Integrable dissipative spin chains

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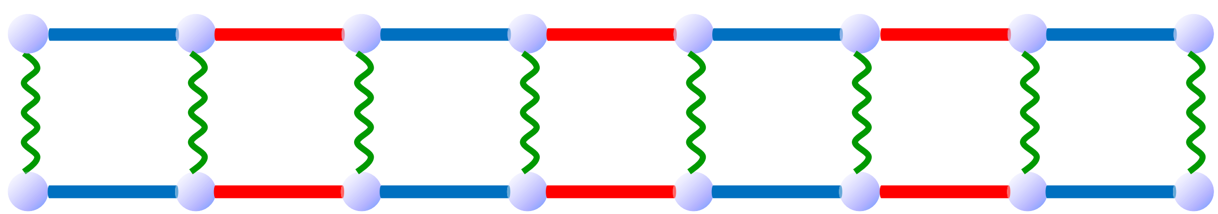


Outline

Introduction

- Open quantum systems, Lindblad equation
- NESS, Liouvillian gap
- Dissipative quantum spin chains

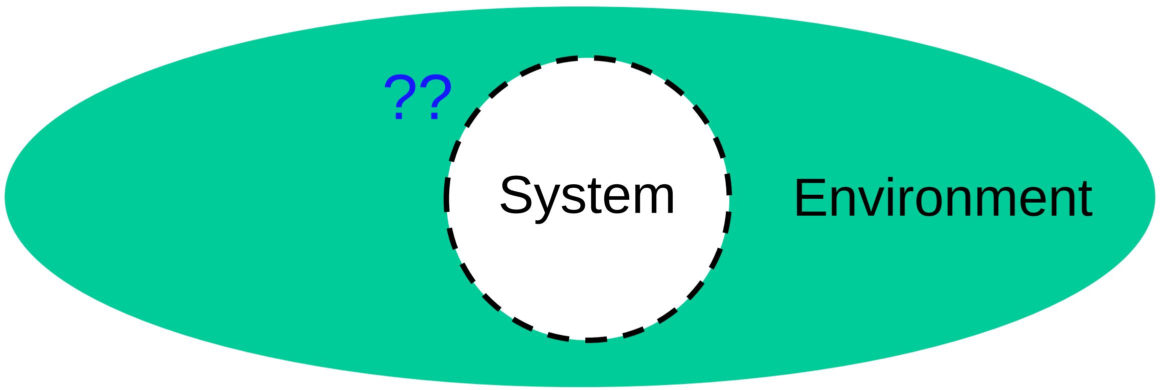
Model I: Quantum compass chain + dissipation

Model II: Quantum Ising chain + dissipation

Summary

Open quantum systems

- System & environment



What's the equation that describes the system dynamics?

- Lindblad (or GKSL) equation [Lindblad, *CMP* **48**, 119 (1976)]
  - Markovian, CPTP (completely positive trace-preserving)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}[\rho] := -\mathrm{i}[H, \rho] + \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

Liouvillian

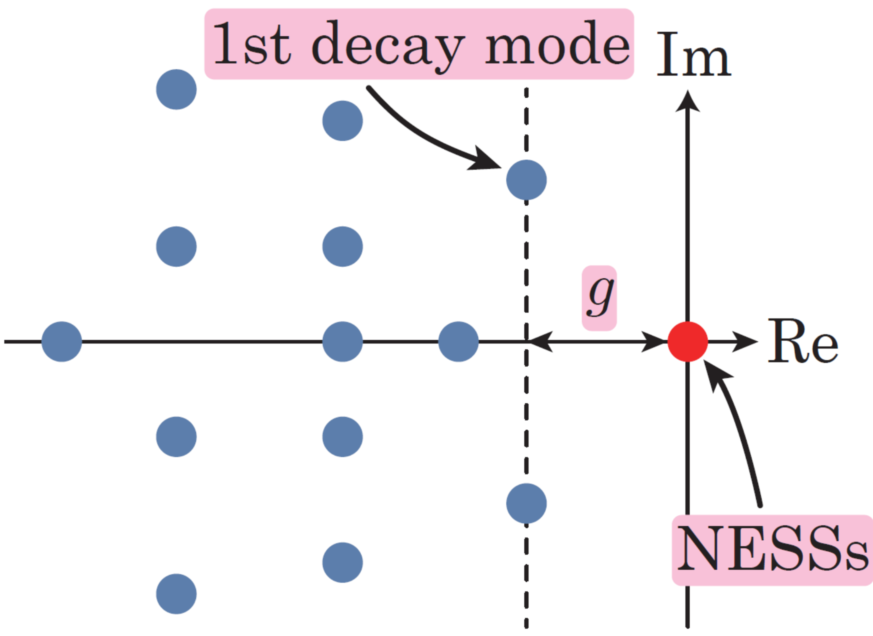
ρ : density matrix $\qquad L_i$: Lindblad operator

NESS and Liouvillian gap

■ Eigenvalue problem

$$\mathcal{L}[\rho_i] = \lambda_i \rho_i$$

1. $\mathrm{Re}(\lambda_i) \leq 0 \quad \forall i$, 2. $\mathcal{L}[\rho_i] = \lambda_i \rho_i \Rightarrow \mathcal{L}[\rho_i^\dagger] = \lambda_i^* \rho_i^\dagger$



- Non-Equilibrium Steady State

$$\mathcal{L}[\rho_0] = 0 \quad (\lambda_0 = 0)$$

- Liouvillian gap

$$g := -\max_{\substack{i \\ \lambda_i \neq 0}} \mathrm{Re}(\lambda_i)$$

determined by the 1st decay mode

Relaxation time: $\tau = 1/g$

Two-level examples

- **Dephasing** $H=0, L=\sqrt{\gamma}\sigma^z \qquad (\gamma>0)$

$$\mathcal{L}[\rho]=\frac{\mathrm{d}\rho}{\mathrm{d}t}=\gamma(\sigma^z\rho\sigma^z-\rho)$$

