Ljubljana PhD School on Quantum Physics
2024 June 21

Frustration-free Models and beyond

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Outline of my lectures

- Day 1 (June 19)
  - Introduction to frustration-free systems
  - Systematic construction of models
- Day 2 (June 20)
  - Non-interacting Kitaev chain
  - Interacting Kitaev chain
- Day 3 (June 21)
  - Divergence-free conditions
  - Application to quantum many-body scars

Ground-state Physics (Day 1–2)

Dynamics (Day 3)

Frustration-free systems (recap)

■ Universal form

$$H = \sum_j h_j, \quad h_j = L_j^\dagger L_j$$

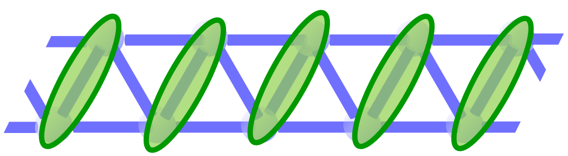
- Positive semi-definite
- Zero-energy ground state $|\psi\rangle$ s.t. $L_j|\psi\rangle = 0\ \forall j$
- ψ saturates Anderson's bound, i.e.,

$$h_j|\psi\rangle = 0 \text{ for all } j$$

Can we generalize this idea to excited states?

■ Examples

- Ferromagnetic Heisenberg model
- Majumdar-Ghosh model
- AKLT model
- Kitaev's toric code



Local divergence condition (1)

■ Baxter's telescoping trick

Ann. Phys. **76**, 1 (1973)

- XYZ chain Hamiltonian

$$H = \sum_{j=1}^{L} h_j, \quad h_j = J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z$$

- Product eigenstate $\phi_j \in \mathbb{C}^2$

$$\psi = \phi_1 \otimes \cdots \otimes \phi_{j-1} \otimes \underline{\phi_j \otimes \phi_{j+1}} \otimes \phi_{j+2} \otimes \cdots \otimes \phi_L$$

  - ➢ Not annihilated by h_j

- Local divergence condition

$$\begin{aligned} h_j \psi = {} & \phi_1 \otimes \cdots \otimes \phi_{j-1} \otimes \underline{s_j \otimes \phi_{j+1}} \otimes \phi_{j+2} \otimes \cdots \otimes \phi_L \\ & - \phi_1 \otimes \cdots \otimes \phi_{j-1} \otimes \underline{\phi_j \otimes s_{j+1}} \otimes \phi_{j+2} \otimes \cdots \otimes \phi_L \end{aligned}$$

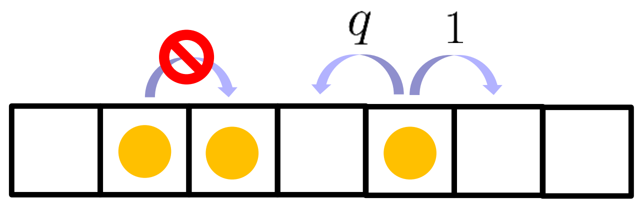
  - ➢ ψ is a zero-energy state of H
  - ➢ But not a ground state

Limited to product states?

Local divergence condition (2)

■ Asymmetric simple exclusion process (ASEP)

- Master eq.



$$\frac{\mathrm{d}}{\mathrm{d}t}|P\rangle\rangle = -M|P\rangle\rangle$$

  - ➢ Probability vector $|P\rangle\rangle = \sum_{\boldsymbol{\sigma}} P(\sigma_1, ..., \sigma_L)|\sigma_1, ..., \sigma_L\rangle, \quad \sigma_j = 0 \text{ or } 1$
  - ➢ Transition-rate matrix

$$M = \sum_j m_j, \quad m_j = \begin{pmatrix} 0 & & & \\ & q & -1 & \\ & -q & 1 & \\ & & & 0 \end{pmatrix}_{j,j+1}$$

■ Matrix-product steady state

Derrida et al., *JPA* **26** 1493 (1993)

$$|P\rangle\rangle = \frac{1}{Z}\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle, \quad DE - qED = (1-q)(D+E)$$

- Local divergence condition

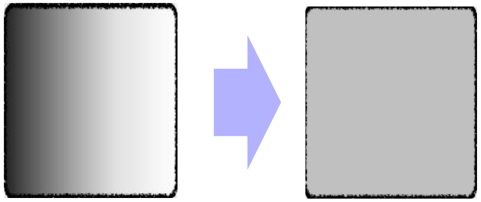
$$\frac{m}{1-q}\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} - \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow M|P\rangle\rangle = 0$$

Outline of today's lecture

1. Introduction and motivation
   - Eigenstate thermalization hypothesis (ETH)
   - Violation of ETH
   - Rydberg atom arrays and PXP model
   - Quantum many-body scars (QMBS)
2. Onsager scars
3. Other examples
4. Summary

Foundation of equilibrium stat-mech

An isolated macro classical/quantum system relaxes towards a steady state at late times.



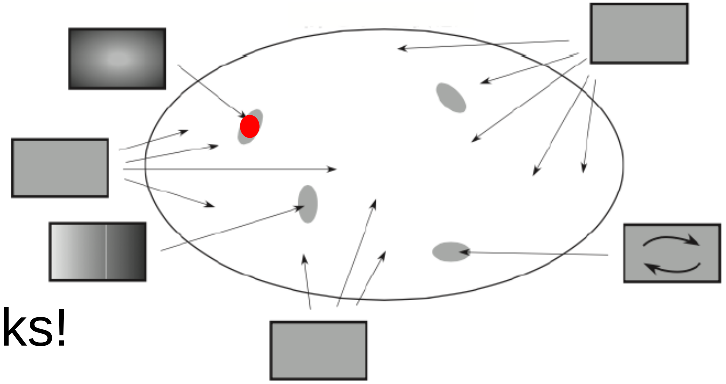
But why?

Fundamental problem since von Neumann's work (1929)

- Typicality

A great majority of states with the same energy are indistinguishable by macroscopic observables!

H. Tasaki, J. Stat. Phys. **163** (2016) and his book



"thermal equilibrium"
= common properties shared by the majority of states

→ Microcanonical (MC) ensemble works!

