Frustration-free Models and beyond

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Outline of my lectures

- Day 1 (June 19)
  - Introduction to frustration-free systems
  - Systematic construction of models
- Day 2 (June 20)
  - Non-interacting Kitaev chain
  - Interacting Kitaev chain
- Day 3 (June 21)
  - Divergence-free conditions
  - Application to quantum many-body scars

Ground-state Physics (Days 1–2)

Dynamics (Day 3)

Frustration-free systems (recap)

Setup

- Total Hamiltonian $H = \sum_j h_j$
- Sub-Hamiltonian h_j satisfies $h_j \geq E_j^{(0)} \mathbf{1}$

Frustration-free Hamiltonian

- Definition

> $H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j|\psi\rangle = E_j^{(0)}|\psi\rangle$ for all *j*.

- *ψ* saturates Anderson's bound
- Universal form

$$H = \sum_j L_j^\dagger L_j$$

⇦ Positive semidefinite

- Zero-energy manifold

  $G = \mathrm{span}\{|\Psi_1\rangle, ..., |\Psi_n\rangle\}$, where $L_j|\Psi_i\rangle = 0$ for all *j*

Recipe for new models (recap)

- **Conjugation**
  - Deformed *L* operators $\widetilde{L}_j := M L_j M^{-1} \quad (M\colon \text{ invertible})$
  - Deformed Hamiltonian

$$\widetilde{H} = \sum_j \widetilde{L}_j^\dagger \widetilde{L}_j$$

  - Ground-state manifold $\widetilde{G} = \mathrm{span}\{M|\Psi_1\rangle, \ldots, M|\Psi_n\rangle\}.$
- **Sandwiching**
  - Positive definite operators $C_j > 0$
  - Further deformation of *H*

$$H_{\mathrm{new}} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j$$

  - Ground-state manifold $\widetilde{G} = \mathrm{span}\{M|\Psi_1\rangle, \ldots, M|\Psi_n\rangle\}.$
  - H_{new} and H have the same number of g.s.

Outline of today's lecture

1. Duality in Ising model
   - Classical Ising model
   - Quantum Ising model
2. Non-interacting Kitaev chain
3. Frustration-free Kitaev chain
4. Frustration-free quantum Potts chain
5. Summary

2D classical Ising model

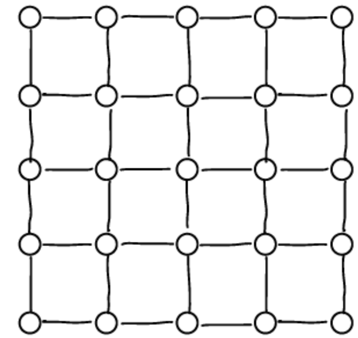
■ Model

- Spin configuration $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, ..., \sigma_N\}, \quad \sigma_i = \pm 1$
- Hamiltonian $H(\boldsymbol{\sigma}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad (J > 0)$
- Partition function

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}$$

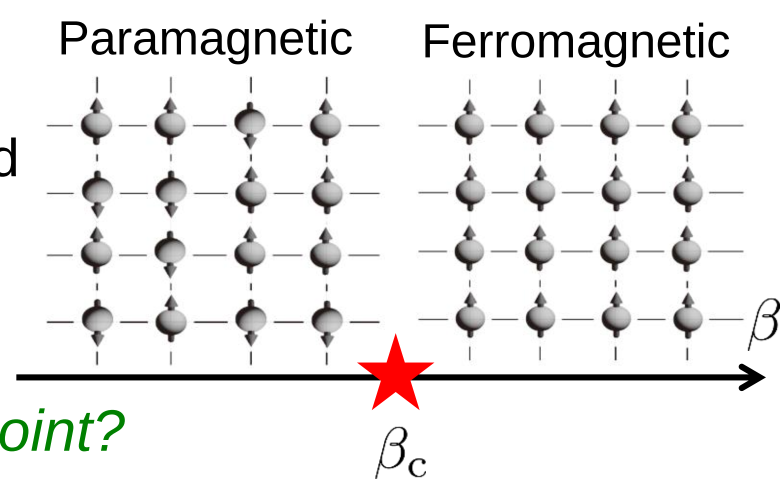
Solved by Onsager, Phys. Rev. **65**, 117 (1944)
Majorana-fermion trick by Kauffmann (1949)

Square lattice



■ Phases

- Zero temperature ($\beta = \infty$)
  All-up and all-down states are realized
- Infinite temperature ($\beta = 0$)
  All states occur with equal probability



Where is the transition (critical) point?

Kramers-Wannier duality (1)

Phys. Rev. **60**, 252 (1941)

■ "High-temperature" expansion

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \exp\left(\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \right)$$

$\beta J \to K$

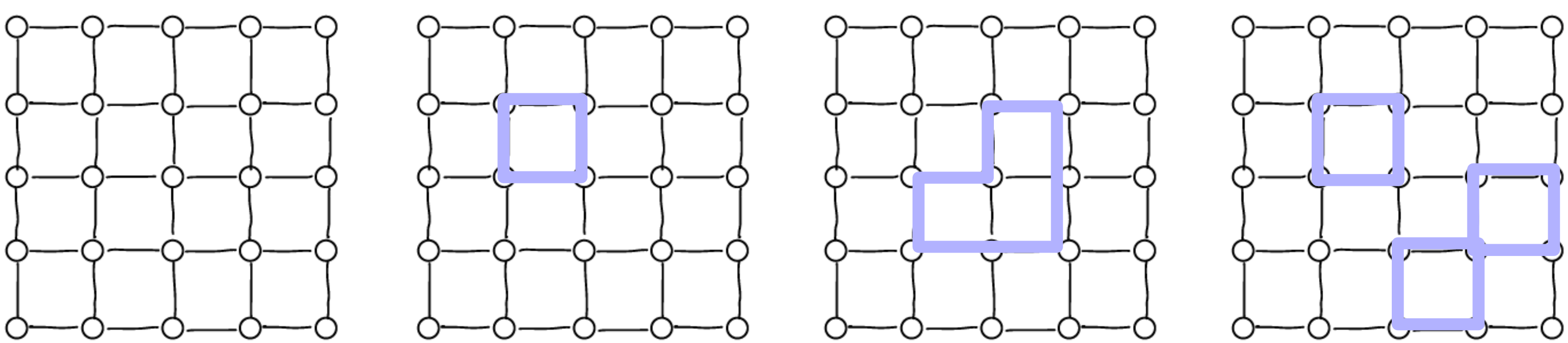
- Useful identity: $e^{K\sigma_i\sigma_j} = \cosh K(1 + \sigma_i\sigma_j \tanh K)$