  - Eigen-operators

$$\mathcal{L}[1_2]=\mathcal{L}[\sigma^z]=0,$$
$$\mathcal{L}[\sigma^x]=-2\gamma\sigma^x, \quad \mathcal{L}[\sigma^y]=-2\gamma\sigma^y$$

→ Two NESS
Liouvillian gap = 2γ

- **Spontaneous emission** $H=\sigma^z, L=\sqrt{\gamma}\sigma^- \qquad \sigma^\pm=\frac{\sigma^x\pm\mathrm{i}\sigma^y}{2}$

$$\mathcal{L}[\rho]=\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\mathrm{i}[\sigma^z,\rho]+\gamma\left(\sigma^-\rho\sigma^+-\frac{1}{2}\{\sigma^+\sigma^-,\rho\}\right)$$

  - Eigen-operators

$$\mathcal{L}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]=0, \quad \mathcal{L}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]=-\gamma\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\mathcal{L}[\sigma^\pm]=\left(-\frac{\gamma}{2}\pm 2\mathrm{i}\right)\sigma^\pm$$

→ Single NESS
Liouvillian gap = $\gamma/2$

Super-operator formalism

■ Vectorization of density matrix

$$\rho = \sum_{m,n} \langle m|\rho|n\rangle\, |m\rangle\langle n| \qquad \{|m\rangle\}:\ \text{orthonormal basis}$$

$$|\sigma\rangle\langle\tau| \Leftrightarrow |\sigma\rangle \otimes |\tau\rangle$$

$$\rho \mapsto |\rho\rangle = \sum_{m,n} \langle m|\rho|n\rangle\, |m\rangle|n\rangle$$

$$P\rho Q \mapsto \sum_{m,n} \langle m|\rho|n\rangle\, (P|m\rangle)(Q^{\mathrm{T}}|n\rangle)$$

■ Example: dephasing $\mathcal{L}[\rho] = \dfrac{\mathrm{d}\rho}{\mathrm{d}t} = \gamma(\sigma^z\rho\sigma^z - \rho)$

$$\mathrm{vec}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} = \begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix} = \underline{\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & -2\gamma & 0 & 0\\ 0 & 0 & -2\gamma & 0\\ 0 & 0 & 0 & 0\end{pmatrix}}\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix}$$

Clearly, the eigenvalues of $\mathcal{L}$ are 0 and -2γ.

Dissipative quantum spin chains

Previous exact results

- Boundary dissipation: Prosen, *PRL* **107**, 137201 (2011)
- Reducible to free fermions: Prosen, *NJP* **10**, 043026 (2008)
- Reducible to imaginary-*U* Hubbard: Medvedyeva, Essler, Prosen, *PRL* **117**, 137202 (2016)
- Richardson-Gaudin: Rowlands, Lamacraft, *PRL* **120**, 090401 (2018)

Our results

Two spin-chain models with bulk dissipation.

Integrable! `Phase transition' in the 1st decay mode.

	Hamiltonian	Dissipation	Liouvillian (after vec.)
I	$\sigma^x_{2j-1}\sigma^x_{2j},\ \sigma^y_{2j}\sigma^y_{2j+1}$	σ^z_j (dephasing)	Non-Hermitian Kitaev Ladder
II	$\sigma^z_j,\ \sigma^x_j\sigma^x_{j+1}$	$\sigma^z_j,\ \sigma^x_j\sigma^x_{j+1}$	Non-Hermitian Ashkin-Teller (→ XXZ)

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Introduction

Model I: Quantum compass chain + dissipation

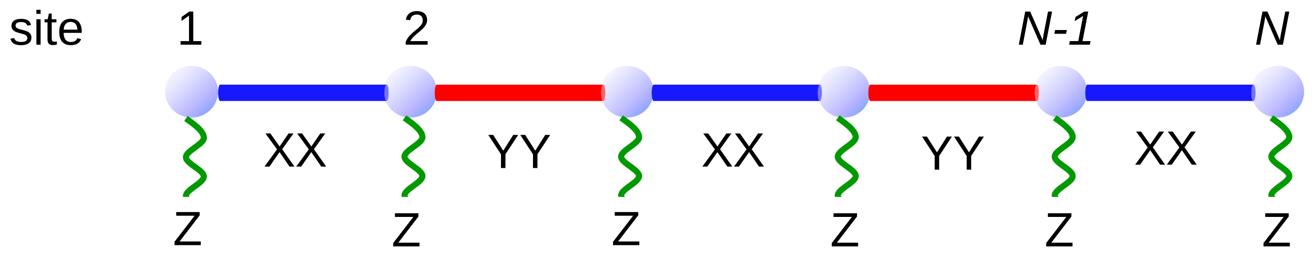
- Hamiltonian, dissipation
- Liouvillian → Kitaev-ladder → non-Hermitian SSH
- Liouvillian gap, autocorrelator of the edge spin

Model II: Quantum Ising chain + dissipation

Summary

Model I

■ Setup: spin-1/2 chain with OBC



■ Hamiltonian

- Quantum compass chain
  Brzezicki, Dziarmaga, and Oles, *PRB* **75**, 134415 (2007)

$$H = H_{\mathrm{XX}} + H_{\mathrm{YY}} = -J_x \sum_{j=1}^{N/2} \sigma_{2j-1}^x \sigma_{2j}^x - J_y \sum_{j=1}^{N/2-1} \sigma_{2j}^y \sigma_{2j+1}^y$$

Half of the XY chain. Reducible to free fermions.

