}_A \leq \mathrm{const.}$)

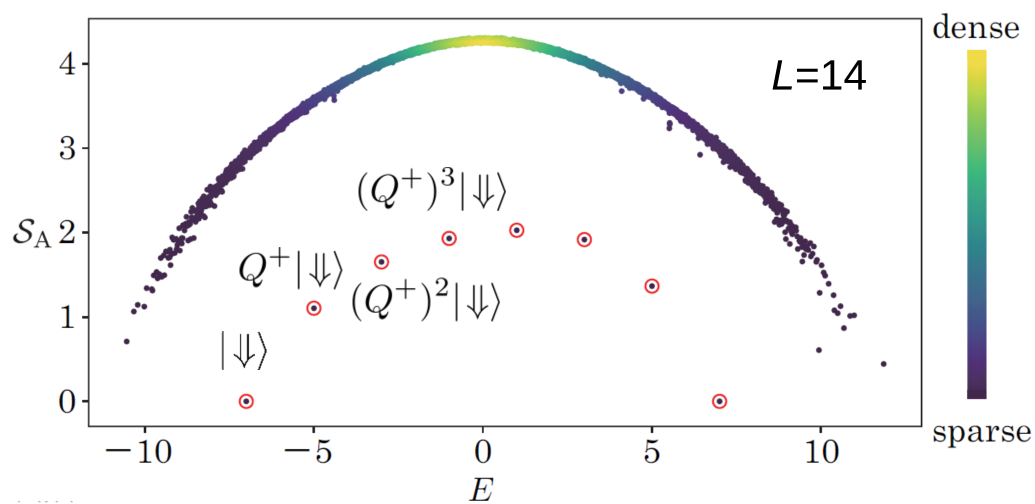
■ Results

- Coherent state $|\psi(\beta)\rangle$

  MPS, area-law EE

- Energy eigenstates?

$$H = H_{\mathrm{int}} + H_{\mathrm{pert}} + Q,$$
$$c_j^{(i)} \in [-1, 1]$$



QMBS: $(Q^+)^k|\Downarrow\rangle$

Rigorous result: EE of QMBS ≦ O (ln *L*)

Dynamics

■ Initial state = coherent state

- Hamiltonian $H = H_{\text{int}} + H_{\text{pert}} + hQ,$
- Coherent state $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle$
- Time evolution

$$|\psi_t(\beta)\rangle = \exp(-\mathrm{i}Ht)|\psi(\beta)\rangle \propto |\psi(\beta e^{-\mathrm{i}ht})\rangle$$

Revival at

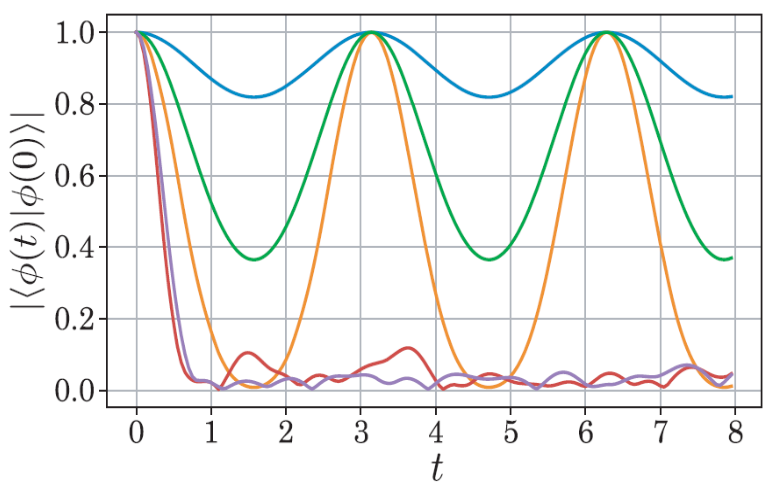
$$t = t_k = \frac{\pi k}{h}, \quad k \in \mathbb{N}$$

