Constructing frustration-free models via Witten's conjugation

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Outline

1. Introduction
   - Exactly solvable models
   - Frustration-free systems

2. Witten's conjugation

3. Examples

4. Super-frustration-free systems

5. Summary

Exactly solvable models

- (Crude) Classification

  - Integrable systems
    Free fermions/bosons, Bethe ansatz
    Many conserved charges

  - Frustration-free systems
    Ground state (g.s.) minimizes each local Hamiltonian
    Explicit g.s., but hard to obtain excited states

  NOTE) Not exclusive! S=1/2 ferromagnetic Heisenberg chain is integrable and frustration-free at the same time.

- Today's subject

  - A systematic way to construct frustration-free models

  - Witten's conjugation plays a key role

A crash course in inequalities

Positive semidefinite operators

Appx. in Tasaki, *Prog. Theor. Phys.* **99**, 489 (1998) or his book

$\mathcal{H}$: finite-dimensional Hilbert space.

A, *B*: Hermitian operators on $\mathcal{H}$

- **Definition 1.** We write $A \geq 0$ and say *A* is positive semidefinite (p.s.d.) if $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}$.
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0$.

Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0$.
  Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0$.

Frustration-free systems

■ Anderson's bound (*Phys. Rev*. **83**, 1260 (1951).)

- Total Hamiltonian: $H = \sum_j h_j$
- Sub-Hamiltonian: h_j that satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
  ($E_j^{(0)}$: the lowest eigenvalue of h_j)

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Used to obtain a lower bound on the g.s. energy of AFM Heisenberg model

■ Frustration-free Hamiltonian

The case where the *equality* holds.

Definition. $H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j |\psi\rangle = E_j^{(0)} |\psi\rangle$ for all *j*.

Ex.) *S*=1 Affleck-Kennedy-Lieb-Tasaki (AKLT), toric code, …

$$h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3} (\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2$$

Field-theory counterpart
Lifshitz field theory

Q-info example

■ Trivially solvable model

Local basis

$$Z|0\rangle = +|0\rangle$$
$$Z|1\rangle = -|1\rangle$$
$$X|\pm\rangle = \pm|\pm\rangle$$

- Pauli operators $X_j, Y_j, Z_j \quad (j = 1, ..., N)$
- Hamiltonian $H = \sum_{j=1}^{N}(1 - X_j) \;\geq 0 \quad (\text{p.s.d})$
- Ground state $|\Psi_+\rangle = |+\rangle_1|+\rangle_2 \cdots |+\rangle_N$ No entanglement

■ Cluster model

Briegel & Raussendorf, *PRL* **86**, 910 (2001)

- Hamiltonian $\widetilde{H} = \sum_{j=1}^{N}(1 - Z_{j-1}X_j Z_{j+1}) \;= UHU^\dagger$
- Unitary tr. $U = (CZ)_{1,2}(CZ)_{2,3} \cdots (CZ)_{N,1}$

  $(CZ)|a,b\rangle = (-1)^{ab}|a,b\rangle, \quad (a,b = 0,1)$
- Ground state $|\widetilde{\Psi}_+\rangle = U|\Psi\rangle$ Entangled!

Two models are unitarily equivalent. Something beyond unitary?

Outline

1. Introduction

2. Witten's conjugation

- N=2 supersymmetric QM
- Zero-energy states
- Application to frustration-free systems

3. Examples

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N=2 supersymmetric (SUSY) QM