■ Dissipation: dephasing $L_j = \sqrt{\gamma}\sigma_j^z \quad (j = 1, ..., N)$

N.B. $L_j^\dagger = L_j, \quad (L_j)^2 = \gamma$

- XY and XXZ with dephasing: Znidaric, *PRE* **92**, 042143 (2015)

Steady states

- Z2 parity $$P = \prod_{j=1}^{N} \sigma_j^z \qquad [P, H] = 0, \quad [P, L_j] = 0 \ \forall j$$

- NESS are prop. to the projections to the parity sectors

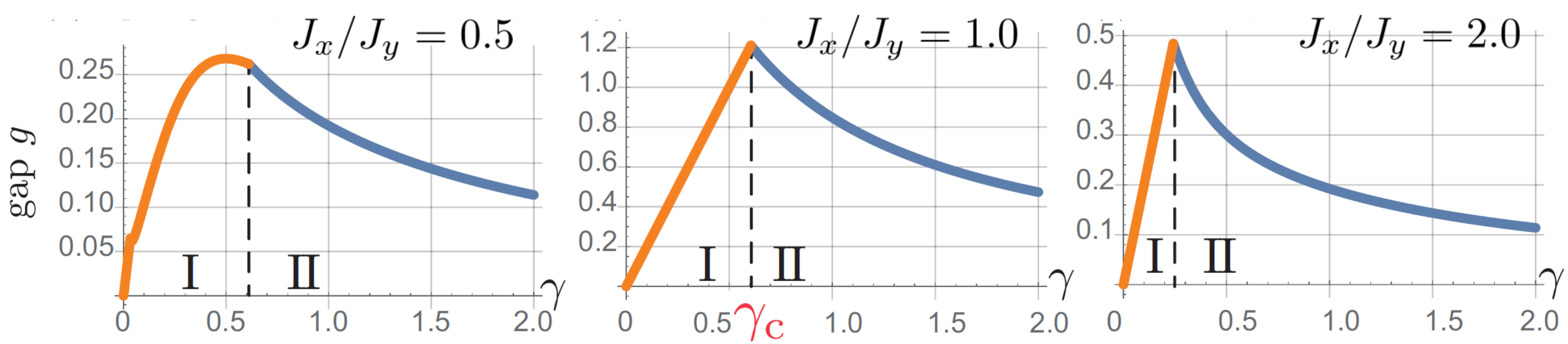
$$\rho_\pm = \frac{1 \pm P}{2^N}$$

Ex.) *N*=2 $$P_+ = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$$

- Proof $$\mathcal{L}[\rho_\pm] = -\mathrm{i}[H, \rho_\pm] + \sum_j [L_j, \rho_\pm] L_j \ = 0$$

- Unique NESS: Numerically checked uniqueness for small *N*

Liouvillian gap



Mapping to Kitaev ladder

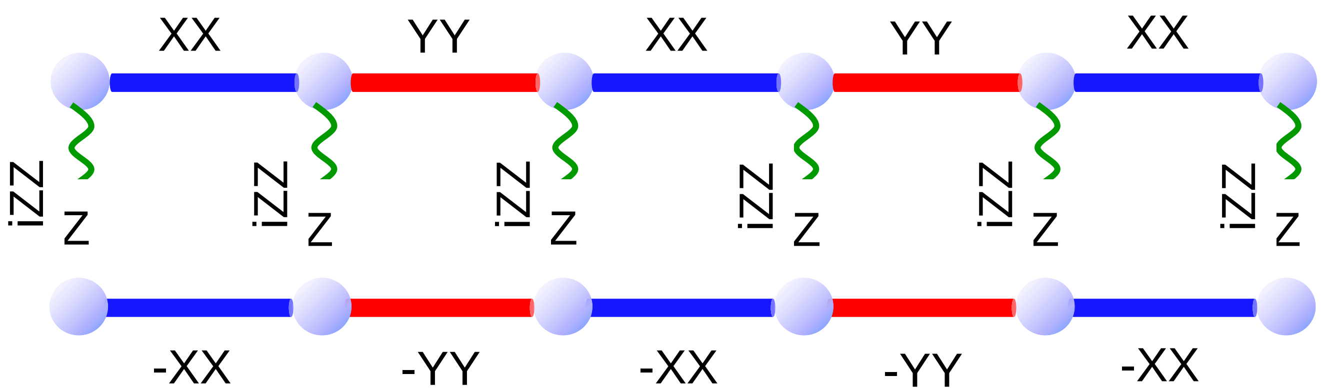
■ Vectorization

$$|\sigma_1, \ldots, \sigma_N\rangle\langle\tau_1, \ldots, \tau_N| \Leftrightarrow |\sigma_1, \ldots, \sigma_N\rangle \otimes |\tau_1, \ldots, \tau_N\rangle$$

With this identification, ρ can be thought of as a state on 2-leg ladder.

■ Non-Hermitian `Hamiltonian'

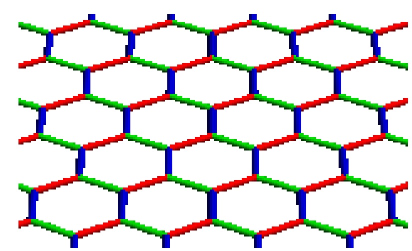
$$\mathcal{H} = \mathrm{i}\mathcal{L} + \mathrm{const.}$$



Kitaev model on a ladder!

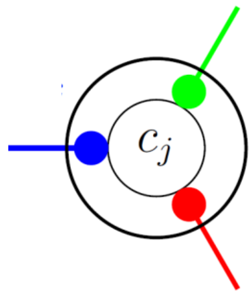
A. Kitaev, *Ann. Phys.* **321**, 2 (2006)

Solvable in the same manner as Kitaev honeycomb.



From Kitaev to SSH

- Spin → Majorana → complex fermions



mobile immobile

$(\sigma_j^x, \sigma_j^y, \sigma_j^z)$ → $(c_j, b_j^x, b_j^y, b_j^z)$ + other sublattice

Gauge fixing → (c_j, d_j) → $(f_j, f_j^\dagger)$

$\{f_i, f_j^\dagger\} = \delta_{i,j},$

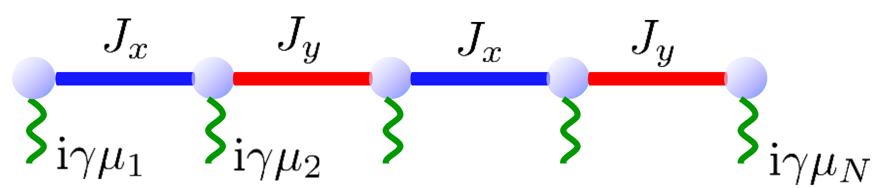
$\{f_i, f_j\} = \{f_i^\dagger, f_j^\dagger\} = 0$

- Non-Hermitian tight-binding chain

$$\mathcal{H} = 2J_x \sum_{i=1}^{N/2} (f_{2i-1}^\dagger f_{2i} + \text{h.c.}) + 2J_y \sum_{i=1}^{N/2-1} (f_{2i}^\dagger f_{2i+1} * \text{h.c.})$$

$$+ 2\mathrm{i}\gamma \sum_{j=1}^{N} \mu_j \left(f_j^\dagger f_j - \frac{1}{2} \right)$$

$\mu_j = \pm 1$



Su-Schrieffer-Heeger chain (*PRL*1979) with imaginary *μ*!

