Ferromagnetism in the SU(n) Hubbard model with a nearly flat band

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- K. Tamura and H. Katsura, arXiv:1908.06286

Outline

1. Introduction

- Origin of magnetism
- Hubbard models (SU(2) and SU(n))
- Frustration-free systems

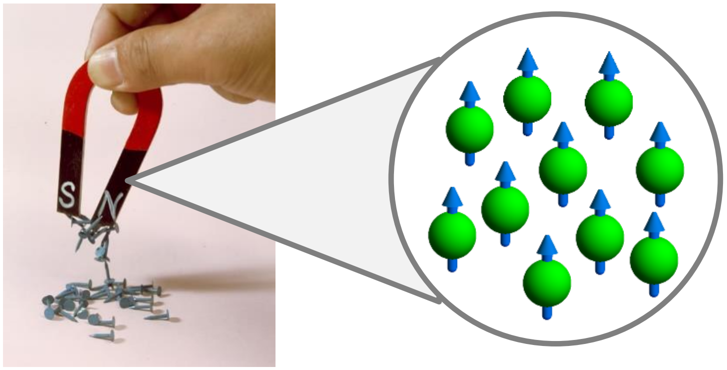
2. SU(n) Hubbard model

3. 1D model and results

Summary

Origin of magnetism

What's the mechanism behind ferromagnetism?



Macroscopic number of spins (carried by electrons) are aligned in the same direction.

But why?

- Coupling between spins
  - Dipole-dipole interaction $$U_{\rm dip}(\boldsymbol{r}) \propto -\frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3} + 3\frac{(\boldsymbol{\mu}_1 \cdot \hat{r})(\boldsymbol{\mu}_2 \cdot \hat{r})}{r^3}$$

    Usually, too small (< 1K) to explain transition temperatures…

  - Exchange interaction

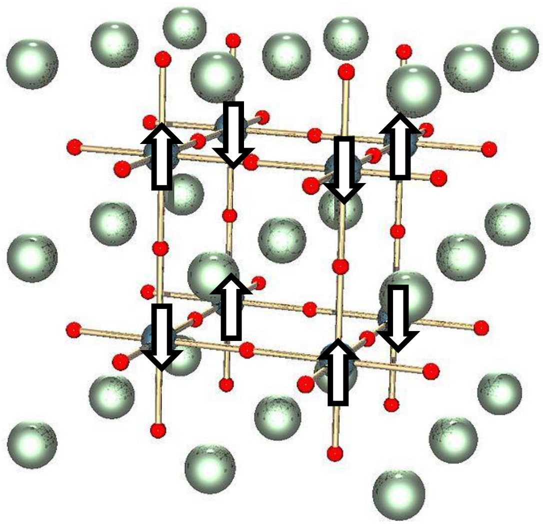
    $$H_{\rm int} = J\boldsymbol{S}_i \cdot \boldsymbol{S}_j$$ ($\boldsymbol{S}_i$: spin at site i)

    Direct exchange: $J < 0$ → ferromagnetic (FM)
    Super-exchange: $J > 0$ → antiferromagnetic (AFM)

Heisenberg

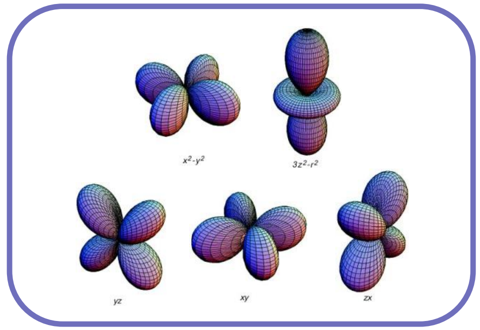
Magnetism in ionic crystals

- Kanamori-Goodenough rules

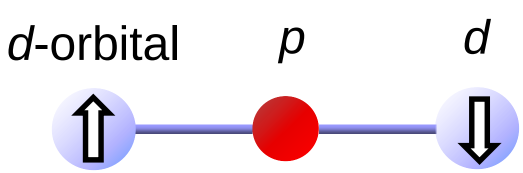


[From *Wikipedia* (perovskite)]

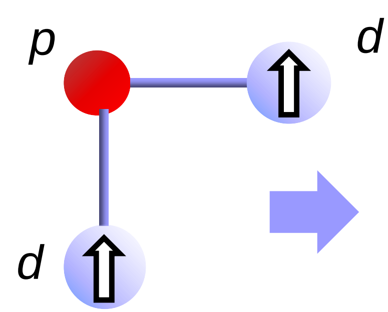
Magnetic ions (cations) do not directly couple each other. They interact via anions.



• Simple examples (neglect orbital order)



⇒ **Antiferromagnetic** int. between mag. ions



⇒ **Ferromagnetic** (except for d^5.)

Symmetry argument… Is there a more *rigorous* approach?

Hubbard model

Paradigmatic model of correlated electrons in solids

- Operators $\bullet^\dagger(\mathrm{phys}) = \bullet^*(\mathrm{math})$

$c_{x,\sigma}$ $(c^\dagger_{x,\sigma})$: Creation (annihilation) op. of electron with spin σ=↑ *or* ↓ at site *x*

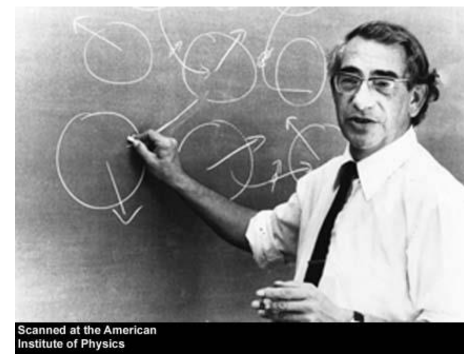
$$\{c_{x,\sigma}, c_{y,\sigma}\} = \{c^\dagger_{x,\sigma}, c^\dagger_{y,\sigma'}\} = 0, \quad \{c_{x,\sigma}, c^\dagger_{y,\sigma'}\} = \delta_{x,y}\delta_{\sigma,\sigma'}$$