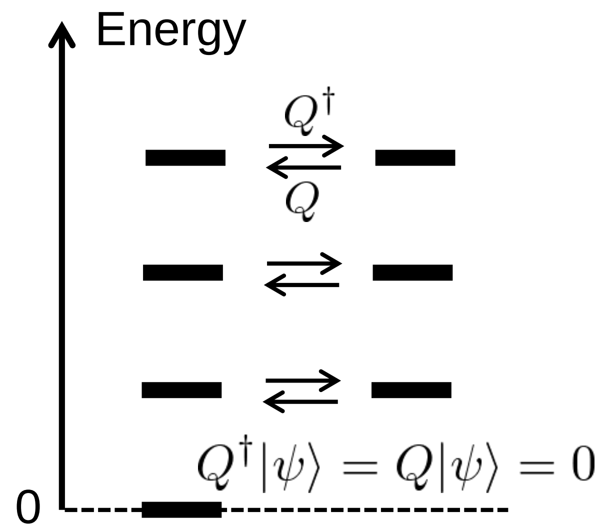
■ Algebra

- Supercharges: $Q,\ Q^\dagger,\quad Q^2=0,\ (Q^\dagger)^2=0$
- Fermionic parity: $\{Q,(-1)^F\}=\{Q^\dagger,(-1)^F\}=0$
- Hamiltonian: $H=\{Q,Q^\dagger\}=QQ^\dagger+Q^\dagger Q$
- Symmetry: $[H,Q]=[H,Q^\dagger]=[H,(-1)^F]=0$

■ Spectrum of H

- $E \geqq 0$ for all states, as H is p.s.d
- $E > 0$ states come in pairs $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
- $E = 0$ iff a state is a SUSY singlet

Ground-state energy = 0 → SUSY ***unbroken***
Ground-state energy > 0 → SUSY ***broken***



Zero-energy states (ZES)

■ Cohomology

- Zero-energy states (ZESs) are in 1-to-1 correspondence with nontrivial cohomology classes of Q.

  Proof) Any ZES $|\psi\rangle$ is annihilated by both Q and $Q^\dagger$.
  But $|\psi\rangle$ cannot be written as $|\psi\rangle = Q|\phi\rangle$ for any $|\phi\rangle$
  since this would imply that $\langle\psi|\psi\rangle = \langle\phi|Q^\dagger|\psi\rangle = 0$.

(ZES of H) = $\dim\left(\mathrm{Ker}\, Q/\mathrm{Im}\, Q\right)$

■ Witten's conjugation (*Nucl. Phys*. B **202**, 253 (1982).)

- Invertible operator M
- New supercharge & Hamiltonian

$$\widetilde{Q} := MQM^{-1}, \qquad \widetilde{H} := \{\widetilde{Q}, \widetilde{Q}^\dagger\}$$

- $\dim\left(\mathrm{Ker}\, \widetilde{Q}/\mathrm{Im}\, \widetilde{Q}\right) = \dim\left(\mathrm{Ker}\, Q/\mathrm{Im}\, Q\right)$

The deformation preserves the number of zero-energy states.

Conjugation argument (non-SUSY ver.)

■ Universal form of frustration-free systems

- Set the ground-state energy to zero
- Hamiltonian (p.s.d.)

$$H = \sum_j L_j^\dagger L_j$$

- Zero-energy ground state

$$|\psi\rangle \text{ s.t. } L_j|\psi\rangle = 0 \ \forall j$$

Can we cook up a new model from a known model? YES!

■ Recipe for the construction

- New *L* operators $\widetilde{L}_j := M L_j M^{-1}$ $\quad (M\text{: invertible})$
- New Hamiltonian

$$\widetilde{H} = \sum_j \widetilde{L}_j^\dagger \widetilde{L}_j$$

- Zero-energy ground state

$$|\widetilde{\psi}\rangle = M|\psi\rangle, \quad \widetilde{L}_j|\widetilde{\psi}\rangle = 0 \ \forall j$$

- $\widetilde{H}$ is inequivalent to H unless *M* is unitary.
- $\widetilde{H}$ and H have the same number of the ground states.
- Similar to the idea of ``Doob transform”

Sandwiching method

A slight generalization

- Inserting a positive definite operator C_j between $\widetilde{L}_j^\dagger$ and $\widetilde{L}_j$ does not change the ground-state (g.s.) manifold.

- Newer Hamiltonian

$$\widetilde{H} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j$$

- Zero-energy g.s.

$$|\widetilde{\psi}\rangle = M|\psi\rangle,$$

$$\widetilde{L}_j = M L_j M^{-1}$$

Theorem

Let $|\Psi_k\rangle$ $(k = 1, \ldots, n)$ be linearly independent zero-energy g.s. of H. The ground-state manifold of $\widetilde{H}$ is given by

$$\widetilde{G} = \mathrm{span}\{M|\Psi_1\rangle, \ldots, M|\Psi_n\rangle\}.$$

Thus, the g.s. degeneracies of H and $\widetilde{H}$ are identical.