Solvable sector-by-sector (Sector~configuration of *μ*s)

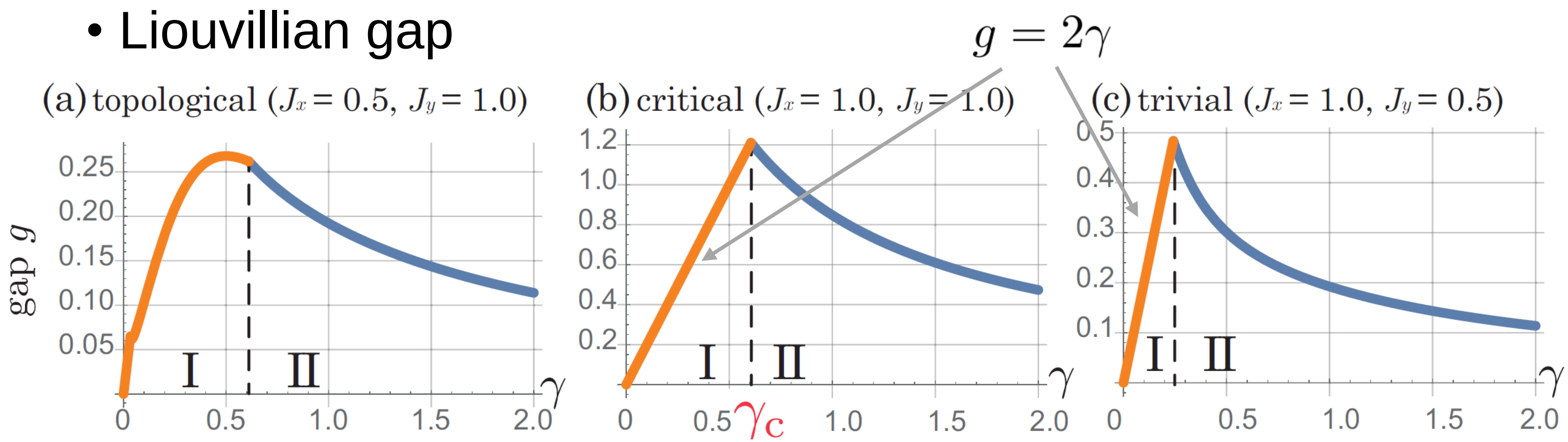
Liouvillian gap

■ Fermionic picture

- Steady state: fully filled state in sector $\boldsymbol{\mu} = (+, +, \cdots, +)$
- 1st decay modes: must be in the other sectors

■ Results

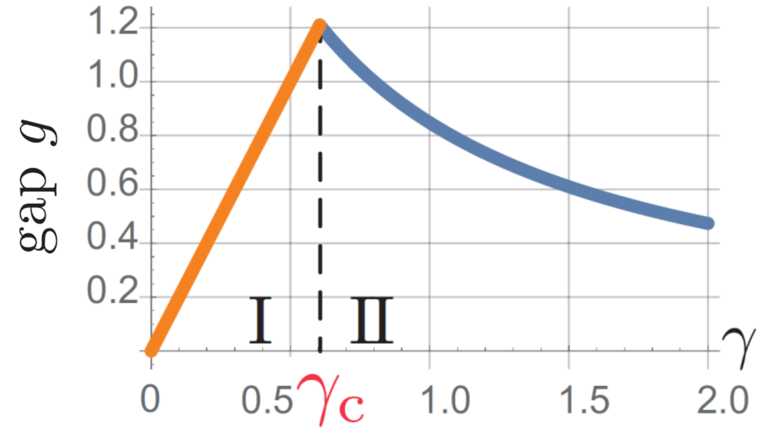
- Three regimes
  1. Topological (Jx<Jy), 2. critical (Jx=Jy), 3. Trivial (Jx>Jy)
- Liouvillian gap



Config. of μs for the 1st decay mode in I is different from that in II.
→ Nonmonotonic behavior in the Liouvillian gap g!

Liouvillian gap at `critical' case (Jx=Jy)

- Numerical result for N=10
- Jx = Jy =1.0



■ Analytical results

- 1st decay mode lives in the sector $\boldsymbol{\mu} = (+, \cdots, +, -, +, \cdots, +)$ or $\boldsymbol{\mu} = (-, -, +, \cdots, +)$.
- Exact result for g (Jx=Jy=1.0)

$$g = \begin{cases} 2\gamma & (0 \leq \gamma \leq \gamma_c) \\ \dfrac{6^{1/3}\left(9\gamma^2 + \sqrt{48\gamma^6 + 81\gamma^4}\right)^{2/3} - 2 \cdot 6^{2/3}\gamma^2}{3\gamma(9\gamma^2 + \sqrt{48\gamma^6 + 81\gamma^4})^{1/3}} & (\gamma_c \leq \gamma) \end{cases}$$

- Critical dissipation strength $$\gamma_c = \sqrt{\frac{\sqrt{3}-1}{2}} = 0.605...$$