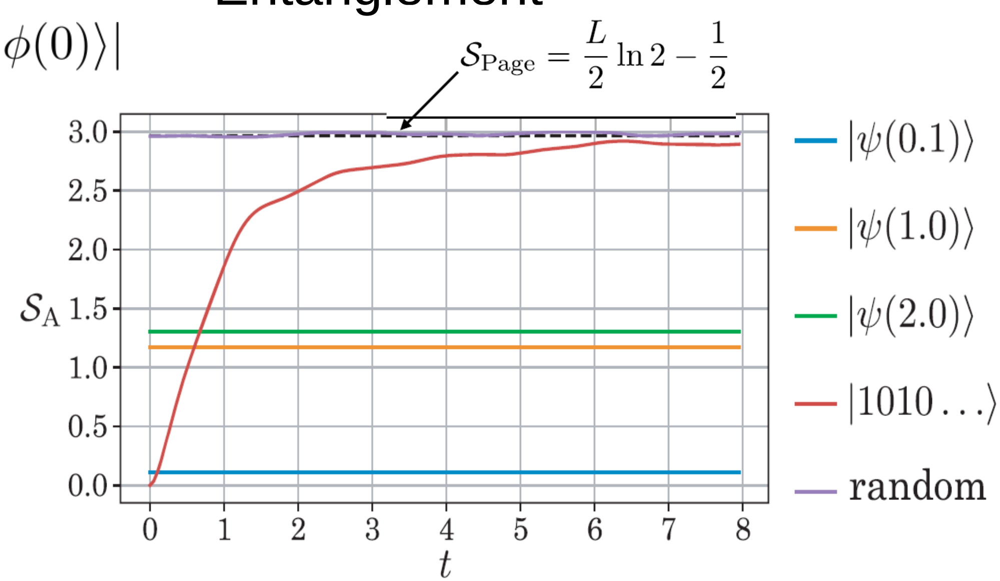
■ Numerical results $L = 10,\ h = 1.0, c_j^{(i)} \in [-1, 1]$ (random)

- Fidelity

$$|\langle\phi(t)|\phi(0)\rangle| = |\langle\phi(0)|\exp(\mathrm{i}Ht)|\phi(0)\rangle|$$

- Entanglement





Onsager algebra

- Hamiltonian $H_2 = \mathrm{i}\sum_{j=1}^{L}(S_j^+ S_{j+1}^- - S_j^- S_{j+1}^+)$ Unitarily equivalent to H_{int}
- Commuting operators

$$Q = \sum_{j=1}^{L} S_j^z, \quad \hat{Q} = 2\sum_{j=1}^{L} S_j^x S_{j+1}^x$$

(Quantum) Ising!

$$H_{\mathrm{QI}} = Q + \lambda\hat{Q}$$

Phys. Rev. 65 (1944)

$$[H_2, Q] = [H_2, \hat{Q}] = 0$$

Any polynomial in $Q, \ \hat{Q}$ commutes with H_2

- Dolan-Grady relation

$$[Q,[Q,[Q,\hat{Q}]]] = 4[Q,\hat{Q}]$$

$$[\hat{Q},[\hat{Q},[\hat{Q},Q]]] = 4[\hat{Q},Q]$$

$$Q = Q_0^0/2, \quad \hat{Q} = (Q_1^0 + Q_1^+ + Q_1^-)/2$$

$$Q_1^0 \propto H_{\mathrm{int}}, \quad Q_1^\pm \propto \sum_{j=1}^{L} S_j^\pm S_{j+1}^\pm$$

- Defining relations of algebra

$$[Q_l^r, Q_m^r] = 0 \quad (r = 0, \pm)$$

$$[Q_l^-, Q_m^+] = Q_{m+l}^0 - Q_{m-l}^0$$

$$[Q_l^\pm, Q_m^0] = \mp 2(Q_{m+l}^\pm - Q_{m-l}^\pm)$$

All Q_m^r commute with H_2

$$Q_m^+ \propto \sum_{j=1}^{L} S_j^+ S_{j+1}^z \cdots S_{j+m-1}^z S_{j+m}^+$$

Allows for scarred models with longer-range interactions!

What about S >1/2 ?

■ Self-dual U(1)-invariant clock model

Vernier, O'Brien, Fendley, J. Stat. Mech. (2019)

- Matrices $\omega = \exp(2\pi \mathrm{i}/n)$

$$\tau = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad S^- = (S^+)^\dagger$$

- Hamiltonian

Truly interacting for *n*>2!

$$H_n = \mathrm{i} \sum_{j=1}^{L} \sum_{a=0}^{n-1} \frac{1}{1-\omega^{-a}} \left[(2a-n)\tau_j^a + n(S_j^+ S_{j+1}^-)^{n-a} - n(S_j^- S_{j+1}^+)^a \right]$$