$$Z_N(K) = (\cosh K)^{2N} \sum_{\boldsymbol{\sigma}} \prod_{\langle i,j \rangle} (1 + \sigma_i\sigma_j \tanh K)$$

$$\sum_{\sigma=\pm 1} \sigma^n = 1 + (-1)^n$$

$$= 2^N (\cosh K)^{2N} \sum_P (\tanh K)^{\ell(P)}$$

P: Sum over loop configurations

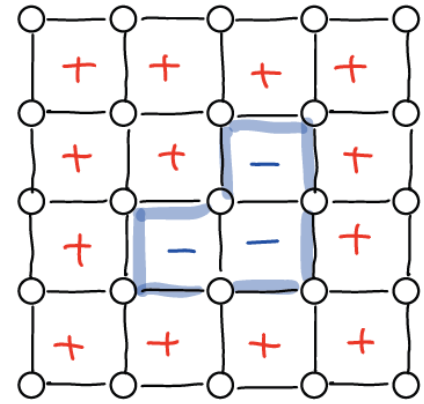


$\ell(P) = 0$ $\ell(P) = 4$ $\ell(P) = 8$ $\ell(P) = 12$

Kramers-Wannier duality (2)

Z in terms of dual variables

- Label dual lattice sites by $a = 1, 2, \ldots$
- Dual spin config. $\boldsymbol{\mu} = \{\mu_1, \mu_2, \ldots\}$
  μ=-1 (+1) inside (outside) a loop



$$\ell(P) = \sum_{\langle a,b \rangle} \frac{1 - \mu_a \mu_b}{2}$$

$$Z_N(K) = (\sinh K)^N \sum_{\boldsymbol{\mu}} \prod_{\langle a,b \rangle} \exp(\tilde{K} \mu_a \mu_b)$$

- Dual coupling $\tilde{K} = \frac{1}{2} \ln \coth K$

Critical temperature

- Assumption: a single critical temperature
- Then $K = \tilde{K} = \beta_c J$ must hold

  Solving $K = \frac{1}{2} \ln \coth K$ leads to $\beta_c J = \frac{1}{2} \ln(1 + \sqrt{2}) \sim 0.44$

Quantum Ising chain

■ Spin operators

- Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Spin op. at site *j* : $\sigma_j^\alpha = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes \sigma^\alpha \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j},$

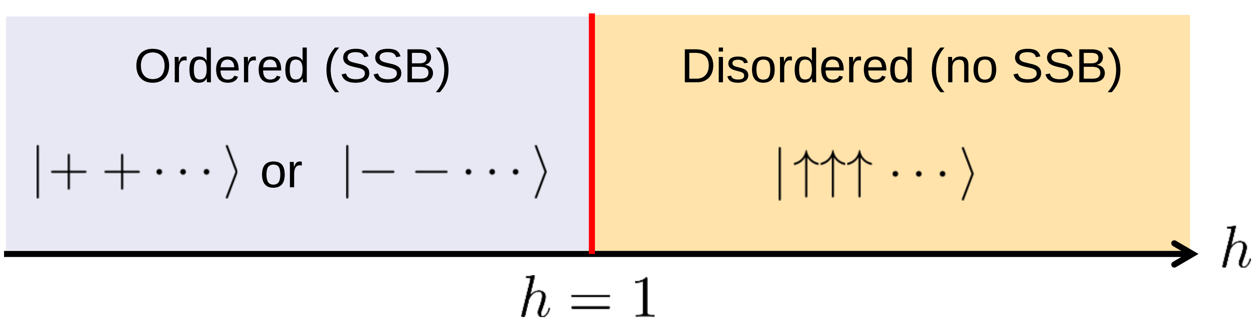
■ Hamiltonian (J, h >0)

$$H_{\text{Ising}} = -J \sum_j \sigma_j^x \sigma_{j+1}^x - h \sum_j \sigma_j^z$$

Local basis

$$\sigma^z |\uparrow\rangle = +|\uparrow\rangle$$
$$\sigma^z |\downarrow\rangle = -|\downarrow\rangle$$
$$\sigma^x |\pm\rangle = \pm|\pm\rangle$$

■ Phase diagram (J =1)



h =1 is a critical point described by Ising CFT (c=1/2)

Kramers-Wannier Duality

■ Unitary transformation

Huang & Chen, *PRB* **91**, 195143 (2015)

$$U = \prod_j \frac{1 + \mathrm{i}\,\sigma_j^z}{\sqrt{2}}, \quad V = \prod_j \frac{\sigma_j^x \sigma_{j+1}^x + \sigma_{j+1}^z}{\sqrt{2}}$$

$$(UV)\,\sigma_j^z\,(UV)^\dagger = \sigma_j^x \sigma_{j+1}^x$$

$$(UV)\,\sigma_j^x \sigma_{j+1}^x\,(UV)^\dagger = \sigma_{j+1}^z$$

■ Dual Hamiltonian

$$\widetilde{H}_{\mathrm{Ising}} = (UV) H_{\mathrm{Ising}} (UV)^\dagger = -h \sum_j \sigma_j^x \sigma_{j+1}^x - J \sum_j \sigma_j^z$$

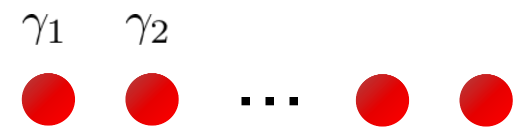
- The roles of *J* and *h* are interchanged
- Must have the same spectrum
  NOTE) Ignore the boundary terms. cf) Non-invertible sym.
- If there exists a single gapless point, it should be at $h = J$

Anything to do with topology?

Outline of today's lecture

1. Duality in Ising model
2. Non-interacting Kitaev chain
   - Majorana fermions
   - Trivial and topological phases
   - Topological invariant
   - Edge zero modes
3. Frustration-free Kitaev chain
4. Frustration-free quantum Potts chain
5. Summary

(Lattice) Majorana fermions



- Majorana ops. $\gamma_j \quad (j = 1, 2, ..., 2L)$
  - Their own Hermitian conjugates $\gamma_j^\dagger = \gamma_j$
  - Anticommute with each other $\{\gamma_i, \gamma_j\}$
  - Fermionic parity $(-1)^F = (-\mathrm{i})^L \gamma_1 \gamma_2 \cdots \gamma_{2L}$
- Connection to spins
  - Jordan-Wigner transformation

$$\gamma_{2j-1} = \left(\prod_{k<j} \sigma_k^z\right) \sigma_j^x, \quad \gamma_{2j} = \left(\prod_{k<j} \sigma_k^z\right) \sigma_j^y$$

  - Fermionic parity → $(-1)^F = \sigma_1^z \cdots \sigma_L^z$
- Spinless (complex) fermions

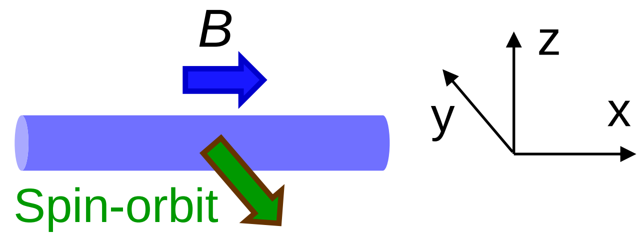
$$c_j = \frac{1}{2}(\gamma_{2j-1} + \mathrm{i}\gamma_{2j}), \quad c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - \mathrm{i}\gamma_{2j})$$

  - They obey $\{c_i, c_j^\dagger\} = \delta_{i,j}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$

How to get rid of spins?