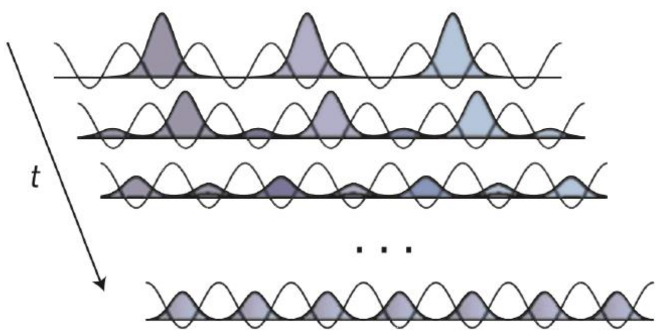
- Thermalization

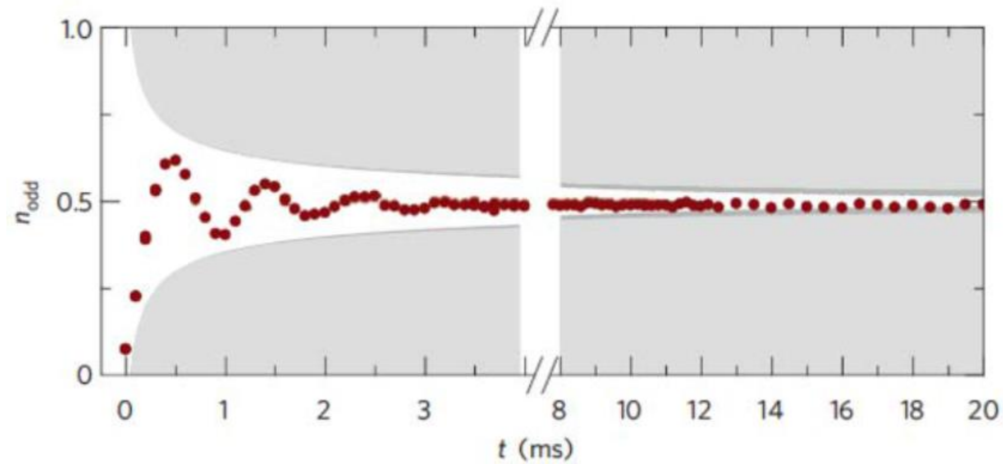
The approach to these typical states

Experimental verification

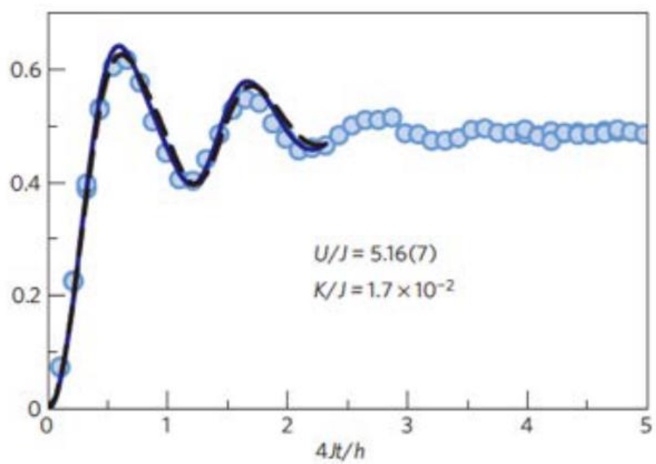
S. Trotzky *et al.*, Nat. Phys. **8** (2012)

1d Bose-Hubbard, ^{87}Rb





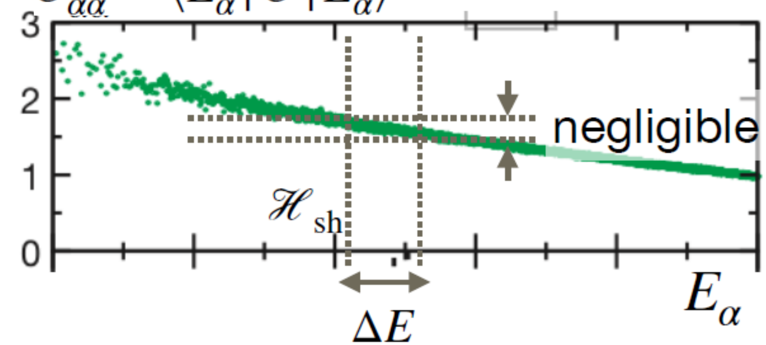
Comparison with t-DMRG result



Numerical verification

M. Rigol *et al.*, Nature **452** (2008)

$$O_{\alpha\alpha} = \langle E_\alpha | \hat{O} | E_\alpha \rangle$$



Eigenstate thermalization hypothesis (ETH)

- Setup

H: Hamiltonian; $|E_n\rangle$: (normalized) energy eigenstate,
O: macroscopic observable, ρ_{mc}: MC ensemble,
Energy shell: $\mathrm{span}\{|E_n\rangle : H|E_n\rangle = E_n|E_n\rangle, E_n \in [E-\Delta E, E)\}$

- Thermal states

A state $|E_n\rangle$ is said to be thermal if $\langle E_n|O|E_n\rangle \simeq \mathrm{Tr}[\rho_{\mathrm{mc}} O]$.

- Strong ETH: All $|E_n\rangle$ in the energy shell are thermal.

Believed to be true for a large class of non-integrable systems

Concept: von Neumann, Deutsch, Srednicki, Tasaki, …
Numerical evidence: D'Alessio et al., Adv. Phys. **65** (2016).

- Weak ETH: Almost all $|E_n\rangle$ in the energy shell are thermal.

Proved under certain conditions

Biroli, Kollath, Lauchli, PRL **105** (2010);
Iyoda, Kaneko, Sagawa, PRL **119** (2017)

Exceptions of strong ETH

1. Integrable systems
   Many conserved charges
   Strong ETH , Weak ETH

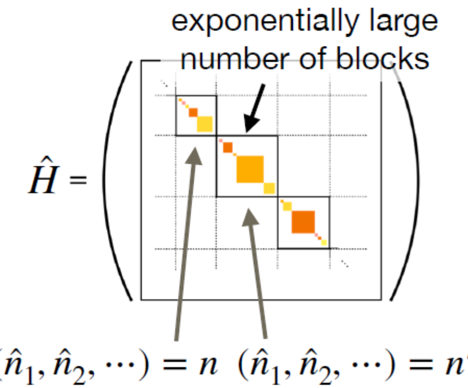
Ex.) S=1/2 Heisenberg chain

$$H_{\mathrm{Hei}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

2. Many-body localized (MBL) systems
   Emergent local integrals of motion
   Strong ETH , Weak ETH

$$H_{\mathrm{MBL}} = H_{\mathrm{Hei}} + \sum_{j=1}^{L} h_j S_j^z$$

3. Hilbert-space fragmentation
   Hilbert space splits into exp. many sectors
   Strong ETH , Weak ETH &



[From Hamazaki's slides]

4. Quantum many-body scarred systems
   Strong ETH , Weak ETH
   Non-integrable but have *scarred* states which do not thermalize for an anomalously long time!

What are scars?

- A very nice blog article

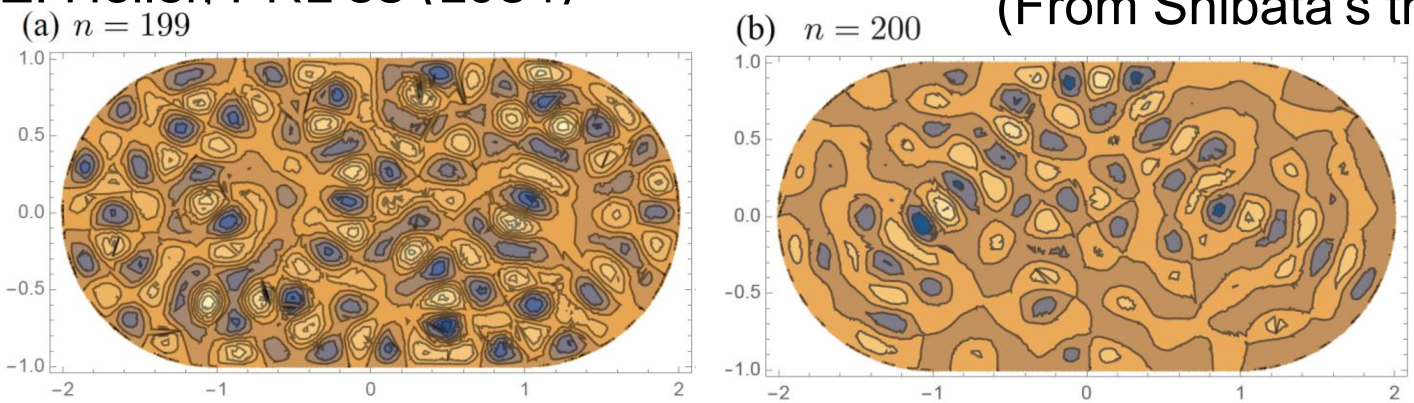
"**Quantum Machine Appears to Defy Universe's Push for Disorder**",
Marcus Woo, Quanta magazine, March 2019

Recommendation:
15-puzzle and Nagaoka ferromagnetism
Quanta magazine, January 2019.