$n_{x,\sigma} = c^\dagger_{x,\sigma} c_{x,\sigma}$: Number op. Λ : Finite lattice

Hamiltonian $H = H_{\mathrm{hop}} + H_{\mathrm{int}}$

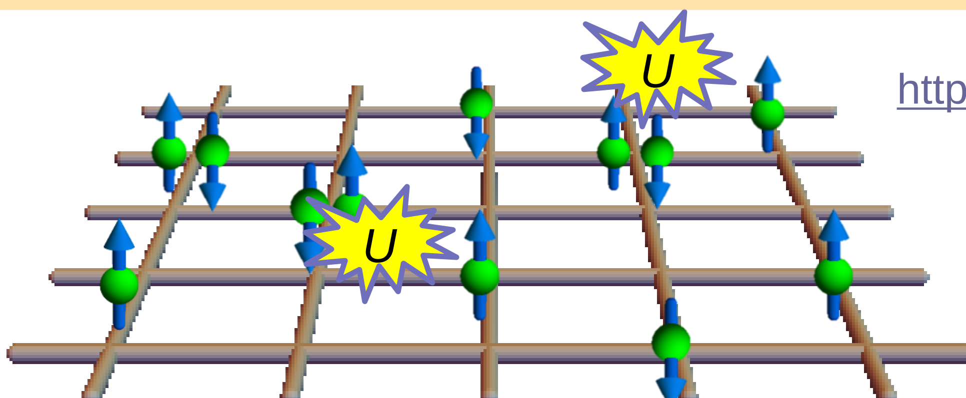
Hopping term $H_{\mathrm{hop}} = \sum_{\sigma=\uparrow,\downarrow} \sum_{x,y\in\Lambda} t_{x,y} c^\dagger_{x,\sigma} c_{y,\sigma}$

On-site repulsion $H_{\mathrm{int}} = U \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow}, \quad (U>0)$



Hubbard (1963)
Kanamori, Gutzwiller

http://theor.jinr.ru/~kuzemsky/jhbio.html

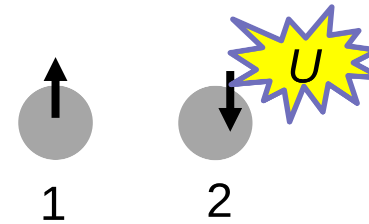


Manifestly SU(2) inv.

Hopping and interaction terms do not commute…

(Crude) derivation of super-exchange

2-site Hubbard model



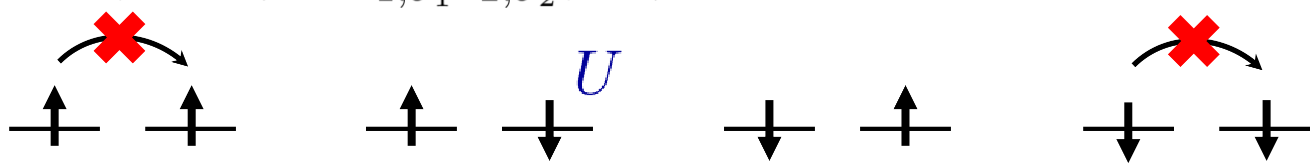
- Hamiltonian

$$H = -t \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{1,\sigma} c_{2,\sigma} + c^\dagger_{2,\sigma} c_{1,\sigma}) + U \sum_{i=1,2} n_{i,\uparrow} n_{i,\downarrow}$$

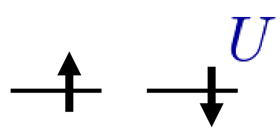
- 2nd order perturbation at half-filling, $U \gg t$

Basis states: $|\sigma_1, \sigma_2\rangle = c^\dagger_{1,\sigma_1} c^\dagger_{2,\sigma_2} |\mathrm{vac}\rangle$

$t = 0$



Pauli's exclusion



$$\begin{array}{cc} \begin{pmatrix} 0 & & & \\ & -\frac{2t^2}{U} & +\frac{2t^2}{U} & \\ & +\frac{2t^2}{U} & -\frac{2t^2}{U} & \\ & & & 0 \end{pmatrix} & \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \end{array}$$

$$\boxed{H_{\mathrm{eff}} = \frac{4t^2}{U}\left(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 - \frac{1}{4}\right)}$$

$$S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$$

Origin of exchange int. = electron correlation!

Can explain antiferromagnetic int. What about ferromagnetism?

Scarcity of exact/rigorous results

Hubbard model lacks general solutions. Numerically demanding…

- Nagaoka ferromagnetism (Infinite-U, 1 hole)
  Nagaoka, *Phys. Rev.* **147** (1966); Tasaki, *PRB* **40** (1989)
- 1D Hubbard chain (Bethe ansatz)
  Lieb-Wu, *PRL*, **20** (1968), "Absence of Mott transition …"
- Ferrimagnetism (spin-reflection positivity)
  Lieb, *PRL*, **62**; Erratum *PRL* **62** (1989).
  G.S. on a bipartite at half-filling has S_{tot} = ||A|-|B||/2.
  Recent extensions by Miyao, arXiv:1712.05529
- Brandt-Giesekus, *PRL* **68** (1992)   Infinite-U Hubbard, RVB

