Fradkin, Fredkin or Fridkin?

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- O. Salberger *et al.*, *J. Stat. Mech*, 063103 (2018)
- T. Udagawa and H. Katsura, *J. Phys. A*, **50**, 405002 (2017)

Outline

1. Introduction

- Frustration-free systems
- Ferromagnetic XXX chain
- t (=q) deformed model

2. Fredkin spin chain

3. Main results

4. Super-frustration-free systems

5. Summary

Jewels of theoretical physics

■ Classification of solvable models

- Integrable systems
  Free fermions/bosons, Bethe ansatz
  Infinitely many conserved charges

- Frustration-free systems
  Ground state (g.s.) minimizes each local Hamiltonian
  Explicit g.s., but hard to obtain excited states

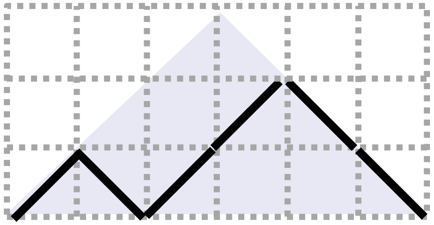
Not exclusive. Ferromagnetic Heisenberg chain is
integrable & ***Frustration-free***! (H. Bethe, 1931)

■ Today's subject

- Frustration-free spin chains related to combinatorics

- Peculiar entanglement properties
  Non-area law behavior of EE (volume-law, …)

- SUSY and super-frustration-free systems?



A crash course in inequalities

■ Positive semidefinite operators

Appendix in H.Tasaki, *Prog. Theor. Phys.* **99**, 489 (1998).

$\mathcal{H}$: finite-dimensional Hilbert space.

A, *B*: Hermitian operators on $\mathcal{H}$

- **Definition 1.** We write $A \geq 0$ and say *A* is positive semidefinite (p.s.d.) if $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}.$
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0.$

■ Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0.$
  Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0.$

Frustration-free systems

- **Anderson's bound** (*Phys. Rev*. **83**, 1260 (1951).
  - Total Hamiltonian: $H = \sum_j h_j$
  - Sub-Hamiltonian: h_j that satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
    ($E_j^{(0)}$ is the lowest eigenvalue of h_j)

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Gives a lower bound on the g.s. energy of AFM Heisenberg model.

- **Frustration-free Hamiltonian**

The case where the *equality* holds.

(Pseudo-)Definition. $H = \sum_j h_j$ is said to be *frustration-free* when the g.s. minimizes individual sub-Hamiltonians h_j.

Ex.) *S*=1 Affleck-Kennedy-Lieb-Tasaki (AKLT), Kitaev's toric code, …

$$H = \sum_j h_j, \quad h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2$$

Ferromagnetic Heisenberg chain

■ Hamiltonian (S=1/2, OBC)

$$H = \sum_{j=1}^{N-1} h_j, \quad \boxed{h_j = -\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{4}}$$

- SU(2) symmetry $[H, S^{\alpha}_{\mathrm{tot}}] = 0, \qquad S^{\alpha}_{\mathrm{tot}} = \sum_{j=1}^{N} S^{\alpha}_j \quad (\alpha = z, +, -)$
- h_j is p.s.d. as it is a projector to singlet

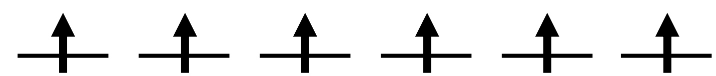
$$h_j = |S_{j,j+1}\rangle\langle S_{j,j+1}|, \quad |S_{j,j+1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_j \downarrow_{j+1}\rangle - |\downarrow_j \uparrow_{j+1}\rangle)$$

Spin-singlet state

■ Ground states

- All-up state

$$|\Uparrow\rangle := |\uparrow_1 \uparrow_2 \cdots \uparrow_N\rangle$$



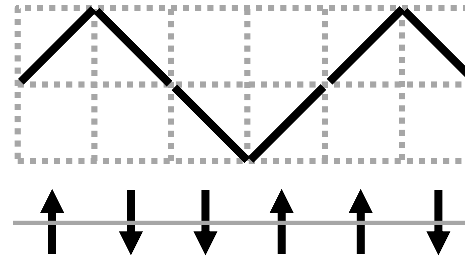
is a zero-energy state of of each h_j → All-up state is a g.s. of H

- Other g.s.: $(S^{-}_{\mathrm{tot}})^k |\Uparrow\rangle \qquad (k = 0, 1, ..., N)$
- Unique in each total S^z sector (due to Perron-Frobenius thm.)

Graphical representation

■ Spin configs ←→ lattice paths

Example (N=6)



- Spin state at site j: $\mathsf{m}_j = \pm 1/2$

  Height between j and j+1: $\mathsf{h}_{j+1/2} = 2(\mathsf{m}_1 + \mathsf{m}_2 + \cdots + \mathsf{m}_j)$
- Eigenvalue of total S^z = (the last height)/2

■ Graphical reps. of G.S.