- One-body scars

1-particle wave function in a Bunimovich stadium
E. Heller, PRL **53** (1984)

(From Shibata's thesis)

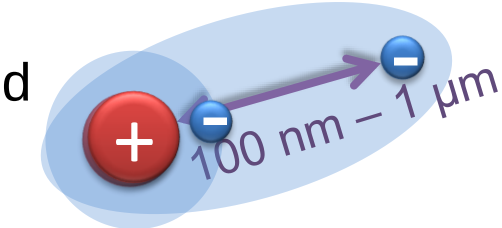


Experiment on Rydberg atom arrays

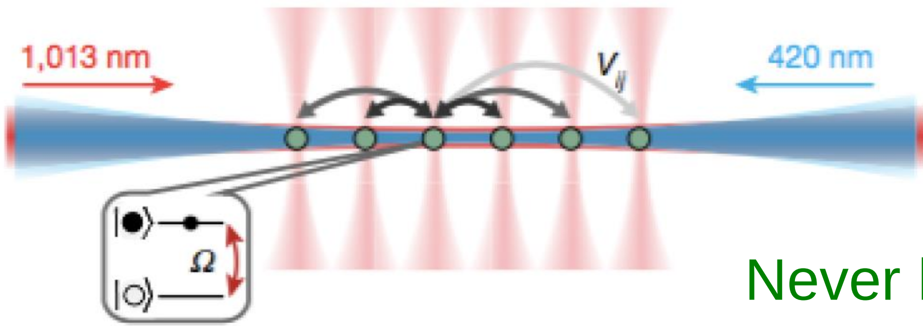
Bernien *et al*., Nature **551** (2017)

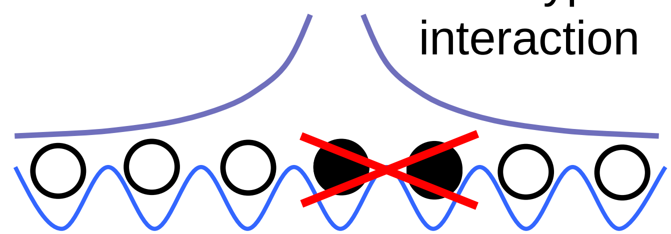
- Rydberg atoms

  Atoms in which one of the electrons is in an excited state with a very high principal quantum number.



- Rydberg blockade





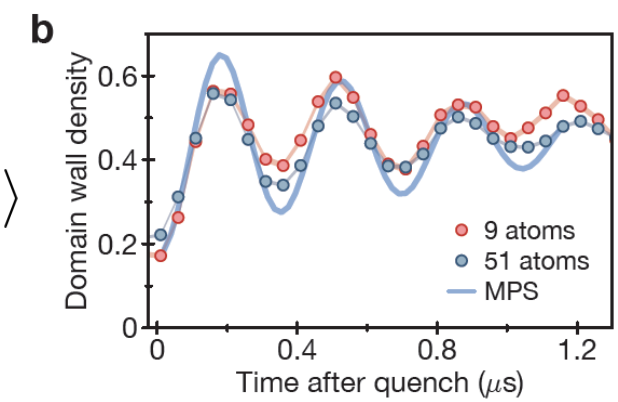
Never have adjacent excited states

- A surprising finding!

  Special initial states

$$|\mathbf{Z}_2\rangle = |\bullet \circ \bullet \circ \cdots\rangle, \ |\mathbf{Z}_2'\rangle = |\circ \bullet \circ \bullet \cdots\rangle$$

  Exhibit robust oscillations. Other initial states thermalize much more rapidly.

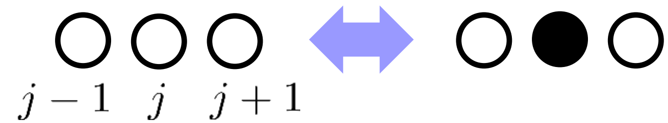


PXP model (1)

- Hamiltonian

Turner *et al.*, Nat. Phys. **14**, 745 (2018)

$$H_{\mathrm{PXP}} = \sum_j P_{j-1} X_j P_{j+1},$$



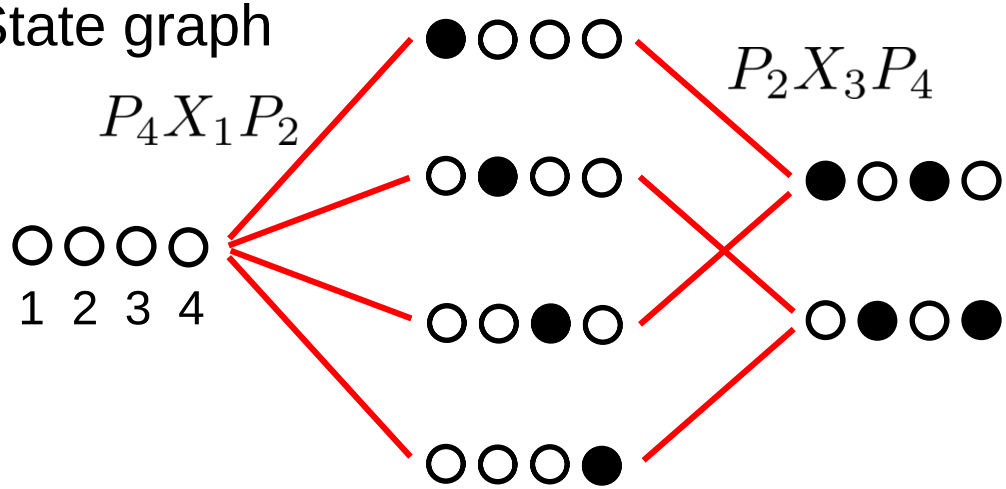
$$P = |\circ\rangle\langle\circ|, \;\; X = |\circ\rangle\langle\bullet| + |\bullet\rangle\langle\circ|$$

➢ Lesanovsky & Katsura, *PRB* **86** (2012)

- Example: 4-site with PBC

Dimension of Hilbert space: $F_3 + F_5 = 7$

State graph

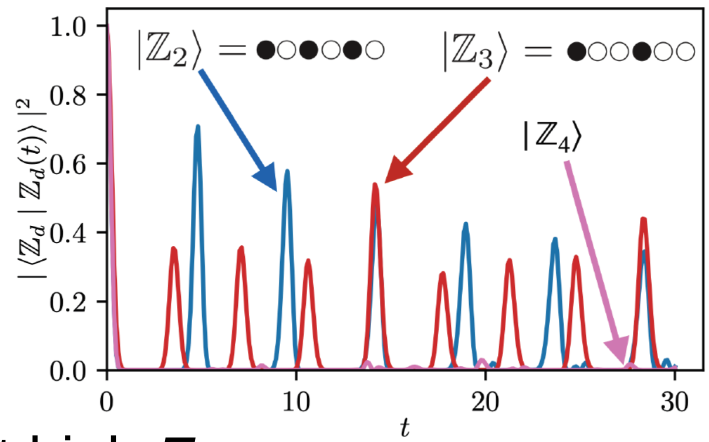


Hamiltonian

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

PXP model (2)

Revivals of fidelity

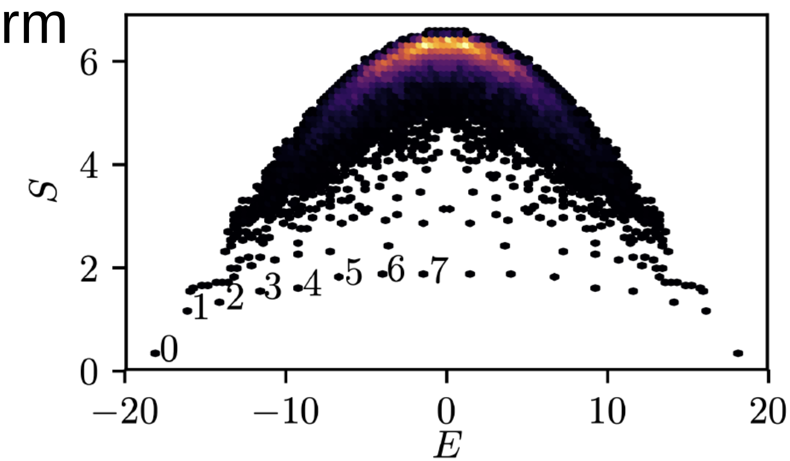


- Properties
  1. Level statistics
     → Wigner-Dyson, non-integrable
  2. Long-time oscillations are observed
  3. Energy (E) v.s. entanglement entropy (S) → Anomalously low S at high E

- Exact QMBS

  Lin and Motrunich, PRL **122**, 173401 (2019).