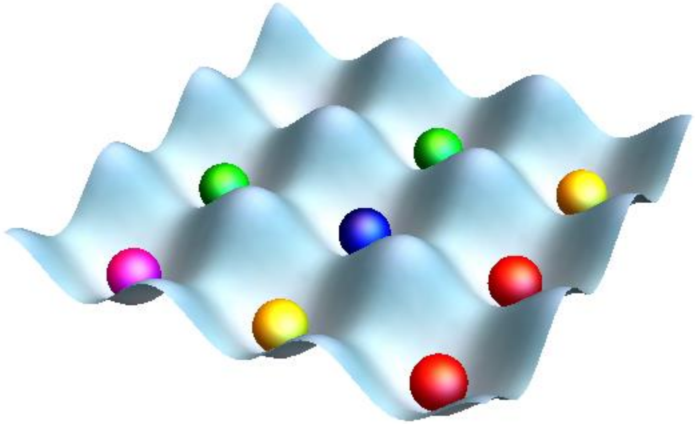
- Flat-band ferromagnetism (Frustration-free)
  Mielke, *JPA* **24**, L73, 3311 (1991); Tasaki, *PRL* **69**, 1608 (1992).
  Review: H.Tasaki, *Prog. Theor. Phys.* **99**, 489 (1998).
  Ferromagnetic states minimize H_{hop} & H_{int} simultaneously!

Multi-component generalization

■ SU(n) Hubbard model

- Fermions carry flavor (α=1, …, n)

- Realization in cold-atom systems
  Taie *et al.*, *Nat. Phys*. **8** (2012).



■ Rigorous results

- Nagaoka ferromagnetism
  in SU(n) Hubbard model

  H.K. and A. Tanaka,
  Phys. Rev. A **87**, 013617 (2013).

  Underlying mechanism is the same as Puzzle & Dragons!

  [See 理学部ニュース 2019]

- Ferromagnetism in a model with completely or nearly flat band

  H.K. & K. Tamura, arXiv:1908.06286

A crash course in inequalities

Positive semidefinite operators

Appendix in H.Tasaki, *Prog. Theor. Phys.* **99**, 489 (1998).

$\mathcal{H}$: finite-dimensional Hilbert space.

A, *B*: Hermitian operators on $\mathcal{H}$

- **Definition 1.** We write $A \geq 0$ and say *A* is positive semidefinite (p.s.d.) if $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}.$
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0.$

Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0.$
  Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0.$

Frustration-free systems

■ Anderson's bound (*Phys. Rev*. **83**, 1260 (1951).)

- Total Hamiltonian: $H = \sum_j h_j$
- Sub-Hamiltonian: h_j that satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
  ($E_j^{(0)}$ is the lowest eigenvalue of h_j)

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Used to obtain a lower bound on the g.s. energy of AFM Heisenberg model

■ Frustration-free Hamiltonian

The case where the *equality* holds.

Definition. $H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j |\psi\rangle = E_j^{(0)} |\psi\rangle$ for all *j*.

Ex.) *S*=1 Affleck-Kennedy-Lieb-Tasaki (AKLT), toric code, …

$$H = \sum_j h_j, \quad h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2$$

Flat-band ferro. is another example.

Outline

1. Introduction

2. SU(n) Hubbard model

- Hamiltonian and symmetry
- What are flat bands?
- Frustration-free case

3. 1D model and results

Summary

SU(n) Hubbard model

Operators and Fock space

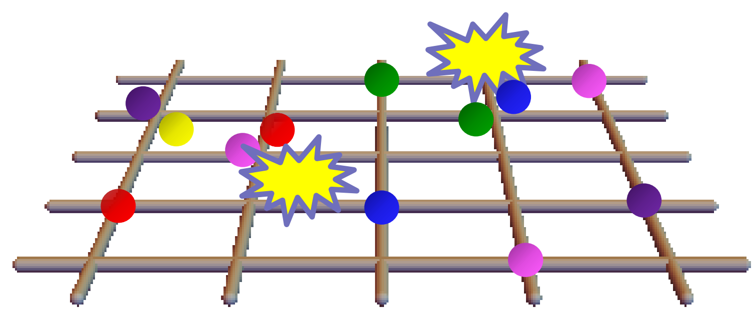
- Finite lattice: Λ
- Creation & annihilation operators at site *x* with color *α*

$$c_{x,\alpha}^\dagger,\ c_{x,\alpha} \quad (x \in \Lambda, \alpha = 1, 2, \cdots, n)$$

$$\{c_{x,\alpha}, c_{y,\beta}\} = \{c_{x,\alpha}^\dagger, c_{y,\beta}^\dagger\} = 0, \quad \{c_{x,\alpha}, c_{y,\beta}^\dagger\} = \delta_{x,y}\delta_{\alpha,\beta}$$

- Number operator: $n_{x,\alpha} = c_{x,\alpha}^\dagger c_{x,\alpha}$
- Vacuum: $c_{x,\alpha}|\Phi_{\text{vac}}\rangle = 0, \ \forall x, \alpha$
- Many-particle states: $c_{x,\alpha}^\dagger c_{y,\beta}^\dagger c_{z,\gamma}^\dagger \cdots |\Phi_{\text{vac}}\rangle$

Hamiltonian $H = H_{\text{hop}} + H_{\text{int}}$



- Hopping term

$$H_{\text{hop}} = \sum_{\alpha=1}^{n} \sum_{x,y\in\Lambda} t_{x,y} c_{x,\alpha}^\dagger c_{y,\alpha}$$

- Interaction term $H_{\text{int}} = U \sum_{1\le\alpha<\beta\le n} \sum_{x\in\Lambda} n_{x,\alpha} n_{x,\beta}, \quad (U > 0)$

Symmetry of the model

Generators

- Total fermion number $N_\mathrm{f} = \sum_{\alpha=1}^{n} \sum_{x\in\Lambda} n_{x,\alpha}$
- Color operators

$$F^{\alpha,\alpha} := \sum_{x\in\Lambda} c^\dagger_{x,\alpha} c_{x,\alpha} \qquad N_\mathrm{f} = \sum_{\alpha=1}^{n} F^{\alpha,\alpha}$$

  Denote their eigenvalues by N_α.