- The states in $S^z = M$ sector

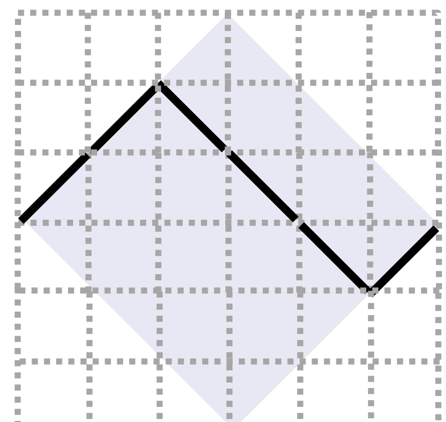
  → starting from height zero

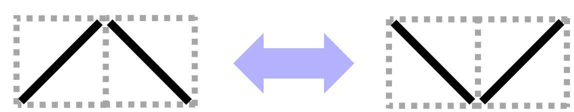
  ending at $\mathsf{h}_{N+1/2} = 2M$
- G.s. in $S^z = M$ sector

  = Equal-weight superposition

  of all such states
- Local transition rule

$$|\Psi_0\rangle = \sum_{\text{path}}$$





t-deformed model

- **Hamiltonian** (S=1/2, OBC)

$$|S_{j,j+1}(t)\rangle = \frac{1}{\sqrt{1+t^2}}(|\uparrow_j\downarrow_{j+1}\rangle - t|\downarrow_j\uparrow_{j+1}\rangle)$$

$$H = \sum_{j=1}^{N-1} h_j, \quad \boxed{h_j = |S_{j,j+1}(t)\rangle\langle S_{j,j+1}(t)|}$$

t-deformed singlet (*t*>0)
t→1 SU(2) spin singlet

- XXZ chain with boundary field

$$h_j \propto -\left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \frac{t+t^{-1}}{2} S_j^z S_{j+1}^z + \frac{t-t^{-1}}{4}(S_j^z - S_{j+1}^z)\right] + \text{const.}$$

- $U_q(sl_2)$ with $q=t$. H commutes with

$$J^z(t) = 2S^z_{\text{tot}}, \quad J^+(t) = \sum_{j=1}^{N} t^{\sigma_1^z} \cdots t^{\sigma_{j-1}^z} \sigma_j^+, \quad J^-(t) = (J^+(t))^\dagger$$

Alcaraz *et al*., *JPA* **20** (1987), Pasquier-Saleur, *NPB* **330** (1990)

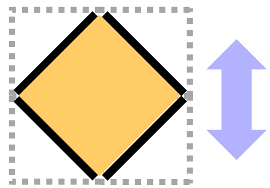
- **Ground states**
  - $|\Uparrow\rangle := |\uparrow_1\uparrow_2\cdots\uparrow_N\rangle$ annihilated by each h_j is a g.s. of H.
  - Other g.s.: $J^-(t)^k|\Uparrow\rangle \qquad (k=0,1,\ldots,N)$ Unique in each M sector

Alcaraz, Salinas, Wreszinski, *PRL* **75** (1995), Gottstein, Werner (1995).

Graphical representation

■ Graphical g.s.

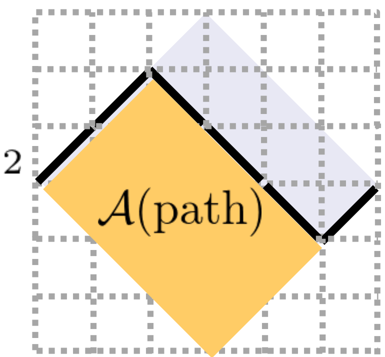
- Local g.s. of h_j $\quad |\uparrow_j\uparrow_{j+1}\rangle, \quad |\downarrow_j\downarrow_{j+1}\rangle, \quad t|\uparrow_j\downarrow_{j+1}\rangle + |\downarrow_j\uparrow_{j+1}\rangle$
- Transition rule



Area difference
→ coefficient *t*

- G.s. in $S^z = M$ sector
= Area-weighted superposition
of states s.t. $\mathsf{h}_{N+1/2} = 2M$

$$|\Psi_0\rangle = \sum_{\text{path}} t^{\mathcal{A}(\text{path})/2}$$



- $\langle \Psi_M | \Psi_M \rangle$ as a *q*-binomial $\begin{bmatrix} n \\ m \end{bmatrix}_q$

Anything to do with Fridkin?

- Not that much…
- Studied a non-frustration-free case $t^3 = q^3 = -1$
Fridkin, Stroganov, Zagier, *JPA* **33** (2000); *JSP* **102** (2001)
- G.s. energy takes a very simple form! No finite-size effect.

Outline

1. Introduction

2. **Fredkin spin chain**

- Colorless (spin-1/2) model w/ & w/o deformation
- Colorful (higher-spin) model w/ & w/o deformation

3. Main results

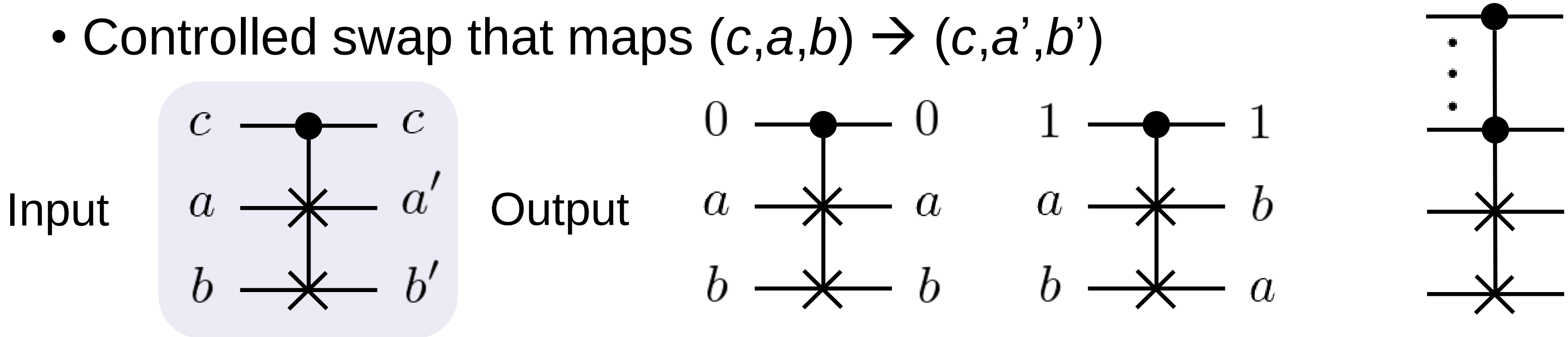
4. Super-frustration-free systems

5. Summary

Fredkin's work

- **Edward Fredkin (1934-)**
  - Physicist and computer scientist.
    Early pioneer of Digital Physics.
  - Primary contributions to reversible computing and cellular automata (from *Wikipedia*)
- **Fredkin gate**
  - Controlled swap that maps (*c*,*a*,*b*) → (*c*,*a*',*b*')