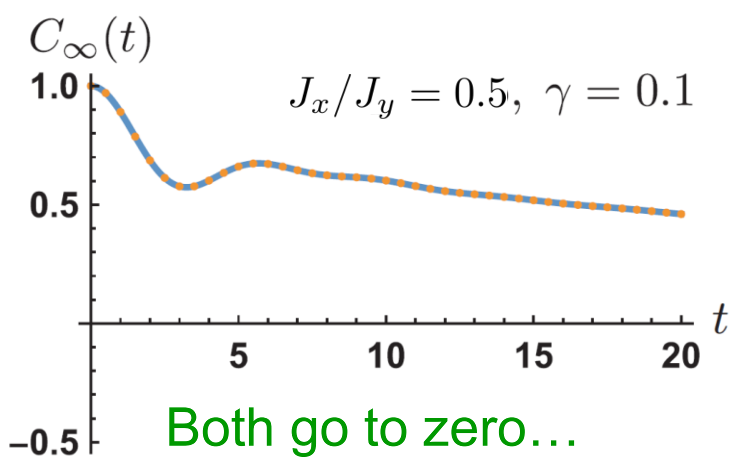
Autocorrelator of the edge spin

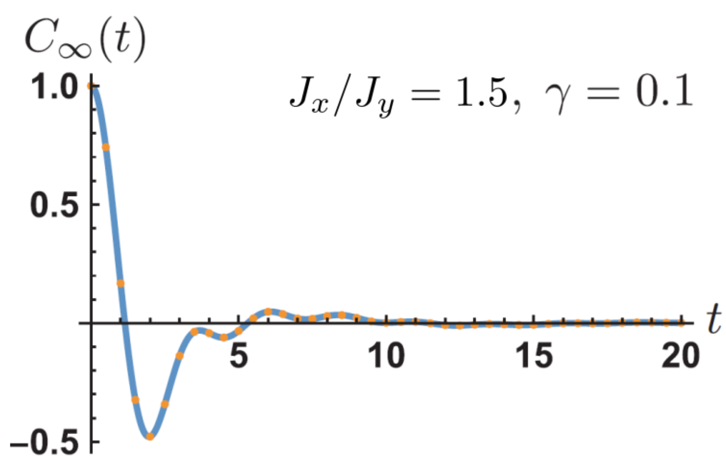
- Signature of long-time coherence

$$C_\infty(t) = \langle \sigma_1^z(t)\sigma_1^z(0)\rangle_{T=\infty} = \frac{1}{2^N}\mathrm{tr}\left(e^{t\mathcal{L}^*}[\sigma_1^z]\sigma_1^z\right)$$

  - XYZ chain: Kemp, Yao, Laumann, Fendley, *JSM* (2017) 063105
  - Dissipative quantum Ising: Vasiloiu *et al*., *PRB* **98**, 094308 (2018)

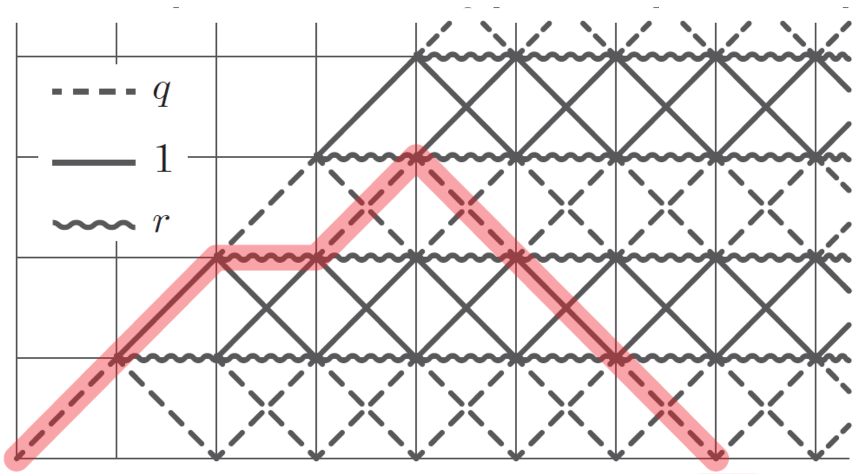
- Exact results
  - Tour de force of 19C math.
  - Topological regime
  - Trivial regime

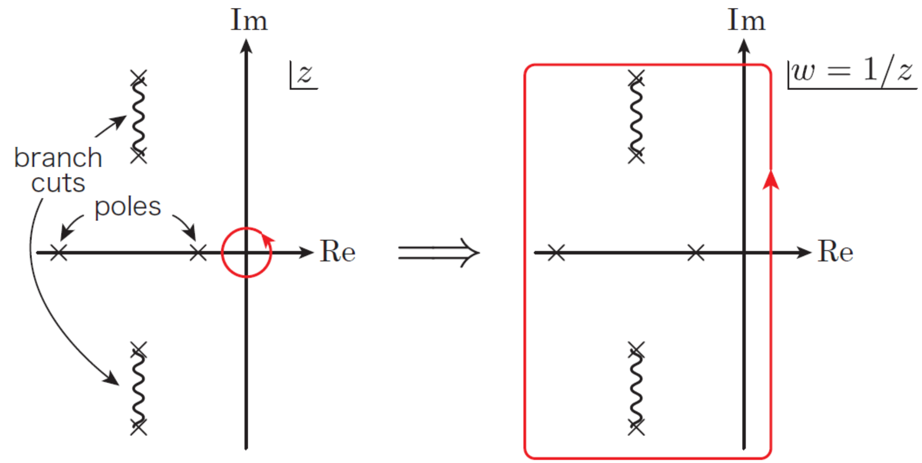




Both go to zero…
Decay is suppressed in topo. regime

Autocorrelator of the edge spin





- Tour de force of 19C math.