  Exact eigenstates of H_{PXP} in the form of matrix product states (MPS)
  → Low entanglement states at high energy



Exact QMBS

- Embedding method
  Shiraishi & Mori, PRL **119** (2017)
- AKLT models
  Moudgalya, Regnault & Bernevig, PRB **98** (2018)
  Mark, Lin & Motrunich, PRB **101** (2020)
- Ising and XY-like models
  Iadecola & Schecter, PRB **101** (2020)
  Chattopadhyay, Pichler, Lukin, Ho & PRB **101** (2020)
- Floquet scars
  Driven PXP: Sugiura, Kuwahara, Saito, PRR **3** (2021)
  Mizuta, Takasan & Kawakami, PRR **2** (2020)
- Recent reviews
  Serbyn, Abanin & Papic, Nat. Phys. **17** (2021)
  Moudgalya, Bernevig & Regnault, Rep. Prog. Phys. (2022)
  Chandran, Iadecola, Khemani & Moessner, ARCMP **14** (2023)

(Generalized) Shiraishi-Mori

■ Sandwiching method

- Frustration-free Hamiltonian

$$H = \sum_j L_j^\dagger L_j$$

  - Zero-energy ground state $|\psi\rangle$ s.t. $L_j|\psi\rangle = 0, \forall j$

- New Hamiltonian

$$H_{\mathrm{new}} = \sum_j L_j^\dagger C_j L_j \quad (C_j: \text{ Hermitian})$$

  - ψ is a zero-energy state for arbitrary C_j
  - But it may not be a ground state of H_{new}

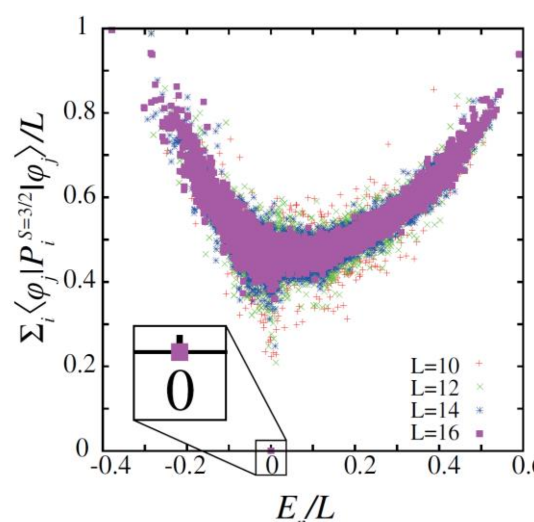
■ Shiraishi-Mori embedding

- particular case where

$$L_j = L_j^\dagger = P_j \text{ (projection)}.$$

  - Example: embedding the g.s. of Majumdar-Ghosh model

PRL **119**, 030601 (2017)



Spin-1 XY chain

■ Hamiltonian M. Schecter and T. Iadecola, PRL **123**, 147201 (2019)

$$H_{\mathrm{XY}} = J\sum_{j=1}^{L}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + D\sum_{j=1}^{L}(S_j^z)^2$$

$S_j^\alpha (\alpha = x, y, z)$
Spin-1 operator at site *j*

■ su(2) algebra

$$J^\pm = \frac{1}{2}\sum_{j=1}^{L}(-1)^j (S_j^\pm)^2, \quad J^z = \frac{1}{2}\sum_{j=1}^{L} S_j^z$$

$$S_j^\pm = S_j^x \pm \mathrm{i} S_j^y$$

- $[J^+, J^-] = 2J^z, \quad [J^z, J^\pm] = \pm J^\pm$
- They do not commute with H_{XY}
- Nevertheless…

■ Tower of eigenstates

$$|\Psi_k\rangle = (J^+)^k |-,-,\cdots,-\rangle \quad (k = 0, 1, 2, ..., L)$$

← Does not contain 0 states!

with eigenenergy $E = DL$

Today's subject

- Quantum many-body scars (QMBS) (recap)
  - ✓ Non-thermal eigenstates of non-integrable Hamiltonians
  - ✓ Finite-energy density
  - ✓ Entanglement entropy does not obey a volume law
- Constructing models with exact QMBS
  - ✓ Using Onsager algebra

    2d Ising model:
    Phys. Rev. 65 (1944)
  - ✓ Using integrable boundary states
  - ✓ Using (restricted) spectrum generating algebra
  - ✓ …

Outline of today's lecture

1. Introduction and motivation
2. Onsager scars
   - Strategy
   - Perturbed S=1/2 XY chain
   - Properties
   - Higher-spin models
     - N. Shibata, N. Yoshioka, HK, *PRL* **124**, 180604 (2020)
3. Other examples
4. Summary

Strategy

1. Starting point:
   Integrable model with conserved charges $Q_1, Q_2, \ldots$
   They commute with the Hamiltonian H_{int}
2. Take a subalgebra $\{Q_1, Q_2, \ldots\}$
3. Find a reference eigenstate $H_{\mathrm{int}}|\psi_0\rangle = E_0|\psi_0\rangle$
   ψ_0: simple state, e.g., product state or MPS
4. Find a tower of eigenstates generated by acting with the subalgebra on the reference state:

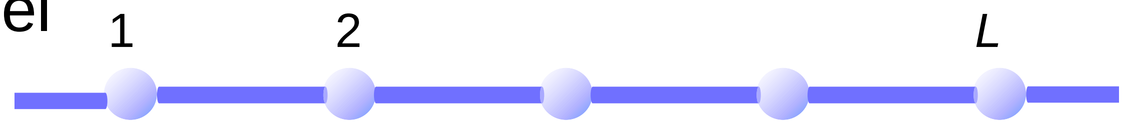
   $$(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle$$ ← QMBS in non-integrable *H*

   They have the same energy as ψ_0
5. Add perturbations that break the integrability of H_{int} but do not hurt the tower of states

   $$H = H_{\mathrm{int}} + H_{\mathrm{pert}}, \quad \text{e.g.,} \; H_{\mathrm{pert}} \, (Q_1)^m (Q_2)^n \cdots |\psi_0\rangle = 0$$

Example: *S*=1/2 XY chain

■ Model



L: even
Periodic chain

- Hamiltonian

$$H_{\rm int} = \sum_{j=1}^{L} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \qquad S_j^{\pm} := \frac{S_j^x \pm {\rm i} S_j^y}{2}$$

Can be mapped to free fermions via Jordan-Wigner
Lieb-Schultz-Mattis (1961), Katsura (1962)

- Conserved charges

$$[H_{\rm int}, Q^{\pm}] = 0$$

Total S^z: $$Q = \sum_{j=1}^{L} S_j^z$$

"bi-magnon" operator: $$Q^{\pm} = \sum_{j=1}^{L} (-1)^{j+1} S_j^{\pm} S_{j+1}^{\pm}$$

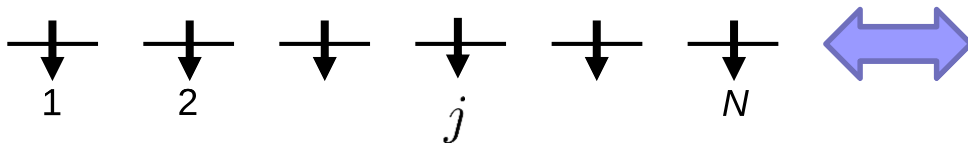
An element of Onsager's algebra! Infinitely many such.