- Color raising & lowring operators $F^{\alpha,\beta} := \sum_{x\in\Lambda} c^\dagger_{x,\alpha} c_{x,\beta}\ (\alpha \neq \beta)$

  They commute with the Hamiltonian. $[H, N_\alpha] = [H, F^{\alpha,\beta}] = 0$

  NOTE) SU(n) symmetry for fixed N_f

Subspaces

- Hamiltonian is block-diagonal w.r.t. $(N_1, \cdots, N_n)$
- Degenerate eigenstates in different subspaces are related to one another by $F^{\alpha,\beta}(\alpha \neq \beta)$.

Hopping term

■ Diagonalization

Boils down to the diagonalization of $T=(t_{x,y})$

$$H_{\mathrm{hop}} = \sum_{\alpha=1}^{n} \sum_{x,y\in\Lambda} t_{x,y} c^{\dagger}_{x,\alpha} c_{y,\alpha}$$

- Eigen-operators

Let ***v*** be an eigenvector of *T* with eigenvalue *ε*.
Then, the operator

$$\psi^{\dagger}_{\alpha} = \sum_{x\in\Lambda} v_x c^{\dagger}_{x,\alpha} \quad \text{satisfies} \quad [H_{\mathrm{hop}}, \psi^{\dagger}_{\alpha}] = \epsilon \psi^{\dagger}_{\alpha}.$$

Acting with $\psi^{\dagger}_{\alpha}$ on an eigenstate of H_{hop} raised energy by *ε*.

- Eigenstates

$|\Phi_{\mathrm{vac}}\rangle$ is an eigenstate of H_{hop} with energy 0.

General eigenstates take the form: $\psi^{\dagger(1)}_{\alpha} \psi^{\dagger(2)}_{\beta} \psi^{\dagger(3)}_{\gamma} \cdots |\Phi_{\mathrm{vac}}\rangle$
where $T\boldsymbol{v}^{(k)} = \epsilon^{(k)} \boldsymbol{v}^{(k)}, \ \psi^{\dagger(k)}_{\alpha} = \sum_{x\in\Lambda} v^{(k)}_x c^{\dagger}_{x,\alpha}$

Interaction Term

- Diagonalization

Already diagonal in the number basis!

$$H_{\text{int}} = U \sum_{1\le\alpha<\beta\le n} \sum_{x\in\Lambda} n_{x,\alpha} n_{x,\beta}$$

- Eigenstates

$c^\dagger_{x,\alpha} c^\dagger_{y,\beta} c^\dagger_{z,\gamma} \cdots |\Phi_{\text{vac}}\rangle$ is an eignstate of H_{int}.

For example, $c^\dagger_{x,1} c^\dagger_{x,2} c^\dagger_{y,3} c^\dagger_{z,1} c^\dagger_{z,3} |\Phi_{\text{vac}}\rangle$ has energy 2*U*.

What about the full Hamiltonian?

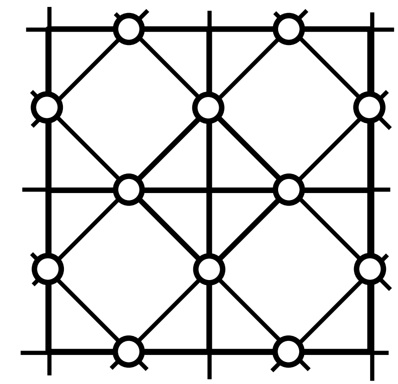
- Hopping and interaction terms do not commute!

$$[H_{\text{hop}}, H_{\text{int}}] \neq 0$$

- Not even frustration-free in general…

But for a hopping term with a *flat band* (at the bottom), the full Hamiltonian becomes frustration-free!

What are flat bands?



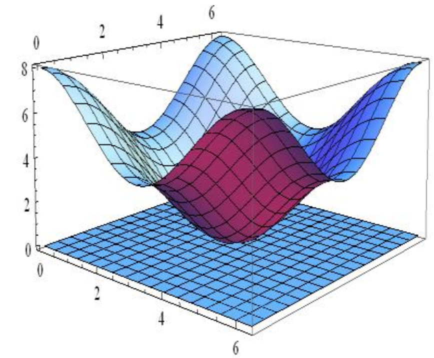
■ Single-particle eigenstates of $H_{\rm hop}$

$$H_{\rm hop} = \sum_{\alpha=1}^{n} \sum_{x,y\in\Lambda} t_{x,y} c^\dagger_{x,\alpha} c_{y,\alpha}$$

- Energy bands

  In systems with translation symmetry, wave-num. ***k*** is a good quantum number.

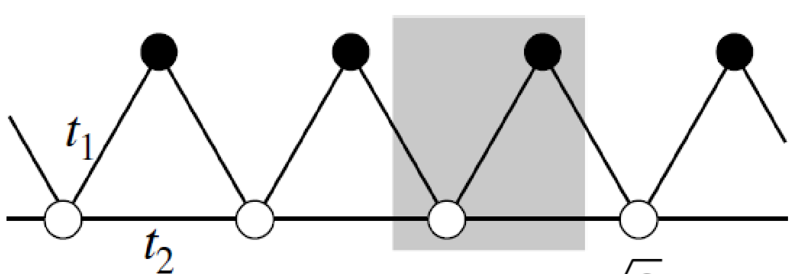
$$H_{\rm hop}\psi^\dagger_\alpha(\boldsymbol{k})|\Phi_{\rm vac}\rangle = \epsilon(\boldsymbol{k})\psi^\dagger_\alpha(\boldsymbol{k})|\Phi_{\rm vac}\rangle$$



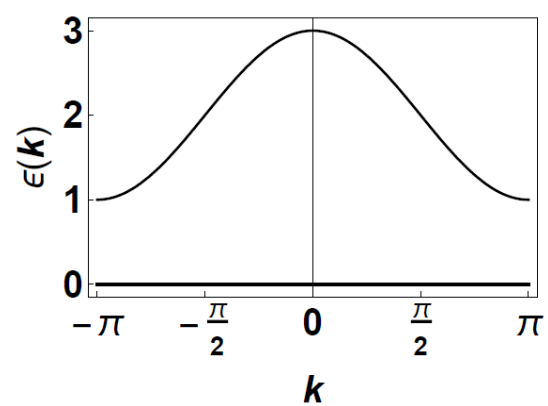
- Flat band

  Single-particle energy $\epsilon(\boldsymbol{k})$ is independent of ***k***.