$$
C_\infty(t) = \begin{cases}
\dfrac{-\eta_+^2+1-q^2+r^2}{r(\eta_+-\eta_-)}\mathrm{e}^{\eta_+ t} \\
\quad + \dfrac{\mathrm{e}^{-rt}}{\pi r}\displaystyle\int_{1-q}^{1+q} f(y,q)\,\frac{(r^{-1}-r)y\cos(yt)+\left[y^2-(r+\eta_+)(r+\eta_-)\right]\sin(yt)}{[y^2+(r+\eta_+)^2][y^2+(r+\eta_-)^2]}\,\mathrm{d}y & (0<q\le 1) \\
\dfrac{\mathrm{e}^{-rt}}{\pi r}\displaystyle\int_{q-1}^{q+1} f(y,q)\,\frac{(r^{-1}-r)y\cos(yt)+\left[y^2-(r+\eta_+)(r+\eta_-)\right]\sin(yt)}{[y^2+(r+\eta_+)^2][y^2+(r+\eta_-)^2]}\,\mathrm{d}y & (q\ge 1,\ 0\le r<1) \\
\mathrm{e}^{-t}\cos\left(\sqrt{q^2-1}\,t\right) + \dfrac{\mathrm{e}^{-t}}{\pi}\displaystyle\int_{q-1}^{q+1}\frac{f(y,q)}{y^2-q^2+1}\sin(yt)\,\mathrm{d}y & (q\ge 1,\ r=1) \\
\dfrac{-\eta_+^2+1-q^2+r^2}{r(\eta_+-\eta_-)}\mathrm{e}^{\eta_+ t} - \dfrac{-\eta_-^2+1-q^2+r^2}{r(\eta_+-\eta_-)}\mathrm{e}^{\eta_- t} \\
\quad + \dfrac{\mathrm{e}^{-rt}}{\pi r}\displaystyle\int_{1-q}^{1+q} f(y,q)\,\frac{(r^{-1}-r)y\cos(yt)+\left[y^2-(r+\eta_+)(r+\eta_-)\right]\sin(yt)}{[y^2+(r+\eta_+)^2][y^2+(r+\eta_-)^2]}\,\mathrm{d}y & (q\ge 1,\ r>1)
\end{cases},
$$

where

$$
\eta_\pm(q,r) := \left(-1-r^2 \pm \sqrt{(1+r^2)^2-4q^2r^2}\right)/(2r)
$$

$$
f(y,q) = \sqrt{[(q+1)^2-y^2][y^2-(q-1)^2]}.
$$

Outline

Introduction

Model I: Quantum compass chain + dissipation

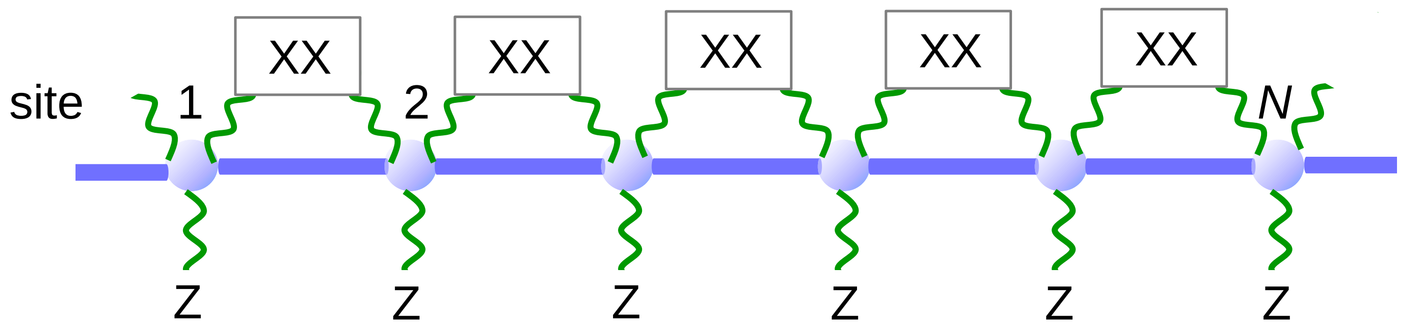
Model II: Quantum Ising chain + dissipation

- Hamiltonian, dissipation
- Liouvillian → Ashkin-Teller → Non-Hermitian XXZ
- Liouvillian gap

Summary

Model II

- Setup: spin-1/2 chain with PBC



- Hamiltonian
  - Quantum Ising chain

$$H = -J\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x - h\sum_{j=1}^{N}\sigma_j^z$$

Can be mapped to free Majorana chain. Critical at $J=h$.

- Dissipation: dephasing + XX

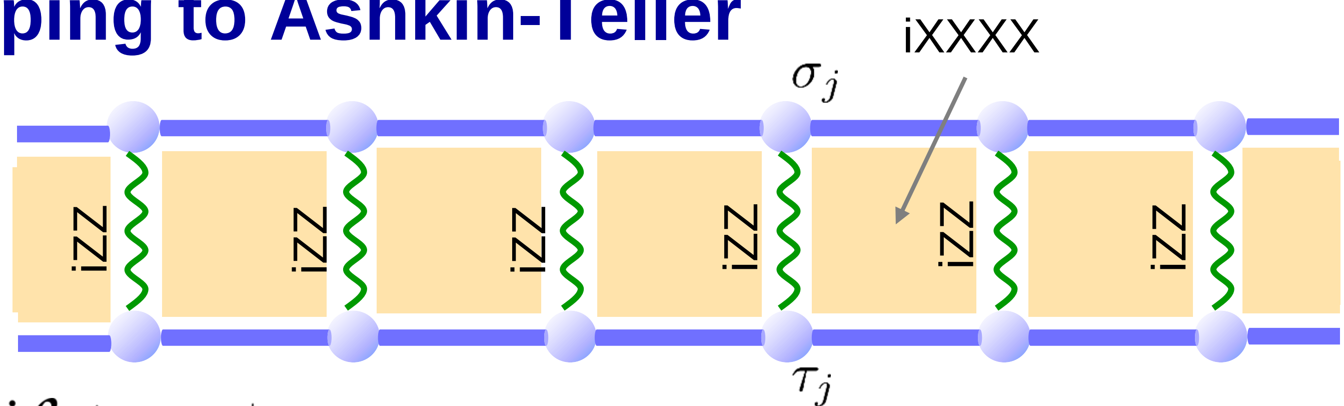
$$L_j^{(1)} = \sqrt{\Delta_1}\sigma_j^z, \quad L_j^{(2)} = \sqrt{\Delta_2}\sigma_j^x\sigma_{j+1}^x \quad (j=1,\ldots,N)$$

N.B. $(L^{(a)})^\dagger = L_j^{(a)}, \quad (L_j^{(a)})^2 = \Delta_a \quad (a=1,2)$

Steady states

- Parity op. $P = \sigma_1^z \sigma_2^z \cdots \sigma_N^z$ commutes with *H* and $L_j^{(1)}, L_j^{(2)} \ \forall j$
- NESS: $\rho_\pm = \dfrac{1 \pm P}{2^N}$ (Uniqueness can be proved.)