Generalization



  - (Classically) universal, i.e., any logical or arithmetic operation can be constructed entirely of Fredkin gates.
  - Quantum version = unitary logic gate
    Implementation using photons: R.B.Patel *et al*., *Sci. Adv*. **2** (2016)

Fredkin spin chain

■ Hamiltonian (S=1/2, OBC, *N* even)

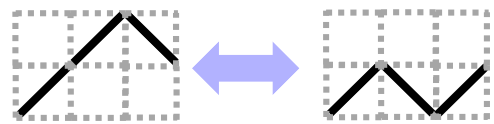
$$H = H_\partial + \sum_{j=1}^{N-1} h_j$$

Salberger and Korepin,
Rev. Math. Phys. **29** (2017)

- Boundary term

$$H_\partial = |\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_N\rangle\langle\uparrow_N|$$

Fredkin moves



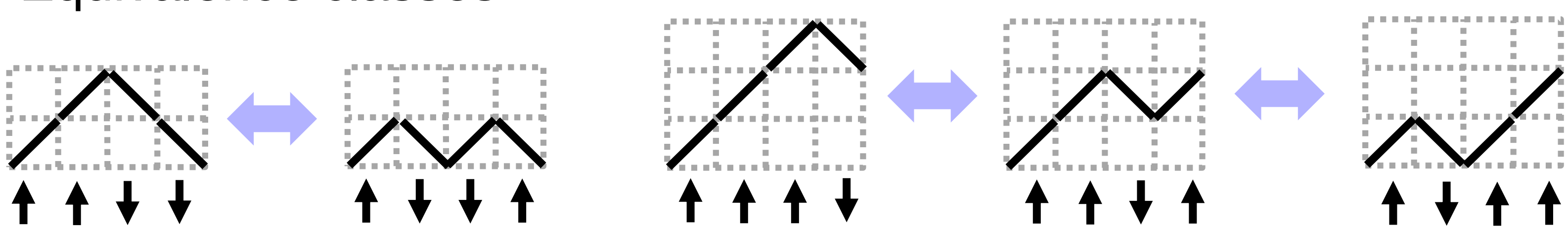
& reflected one

- Bulk terms ~ Fredkin gates

$$h_j = |\uparrow_j\rangle\langle\uparrow_j| \otimes |S_{j+1,j+2}\rangle\langle S_{j+1,j+2}| + |S_{j,j+1}\rangle\langle S_{j,j+1}| \otimes |\downarrow_{j+2}\rangle\langle\downarrow_{j+2}|$$

- Lacks SU(2) symmetry, but preserves U(1). PT-like symmetry

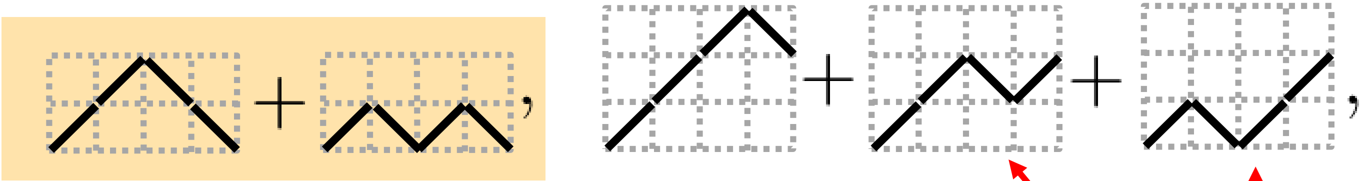
- Equivalence classes

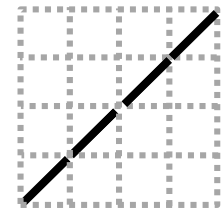


H is block-diagonal w.r.t. disconnected sectors

Ground states

- Importance of $H_\partial = |\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_N\rangle\langle\uparrow_N|$
  - G.S. of the bulk term → *highly degenerate!*



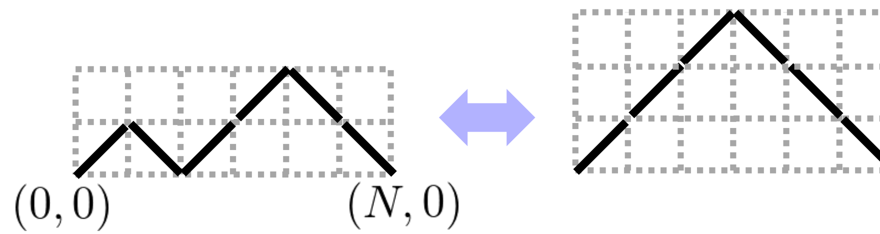


$, \cdots$

  - Only one of them is annihilated by H_∂

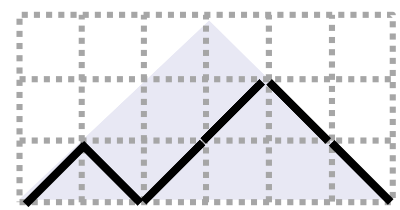
Penalized by H_∂

- Dyck paths
  - Paths from (0,0) to (N,0)
  - Never pass below the x axis



- Graphical reps. of g.s.
  - Equal-weight superposition of all Dyck paths

$$|\Psi_0\rangle = \sum_{\text{path}}$$



  - FM state with M =0 projected to the Dyck sector
  - Catalan number! $\langle\Psi_0|\Psi_0\rangle = \frac{1}{n+1}\binom{2n}{n}, \quad N = 2n$

t-deformed model

■ Hamiltonian (S=1/2, OBC) Salberger *et al., J.Stat.Mech.* (2017)

$$H = H_\partial + \sum_{j=1}^{N-1} h_j$$

- Boundary term

$$H_\partial = |\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_N\rangle\langle\uparrow_N|$$