- Reference eigenstate

All down state: $$|\Downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \quad H_{\rm int} |\Downarrow\rangle = 0$$

Magnon eigenstates

■ ``Motion" of flipped spin



$|j\rangle = S_j^+ |\Downarrow\rangle$

not an eigenstate of $H_{\rm int}$

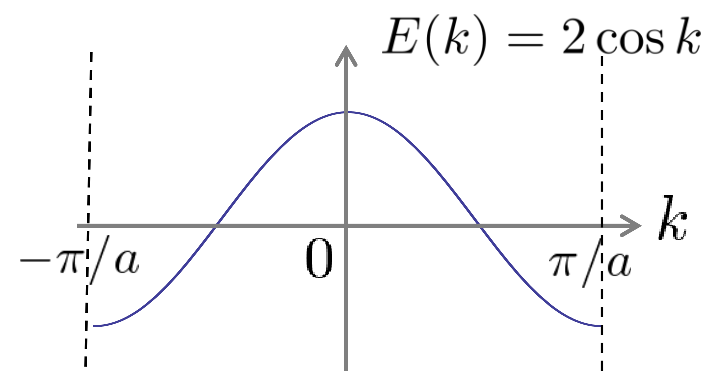
$$(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+)|j\rangle = |j+1\rangle$$

Flipped spin hops to the neighboring sites

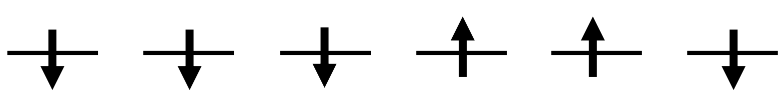
■ Bloch state

$$|\psi_k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} |j\rangle$$

is an exact eigenstate of $H_{\rm int}$



■ Bi-magnon state with momentum π



$|j, j+1\rangle = S_j^+ S_{j+1}^+ |\Downarrow\rangle$

$$Q^+ |\Downarrow\rangle = \sum_{j=1}^{L} (-1)^{j+1} |j, j+1\rangle$$ is an exact zero-energy state

Local divergence condition

- Tower of exact eigenstates of $H_{\rm int}$

$$|\Downarrow\rangle,\ Q^+|\Downarrow\rangle, ..., (Q^+)^k|\Downarrow\rangle, ..., (Q^+)^{L/2}|\Downarrow\rangle \qquad (\,(Q^+)^{L/2+1}=0\,)$$

  - “Coherent state”

$$|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle \ = \sum_{k=0}^{L/2} \frac{\beta^{2k}}{k!}(Q^+)^k|\Downarrow\rangle$$

  - Can be written as an MPS

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[M_1 \cdots M_j M_{j+1} \cdots M_L\right],$$

$$M_j = \begin{pmatrix} |\downarrow\rangle_j & (-1)^{j+1}\beta|\uparrow\rangle_j \\ \beta|\uparrow\rangle_j & 0 \end{pmatrix}$$

    - ex.) Prove this

- Telescoping trick
  - Local Hamiltonian $h_j = S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+$

$$h_j M_j M_{j+1} = L_j M_{j+1} - M_j L_{j+1},$$

$$L_j = \begin{pmatrix} 0 & 0 \\ 0 & |\downarrow\rangle_j \end{pmatrix}$$

    - ex.) Prove this
  - We get $H_{\rm int}|\psi(\beta)\rangle = 0.$ $[H_{\rm int}, Q^+] = 0$ isn't so important(?)

Designed perturbations

- Find H_{pert} such that
  - $H_{\text{pert}}|\psi(\beta)\rangle = 0$ (annihilates the coherent state)
  - Breaks the integrability of H_{int}
- Local structure of MPS

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[\begin{pmatrix} |\downarrow\rangle_1 & \beta|\uparrow\rangle_1 \\ \beta|\uparrow\rangle_1 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_2 & -\beta|\uparrow\rangle_2 \\ \beta|\uparrow\rangle_2 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_3 & -\beta|\uparrow\rangle_3 \\ \beta|\uparrow\rangle_3 & 0 \end{pmatrix}\cdots\right]$$

$$\begin{pmatrix} |\downarrow\downarrow\downarrow\rangle - \beta^2(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) & \beta|\downarrow\downarrow\uparrow\rangle + \beta^3|\uparrow\uparrow\uparrow\rangle \\ \beta|\uparrow\downarrow\downarrow\rangle - \beta^3|\uparrow\uparrow\uparrow\rangle & \beta^2|\uparrow\downarrow\uparrow\rangle \end{pmatrix}_{1,2,3}$$

  - $|\downarrow\uparrow\downarrow\rangle$ and $(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)/\sqrt{2}$ never appear
- Possible perturbations

$$H_{\text{pert}} = \sum_j c_j^{(1)}(|\downarrow\uparrow\downarrow\rangle\langle\downarrow\uparrow\downarrow| + \frac{c_j^{(2)}}{2}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|)$$
$$+ c_j^{(3)}[|\downarrow\uparrow\downarrow\rangle(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|) + \mathrm{H.c.}])_{j-1,j,j+1}$$

c's can be made random!

Is the perturbed model non-integrable?

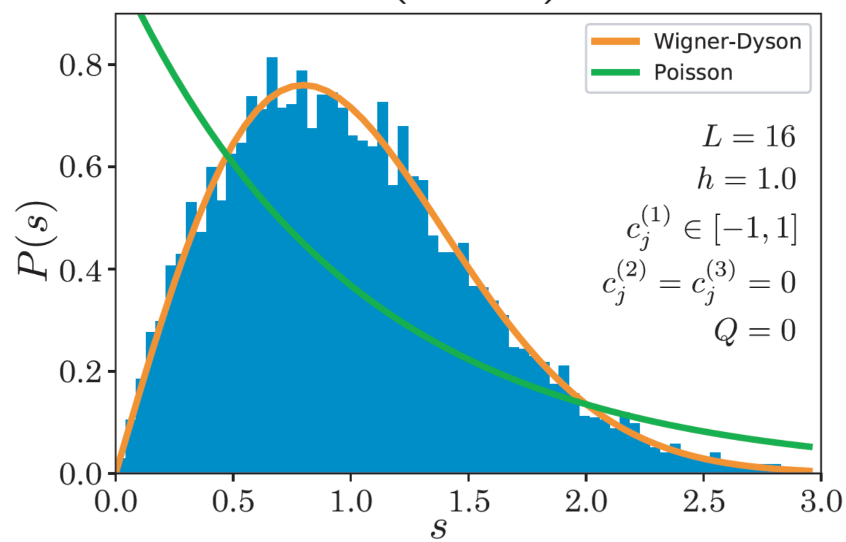
- Level spacing statistics
  - Perturbed Hamiltonian $H = H_{\rm int} + H_{\rm pert} + hQ,$
  - Energy levels $E_1 \leq E_2 \leq E_3 \leq \cdots$ $\Delta E_i = E_{i+1} - E_i$
  - Level spacing $$s_i := \frac{\Delta E_i}{\langle \Delta E_i \rangle}$$ $\langle \Delta E_i \rangle$: average
  - *H* is integrable $P(s) = \exp(-s)$
    → Poisson distribution

Casati *et al*, PRL **54** (1985),
Pal, Huse, PRB **82** (2010)

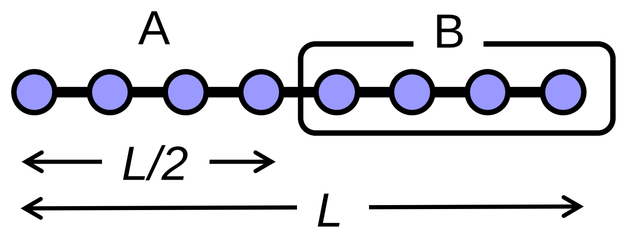
  - *H* is non-integrable (GOE) $$P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi s^2}{4}\right)$$
    → Wigner-Dyson distribution

- Numerical result
  - System size: *L*=16
  - Only diagonal pertubations
  - Zero-magnetization sector

H is non-integrable!