■ 1D example (Tasaki lattice)



$$t_1 = \sqrt{2}t,\ \ t_2 = t$$



Why frustration-free

- Positive-semi-definite Hopping matrix $T \geq 0$
- Kernel of T spanned by orthonormal $\boldsymbol{v}^{(j)}\ (j = 1, \cdots, D_0),\ T\boldsymbol{v}^{(j)} = 0$
- Zero-energy eigen-operators $a^\dagger_{j,\alpha} = \sum_{x \in \Lambda} v^{(j)}_x c^\dagger_{x,\alpha}$ $\quad [H_{\text{hop}}, a^\dagger_{j,\alpha}] = 0$
- Interaction term is p.s.d.
- Many-body zero-energy state (for fermion num. = D_0) $|\Phi_{\text{ferro},\alpha}\rangle = \left(\prod_{j=1}^{D_0} a^\dagger_{j,\alpha} \right) |\Phi_{\text{vac}}\rangle$

Because of the Pauli principle $(c^\dagger_{x,\alpha})^2 = 0$,

$$H_{\text{hop}}|\Phi_{\text{ferro},\alpha}\rangle = H_{\text{int}}|\Phi_{\text{ferro},\alpha}\rangle|\Phi_{\text{vac}}\rangle = 0$$

Frustration-free!

Are they unique (up to trivial degeneracy)?

- In the SU(2) case, Mielke established a necessary and sufficient condition for the uniqueness [Mielke, Phys. Lett. A **174**, 443 (1993)]
- Related to irreducibility of $(P_0)_{x,y} := \sum_{j=1}^{D_0} (v^{(j)}_x)^* v^{(j)}_y$

Outline

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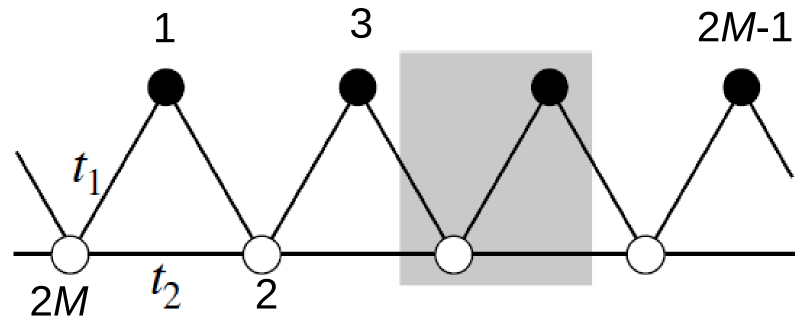
2. SU(n) Hubbard model

3. 1D Model and results

- Lattice and Hamiltonian
- Model with completely flat band
- Model with nearly flat band

Summary

Model on 1D Tasaki lattice



$t_1 = \nu t,\ t_2 = \nu^2 t,$

$t_{\bullet\bullet} = t,\ t_{\circ\circ} = 2\nu^2 t$

■ Lattice and hopping term

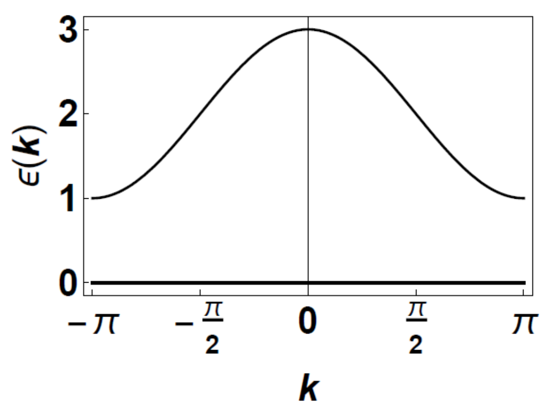
- Lattice: $\Lambda = \{1, 2, ..., 2M\}$

  $\mathcal{O} = \{1, 3, 5, ...\},\ \mathcal{E} = \{2, 4, 6, ...\}$
- Periodic boundary conditions: Identify site *j* with *j*+2*M*.
- Hopping term

$$H_{\rm hop} = \sum_{\alpha=1}^{n} \sum_{x,y\in\Lambda} t_{x,y} c^\dagger_{x,\alpha} c_{y,\alpha} = t \sum_{\alpha=1}^{n} \sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha} b_{x,\alpha}$$

$$b_{x,\alpha} := \nu c_{x-1,\alpha} + c_{x,\alpha} + \nu c_{x+1,\alpha}, \quad x \in \mathcal{O}$$

$t = 1,\ \nu = 1/\sqrt{2}$



■ Localized eigen-operators of $H_{\rm hop}$

$$a_{x,\alpha} = -\nu c_{x-1,\alpha} + c_{x,\alpha} - \nu c_{x+1,\alpha}, \quad x \in \mathcal{E}$$

$$[H_{\rm hop}, a^\dagger_{x,\alpha}] = 0 \quad (\because \{a^\dagger_{x,\alpha}, b_{y,\beta}\} = 0)$$

The flat band is spanned by *a*-operators.