Proof) Just follows from $\widetilde{G} \subseteq \bigcap_j \ker(\widetilde{L}_j)$ and $\widetilde{G} \supseteq \bigcap_j \ker(\widetilde{L}_j)$.

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3. Examples

- Classical Ising → XY, XYZ
- S=1/2 Heisenberg → *q*-deformed Heisenberg
- Other examples

4. Super-frustration-free systems
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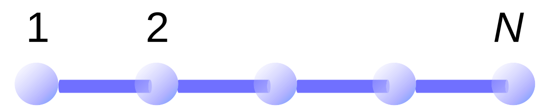
From Ising to XY (1)



- **Spin operators acting on** $(\mathbb{C}^2)^{\otimes N}$
  - Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - Spin op. at site *j* : $\sigma_j^\alpha = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes \sigma^\alpha \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j}, \qquad (\sigma_j^\alpha)^2 = 1$
  - Local basis: $\sigma^z|\uparrow\rangle = +|\uparrow\rangle, \ \sigma^z|\downarrow\rangle = -|\downarrow\rangle$

- **(Classical) Ising chain (with OBC)**
  - Hamiltonian $H = \sum_{j=1}^{N-1} h_j, \quad \boxed{h_j = -2\sigma_j^x \sigma_{j+1}^x + 2 \geq 0}$

    Diagonal in X-basis.
  - *L* operators

$$h_j = L_j^\dagger L_j, \quad L_j = L_j^\dagger = \sigma_j^x - \sigma_{j+1}^x$$

  - Zero-energy ground states $\boxed{|\Psi_\pm\rangle = \bigotimes_{j=1}^{N} (|\uparrow\rangle_j \pm |\downarrow\rangle_j)}$

    NOTE) $\sigma^x(|\uparrow\rangle \pm |\downarrow\rangle) = \pm(|\uparrow\rangle \pm |\downarrow\rangle)$

From Ising to XY (2)

■ Conjugation method

- *M* operator $$M = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}, \quad 0 < r < \infty$$
- XY Hamiltonian

$$H_{\mathrm{XY}} = \sum_{j=1}^{N-1} \widetilde{h}_j^{(1)}, \quad \widetilde{h}_j^{(1)} = \widetilde{L}_j^\dagger \widetilde{L}_j, \quad \widetilde{L}_j = M L_j M^{-1}$$

Simple calculation shows that

$$\boxed{\widetilde{h}_j^{(1)} = -J_x \sigma_j^x \sigma_{j+1}^x - J_y \sigma_j^y \sigma_{j+1}^y + \frac{B_1}{2}(\sigma_j^z + \sigma_{j+1}^z) + \epsilon_1}$$

XY chain in a field! Can be mapped to *free-fermion* chain.

where $J_x = \dfrac{(r + r^{-1})^2}{2}, \quad J_y = \dfrac{(r - r^{-1})^2}{2}, \quad B_1 = r^2 - r^{-2}, \quad \epsilon_1 = r^2 + r^{-2}$

■ Ground states

$$|\widetilde{\Psi}_\pm\rangle = M|\Psi_\pm\rangle = \bigotimes_{j=1}^{N} (|\uparrow\rangle_j \pm r|\downarrow\rangle_j)$$

- Reproduces old results [Barouch & McCoy, *PRA* **3**, 786 (1971)]
- Gapped for all $0 < r < \infty$. [Can be proved using free-fermion method]

From XY to XYZ

Sandwiching method

- C operators $C_j = \frac{r^2}{2} \overbrace{\mathbb{1} \otimes \cdots \mathbb{1}}^{j-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j-1}$
- XZ Hamiltonian

$$H_{\mathrm{XZ}} = \sum_{j=1}^{N-1} \widetilde{h}_j^{(2)}, \quad \widetilde{h}_j^{(2)} = \widetilde{L}_j^\dagger C_j \widetilde{L}_j, \quad \widetilde{L}_j = M L_j M^{-1}$$

$$\widetilde{h}_j^{(2)} = -\sigma_j^x \sigma_{j+1}^x - J_z \sigma_j^z \sigma_{j+1}^z + \frac{B_2}{2}(\sigma_j^z + \sigma_{j+1}^z) + \epsilon_2$$

where $J_z = \frac{(r - r^{-1})^2}{2}, \quad B_2 = \frac{r^2 - r^{-2}}{2}, \quad \epsilon_2 = \frac{(r + r^{-1})^2}{4}$