Entanglement diagnosis

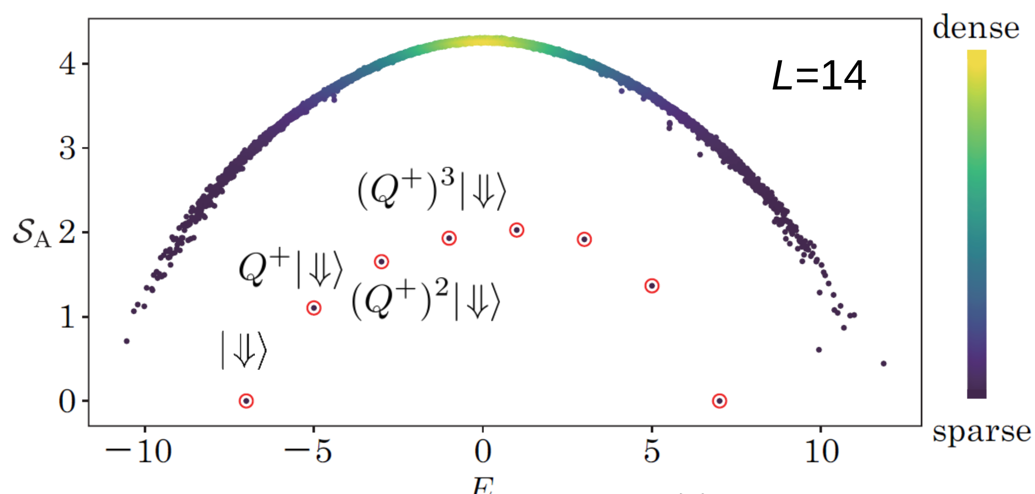


- **Half-chain entanglement**
  - Reduced density matrix

$$\rho = |\psi\rangle\langle\psi|, \quad \rho_A = \mathrm{Tr}_B[\rho]$$

  - Entanglement entropy (EE) $\mathcal{S}_A = -\mathrm{Tr}_A[\rho_A \ln \rho_A]$
  - Thermodynamic entropy ~ EE   Mori *et al.*, J. Phys. B **51** (2018)
    - Volume law $\mathcal{S}_A \propto L$ → Thermal
    - Sub-volume law (e.g., area law $\mathcal{S}_A \leq \mathrm{const.}$) → non-thermal

- **Results**
  - QMBS states $(Q^+)^k|\Downarrow\rangle$
  - Their EE obey sub-volume law
  - Rigorous bound
    - EE of QMBS ≦ O (ln *L*)



$$H = H_{\mathrm{int}} + H_{\mathrm{pert}} + Q, \quad c_j^{(i)} \in [-1, 1]$$

Dynamics

- Hamiltonian $H = H_{\rm int} + H_{\rm pert} + hQ,$
- Initial state = coherent state $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle$
- Time evolution

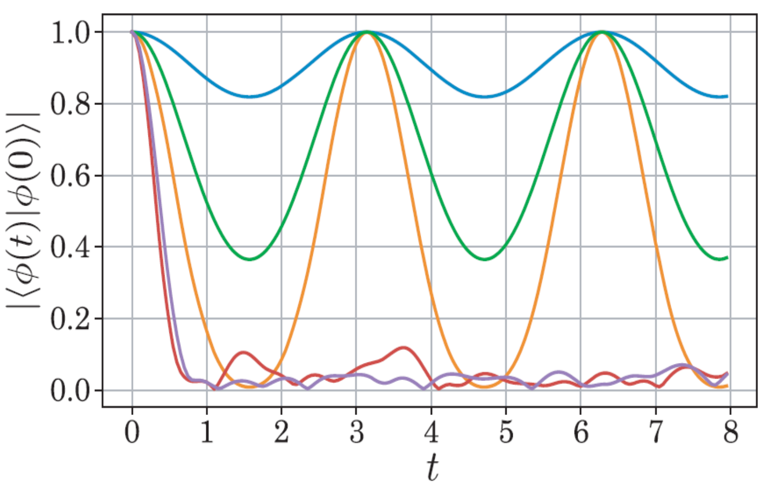
$$|\psi_t(\beta)\rangle = \exp(-\mathrm{i}Ht)|\psi(\beta)\rangle \propto |\psi(\beta e^{-\mathrm{i}ht})\rangle$$

Revival at $t = t_k = \frac{\pi k}{h}, \quad k \in \mathbb{N}$

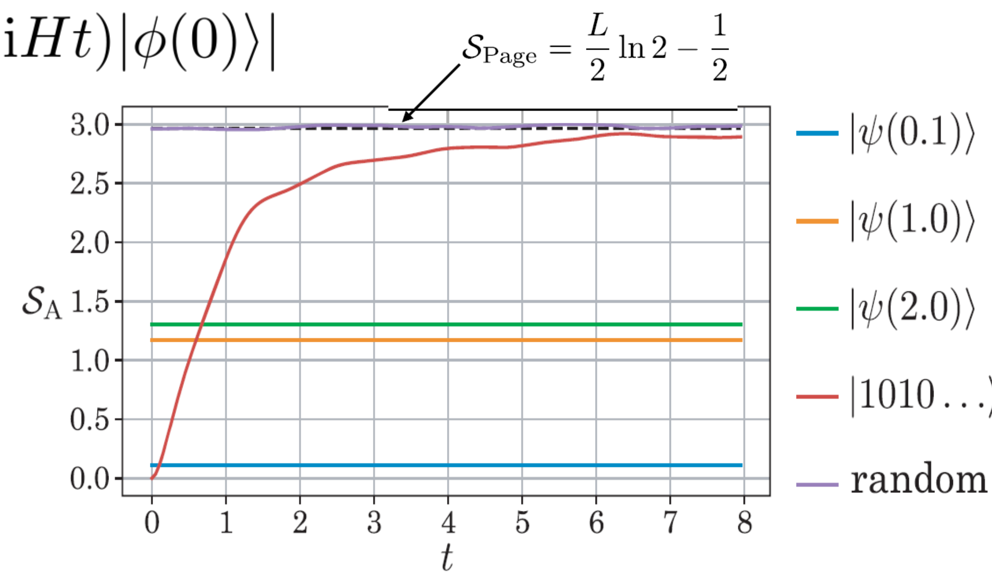
■ Numerical results $L = 10,\ h = 1.0, c_j^{(i)} \in [-1, 1]$ (random)

- Fidelity

$$|\langle\phi(t)|\phi(0)\rangle| = |\langle\phi(0)|\exp(\mathrm{i}Ht)|\phi(0)\rangle|$$



- Entanglement



Onsager algebra

- XY Hamiltonian

$$H_2 = \mathrm{i}\sum_{j=1}^{L}(S_j^+ S_{j+1}^- - S_j^- S_{j+1}^+)$$

Unitarily equivalent to H_{int}

- Commuting operators

$$Q = \sum_{j=1}^{L} S_j^z, \quad \hat{Q} = 2\sum_{j=1}^{L} S_j^x S_{j+1}^x$$

(Quantum) Ising!

$$H_{\mathrm{QI}} = Q + \lambda \hat{Q}$$

Phys. Rev. 65 (1944)

Any polynomial in $Q, \ \hat{Q}$ commutes with H_2

- Dolan-Grady relations

$$[Q,[Q,[Q,\hat{Q}]]] = 4[Q,\hat{Q}]$$

$$[\hat{Q},[\hat{Q},[\hat{Q},Q]]] = 4[\hat{Q},Q]$$

$$\hat{Q} = (Q_1^0 + Q_1^+ + Q_1^-)/2$$

$$Q_1^0 \propto H_{\mathrm{int}}, \quad Q_1^{\pm} \propto \sum_{j=1}^{L} S_j^{\pm} S_{j+1}^{\pm}$$

- Higher-order generators

$$Q_m^+ \propto \sum_{j=1}^{L} S_j^+ S_{j+1}^z \cdots S_{j+m-1}^z S_{j+m}^+$$

  - Commutes with H_2. Allows for scarred model with longer-range int.