- 1+1 D quantum wire setup

Lutchyn *et al*.,; Oreg *et al*., *PRL* (2010)
Mourik *et al*., *Science* **336**, 1003 (2012)
Review: Elliott & Franz,
Rev. Mod. Phys. **87**, 139 (2015).

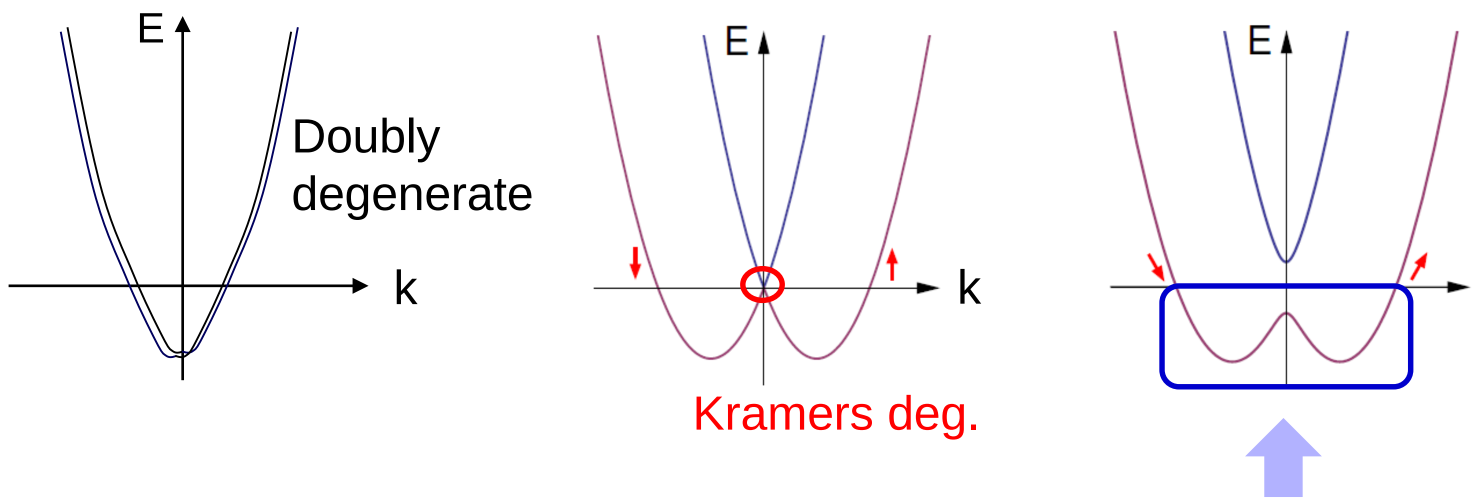


- Hamiltonian in *k*-space

Spin-orbit

Zeeman

$$H_0(k) = \frac{k^2}{2m} - \mu \qquad H_0(k) + \alpha\sigma_y k \qquad H_0(k) + \alpha\sigma_y k + V_x\sigma_x$$



Low-energy physics is effectively described by spinless fermions!

Quantum Ising chain → Kitaev chain

■ Majorana rep.

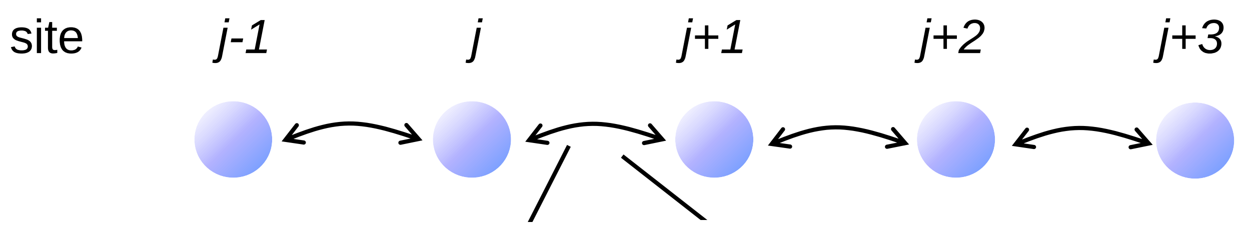
Kitaev, *Phys. Usp*. (2001)
cond-mat/0010440

- Hamiltonian

$$H_{\text{Ising}} = \mathrm{i}\,J \sum_j \gamma_{2j}\gamma_{2j+1} - \mathrm{i}\,h \sum_j \gamma_{2j-1}\gamma_{2j}$$

- Duality = Translation by 1 Majorana site
- Self-duality = Emergent translation symmetry at *J*=*h*

■ Spinless fermion rep.



- Hamiltonian

hopping pairing Chemical potential

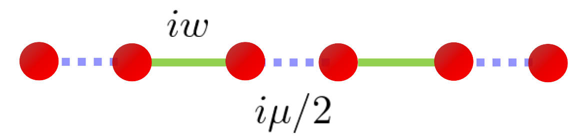
$$H_0 = -w \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + w \sum_j (c_j c_{j+1} + c_{j+1}^\dagger c_j) - \mu \sum_j (c_j^\dagger c_j - 1/2)$$

➢ H_{Ising} and H_0 are identical when $J=w$ and $h=\mu/2$

Phases of Kitaev chain

- Hamiltonian (with OBC; *w*, *μ*>0)

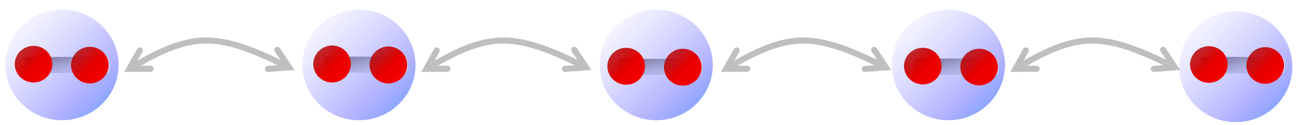
$$H_0 = \mathrm{i}w \sum_{j=1}^{L-1} \gamma_{2j}\gamma_{2j+1} - \frac{\mathrm{i}}{2}\mu \sum_{j=1}^{L} \gamma_{2j-1}\gamma_{2j}$$