XYZ chain

- H_{XY} and H_{XZ} share the same g.s. $|\widetilde{\Psi}_\pm\rangle$
- $H_{\mathrm{XYZ}} = \alpha_1 H_{\mathrm{XY}} + \alpha_2 H_{\mathrm{XZ}}$ also shares the same g.s. as long as $\alpha_1 > 0, \ \alpha_2 > 0$. Reproduces Müller & Shrock, *Phys. Rev. B* **32** (1985).
- Gapped for all $0 < r < \infty$ and $\alpha_1, \alpha_2 > 0$. [Min-max theorem]

Fermionic rephrasing

From spins to fermions

- Jordan-Wigner transformation

$$c_j := \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+, \quad c_j^\dagger := \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^- \qquad \left(\sigma_j^\pm := \frac{\sigma_j^x \pm \mathrm{i}\sigma_j^y}{2}\right)$$

- They obey CAR: $\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$

- Number op.: $n_j = c_j^\dagger c_j$

Fermionic model

- Hamiltonian $H_{\mathrm{fXYZ}} = \sum_{j=1}^{N-1} h_j^{(3)}$

Frustration-free condition

$$\left(\frac{\Delta}{t+2U}\right)^2 + \left(\frac{\mu}{2(t+2U)}\right)^2 = 1$$

$$\begin{aligned} h_j^{(3)} =& -t(c_j^\dagger c_{j+1} + \mathrm{h.c.}) + \Delta(c_j c_{j+1} + \mathrm{h.c.}) \\ & - \frac{\mu}{2}(n_j + n_{j+1} - 1) + U(2n_j - 1)(2n_{j+1} - 1) + \mathrm{const.} \end{aligned}$$

Interacting version of Kitaev chain!

- Ground states: $|\widetilde{\Psi}_\pm\rangle = (1 \pm r c_1^\dagger)\cdots(1 \pm r c_N^\dagger)|\mathrm{vac}\rangle$

Ferromagnetic Heisenberg chain

■ Hamiltonian (*S*=1/2, OBC)

$$H = \sum_{j=1}^{N-1} h_j, \quad \boxed{h_j = 1 - \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} \geq 0} \qquad \boldsymbol{\sigma}_j := (\sigma_j^x, \sigma_j^y, \sigma_j^z)$$

- SU(2) symmetry $\quad [H, S_{\mathrm{total}}^\alpha] = 0, \quad S_{\mathrm{tot}}^\alpha = \frac{1}{2}\sum_{j=1}^{N} \sigma_j^\alpha, \quad (\alpha = z, +, -)$
- h_j is proportional to a projector

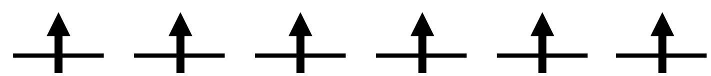
$$h_j = 4\,|\mathrm{sing}\rangle_{j,j+1}\langle\mathrm{sing}|, \quad |\mathrm{sing}\rangle_{j,j+1} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_j|\downarrow\rangle_{j+1} - |\downarrow\rangle_j|\uparrow\rangle_{j+1})$$

Spin-singlet

■ Ground states & excitations

- All-up state

$$|\Uparrow\rangle := |\uparrow\rangle_1|\uparrow\rangle_2\cdots|\uparrow\rangle_N$$



is a zero-energy state of of each h_j → A g.s. of *H*

- Other g.s.: $\quad (S_{\mathrm{tot}}^-)^k|\Uparrow\rangle \qquad (k = 0, 1, \ldots, N)$
- Unique in each total S^z sector (due to Perron-Frobenius thm.)
- Gapless spin-wave excitations (gap $\propto 1/N^2$ for large *N*)