Mapping to Ashkin-Teller



$$\mathcal{H} = \mathrm{i}\mathcal{L} + \mathrm{const.}$$

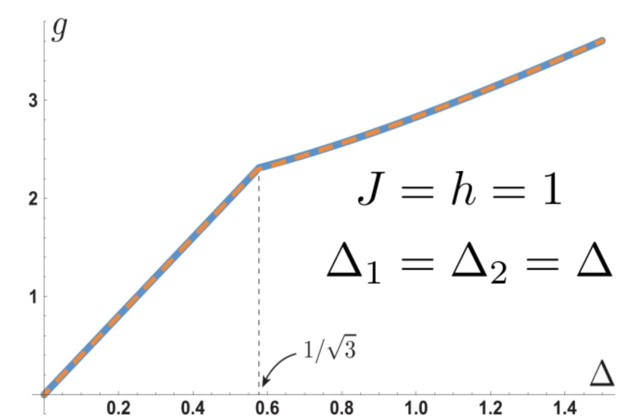
$$= -\sum_j (J\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z) + \sum_j (J\tau_j^x \tau_{j+1}^x + h\tau_j^z)$$

$$+ \mathrm{i}\Delta_1 \sum_j \sigma_j^z \tau_j^z + \mathrm{i}\Delta_2 \sum_j \sigma_j^x \sigma_{j+1}^x \tau_j^x \tau_{j+1}^x$$

Quantum Ashkin-Teller model!

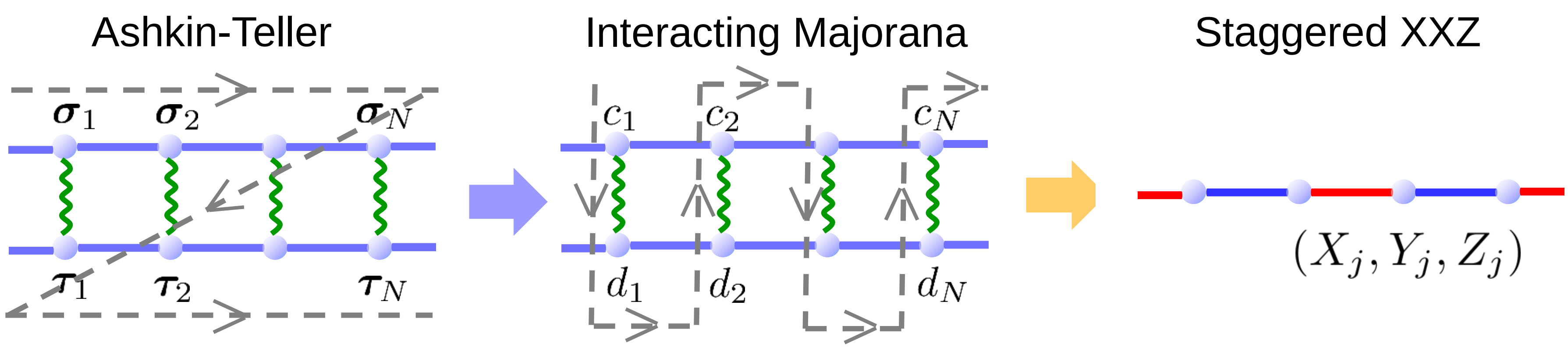
But non-Hermitian interchain coupling.

- Liouvillian gap



From Ashkin-Teller to staggered XXZ

■ Spin → Majorana → Spin



Modern rephrasing of
Kohmoto, den Nijs, Kadanoff, *PRB* **24**, 5229 (1981).

■ Non-Hermitian Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N}[h(X_{2i-1}X_{2i} + Y_{2i-1}Y_{2i}) + \mathrm{i}\Delta_1 Z_{2i-1}Z_{2i}]$$

$$+ \sum_{i=1}^{N-1}[J(X_{2i}X_{2i+1} + Y_{2i}Y_{2i+1}) + \mathrm{i}\Delta_2 Z_{2i}Z_{2i+1}] + \mathcal{H}_{\mathrm{boudary}}$$

$|\Uparrow\rangle = |\uparrow\cdots\uparrow\rangle$ and $|\Downarrow\rangle = |\downarrow\cdots\downarrow\rangle$ correspond to 2 NESS.

Uniqueness of NESS

■ Lemma

Let T and K be (finite) Hermitian matrices. Let κ_i and λ_i (i=1,2,…) be eigenvalues of K and those of $\mathcal{H} = T + \mathrm{i}K$, respectively. Then, we have

$$\max\left(\mathrm{Im}\lambda_i\right) \leq \max \kappa_i .$$

■ Proof

Let $|\psi_i>$ be a right-eigenvector of H with eigenvalue λ_i.
Assume that $|\psi_i>$ is normalized as $<\psi_i|\psi_i>=1$. Then,

$$\lambda_i = \langle \psi_i | \mathcal{H} | \psi_i \rangle = \underline{\langle \psi_i | T | \psi_i \rangle} + \mathrm{i} \underline{\langle \psi_i | K | \psi_i \rangle}$$

real

$$\Rightarrow \quad \mathrm{Im}\lambda_i = \langle \psi_i | K | \psi_i \rangle \leq \max \kappa_i$$

■ Application

$$K = \sum_{i=1}^{N} (\Delta_1 Z_{2i-1} Z_{2i} + \Delta_2 Z_{2i} Z_{2i+1}) + \Delta_2 Z_2 \cdots Z_{2N-1}$$

$|\Uparrow\rangle$ and $|\Downarrow\rangle$ are eigenvectors of K with the maximum eigenvalue.
(No other states with the same e.v.) They are annihilated by T.