- Trivial phase $(w \ll \mu)$

$$H_0 = -\frac{\mathrm{i}}{2}\mu \sum_{j=1}^{L} \gamma_{2j-1}\gamma_{2j} = -\mu \sum_{j=1}^{L} (c_j^\dagger c_j - 1/2)$$

The fully filled state is the unique g.s. (*μ*>0)

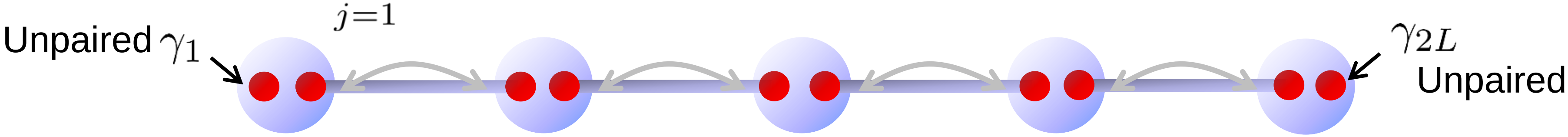


- Topological phase $(w \gg \mu)$

$$H_0 \sim \mathrm{i}w \sum_{j=1}^{L-1} \gamma_{2j}\gamma_{2j+1}$$

New, non-local fermion

$$f = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_{2L})$$



presence/absence of *f* ⇔ two-fold degenerate g.s. $|0\rangle, \quad |1\rangle = f^\dagger|0\rangle$

Edge zero modes exist as long as *w*>*μ*.

Bulk topological invariant

- BdG Hamiltonian in *k*-space

$$H = \frac{1}{2}\sum_k \Psi^\dagger(k)\,\mathcal{H}(k)\,\Psi(k), \qquad \Psi(k) = \begin{pmatrix} \psi_k \\ \psi^\dagger_{-k} \end{pmatrix}$$

$$\mathcal{H}(k) = \begin{pmatrix} h(k) & \Delta(k) \\ \Delta^*(k) & -h(k) \end{pmatrix} = \boldsymbol{d}(k)\cdot\boldsymbol{\sigma}$$

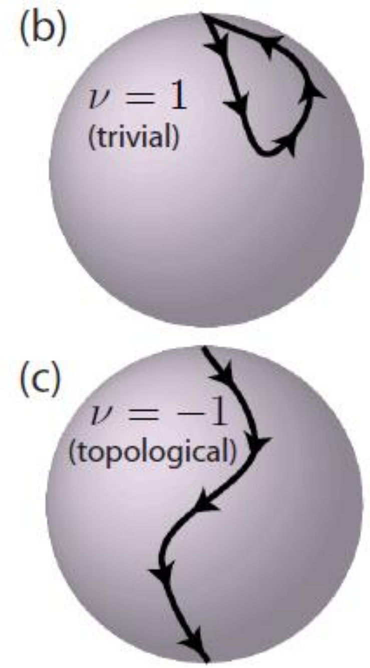
  - Symmetry

$$d_{x,y}(k) = -d_{x,y}(-k), \quad d_z(k) = d_z(-k)$$

  - *n*-vector

$$\boldsymbol{n}(k) := \frac{\boldsymbol{d}(k)}{|\boldsymbol{d}(k)|}$$

Nonzero gap
$\Leftrightarrow$ $|\boldsymbol{d}(k)| > 0$



J. Alicea, *Rep. Prog. Phys*. **75**, 076501 (2012)

- Topological number
  - *k*= 0 and *π* are special!

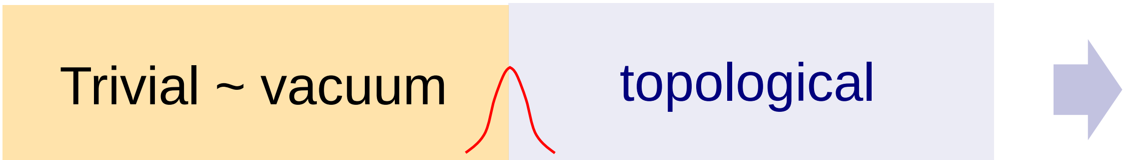
$$\boldsymbol{n}(0) = s_0\hat{z}, \quad \boldsymbol{n}(\pi) = s_\pi\hat{z}$$

$$\nu = s_0 s_\pi \in \{+1, -1\}$$

  - There must be a critical point between (b) and (c)

Majorana edge zero modes

- Implication of topo. Invariant

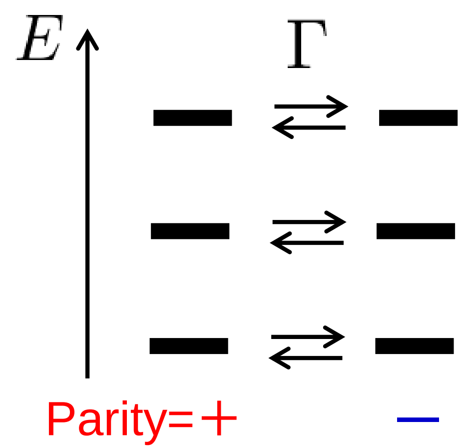


Robust zero mode in presence of open edges

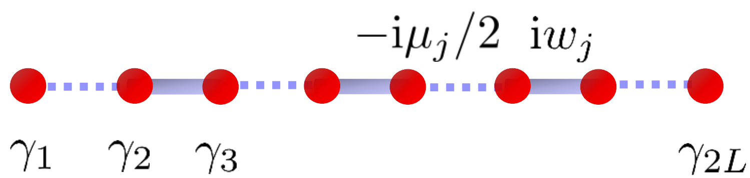
Gap should close at the boundary!

- Majorana edge zero modes are s.t.
  - $\Gamma^\dagger = \Gamma$
  - $[H_0, \Gamma] = 0$
  - $\{(-1)^F, \Gamma\} = 0$
  - localized near the edge, and normalizable as $\Gamma^2 = 1$ even in the infinite-size limit

P. Fendley, *J. Stat. Mech.* P11020 (2012)



- Robust against disorder



$$\Gamma_L \propto \gamma_1 - \frac{\mu_1}{2w_1}\gamma_3 + \frac{\mu_1\mu_2}{4w_1w_2}\gamma_5 + \cdots$$

Outline of today's lecture

1. Duality in Ising model
2. Non-interacting Kitaev chain
3. Frustration-free Kitaev chain
   - Decoupling limit
   - Witten's conjugation
   - Sandwiching method
   - Spectral gap
4. Frustration-free quantum Potts chain
5. Summary

- H.K., Schuricht, Takahashi, *PRB* **92**, 115137 (2015)
- Wouters, H.K., Schuricht, *PRB* **98**, 155119 (2018)

Decoupling limit is frustration-free!