- Bulk terms

$$h_j = |\uparrow_j\rangle\langle\uparrow_j| \otimes |S_{j+1,j+2}(t)\rangle\langle S_{j+1,j+2}(t)| + |S_{j,j+1}(t)\rangle\langle S_{j,j+1}(t)| \otimes |\downarrow_{j+2}\rangle\langle\downarrow_{j+2}|$$

t-deformed singlet (t>0)

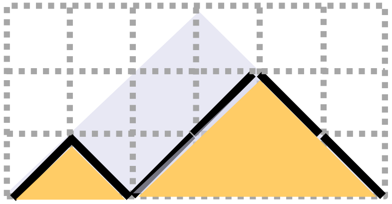
$$|S_{1,2}(t)\rangle = \frac{|\uparrow_1\downarrow_2\rangle - t|\downarrow_1\uparrow_2\rangle}{\sqrt{1+t^2}}$$

- The same equivalence classes as t=1(undeformed)

■ Graphical g.s.

- Area-weighted superposition of all Dyck paths

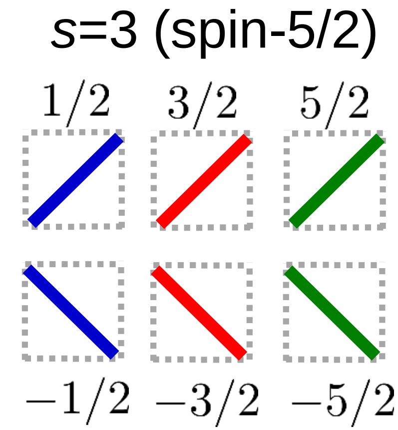
$$|\Psi_0\rangle = \sum_{\text{path}} t^{\mathcal{A}(\text{path})/2}$$



- Unique g.s.
- Projection of the g.s. of DW XXZ
- Carlitz-Riordan q(=t) Catalan number

$$\langle\Psi_0|\Psi_0\rangle = \sum_{\text{path}} t^{\mathcal{A}(\text{path})}$$

Colorful (higher-spin) model



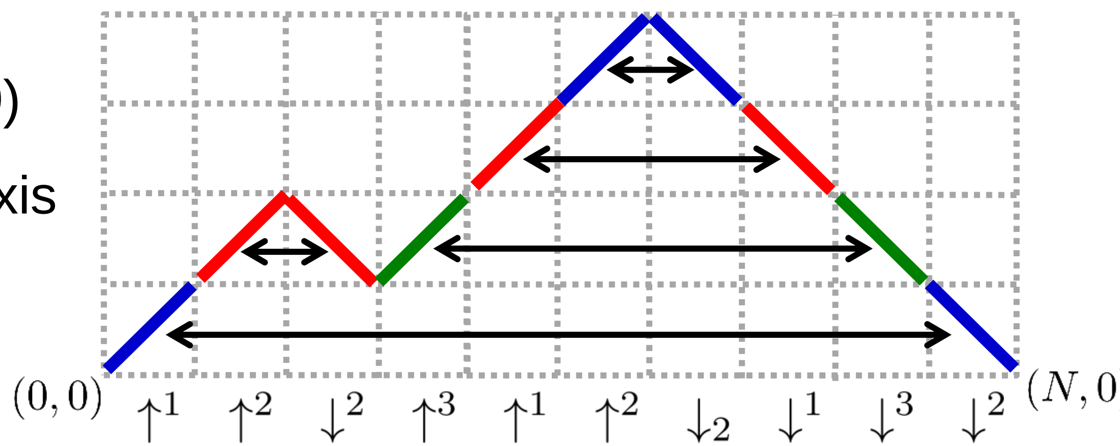
■ Spin states←→colored steps $(c = 1, 2, \ldots, s)$

$+\left(c-\frac{1}{2}\right)$ ⟺ up step with color c $=:\uparrow^{c}$

$-\left(c-\frac{1}{2}\right)$ ⟺ down step with color c $=:\downarrow^{c}$

■ Colored Dyck paths

- Paths from (0,0) to (N,0)
- Never go below the x axis
- *Matched* ↑ and ↓ steps have the same color



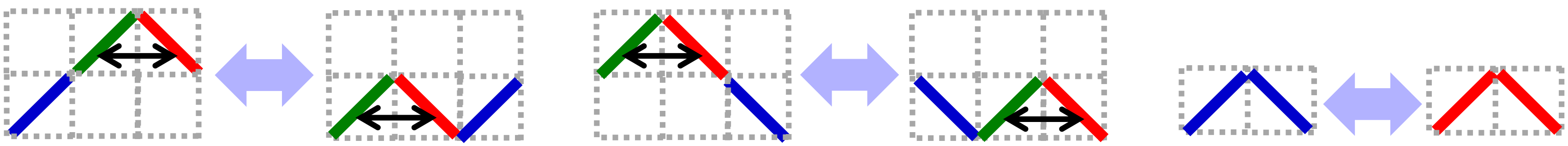
■ Frustration-free models

- Undeformed model: Salberger and Korepin, *Rev. Math. Phys.* **29** (2017)
- Deformed model: Salberger *et al., J.Stat.Mech.* (2017)