q-deformed Heisenberg chain

Conjugation of *L*

- *L* operators $h_j = L_j^\dagger L_j, \quad L_j = 2|\uparrow,\downarrow\rangle_{j,j+1}\langle \mathrm{sing}|$
- *M* operator
$$M = q^{-\sigma_1^z/2}\cdots q^{-j\sigma_j^z/2}\cdots q^{-N\sigma_N^z/2} \qquad (q\in\mathbb{R})$$
$$= \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2}\end{pmatrix} \otimes\cdots\otimes \begin{pmatrix} q^{-j/2} & 0 \\ 0 & q^{j/2}\end{pmatrix} \otimes\cdots\otimes \begin{pmatrix} q^{-N/2} & 0 \\ 0 & q^{N/2}\end{pmatrix}$$
- Conjugated *L* $\widetilde{L}_j = M L_j M^{-1} = \sqrt{2(1+q^2)}\,|\uparrow\downarrow\rangle_{j,j+1}\langle \mathrm{sing}(q)|$

$$|\mathrm{sing}(q)\rangle_{j,j+1} = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{-1/2}|\uparrow\downarrow\rangle_{j,j+1} - q^{1/2}|\downarrow\uparrow\rangle_{j,j+1}\right)$$

q-deformed singlet

New Hamiltonian

- XXZ chain with boundary magnetic fields

$$H_{\mathrm{XXZ}} = \sum_{j=1}^{N-1} \widetilde{h}_j,$$
$$\tilde{h}_j = q^{-1}\widetilde{L}_j^\dagger \widetilde{L}_j = -\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{q+q^{-1}}{2}\left(\sigma_j^z\sigma_{j+1}^z - 1\right) + \frac{q-q^{-1}}{2}\left(\sigma_j^z - \sigma_{j+1}^z\right)\right]$$

Turns out to be $U_q(sl_2)$ symmetric XXZ chain!

Alcaraz et al., *J. Phys. A* **20** (1987), Pasquier & Saleur, *Nucl. Phys. B* **330** (1990)

Ground states and excitations

■ Ground states

- Conjugated lowering operator

$$\tilde{S}^-_{\text{tot}} = M S^-_{\text{tot}} M^{-1} = \sum_{j=1}^{N} q^j \sigma_j^- \notin U_q(sl_2), \qquad \sigma_j^- := \frac{\sigma_j^x - \mathrm{i}\sigma_j^y}{2}$$

- Ground states

Because $M|\Uparrow\rangle \propto |\Uparrow\rangle$, we have

$$M(S^-_{\text{tot}})^k|\Uparrow\rangle = (M S^-_{\text{tot}} M^{-1})^k M|\Uparrow\rangle \propto (\widetilde{S}^-_{\text{tot}})^k|\Uparrow\rangle \qquad (k = 0, 1, ..., N)$$

They are the g.s. of H_{XXZ}. Unique in each total S^z sector.
They are the same as those generated by $S_q^- \in U_q(sl_2)$.
Alcaraz, Salinas & Wreszinski, *Phys. Rev. Lett* **75** (1995),
Gottstein, Werner (1995).

■ Excited states

- Energy gap is lower bounded by $\gamma = 2(q^{1/2} - q^{-1/2})^2 > 0$ for all $q \neq 1$ and *N*. Koma & Nachtergaele, *Lett. Math. Phys.* **40**, 1 (1997)

Fredkin spin chain

■ Hamiltonian (S=1/2, OBC)

Salberger & Korepin, *Rev. Math. Phys.* **29** (2017)

$$H = H_\partial + \sum_{j=1}^{N-1} h_j$$

- Boundary term

$$H_\partial = |\uparrow\rangle_1\langle\uparrow| + |\downarrow\rangle_N\langle\downarrow|$$

- Bulk terms ~ Fredkin gates

$$h_j = |\uparrow\rangle_j\langle\uparrow| \otimes |\mathrm{sing}\rangle_{j+1,j+2}\langle\mathrm{sing}| + |\mathrm{sing}\rangle_{j,j+1}\langle\mathrm{sing}| \otimes |\downarrow\rangle_{j+2}\langle\downarrow|$$