- Hamiltonian (+ const. shift)

$$H_0 = w\sum_{j=1}^{L-1}(1+\mathrm{i}\,\gamma_{2j}\gamma_{2j+1}) = \frac{w}{2}\sum_{j=1}^{L-1}\overbrace{(-\mathrm{i}\,\gamma_{2j}-\gamma_{2j+1})}^{A_j^\dagger}\overbrace{(\mathrm{i}\,\gamma_{2j}-\gamma_{2j+1})}^{A_j} \geq 0$$

$$A_j = c_j - c_j^\dagger - c_{j+1} - c_{j+1}^\dagger$$

- Coherent states

$c_j|\mathrm{vac}\rangle = 0$ for all j

$$|\Psi_\pm\rangle = e^{\pm c_1^\dagger}e^{\pm c_2^\dagger}\cdots e^{\pm c_L^\dagger}|\mathrm{vac}\rangle = (1\pm c_1^\dagger)(1\pm c_2^\dagger)\cdots(1\pm c_L^\dagger)|\mathrm{vac}\rangle$$

They are annihilated by A_j for all j. → The g.s. of H_0. ex.) Prove this

But they are NOT eigenstates of fermionic parity $(-1)^F$.

L=2

Even $|\Psi_+\rangle + |\Psi_-\rangle \propto |\circ\circ\rangle + |\bullet\bullet\rangle$

Odd $|\Psi_+\rangle - |\Psi_-\rangle \propto |\bullet\circ\rangle + |\circ\bullet\rangle$

○: empty site
●: filled site

L=3

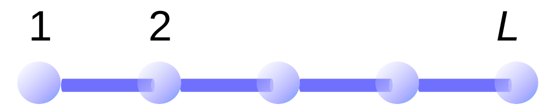
Even $|\Psi_+\rangle + |\Psi_-\rangle \propto |\circ\circ\circ\rangle + |\bullet\bullet\circ\rangle + |\bullet\circ\bullet\rangle + |\circ\bullet\bullet\rangle$

Odd $|\Psi_+\rangle - |\Psi_-\rangle \propto |\bullet\circ\circ\rangle + |\circ\bullet\circ\rangle + |\circ\circ\bullet\rangle + |\bullet\bullet\bullet\rangle$

Obvious in the spin language

- Hamiltonian in terms of spins

$$H_{\text{Ising}} = \sum_{j=1}^{L-1} h_j, \quad h_j = 1 - \sigma_j^x \sigma_{j+1}^x \geq 0$$



  - *L* operators

$$h_j = L_j^\dagger L_j, \quad L_j = L_j^\dagger = (\sigma_j^x - \sigma_{j+1}^x)/\sqrt{2}$$

  - Diagonal in X-basis

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$$

$$\sigma^x |\pm\rangle = \pm|\pm\rangle$$

$$\sigma^z |\uparrow\rangle = |\uparrow\rangle, \ \sigma^z|\downarrow\rangle = -|\downarrow\rangle$$

- Ground states
  - Zero-energy states $|+\rangle_1|+\rangle_2 \cdots |+\rangle_L$ and $|-\rangle_1|-\rangle_2 \cdots |-\rangle_L$
  - No other g.s.
  - (Unnormalized) g.s.

$$|\Psi_\pm\rangle = \bigotimes_{j=1}^{L} (|\uparrow\rangle_j \pm |\downarrow\rangle_j)$$

    - Equivalent to the fermionic g.s. via Jordan-Wigner

Witten's conjugation (1)

- Invertible operator (α: real, num. op.: $n_j := c_j^\dagger c_j$)

$$M = [1+(\alpha-1)n_1]\,[1+(\alpha-1)n_2]\cdots[1+(\alpha-1)n_L]$$

$$M^{-1} = [1+(\alpha^{-1}-1)n_1]\,[1+(\alpha^{-1}-1)n_2]\cdots[1+(\alpha^{-1}-1)n_L]$$

- Conjugations

$$M c_j M^{-1} = \alpha^{-1} c_j, \quad M c_j^\dagger M^{-1} = \alpha c_j^\dagger$$

$$A_j = c_j - c_j^\dagger - c_{j+1} - c_{j+1}^\dagger \;\Rightarrow\; \widetilde{A}_j = \alpha^{-1} c_j - \alpha c_j^\dagger - \alpha^{-1} c_{j+1} - \alpha c_{j+1}^\dagger$$

- New ground states

Using *M* |vac> = |vac>, we have

One-parameter deformation of the decoupling limit!

$$|\widetilde{\Psi}_\pm\rangle = e^{\pm\alpha c_1^\dagger}\cdots e^{\pm\alpha c_L^\dagger}|\mathrm{vac}\rangle = (1\pm\alpha c_1^\dagger)\cdots(1\pm\alpha c_L^\dagger)|\mathrm{vac}\rangle$$

L=3

$$\text{Even}\quad |\widetilde{\Psi}_+\rangle + |\widetilde{\Psi}_-\rangle \propto |\circ\circ\circ\rangle + \alpha^2|\bullet\bullet\circ\rangle + \alpha^2|\bullet\circ\bullet\rangle + \alpha^2|\circ\bullet\bullet\rangle$$

$$\text{Odd}\quad |\widetilde{\Psi}_+\rangle - |\widetilde{\Psi}_-\rangle \propto \alpha|\bullet\circ\circ\rangle + \alpha|\circ\bullet\circ\rangle + \alpha|\circ\circ\bullet\rangle + \alpha^3|\bullet\bullet\bullet\rangle$$

Witten's conjugation (2)

■ Hamiltonian

$$\widetilde{H}_0 = \frac{w}{2} \sum_j \widetilde{A}_j^\dagger \widetilde{A}_j = \sum_{j=1}^{L-1} \widetilde{h}_j$$

$$\widetilde{h}_j = -t(c_j^\dagger c_{j+1} + \text{h.c.}) + \Delta(c_j c_{j+1} + \text{h.c.}) - \frac{\mu}{2}(n_j + n_{j+1} - 1) + \text{const.}$$

with $\frac{\Delta}{t} = \frac{2}{\alpha^2 + \alpha^{-2}}, \quad \frac{\mu}{t} = \frac{2(\alpha^2 - \alpha^{-2})}{\alpha^2 + \alpha^{-2}}$

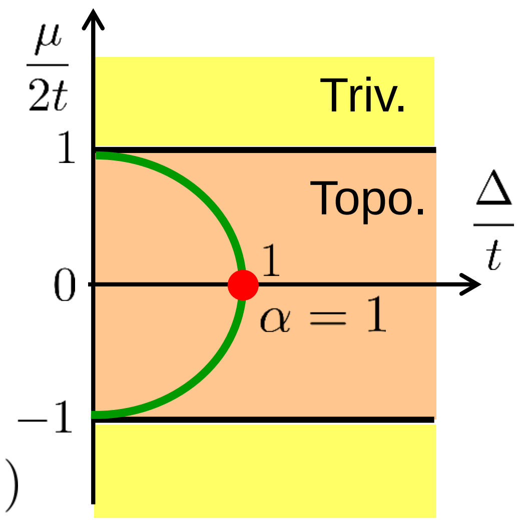
$$\left(\frac{\Delta}{t}\right)^2 + \left(\frac{\mu}{2t}\right)^2 = 1$$

■ Barouch-McCoy circle

- Phase diagram of Kitaev chain
  $|\mu/t| > 2$ → Trivial, $|\mu/t| < 2$ → Topo.