H_2 boils down to (twisted) XY, H_3 → S=1 Fateev-Zamolodchikov

- U(1) symmetry $[H_n, Q] = 0, \quad Q = \sum_{j=1}^{L} S_j^z$
- Self-duality (in the $\sigma - \tau$ rep.)
- Onsager algebra! $Q^+ = \sum_{j=1}^{L} \sum_{a=1}^{n-1} \frac{1}{1-\omega^{-a}} (S_j^+)^a (S_{j+1}^+)^{n-a}, \quad [H_n, Q^+] = 0$

S=1 (n=3) model

- Integrable Hamiltonian

$$H_{\text{int}} = \sqrt{3}\sum_{j=1}^{L}\left[S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ - (S_j^+ S_{j+1}^-)^2 - (S_j^- S_{j+1}^+)^2 - (S_j^z)^2 + \frac{2}{3}\right]$$

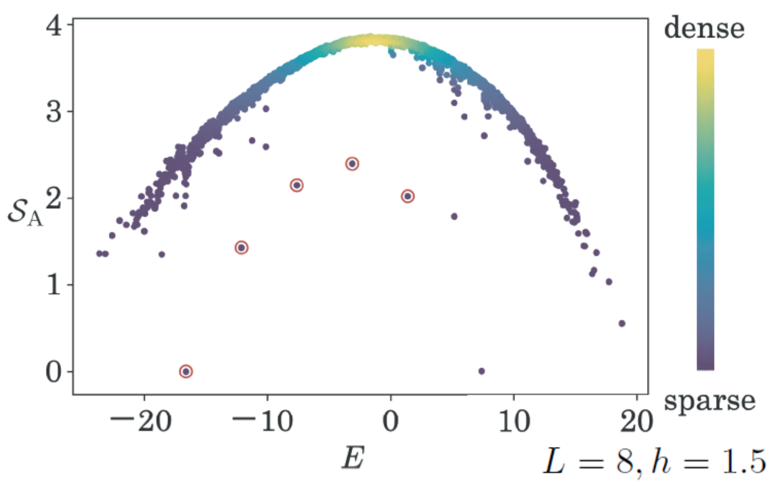
- Coherent state

$$Q^+ = \frac{2}{\sqrt{3}}\sum_{j=1}^{L} S_j^+ (S_j^+ - S_{j+1}^+) S_{j+1}^+, \quad |\psi(\beta)\rangle = \exp(\beta^2 Q^+)|-,-,\cdots,-\rangle$$

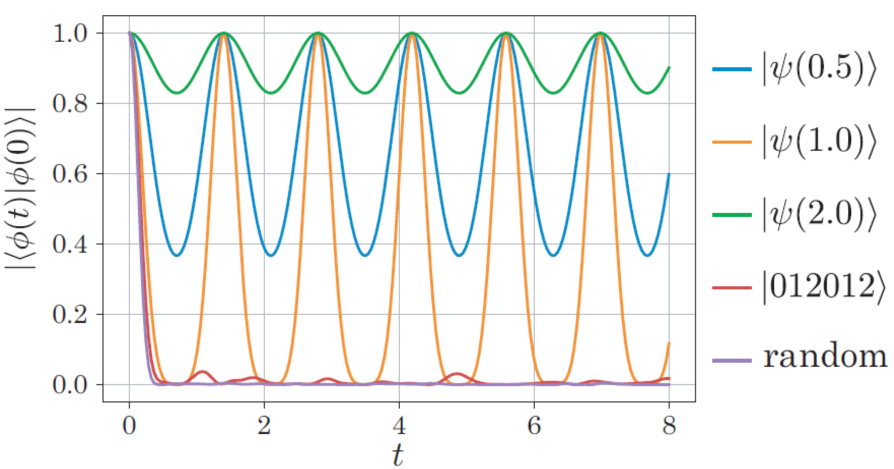
This is again an MPS. The bond dimension is 3.
Desired perturbations can be identified from this MPS.

$$H = H_{\text{int}} + H_{\text{pert}} + hQ,$$

- Half-chain entanglement



- Fidelity



Summary

- **A new construction of QMBS**
  - Perturbed S=1/2 XY chain
  - Use Onsager algebra
  - Level-spacing statistics
    → non-integrability
  - Exact MPS rep. of scarred states
  - Entanglement, fidelity, dynamics, …
- **Generalizations**
  - Use other elements of Onsager algebra
  - Higher-spin models
- **Future directions**
  - Floquet dynamics version
  - Other models with other symmetries