Model on 1D Tasaki lattice

Lattice and hopping term

- Lattice: $\Lambda = \{1, 2, ..., 2M\}$

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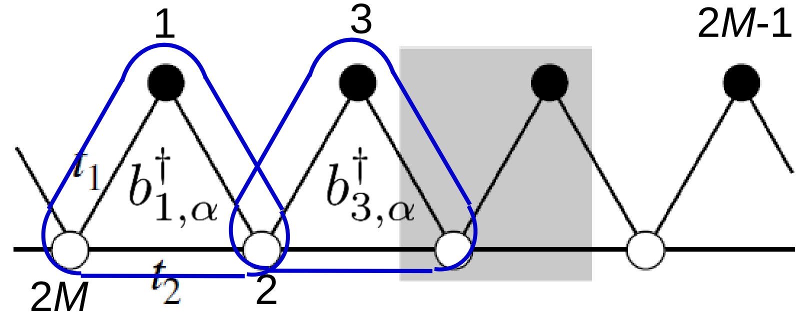

$t_1 = \nu t,\ t_2 = \nu^2 t,$

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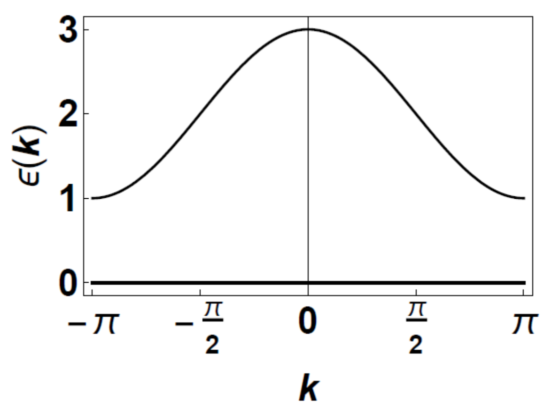


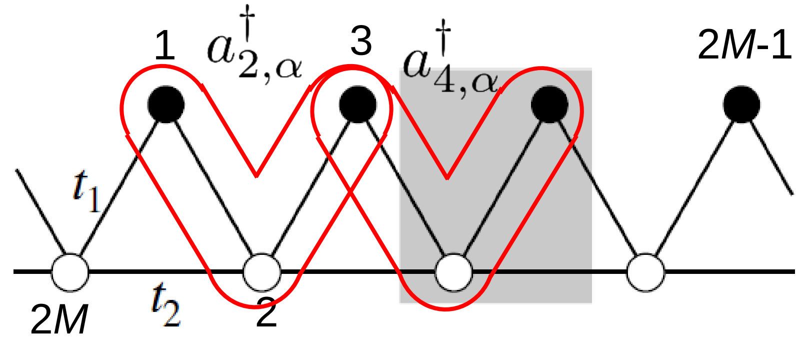
Localized eigen-operators of H_{hop}

$$a_{x,\alpha} = -\nu c_{x-1,\alpha} + c_{x,\alpha} - \nu c_{x+1,\alpha}, \quad x \in \mathcal{E}$$

$$[H_{\mathrm{hop}}, a^\dagger_{x,\alpha}] = 0 \quad (\because \{a^\dagger_{x,\alpha}, b_{y,\beta}\} = 0)$$

The flat band is spanned by *a*-operators.

Model on 1D Tasaki lattice



$t_1=\nu t,\ t_2=\nu^2 t,$

$t_{\bullet\bullet}=t,\ t_{\circ\circ}=2\nu^2 t$

■ Lattice and hopping term

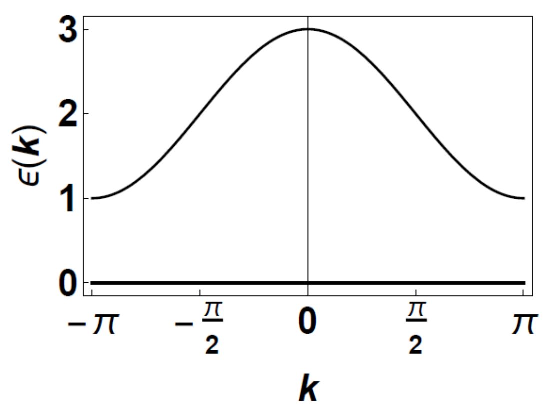
- Lattice: $\Lambda=\{1,2,...,2M\}$

  $\mathcal{O}=\{1,3,5,...\},\ \mathcal{E}=\{2,4,6,...\}$
- Periodic boundary conditions:
  Identify site *j* with *j*+2*M*.
- Hopping term

$$H_{\rm hop}=\sum_{\alpha=1}^{n}\sum_{x,y\in\Lambda}t_{x,y}c^\dagger_{x,\alpha}c_{y,\alpha}=t\sum_{\alpha=1}^{n}\sum_{x\in\mathcal{O}}b^\dagger_{x,\alpha}b_{x,\alpha}$$

$$b_{x,\alpha}:=\nu c_{x-1,\alpha}+c_{x,\alpha}+\nu c_{x+1,\alpha},\quad x\in\mathcal{O}$$

$t=1,\ \nu=1/\sqrt{2}$



■ Localized eigen-operators of $H_{\rm hop}$

$$a_{x,\alpha}=-\nu c_{x-1,\alpha}+c_{x,\alpha}-\nu c_{x+1,\alpha},\quad x\in\mathcal{E}$$

$$[H_{\rm hop},a^\dagger_{x,\alpha}]=0\ \ (\because \{a^\dagger_{x,\alpha},b_{y,\beta}\}=0)$$

The flat band is spanned by *a*-operators.