Ground states

■ Hamiltonian (OBC)

- Boundary term: $H_\partial(s) = \sum_{c=1}^{s}(|\downarrow_1^c\rangle\langle\downarrow_1^c| + |\uparrow_N^c\rangle\langle\downarrow_N^c|)$
- Bulk term: $H_F(s,t) + H_X(s)$ Sum of projectors.
- Transition rules

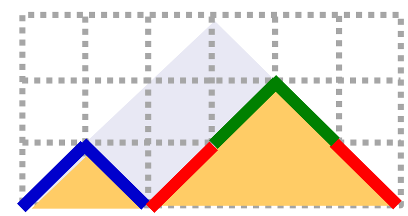


- Invariant under permutation of colors

■ Graphical g.s.

- Area-weighted superposition of all colored Dyck paths

$$|\Psi_0\rangle = \sum_{\mathrm{path}} t^{\mathcal{A}(\mathrm{path})/2}$$



- Unique g.s. of *H*
- Normalization $\langle\Psi_0^{(s)}|\Psi_0^{(s)}\rangle = s^{N/2}\langle\Psi_0^{(1)}|\Psi_0^{(1)}\rangle$

Norm for colorless (s=1) model

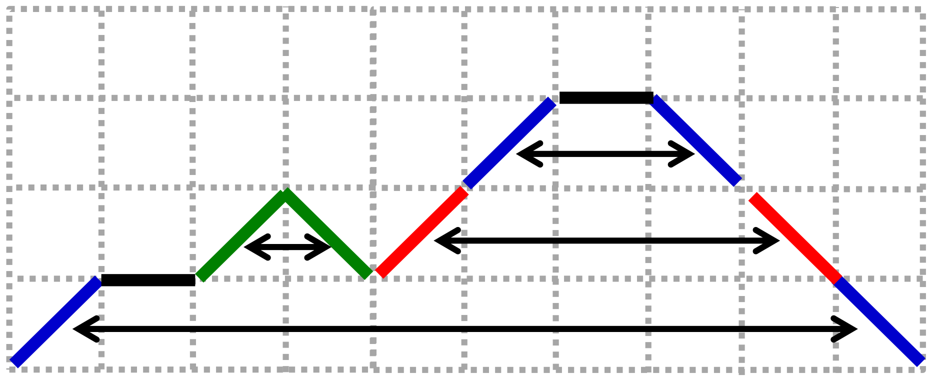
Movassagh-Shor's integer spin chains

■ Frustration-free models

- Spin-1: Bravyi *et al.*, *PRL* **109** (2012)
- Spin-*s*: Movassagh and Shor, *PNAS* **113** (2016)
- Deformed model: Zhang, Ahmadain, Klich, *PNAS* **114** (2017)

■ Colored Motzkin paths

- Flat step → m=0, up/down step with c → $m=\pm c$ (c=1,…,s)
- Paths from (0,0) to (N,0)
- Never go below the x axis
- *Matched* ↑ and ↓ steps have the same color



■ Ground state

- Equal (or weighted) superposition of all colored Motzkin paths
- Peculiar entanglement properties. Critical at t=1 (undeformed case)

Outline

1. Introduction

2. Fredkin spin chain

3. **Main results**

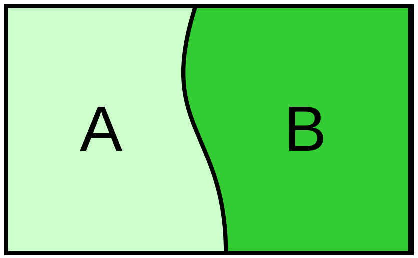
- Half-chain entanglement
- Volume-law when s>1 and t>1
- Other results

4. Super-frustration-free systems

5. Summary

Quantum entanglement

■ Schmidt decomposition



- Many-body g.s. (normalized)

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi} \sqrt{p_\alpha}\, |\phi_\alpha^A\rangle \otimes |\phi_\alpha^B\rangle$$

- Orthonormal states $\phi_\alpha^A \in \mathcal{H}_A, \phi_\alpha^B \in \mathcal{H}_B$ $\{|\phi_\alpha^A\rangle\}, \{|\phi_\alpha^B\rangle\}$

- Schmidt coefficient p_α
  Schmidt rank χ = (The number of $p_\alpha \neq 0$)

■ Reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_{\alpha=1}^{\chi} p_\alpha |\phi_\alpha^A\rangle\langle\phi_\alpha^A|$

- Entanglement (von Neumann) entropy

$$S = -\mathrm{Tr}\rho_A \log \rho_A = -\sum_{\alpha=1}^{\chi} p_\alpha \log p_\alpha$$

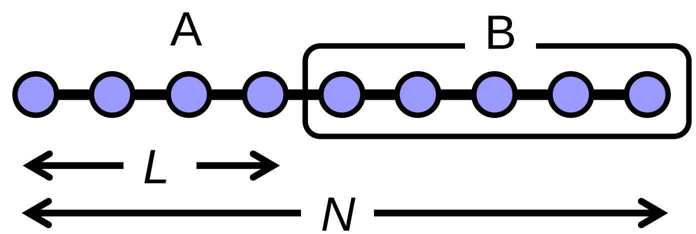
- Entanglement spectrum
  Li and Haldane, *PRL* **101** (2008)

$$p_\alpha = e^{-\xi_\alpha} \qquad (\alpha = 1, 2, \ldots)$$

Entanglement in Fredkin chain

■ Scaling of Entanglement entropy (EE) in 1D

- Area law: *S* is bounded by a constant
- Volume law: $S \propto L$
- CFT scaling: $S \propto \log L$
- Gapped spectrum → area law (Hastings' thm., *J.Stat.Mech.* (2007))