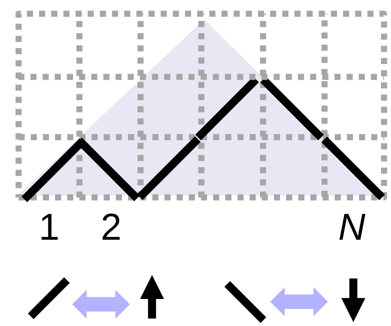
Resonance

$$|\uparrow\uparrow\downarrow\rangle \leftrightarrow |\uparrow\downarrow\uparrow\rangle \qquad |\uparrow\downarrow\downarrow\rangle \leftrightarrow |\downarrow\uparrow\downarrow\rangle$$

■ Ground state and excitations

- G.s. is unique and written as equal-weight superposition of all Dyck paths

$$|\Psi_0\rangle = \sum_{\mathrm{path}}$$

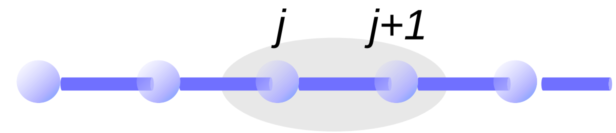


- Logarithmic entanglement, power-law finite-size gap

■ Deformed model

- Conjugation with *M* reproduces Salberger et al., *J.Stat.Mech*. 063103 (2017)
- Exponentially small finite-size gap for *q*>1. Seems gapped for *q*<1.

Other examples



From AKLT to *q*-AKLT

- S=1 AKLT [Affleck et al., *PRL* **59**, 799 (1987)]

$$H = \sum_{j=1}^{N-1} P^{(2)}_{j,j+1}, \quad \boxed{P^{(2)}_{j,j+1} = \frac{1}{3} + \frac{1}{2}\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{6}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2}$$

Projector to total spin 2

- G.s. can be written as a matrix-product state. Gapped.
- $M = \prod_{j=1}^{N} q^{-2jS_j^z} \left(\frac{q+q^{-1}}{2}\right)^{(S_j^z)^2/2}$ and judiciously chosen center terms reproduce *q*-deformed AKLT! [Klümper et al., *JPA* **24** (1991); *ZPB* **87** (1992)]

$\mathbf{Z}_N$ quantum spin models

- Natural *N*-level generalization of Ising and XY chains
- Related to parafermions $\psi_i \psi_j = \omega \psi_j \psi_i \ (i<j), \quad \omega = \exp(2\pi \mathrm{i}/N)$
- Can reproduce known models
  Iemini et al., *PRL* **118** (2017), Mahyaeh & Ardonne, *PRB* **98** (2018)
- Can construct tons of new models. See our SciPost paper!

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4. Super-frustration-free systems

- N=1 SUSY algebra
- Majorana-Nicolai model

5. Summary

Super-frustration-free systems

■ *N*=1 SUSY algebra

- Fermionic parity: $(-1)^F$ (*F*: total fermion num.)
- Supercharge: Q $(Q^\dagger = Q)$ anti-commutes with $(-1)^F$
- Hamiltonian: $H = Q^2$
- Symmetry: $[H, (-1)^F] = [H, Q] = 0.$
- *E* = 0 state, if exists, must be annihilated by *Q*

■ "Local" supercharge

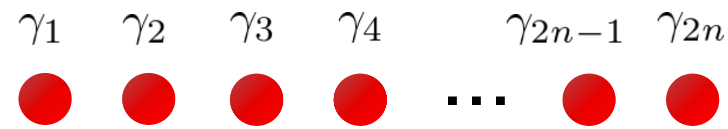
- Total supercharge: $Q = \sum_j q_j$
- Local supercharge: Each q_j anti-commutes with $(-1)^F$

Definition. $Q = \sum_j q_j$ is said to be ***super-frustration-free*** if there exists a state $|\psi\rangle$ such that $q_j|\psi\rangle = 0$ for all *j*.