Flat-band ferromagnetism

■ SU(n) Ferromagnetic (FM) states

- Fix total fermion number: $N_{\mathrm{f}} = M$ (total number of unit cells)
- Fully polarized states $|\Phi_{\mathrm{all},\alpha}\rangle := \left(\prod_{x\in\mathcal{E}} a^{\dagger}_{x,\alpha}\right)|\Phi_{\mathrm{vac}}\rangle, \quad \alpha = 1, \ldots, n$

  are ground states of $H = H_{\mathrm{hop}} + H_{\mathrm{int}}$
  as it makes both H_{hop} and H_{int} vanish.

Frustration-free!

- Other FM ground states: $|\Phi_{N_1,\cdots,N_n}\rangle = (F^{n,1})^{N_n}\cdots(F^{2,1})^{N_2}|\Phi_{\mathrm{all},1}\rangle$
- Total number of FM states: $deg. = \dfrac{(M+n-1)!}{M!(n-1)!}$

■ Theorem 1 (uniqueness of the FM ground states)

Consider the Hubbard Hamiltonian H with the total fermion number $N_{\mathrm{f}}=M$. For arbitrary t >0 and U >0, the ground states of the Hamiltonian are SU(n) ferromagnetic states and unique apart from trivial degeneracy due to the SU(n) symmetry.

A slight generalization of R.-J. Liu *et al*., arXiv:1901.07004.

Outline of proof

- Hamiltonian

$$H = H_{\rm hop} + H_{\rm int} = t\sum_{\alpha=1}^{n}\sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha} b_{x,\alpha} + U\sum_{\alpha<\beta}\sum_{x\in\Lambda}(c_{x,\alpha}c_{x,\beta})^\dagger c_{x,\alpha}c_{x,\beta}$$

Since $H_{\rm hop} \geqq 0$ and $H_{\rm int} \geqq 0$, any ground state of H must be annihilated by $H_{\rm hop}$ and $H_{\rm int}$ simultaneously. This further means

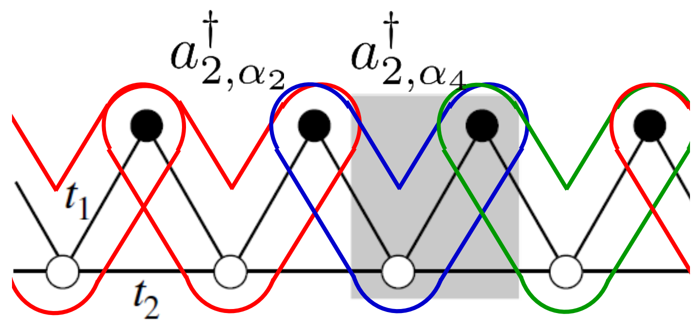
$$b_{x,\alpha}|\Phi_{\rm GS}\rangle = 0,\ \forall x \in \mathcal{O} \text{ and } \alpha = 1, \ldots, n$$

b's do not appear in g.s.

$$c_{x,\alpha}c_{x,\beta}|\Phi_{\rm GS}\rangle = 0,\ \forall x \in \Lambda \text{ and } (\alpha,\beta).$$

- Multiple occupancy of *a*'s are prohibited

$$|\Phi_{\rm GS}\rangle = \sum_{\boldsymbol{\alpha}} C(\boldsymbol{\alpha})\left(\prod_{x\in\mathcal{E}} a^\dagger_{x.\alpha_x}\right)|\Phi_{\rm vac}\rangle$$



- Examining the 2nd condition on top sites, we have $C(\boldsymbol{\alpha}) = C(\boldsymbol{\alpha}_{x\leftrightarrow y})$.

$$C(\boldsymbol{\alpha}) = C(\cdots,\alpha_x,\cdots,\alpha_y,\cdots),\quad C(\boldsymbol{\alpha}_{x\leftrightarrow y}) = C(\cdots,\alpha_y,\cdots,\alpha_x,\cdots)$$

- In a subspace labeled by $(N_1,\cdots,N_n)$, the g.s. is an equal weight superposition of $a^\dagger_{2,w_1}a^\dagger_{4,w_2}\cdots a^\dagger_{2M,w_M}|\Phi_{\rm vac}\rangle,\quad w \in W(N_1,\ldots,N_n)$ (set of possible permutations). This state is equivalent to a FM state.

Model with nearly flat band

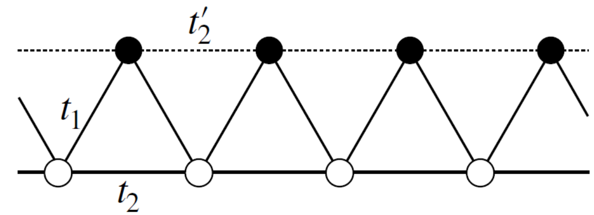
■ Lattice and hopping term

- Hopping term

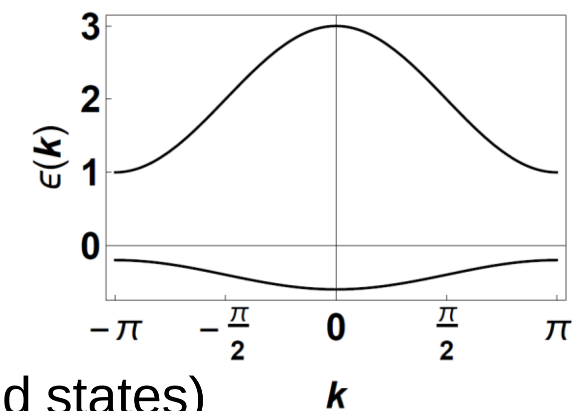
$$H_{\text{hop}} = -\, s \sum_{\alpha=1}^{n} \sum_{x\in\mathcal{E}} a^{\dagger}_{x,\alpha} a_{x,\alpha}$$
$$+\, t \sum_{\alpha=1}^{n} \sum_{x\in\mathcal{O}} b^{\dagger}_{x,\alpha} b_{x,\alpha}$$