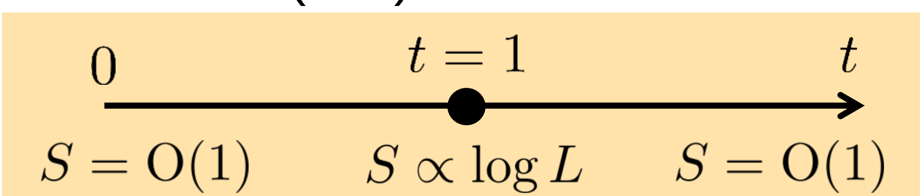
Contraposition

- Non-area law implies gapless spectrum above g.s.
- Volume law when *s*>1, *t*>1 though *H* consists of local terms

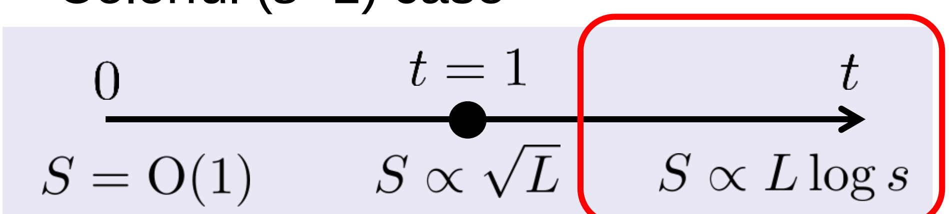
Cf.) Vitagliano *et al.*, *NJP* **12** (2010); Ramirez *et al.*, *J.Stat.Mech* (2014)

■ EE phase diagram

- Colorless (*s*=1) case

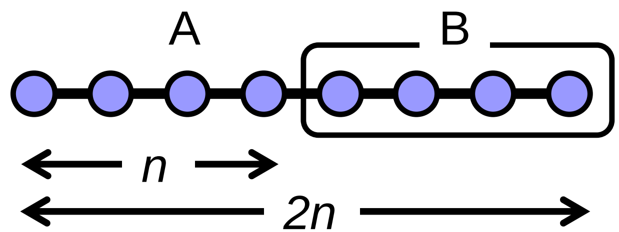


- Colorful (*s*>1) case



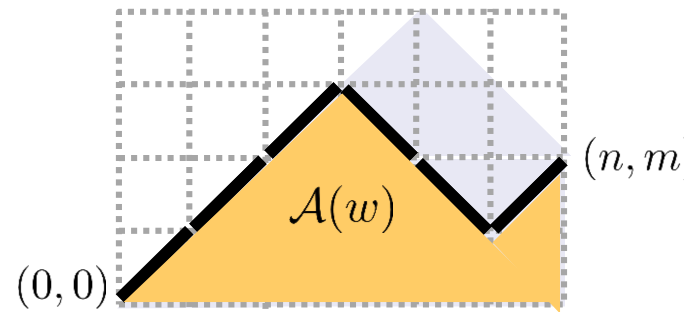
Half-chain entanglement

- System size: $2n$, subsystem=left half
- EE of subsystem A: $S_n(s,t)$



■ *q*-ballot numbers (*q*=*t*)

- $w \in C_{n,m}$: A path from (0,0) to (*n*,*m*)
- *A*(*w*): Area between *w* and the *x*-axis
- $M_{n,m}(t) = \sum_w t^{\mathcal{A}(w)}$



Ex.) $M_{6,2}(t) = t^{14} + t^{12} + 2t^{10} + 2t^8 + 2t^6 + t^4$

$M_{n,m} = 0$ if $n-m$ is odd.

■ EE in terms of $M_{n,m}$

$$S_n(s,t) = -\sum_{m=0}^{n} p_{n,m}(s,t) \log p_{n,m}(s,t)$$

$$p_{n,m}(s,t) = s^{-m} \frac{\{M_{n,m}(t)\}^2}{M_{2n,0}(t)}$$

- Normalization $M_{2n,0}(t)$ → Carlitz-Riordan *q*-Catalan num.

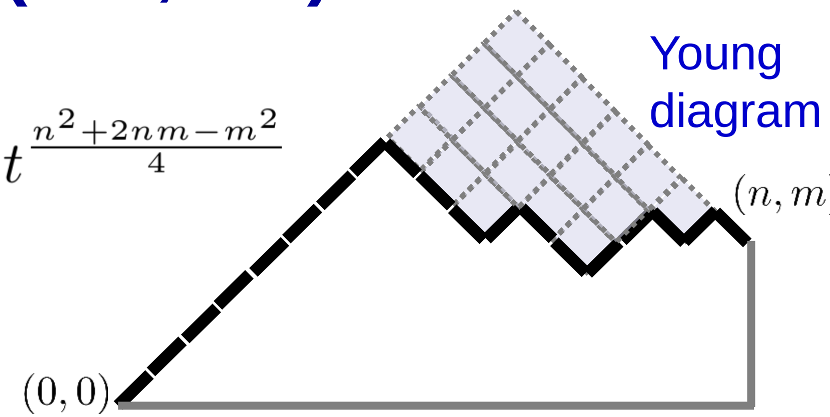
Proof of the volume law (*s*>1, *t*>1)

■ Lemma (bound on $M_{n,m}$)

$$t^{\frac{n^2+2nm-m^2}{4}} \leq M_{n,m}(t) \leq C(t)\, t^{\frac{n^2+2nm-m^2}{4}}$$

where $C(t) = \sum_{k=0}^{\infty} t^{-2k} P(k) < \infty$

with $P(k)$ the integer partition of *k*.