Liouvillian gap on self-dual line

■ Reducible to uniform XXZ

$$J = h(=1), \quad \Delta_1 = \Delta_2 = \Delta$$

- Hamiltonian $$\mathcal{H} = \sum_{j=1}^{N} (X_j X_{j+1} + Y_j Y_{j+1} + \mathrm{i}\Delta Z_j Z_{j+1}) + \mathcal{H}_{\mathrm{boudary}}$$

- Boundary term

$$\mathcal{H}_{\mathrm{boundary}} = (-1)^N Q_X (X_{2N} X_1 + Q_Z Y_{2N} Y_1) + \mathrm{i}\Delta Q_Z Z_{2N} Z_1$$

$Q_Z = \prod_j Z_j, \quad Q_X = \prod_j X_j$ commutes with $\mathcal{H}$.

■ Different sectors

Alcaraz *et al*., *Ann. Phys.* **182**, 280 (1988).

(i) $Q_Z = +1, \quad (-1)^N Q_X = +1$	Perodic	U(1) symm.
(ii) $Q_Z = +1, \quad (-1)^N Q_X = -1$	Anti-perodic	
(iii) $Q_Z = -1, \quad (-1)^N Q_X = +1$	Anti-diagonal twisted	~~U(1) symm.~~
(iv) $Q_Z = -1, \quad (-1)^N Q_X = -1$		

Liouvillian gap in sectors (i) and (ii)

■ Explicit 1st decay mode

- Singular state $$|\chi^{(i)}\rangle = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} (-1)^{j-1} \sigma_j^- \sigma_{j+1}^- | \Uparrow \rangle$$

Avdeev, Vladimirov, *Theor. Math. Phys*. **69**, 1071 (1986)
Essler, Korepin, Schoutens, *JPA* **25**, 4115 (1992)

- Eigenstate of $\mathcal{H}$ $$\frac{1+(-1)^N Q_X}{\sqrt{2}} |\chi^{(i)}\rangle$$

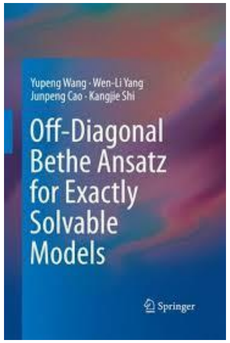
$\text{with eigenvalue } (2N-4)\Delta\mathrm{i}$ Similar eigenstate in (ii)

■ Exact gap

$$g_{(\mathrm{i})} = g_{(\mathrm{ii})} = 4\Delta$$

One can prove independently that the lower bound for *g* is 4Δ.
→ The above state saturates the bound!

Gap in sectors (iii) and (iv)



■ Off-diagonal Bethe ansatz

- Qiao *et. al*., *NJP* **20**, 073046 (2018)
- Maximum modulus eigenvalue $\eta = \mathrm{i}\Delta$

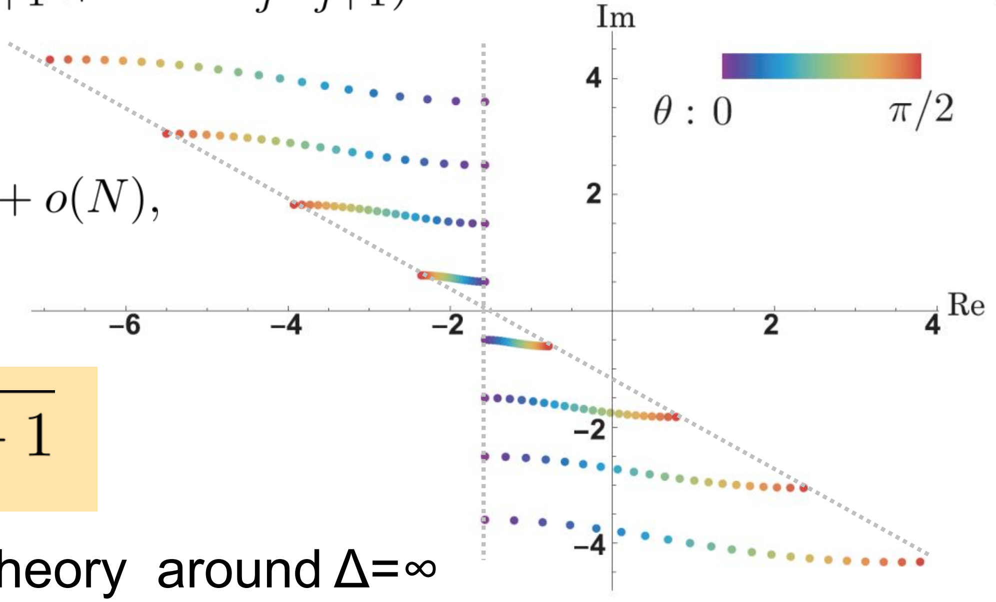
$$E = -2\mathrm{i}\sinh\eta \sum_{j=1}^{2N}\left[\cot\left(u_j + \frac{\mathrm{i}\eta}{2}\right) - \cot\left(u_j - \frac{\mathrm{i}\eta}{2}\right)\right] + 2N\sinh\eta + 2\sinh\eta,$$

- Bethe roots

$$\mathcal{H}(\theta) = \sum (X_j X_{j+1} + Y_j Y_{j+1} + \Delta e^{\mathrm{i}\theta} Z_j Z_{j+1})$$

Tilted string

$$u_j = -\frac{\pi}{2} + \left(\frac{2N+1}{2} - j\right)\mathrm{i}\eta + o(N),$$



■ Exact gap

$$g_{(\mathrm{iii})} = g_{(\mathrm{iv})} = 2\sqrt{\Delta^2 + 1}$$

Consistent with perturbation theory around Δ=∞

Summary

	Hamiltonian	Dissipation	Liouvillian (after vec.)
I	$\sigma^x_{2j-1}\sigma^x_{2j},\ \sigma^y_{2j}\sigma^y_{2j+1}$	σ^z_j (dephasing)	Non-Hermitian Kitaev Ladder
II	$\sigma^z_j,\ \sigma^x_j\sigma^x_{j+1}$	$\sigma^z_j,\ \sigma^x_j\sigma^x_{j+1}$	Non-Hermitian Ashkin-Teller (→ XXZ)

- Mapping to integrable non-Hermitian Hamiltonians
- NESS = completely mixed state in each parity sector
- Exact formulas for Liouvillian gap, transition at 1st decay mode