- Total Hamiltonian

$$H = H_{\text{hop}} + H_{\text{int}}$$



$$t_1 = \nu(t+s),\ t_2 = \nu^2 t,\ t_2' = -\nu^2 s$$
$$t_{\bullet\bullet} = t - 2\nu^2 s,\ t_{\circ\circ} = -s + 2\nu^2 t$$



■ Theorem 2 (uniqueness of the FM ground states)

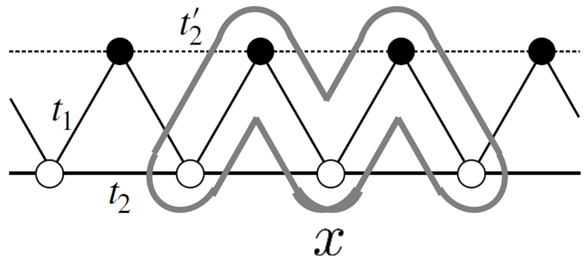
Consider the Hubbard Hamiltonian H with the total fermion number $N_f = M$. For sufficiently large $t/s > 0$ and $U/s > 0$, the ground states of H are SU(n) ferromagnetic states and unique apart from trivial degeneracy due to the SU(n) symmetry.

A natural SU(n) generalization of H. Tasaki, *PRL* **75**, 4678 (1995).

Outline of proof (1)

■ Decoupling of the Hamiltonian

$$H = \lambda H_{\mathrm{flat}} + \sum_{x\in\mathcal{E}} h_x - sM(2\nu^2+1)$$



Flat-band Hamiltonian
(t=U=1)

- Flat part

$$H_{\mathrm{flat}} = \sum_{\alpha=1}^{n}\sum_{x\in\mathcal{O}} b^\dagger_{x,\alpha}b_{x,\alpha} + \sum_{\alpha<\beta} n_{x,\alpha}n_{x,\beta}$$

- Local term

$$\begin{aligned} h_x =& \sum_{\alpha=1}^{n}\left(-sa^\dagger_{x,\alpha}a_{x,\alpha} + \frac{t-\lambda}{2}(b^\dagger_{x-1,\alpha}b_{x-1,\alpha} + b^\dagger_{x+1,\alpha}b_{x+1,\alpha})\right) \\ &+ \frac{\kappa(U-\lambda)}{4}n_{x-2}(n_{x-2}-1) + \frac{U-\lambda}{4}n_{x-1}(n_{x-1}-1) \\ &+ \frac{(1-\kappa)(U-\lambda)}{2}n_x(n_x-1) + \frac{U-\lambda}{4}n_{x+1}(n_{x+1}-1) \\ &+ \frac{\kappa(U-\lambda)}{4}n_{x+2}(n_{x+2}-1) + s(2\nu^2+1), \end{aligned}$$

$$(0<\lambda<\min\{t,U\}, \quad 0<\kappa<1)$$

■ Lemma 1

If each local Hamiltonian h_x is positive semi-definite, then the ground states of H are the same as those of H_{flat}.

Outline of proof (2)

■ Proof of Lemma 1

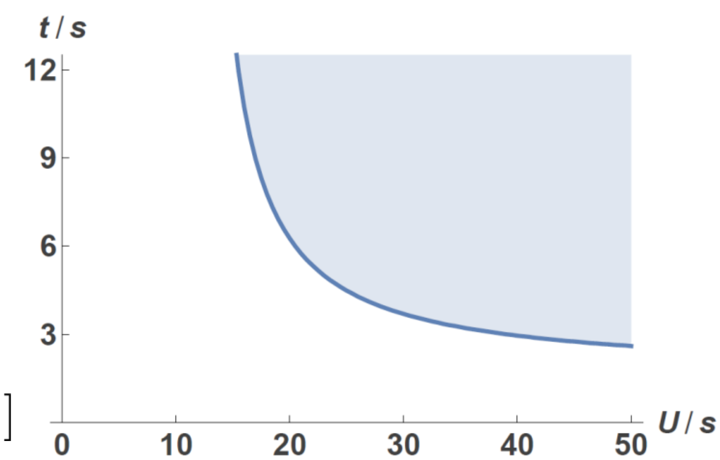
- Frustration-free?
  If each h_x is p.s.d., any state annihilated by H_{flat} and all h_x is a ground states of H.

- Fully polarized states $|\Phi_{\text{all},\alpha}\rangle := \left(\prod_{x\in\mathcal{E}} a^\dagger_{x,\alpha}\right)|\Phi_{\text{vac}}\rangle$ are annihilated by H_{flat} and all h_x.
  → They are eigenstates of H with $E = -sM(2\nu^2+1)$.

- Uniqueness
  Uniqueness of these ground states just follows from Theorem 1.

■ Positive semi-definiteness of h_x

- Computer-assisted proof
  By numerially diagonalizing h_x (5-site Hamiltonian), one can identify the region in which h_x is p.s.d.



$[\text{plot for } n=4, \nu=1/\sqrt{2}, \kappa=0^+]$

■ Lemma 2

Suppose that t, U are infinitely large and $0 < \kappa < 1$. Then, h_x is positive semi-definite.

Proof.) Based on the analysis of projected Hamiltonian Ph_xP (Projected onto the space of finite-energy states.)

Remark. Lemma 2 ensures finite thresholds for t/s and U/s, above which h_x is positive semi-definite.

Summary

- Reviewed rigorous results for Hubbard models
- Introduced SU(n) Hubbard model on 1D Tasaki lattice
- Ferromagnetism in the model with a completely flat band
- Ferromagnetism in the model with a nearly flat band

Established rigorous example in a non-singular situation!