■ Theorem

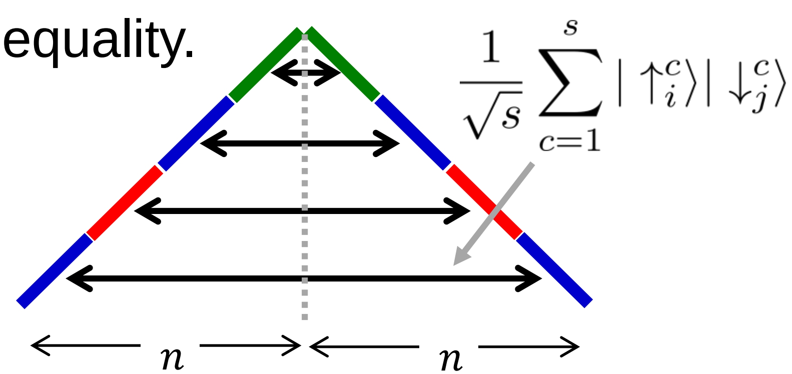
When $t > 1$, the EE $S_n(s,t)$ satisfies

$$n \log s + D_1(s,t) \leq S_n(s,t) \leq n \log s + D_2(t) + D_3(t),$$

where $D_1(s,t)$, $D_2(t)$, and $D_3(t)$ are constants independent of n.

Proof) Based on Lemma and Gibbs inequality.

- *s*=1 → Area law, *s*>1 → Volume law
- For *s*>1 and *t* infinity, each matched pair is maximally entangled.

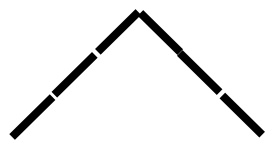


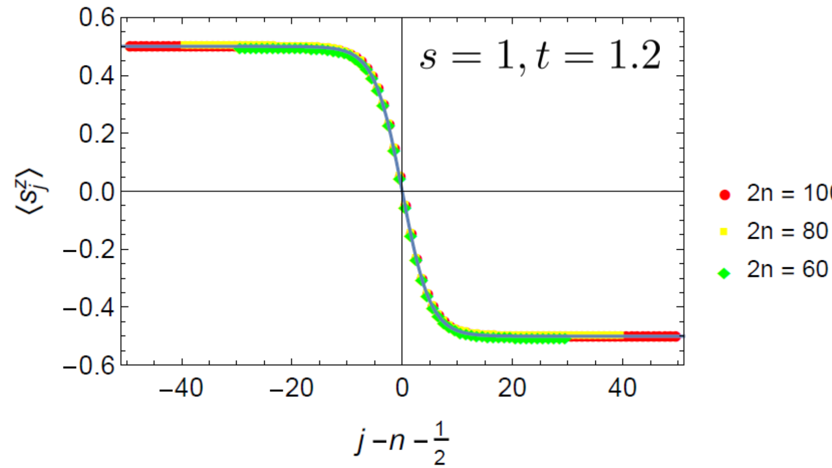
Other results

■ Magnetization

0 — $t=1$ — t

Antiferro-magnetic | Zero bulk M | Domain wall





■ Finite-size gap

- Colorless (s=1) case

0 — $t=1$ — t

gapped | $\propto 1/N^z$ | $< t^{-N/2+1}$

- Colorful (s>1) case

0 — $t=1$ — t

gapped | $\propto 1/N^z$ | $< \mathrm{O}(s^n t^{-N^2/8})$

- Gap for t<<1 can be proved using Knabe's method.
- Power law at t=1. Movassagh, arXiv:1609.09160.
- Exponentially or super-exponentially small gap for t>1.
  Udagawa-Katsura, *JPA* **50** (2017); Zhang-Klich, *JPA* **50** (2017)

Fradkin's work

- Eduardo Fradkin $\neq$ Edward Fredkin
  - Argentinian-American theoretical physicist at University of Illinois at Urbana-Champaign.
  - Working in various areas of cond-mat. physics (FQHE etc.) using QFT approaches (from *Wikipedia*)

- Fradkin's paper on Fredkin chain

  - Chen, Fradkin, Witczak-Krempa, *JPA* **50**, 464002 (2017)

  - Quantum Lifshitz model in 1+1D
    Continuum counterpart (≠ continuum limit) of Fredkin chain
    Dynamical exponent $z = 2$
  - DMRG study on the original (lattice) model
    $z \sim 3.23$ ($z_0 \sim 2.76$) for the lowest excitation with $S^z_{\mathrm{tot}} = \pm 1 \ (0)$
  - Fredkin-Heisenberg chain $H_{\mathrm{bulk}} = \alpha H_F + 2(1-\alpha) H_H$

Outline

1. Introduction
2. Fredkin spin chain
3. Main results
4. **Super-frustration-free systems**

- N=1 SUSY QM
- Local supercharges
- Majorana-Nicolai model

5. Summary

N=1 Supersymmetric (SUSY) QM

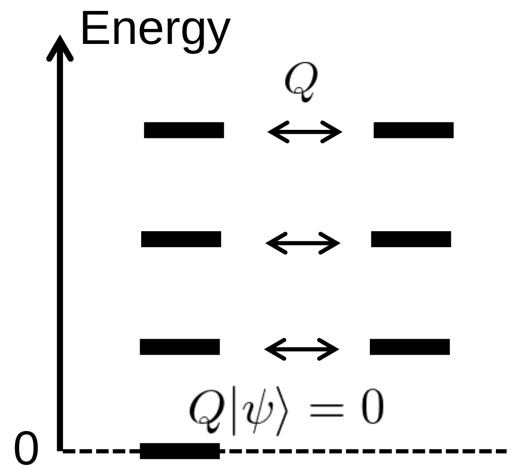
Algebraic structure

- Fermionic parity: $(-1)^F$ (F: total fermion num.)
- Supercharge: Q $(Q^\dagger = Q)$ anti-commutes with $(-1)^F$
- Hamiltonian: $H = Q^2$
- Symmetry: $[H, (-1)^F] = [H, Q] = 0.$