What about *S* >1/2 ?

■ Self-dual U(1)-invariant clock model

Vernier, O'Brien & Fendley, J. Stat. Mech. (2019)

- Matrices $\omega = \exp(2\pi \mathrm{i}/n)$

$$\tau = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad S^- = (S^+)^\dagger$$

- Hamiltonian

Truly interacting for *n*>2!

$$H_n = \mathrm{i} \sum_{j=1}^{L} \sum_{a=0}^{n-1} \frac{1}{1-\omega^{-a}} \left[(2a-n)\tau_j^a + n(S_j^+ S_{j+1}^-)^{n-a} - n(S_j^- S_{j+1}^+)^a \right]$$

H_2 boils down to (twisted) XY, H_3 → S=1 Fateev-Zamolodchikov

- U(1) symmetry $[H_n, Q] = 0, \quad Q = \sum_{j=1}^{L} S_j^z$

$$[H_n, Q^+] = 0$$

- Self-duality (in the $\sigma - \tau$ rep.)
- Onsager algebra! $Q^+ = \sum_{j=1}^{L} \sum_{a=1}^{n-1} \frac{1}{1-\omega^{-a}} (S_j^+)^a (S_{j+1}^+)^{n-a}$

S=1 (n=3) model

- Integrable Hamiltonian

$$H_{\rm int} = \sqrt{3}\sum_{j=1}^{L}\left[S_j^+S_{j+1}^- + S_j^-S_{j+1}^+ - (S_j^+S_{j+1}^-)^2 - (S_j^-S_{j+1}^+)^2 - (S_j^z)^2 + \frac{2}{3}\right]$$

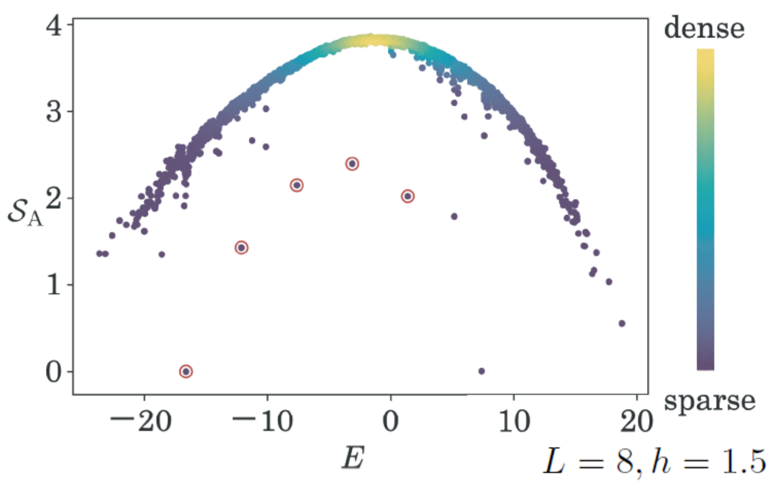
- Coherent state

$$Q^+ = \frac{2}{\sqrt{3}}\sum_{j=1}^{L} S_j^+(S_j^+ - S_{j+1}^+)S_{j+1}^+, \quad |\psi(\beta)\rangle = \exp(\beta^2 Q^+)|-,-,\cdots,-\rangle$$

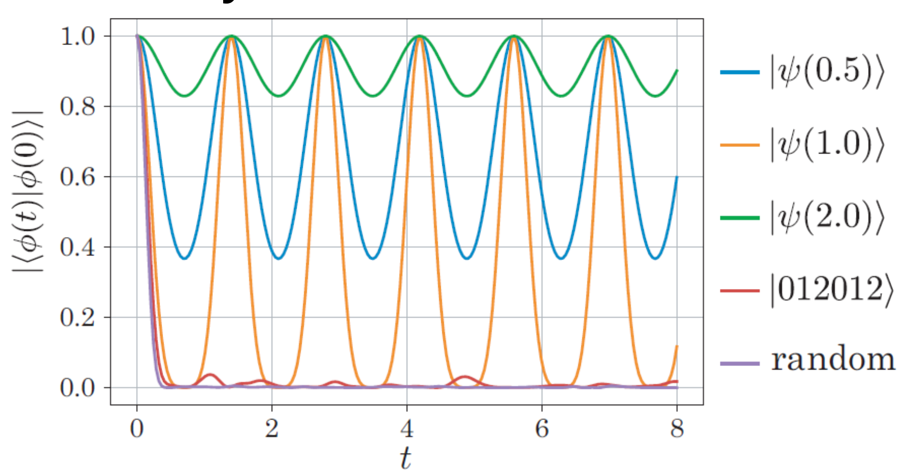
Matrix product state (MPS) with bond dimension 3. Desired perturbations can be identified from this MPS.

$$H = H_{\rm int} + H_{\rm pert} + hQ,$$

- Half-chain entanglement



- Fidelity



Outline of today's lecture

1. Introduction and motivation
2. Onsager scars
3. Other examples
   - Boundary scars and scalar chirality
   - Dzyaloshinskii-Moriya int. + Zeeman
     - K. Sanada, Y. Miao & HK, *PRB* **108**, 155102 (2023)
     - M. Kunimi, T. Tomita, HK & Y. Kato, arXiv:2306.05591
4. Summary

Integrable boundary states

- Integrable Hamiltonian: $H_{\text{int}} = \sum_j H_j$ (1d nearest neighbor int.)
- Boost operator: $B = \sum_j j H_j$
- Conserved charges: $Q_{n+1} = [B, Q_n], \quad Q_2 \propto H_{\text{int}}$

  Q_{2k}/Q_{2k+1} is even / odd under parity $\mathcal{I}$:

  $$\mathcal{I}|\sigma_1, \sigma_2, \cdots, \sigma_{L-1}, \sigma_L\rangle = |\sigma_L, \sigma_{L-1}, \cdots, \sigma_2, \sigma_1\rangle$$

  ➢ Example: *S* =1/2 Heisenberg chain

  Scalar chirality

  $$H_{\text{int}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \Rightarrow Q_3 \propto C_{\text{SC}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot (\boldsymbol{S}_{j+1} \times \boldsymbol{S}_{j+2})$$

- Integrable boundary states: Piroli, Pozsgay & Vernier, NPB **925** (2017)

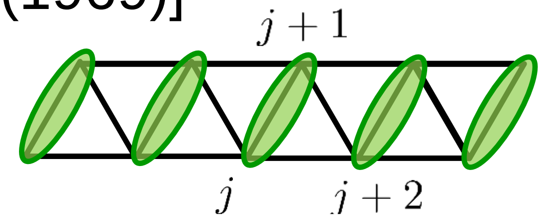
$$|\Psi_0\rangle \text{ such that } Q_{2k+1}|\Psi_0\rangle = 0 \text{ for all } k = 1, 2, 3, \ldots$$

Lattice version of boundary states in integrable QFT:
Ghoshal & Zamolodchikov, IJMP **A9**, 3841 (1994)

Boundary scars

- If $|\Psi_0\rangle$ is an eigenstate of a non-integrable Hamiltonian H_0, then it is an eigenstate of $H_0 + \sum_{k=1}^{\infty} t_k Q_{2k+1} \quad (t_k \in \mathbb{R})$
- Example
  - H_0 : Majumdar-Ghosh model [JMP **10** (1969)]

$$H_{\mathrm{MG}} = \sum_{j=1}^{L} \left[(\boldsymbol{S}_j + \boldsymbol{S}_{j+1} + \boldsymbol{S}_{j+2})^2 - \frac{3}{4} \right]$$