- Can be read off from that of XY spin chain
- Factorized ground states on the circle
  Barouch-McCoy, *PRA* **3**, 786 (1971).

$$|\widetilde{\Psi}_\pm\rangle = (|\uparrow\rangle_1 \pm \alpha|\downarrow\rangle_1) \otimes \cdots \otimes (|\uparrow\rangle_L \pm \alpha|\downarrow\rangle_L)$$



What about the interacting case?

■ Sandwiching method

- New Hamiltonian $H = \frac{w}{2} \sum_j \widetilde{A}_j^\dagger C_j \widetilde{A}_j$

  Center term can be anything as long as $C_j^\dagger = C_j > 0$
  The g.s. of $\widetilde{H}_0$ remain unchanged

- Choice
  $C_j = \alpha_1 P^e_{j,j+1} + \alpha_2 P^o_{j,j+1}$
  $(\alpha_1, \alpha_2 > 0)$

- Projectors
  $P^e = |\circ\, \circ\rangle\langle\circ\, \circ| + |\bullet\, \bullet\rangle\langle\bullet\, \bullet|$ Even sector
  $P^o = |\circ\, \bullet\rangle\langle\circ\, \bullet| + |\bullet\, \circ\rangle\langle\bullet\, \circ|$ Odd sector

■ Explicit Hamiltonian

$$H = \sum_{j=1}^{L-1} h_j$$

$$\left(\frac{\Delta}{t+2U}\right)^2 + \left(\frac{\mu}{2(t+2U)}\right)^2 = 1$$

$$h_j = -t(c_j^\dagger c_{j+1} + \text{h.c.}) + \Delta(c_j c_{j+1} + \text{h.c.})$$

4-Majorana int.

$$- \frac{\mu}{2}(n_j + n_{j+1} - 1) + U(2n_j - 1)(2n_{j+1} - 1) + \text{const.}$$

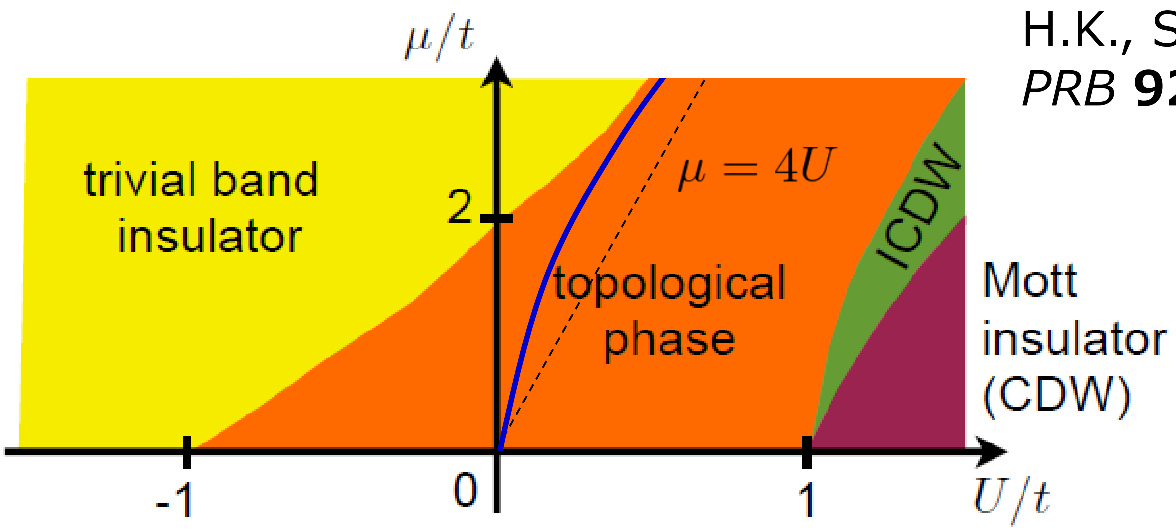
$$= -\gamma_{2j-1}\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2}$$

Phase diagram of interacting Kitaev chain

- **Previous studies**
  - Gangadharaiah et al., *PRL* (2011); Hassler & Schuricht, *NJP* (2012); Thomale et al., *PRB* (2013); Rahmani et al, *PRL* (2015); …
- **Phase diagram for *t*=*Δ***
  - Equivalent to quantum ANNNI model
    - Beccaria *et al*., *PRB* (2007); Sela & Pereira, *PRB* (2011); …

Solvable line $\mu = \mu^* = 4\sqrt{U^2 + tU}$

Exact g.s. $|\widetilde{\Psi}_\pm\rangle = (1 \pm \alpha c_1^\dagger) \cdots (1 \pm \alpha c_L^\dagger)|\text{vac}\rangle$



H.K., Schuricht, Takahashi, *PRB* **92**, 115137 (2015)

Spin chain representation

■ Jordan-Wigner transformation

$$\gamma_{2j-1} = \left(\prod_{k<j} \sigma_k^z \right) \sigma_j^x, \quad \gamma_{2j} = \left(\prod_{k<j} \sigma_k^z \right) \sigma_j^y$$

Fermionic parity

$$(-1)^F = \prod_{j=1}^{L} \sigma_j^z$$

■ XYZ chain in a field

- Fermionic Hamiltonian

$$h_j = -t(c_j^\dagger c_{j+1} + \mathrm{h.c.}) + \Delta(c_j c_{j+1} + \mathrm{h.c.}) - \frac{\mu}{2}(n_j + n_{j+1} - 1) + U(2n_j - 1)(2n_{j+1} - 1) + \mathrm{const.}$$

- Spin Hamiltonian

$$h_j = -J_x \sigma_j^x \sigma_{j+1}^x - J_y \sigma_j^y \sigma_{j+1}^y + \underline{J_z \sigma_j^z \sigma_{j+1}^z} - B_j \sigma_j^z$$

$$J_x = \frac{t+\Delta}{2}, \quad J_y = \frac{t-\Delta}{2}, \quad J_z = U, \quad B_j = -\frac{\mu}{2}$$