Spectrum of H

- $E \geqq 0$ for all states, as H is p.s.d
- $E > 0$ states come in pairs $\{|\psi\rangle, Q|\psi\rangle\}$
- $E = 0$ state must be annihilated by Q

G.S. energy = 0 → SUSY ***unbroken***
G.S. energy > 0 → SUSY ***broken***



Super-frustration-free systems

■ "Local" supercharge

- Total supercharge: $Q = \sum_j q_j$
- Local supercharge: Each q_j anti-commutes with $(-1)^F$

Definition. $Q = \sum_j q_j$ is said to be *super-frustration-free* if there exists a state $|\psi\rangle$ such that $q_j|\psi\rangle = 0$ for all *j*.

Lattice Majorana fermions

$$(\gamma_i)^\dagger = \gamma_i, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

- Fermionic parity: $(-1)^F = \mathrm{i}^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$
- Complex fermions from Majoranas

$$c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - \mathrm{i}\gamma_{2j})$$

Each *γ* fermion carries quantum dimension $\sqrt{2}$

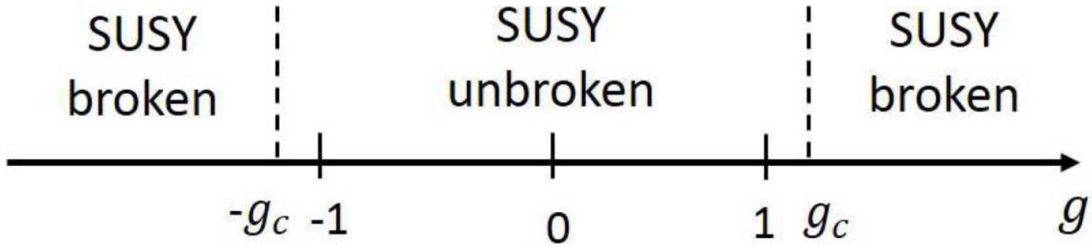
Majorana-Nicolai model

■ Definition

- Supercharge $$Q = \sum_j (g\gamma_j + \mathrm{i}\gamma_{j-1}\gamma_j\gamma_{j+1}), \quad (g \in \mathbb{R})$$
- Hamiltonian
  $H = Q^2$ consists of quadratic and quartic terms in γ

■ Phase diagram

- Sannomiya-Katsura, arXiv:1712.01148
- O'Brien-Fendley, arXiv:1712.0662, *PRL* **120** (2018) [More general]



(NG fermion with cubic dispersion)

- Free-fermionic when g>>1. Rigorous upper bound on g_c.
- Integrable at g=0, super-frustration-free at $g=\pm 1$.

Super-frustration-free at *g*=1

$$Q = \sum_{l=1}^{N/2} (\gamma_{2l-2} + \gamma_{2l+1}) \underbrace{(1 + \mathrm{i}\gamma_{2l-1}\gamma_{2l})}_{h_{2l-1}} = \sum_{l=1}^{N/2} (\gamma_{2l-1} + \gamma_{2l+2}) \underbrace{(1 + \mathrm{i}\gamma_{2l}\gamma_{2l+1})}_{h_{2l}}$$

- h_{2l-1} : Local *H* of Kitaev chain in a trivial phase
- h_{2l} : Local *H* of Kitaev chain in a topological phase
- $H = Q^2$ has two g.s. annihilated by all local *q*. Easy to write down their explicit forms.

Nicolai models with *N*=2 SUSY

- Nicolai, *JPA* **9**, 1497 (1976); *JPA* **10**, 2143 (1977)

$$Q = \sum_{k=1}^{(N-1)/2} c_{2k-1}\, c_{2k}^{\dagger}\, c_{2k+1}$$

- Sannomiya-Katsura-Nakayama, *PRD* **94**, 045014 (2016); *PRD* **95**, 065001 (2016) G.S. degeneracy grows exponentially with system size.

$$Q = \sum_{j}^{N-2} c_j\, c_{j+1}\, c_{j+2}$$

- Schoutens *et al*., in preparation(?), counting the number of g.s.

Summary

- Studied frustration-free Fredkin chains described by Dyck paths
- Rigorous results on entanglement entropy, finite-size gap, etc.

Colorless (s=1) case

0	$t=1$	t
$S=\mathrm{O}(1)$	$S \propto \log L$	$S=\mathrm{O}(1)$
gapped	$\propto 1/N^z$	$< t^{-N/2+1}$

Colorful (s>1) case

Volume law

0	$t=1$	t
$S=\mathrm{O}(1)$	$S \propto \sqrt{L}$	$\boxed{S \propto L \log s}$
gapped	$\propto 1/N^z$	$< \mathrm{O}(s^n t^{-N^2/8})$

- Studied super-frustration-free fermionic systems

What I did not touch on

- Determinant formula for *q*-Carlitz-Riordan, Grothendieck poly?
  Y. Ueno, *J. Alg.* **116**, 261 (1988)
- Stochastic model corresponding to Fredkin chain
- Connection to Temperley-Lieb and Artin group?