Dimer g.s. are annihilated by C_{SC}

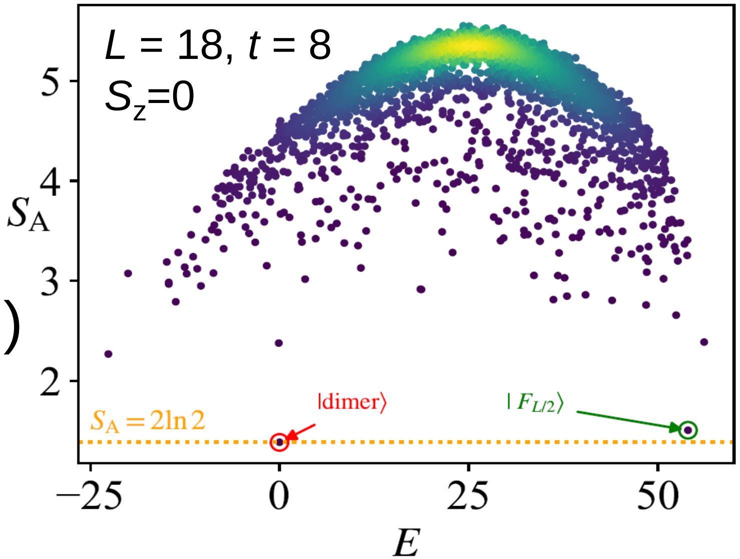
  - Hamiltonian

$$H(t) = H_{\mathrm{MG}} + tC_{\mathrm{SC}}$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot

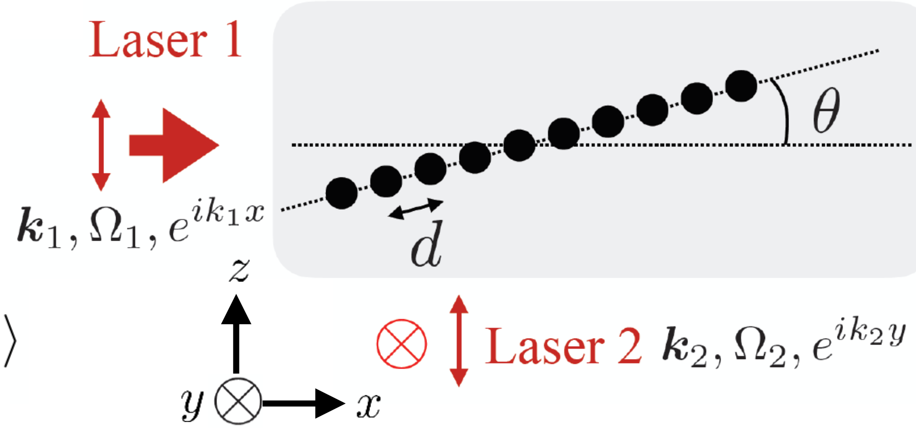
✓ Dimer g.s. is a scar!



DH model

- **Experimental setup**
  - 1d array of Rb atoms
  - Effective spin states

$$|\downarrow\rangle \leftrightarrow |n_1 S_{1/2}\rangle, \ |\uparrow\rangle \leftrightarrow |n_2 S_{1/2}\rangle$$



  - Effective Hamiltonian
    → *S*=1/2 XXZ chain in a rotating magnetic field

$$-\Omega_{\text{eff}}[\cos(qj)S_j^x + \sin(qj)S_j^y] - \tilde{\Delta}S_j^z, \quad q = k_1 d \cos\theta$$

- **Hamiltonian in spin-rotating frame**

$$H_{\text{eff}} = J\cos q \sum_j (S_j^z S_{j+1}^z + S_j^x S_{j+1}^x) + J\delta \sum_j S_j^y S_{j+1}^y - \tilde{\Delta}\sum_j S_j^y$$

$$- J\sin q \sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - \Omega_{\text{eff}} \sum_j S_j^z$$

DH model

  - Tuning *q*, *δ*, etc. → Model with only Dzyaloshinskii-Moriya int. and field in the *z*-direction [Kodama, Kato & Tanaka, PRB **107** (2023)]

QMBS states in DH model

- Hamiltonian $H_{\rm DH} = D\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - H\sum_j S_j^z$ PBC or special OBC
- Raising operator $Q^\dagger = \sum_j P_{j-1} S_j^+ P_{j+1}$ Similar to $Q^\dagger$ in Schecter & Iadecola, PRL **123** (2019).
- They satisfy a restricted spectrum generating algebra (SGA)

$$H_{\rm DH}|\Downarrow\rangle = E_0|\Downarrow\rangle \quad (|\Downarrow\rangle = |\downarrow\cdots\downarrow\rangle)$$

$$[H_{\rm DH}, Q^\dagger]|\Downarrow\rangle = -HQ^\dagger|\Downarrow\rangle$$

$$[[H_{\rm DH}, Q^\dagger], Q^\dagger] = 0$$

See e.g., Moudgalya *et al.*, PRB **102**, 085140 (2020).

- Exact eigenstates

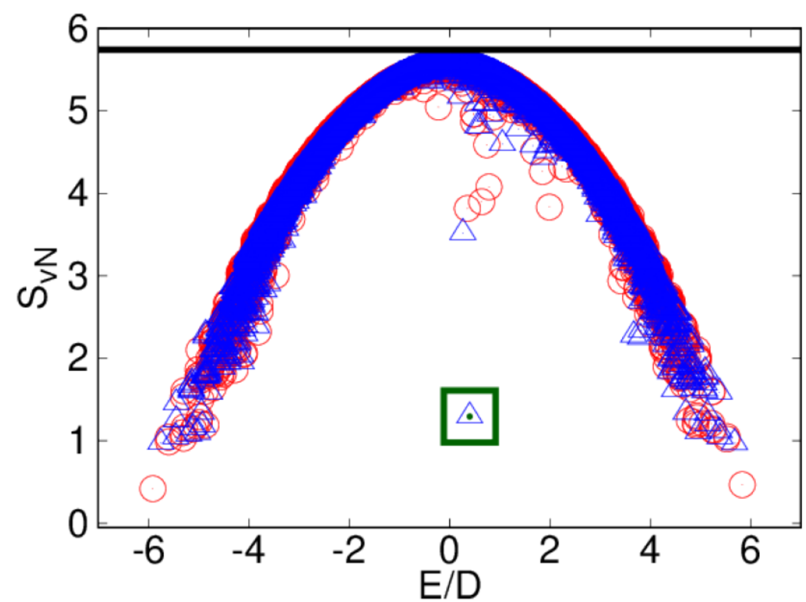
$$|S_n\rangle = (Q^\dagger)^n|\Downarrow\rangle$$

$$H_{\rm DH}|S_n\rangle = (E_0 - nH)|S_n\rangle$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot, fidelity

✓ They are scars!



OBC, *L*=18, *H*=0.1*D*, Soliton num. = 5

Summary

- Local divergence condition
  - Generalization of frustration-freeness
- Constructing models with QMBS
  - Using Onsager algebra
    → Perturbed *S*=1/2 XY chian, higher-spin models
  - Using integrable boundary states
    → Majumdar-Ghosh + scalar chirality
  - Using restricted SGA
    → Dzyaloshinskii-Moriya + Zeeman
- Other models
  - Correlated hopping model: Tamura & HK, *PRB* **106** (2022)
  - Generalization of eta-pairing: Yoshida & HK, *PRB* **105** (2022)
  - *S*=1 AKLT + SU(3) scalar chirality
  - Perturbed *S*=1 scalar chirality in 1d and 2d