Majorana-Nicolai model

■ Lattice Majorana fermions

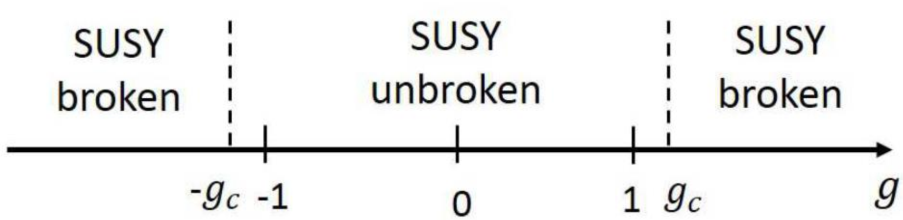
$$(\gamma_i)^\dagger = \gamma_i, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$



- Fermionic parity: $(-1)^F = \mathrm{i}^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$
- Complex fermions $c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - \mathrm{i}\gamma_{2j})$

■ Model

- Supercharge $$Q = \sum_j (g\gamma_j + \mathrm{i}\gamma_{j-1}\gamma_j\gamma_{j+1}),$$ $(g \in \mathbb{R})$ with PBC
- Hamiltonian
  $H = Q^2$ consists of quadratic and quartic terms in *γ*
- Phase diagram



$g_c < 8/\pi = 2.546...$

- Sannomiya & Katsura, *PRD* **99** (2019).
- O'Brien & Fendley, *PRL* **120** (2018).

- Rigorous upper bound on g_c
- Majorana version of the original Nicolai model (g=0) is integrable
- Super-frustration-free at $g=\pm 1$

Exact ground states at $g = 1$

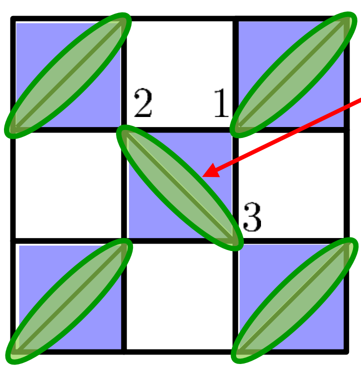
■ Rewriting Q

$$Q = \sum_{l=1}^{N/2} (\gamma_{2l-2} + \gamma_{2l+1}) \underline{h_{2l-1}} = \sum_{l=1}^{N/2} (\gamma_{2l-1} + \gamma_{2l+2}) \underline{h_{2l}}$$

- $h_{2l-1} = 1 + \mathrm{i}\gamma_{2l-1}\gamma_{2l}$: Local term of Kitaev chain in a trivial phase
- $h_{2l} = 1 + \mathrm{i}\gamma_{2l}\gamma_{2l+1}$: Local term of Kitaev chain in a topo. phase
- $H = Q^2$ has two g.s. annihilated by all local q.
  Easy to write down their explicit forms!

Other lattices

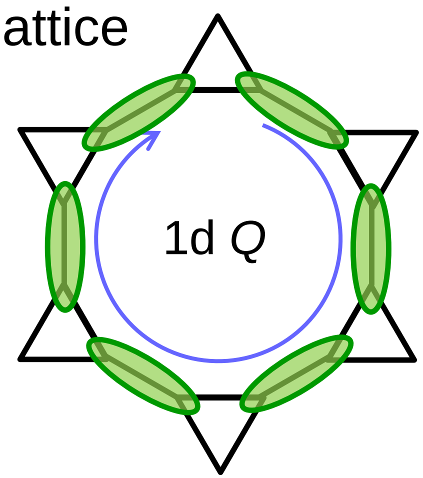
- Shastry-Sutherland



$q_{\mathrm{loc}} = \gamma_1(1 + \mathrm{i}\gamma_2\gamma_3)$

G.S. = Majorana dimer covering

- 3-12 lattice



Summary

- Witten's conjugation in *N*=2 SUSY QM
- Application to frustration-free systems

$$H = \sum_j L_j^\dagger L_j \Rightarrow \widetilde{H} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j$$

$$C_j > 0, \ \widetilde{L}_j = M L_j M^{-1}$$
$$(M: \text{ invertible})$$

- Reproduce many known examples, e.g., *q*-Heisenberg
  Allows for the construction of new models

Future direction

- Does Witten conjugation help prove a gap?
  Gap in 2d AKLT: Pomata & Wei, *PRL* **124**, 177203 (2020)
  Lemm, Sandvik & Wang, *PRL* **124**, 177204 (2020)
  Mapping to an anisotropic model? Niggemann et al., *ZPB* **104** (1997)
- Application to quantum many-body scars?
  A few excited states in AKLT: Moudgalya et al., *PRB* **98**, 235156 (2018)
- Good for dissipative (open quantum) systems?