What we found is a fermionic rephrasing of
Peschel-Emery (1981), Mueller-Schrock (1985), …

Spectral gap

- Min-max theorem (Courant-Fischer-Weyl)

Let A and B be two hermitian matrices.
Let a_i and b_i be the i-th eigenvalues of A and B, respectively.
(Assume the order, $a_1 \leq a_2 \leq \cdots, \; b_1 \leq b_2 \leq \cdots$.)
If $A \geq B$, then we have $a_i \geq b_i, \; \forall i.$

- Upper and lower bounds

$$H = \frac{w}{2} \sum_j \widetilde{A}_j^\dagger (\alpha_1 P^e_{j,j+1} + \alpha_2 P^o_{j,j+1}) \widetilde{A}_j \qquad (\alpha_1 \geq \alpha_2, \quad P^e + P^o = 1)$$

$$= \alpha_2 + (\alpha_1 - \alpha_2) P^e_{j,j+1} = \alpha_1 - (\alpha_1 - \alpha_2) P^o_{j,j+1}$$

$$\Rightarrow \quad \alpha_2 \widetilde{H}_0 \leq H \leq \alpha_1 \widetilde{H}_0$$

$E_n^{(0)}$: n-th eigen-energy of $\widetilde{H}_0$. Note $E_1^{(0)} = E_2^{(0)} = 0.$
Since H and $\widetilde{H}_0$ share the same g.s., the gap is bounded as
$\alpha_2 E_3^{(0)} \leq \Delta E \leq \alpha_1 E_3^{(0)}.$ → *Uniform gap for Δ/t ≠0*

Frustration-free non-interacting model

■ Single-particle spectrum

$$t = 1,\ \Delta = \sin\theta,\ \mu = 2\cos\theta$$

$$\widetilde{H}_0 + \text{const.} = \frac{i}{2}\sum_{j,k} B_{j,k}\gamma_{2j-1}\gamma_{2k} = \sum_{k=1}^{L}\epsilon_k\left(f_k^\dagger f_k - \frac{1}{2}\right)$$

SVD: $B = U\Lambda V^{\mathrm{T}},\ \Lambda = \mathrm{diag}(\epsilon_1, ..., \epsilon_L).$

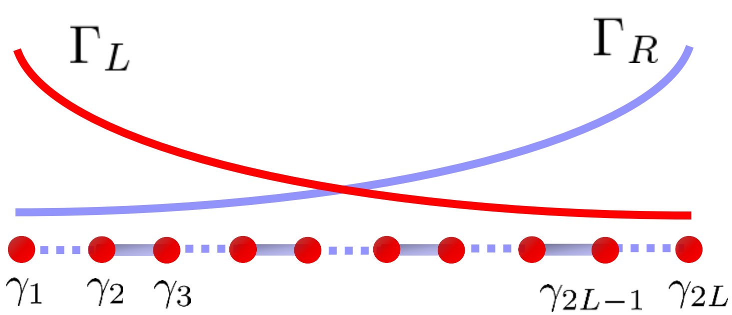
Singular values: $\epsilon = 0,\ 2 + 2\cos\theta\cos\left(\frac{n\pi}{L}\right) \quad (n = 1, 2, ..., L-1)$

■ Exact edge zero modes

$$\mathfrak{c} = \cos\theta,\ \mathfrak{s} = \sin\theta$$

$$\Gamma_L = \mathcal{N}(\gamma_1 + r\gamma_3 + \cdots + r^{L-1}\gamma_{2L-1})$$

$$B = -\begin{pmatrix} \mathfrak{c} & 1-\mathfrak{s} & & & \\ 1+\mathfrak{s} & 2\mathfrak{c} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2\mathfrak{c} & 1-\mathfrak{s} \\ & & & 1+\mathfrak{s} & \mathfrak{c} \end{pmatrix}$$



$$(1,\ r,\ r^2,\ \cdots,\ r^{L-1})B = 0 \qquad r = -\frac{\mathfrak{c}}{1+\mathfrak{s}}$$

Staggered model

Hamiltonian

$$H = -t\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta\sum_{j=1}^{L-1}(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$

$$-\sum_{j=1}^{L}\mu_j(c_j^\dagger c_j - 1/2) + U\sum_{j=1}^{L-1}(2c_j^\dagger c_j - 1)(2c_{j+1}^\dagger c_{j+1} - 1) \ +S$$

- Staggered potential

$$\mu_j = \begin{cases} q, & \text{if } j \text{ odd} \\ 1/q, & \text{if } j \text{ even} \end{cases} \qquad S = q_1(n_1 - 1/2) + q_L(n_L - 1/2)$$

Frustration-free case

- Non-interacting (U=0) model is so for

$$t = \frac{\eta + \eta^{-1}}{2}, \quad \Delta = \frac{\eta - \eta^{-1}}{2}.$$ (parametrization)

- Interacting model was worked out (Wouters, H.K., Schuricht, PRB (2018)) (XYZ spin chain in staggered magnetic field)
- Upper and lower bound on gap, topo. order …

Outline of today's lecture

1. Duality in Ising model
2. Non-interacting Kitaev chain
3. Frustration-free Kitaev chain
4. Frustration-free quantum Potts chain
   - Shift and clock matrices
   - Duality, parafermions, …
   - Deformed models
5. Summary

➢ Wouters, Katsura, Schuricht, *SciPost Phys. Core* **4** (2021)

Quantum Potts chains

■ Shift & clock matrices

$$\sigma|\tau,i\rangle = |\tau,i-1\rangle,\quad \tau|\tau,i\rangle = \omega^i|\tau,i\rangle,\quad i=0,\ldots,N-1,\ \omega = e^{2\pi\mathrm{i}/N}$$

$$\sigma^N = \tau^N = 1,\quad \sigma^\dagger = \sigma^{N-1},\quad \tau^\dagger = \tau^{N-1}\qquad \sigma\tau = \omega\,\tau\sigma$$

■ Hamiltonian (*J*, *h* >0)

$$H_{\text{Potts}} = -J\sum_j(\sigma_j^\dagger\sigma_{j+1} + \text{h.c.}) - h\sum_j(\tau_j + \tau_j^\dagger)$$

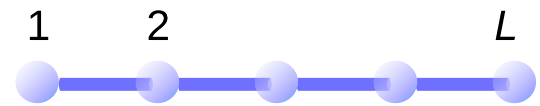
- Duality: $\tau_j \to \sigma_j^\dagger\sigma_{j+1},\quad \sigma_j^\dagger\sigma_{j+1} \to \tau_{j+1}$
- Parafermions Fradkin & Kadanoff, *NPB* **170** (1980)

$$\chi_{2j-1} = \sigma_j\prod_{k<j}\tau_k,\quad \chi_{2j} = -\omega^{1/2}\tau_j\sigma_j\prod_{k<j}\tau_k$$

$$H_{\text{Potts}} = J\sum_j(\omega^{1/2}\chi_{2j-1}^\dagger\chi_{2j} + \text{h.c.}) + h\sum_j(\omega^{1/2}\chi_{2j}^\dagger\chi_{2j+1} + \text{h.c.})$$

- Translation invariant at the self-dual point *h* = *J*
- Gap closing (2nd order for *N*=2, 3, 4, 1st order for *N*>4)

Classical 3-state Potts chain



■ Hamiltonian

$$H = \sum_{j=1}^{L-1} h_j, \quad h_j = 2 - \sigma_j^\dagger \sigma_{j+1} - \sigma_{j+1}^\dagger \sigma_j \geq 0$$

- *L* operators $\quad h_j = L_j^\dagger L_j, \quad L_j = \sigma_j - \sigma_{j+1}$
- Diagonal in *σ*-basis

$$|\sigma, a\rangle = \frac{1}{\sqrt{3}}(|\tau, 0\rangle + \omega^a |\tau, 1\rangle + \omega^{2a} |\tau, 2\rangle) \quad (a = 0, 1, 2)$$

$$\sigma |\sigma, a\rangle = \omega^a |\sigma, a\rangle, \quad \omega = \exp\left(\frac{2\pi \mathrm{i}}{3}\right)$$

■ Ground states

- Zero-energy states
- 3-fold degenerate
- No other ground states

$$|\Psi_a\rangle = \bigotimes_{j=1}^{L} |\sigma, a\rangle_j \quad (a = 0, 1, 2)$$

Deformed Potts chain (1)

■ Conjugation

τ-diagonal basis

- *M* operator $M = m \otimes \cdots \otimes m, \quad m = \begin{pmatrix} 1 & & \\ & e^{\mathrm{i}\theta} r & \\ & & r^2 \end{pmatrix}, \quad r > 0$
- *C* operator $C_j = 1$
- Deformed model $\tilde{H} = \sum_{j=1}^{L-1} \tilde{h}_j, \quad \tilde{h}_j = \tilde{L}_j^\dagger C_j \tilde{L}_j, \quad \tilde{L}_j = M L_j M^{-1}$
- Explicit form of $\tilde{h}_j$

$$\tilde{h}_j = -(1 + b^+ \tau_j + b^- \tau_j^\dagger) \sigma_j \sigma_{j+1}^\dagger (1 + b^- \tau_j + b^+ \tau_j^\dagger) - \frac{f}{2}(\tau_j + \tau_j^\dagger) + \epsilon + \mathrm{H.c.}$$

$$f = \frac{6(1 - 8r^6)}{(r^3 + 2\cos\theta)^2}, \quad b^\pm = \frac{r^3 - \cos\theta \pm \sqrt{3}\sin\theta}{r^3 + 2\cos\theta} \quad \epsilon = \frac{6(r^6 + 2)}{(r^3 + 2\cos\theta)^2}$$

■ Ground states

$$|\tilde{\Psi}_a\rangle = M|\Psi_a\rangle \propto \bigotimes_{j=1}^{L} (|\tau, 0\rangle_j + e^{\mathrm{i}\theta} r \omega^a |\tau, 1\rangle_j + r^2 \omega^{2a} |\tau, 2\rangle_j)$$

- Special case θ=0 previously obtained (Iemini et al., *PRL* 2017)

Deformed Potts chain (2)

- Conjugation
  - *M* operator $M = m \otimes \cdots \otimes m, \quad m = \begin{pmatrix} 1 & & \\ & r & \\ & & r \end{pmatrix}, \quad r > 0$
  - *C* operator

$$C_j = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes k \otimes k \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \quad k = \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix}$$

  - Deformed model

$$\tilde{H} = \sum_{j=1}^{L-1} \tilde{h}_j, \quad \tilde{h}_j = \tilde{L}_j^\dagger C_j \tilde{L}_j, \quad \tilde{L}_j = M L_j M^{-1}$$

$$\tilde{h}_j = -\sigma_j^\dagger \sigma_{j+1} - \frac{f}{2}(\tau_j + \tau_{j+1}) + g_1 \tau_j \tau_{j+1} + g_2 \tau_j^\dagger \tau_{j+1} + \mathrm{H.c.} - \mathrm{const.}$$

- Ground states

$$|\widetilde{\Psi}_a\rangle = M|\Psi_a\rangle \propto \bigotimes_{j=1}^{L} (|\tau, 0\rangle_j + e^{\mathrm{i}\theta} r \omega^a |\tau, 1\rangle_j + r^2 \omega^{2a} |\tau, 2\rangle_j)$$

  - Reproduces the previous result (Mahyaeh & Ardonne, PRB 2018)
  - Can prove the existence of a gap for certain *r* (Knabe's method)

Summary

■ Frustration-free Kitaev chain

- Applied Witten's conjugation & sandwiching

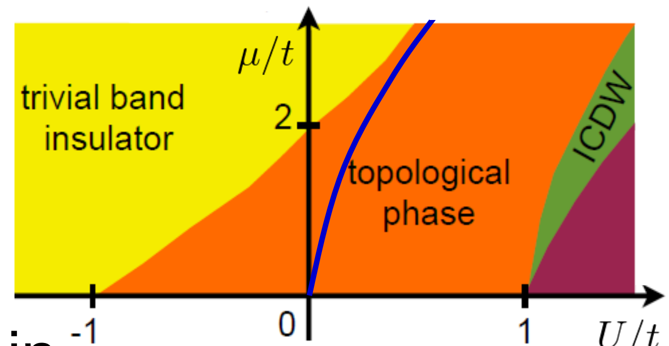
$$H = \sum_j L_j^\dagger L_j \quad \Rightarrow \quad \widetilde{H} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j$$

$$C_j > 0, \ \widetilde{L}_j = M L_j M^{-1}$$
$$(M: \ \text{invertible})$$

- Explicit ground states

$$|\widetilde{\Psi}_\pm\rangle = e^{\pm\alpha c_1^\dagger} \cdots e^{\pm\alpha c_L^\dagger} |\text{vac}\rangle$$

- Proof of a spectral gap



■ Frustration-free quantum Potts chain

- Conjugation & sandwiching work
- Reproduce known examples
- Can produce tons of new examples

➢ H.K., Schuricht, Takahashi, *PRB* **92**, 115137 (2015)
➢ Wouters, H.K., Schuricht, *PRB* **98**, 155119 (2018)
➢ Wouters, Katsura, Schuricht, *SciPost Phys. Core* **4** (2021)