Frustration-free Models and beyond

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Outline of my lectures

- Day 1 (June 19)
  - Introduction to frustration-free systems
  - Systematic construction of models
- Day 2 (June 20)
  - Non-interacting Kitaev chain
  - Interacting Kitaev chain
- Day 3 (June 21)
  - Divergence-free conditions
  - Application to quantum many-body scars

Ground-state Physics (Days 1–2)

Dynamics (Day 3)

Outline of today’s lecture

1. Introduction & Motivation
   - Symmetries in Physics
   - Preliminaries
     - Inequalities
     - Spin operators
     - Anderson bound
   - Definition of frustration-free systems
2. Examples
3. Systematic construction of models
4. Summary

Symmetries in Physics

- Lead to conservation laws
  - Rotational sym. → Conservation of angular momentum
    - Allows for analytical treatments
      e.g. Kepler problem,
      Schrodinger eq. of Hydrogen atom, ...

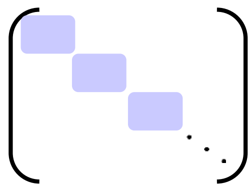
Spherical cow
from *Wikipedia*

- Can be broken spontaneously
  - Result in various phases of matter
    - SO(3) sym. breaking → (Ferro)magnetism
    - U(1) sym. breaking → Superconductivity

- Can enrich symmetry unbroken phases
  - Time-reversal sym. → Topological insulators
  - Symmetry-protected topological phases
    beyond the Landau paradigm

Symmetries in Quantum Physics

- Conserved quantities/charges
  - *H*: Hamiltonian, *A*: Hermitian operator
  - $[H, A] = 0 \Rightarrow$ They can be diagonalized simultaneously
    - *H* is block-diagonal w.r.t. the eigenvalues of *A*

$H =$ 

    - Can reduce the problem
  - (Example) *H*: Hydrogen atom Hamiltonian
    A: one of angular momentum ops. $\boldsymbol{L} = (L_x, L_y, L_z)$
- Infinitely many conserved charges
  - Integrable models e.g. S=1/2 Heisenberg chain
  - Strong constraints on their dynamics

Quiz

- *A, B*: Hermitian operator
- [*A*, *B*] = 0 ⇒ *A* and *B* have a simultaneous eigenstate
- Q1: Does the converse hold?
- A1: No!

- Q2: Can you provide a counterexample?
- A2: *n*s wavefunction of hydrogen atom
  - Angular momentum ops.: $[L_a, L_b] = i\hbar \sum_{c=x,y,z} \epsilon_{abc} L_c$
  - *s* wavefunction $\psi_{0,0}$
  - We have

$$L_x \psi_{0,0} = L_y \psi_{0,0} = L_z \psi_{0,0} = 0, \quad \boldsymbol{L}^2 \psi_{0,0} = 0$$

A baby example of a frustration-free system!

A crash course in inequalities

Positive semidefinite operators

Appendix of Tasaki, *Prog. Theor. Phys.* **99** (1998) or his book

$\mathcal{H}$: finite-dimensional Hilbert space

A, *B*: Hermitian operators on $\mathcal{H}$

- **Definition 1.** We write $A \geq 0$ and say *A* is positive semidefinite (p.s.d.) if $\langle \psi | A | \psi \rangle \geq 0, \ \forall | \psi \rangle \in \mathcal{H}.$
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0.$

Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on $\mathcal{H}$. Then $C^\dagger C \geq 0.$

  Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0.$

Ex.) Prove them.

Spin-1/2 operators and states (1)

Single spin

- Pauli matrices on $\mathbb{C}^2$

$$\mathbb{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma^x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^y=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma^z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- States

$$|\uparrow\rangle=\begin{pmatrix}1\\ 0\end{pmatrix},\quad |\downarrow\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}\qquad \sigma^z|\uparrow\rangle=|\uparrow\rangle,\quad \sigma^z|\downarrow\rangle=-|\downarrow\rangle$$

$$\alpha|\uparrow\rangle+\beta|\downarrow\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in\mathbb{C}^2\quad(\alpha,\beta\in\mathbb{C})$$

- Dual (bra) states $\langle\uparrow|=(1,0)\,,\quad \langle\downarrow|=(0,1)$

Tensor product of vectors

$$|v_1\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\quad |v_2\rangle=\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\ \Rightarrow\ |v_1\rangle\otimes|v_2\rangle=\begin{pmatrix}\alpha\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\\ \beta\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\alpha\gamma\\ \alpha\delta\\ \beta\gamma\\ \beta\delta\end{pmatrix}$$

- Often write it as $|v_1\rangle|v_2\rangle$.

Spin-1/2 operators and states (2)

Tensor product of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

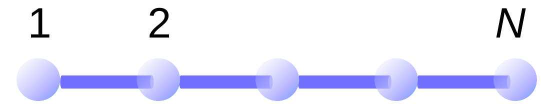
$$\Rightarrow \quad A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix} = \begin{pmatrix} ae & af & * & * \\ ag & ah & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

- Matrix-vector multiplication

$$(A \otimes B)(|v_1\rangle \otimes |v_2\rangle) = (A|v_1\rangle) \otimes (B|v_2\rangle)$$

Many spins

Spin chain



- Spin operators acting on $(\mathbb{C}^2)^{\otimes N}$

$$S_j^a = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes \frac{\sigma^a}{2} \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j}, \quad a = x, y, z$$

  - ex.) $4S_1^x S_2^x + 2S_1^z = ?$

- States $\sigma_j = \uparrow, \downarrow \quad (j = 1, 2, ..., N)$

$$|\sigma_1, \sigma_2, ..., \sigma_N\rangle = |\sigma_1\rangle_1 |\sigma_2\rangle_2 \cdots |\sigma_N\rangle_N = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots \otimes |\sigma_N\rangle$$

Spin-S operators and states

- **Single spin** $\boldsymbol{S} = (S^x, S^y, S^z)$
  - Commutation relations $[S^a, S^b] = \mathrm{i} \sum_{c=x,y,z} \epsilon_{abc} S^c$
  - Raising & lowering ops., total spin

$$S^\pm := S^x \pm \mathrm{i} S^y, \qquad \boldsymbol{S}^2 = (S^x)^2 + (S^y)^2 + (S^z)^2 = S(S+1)\mathbb{1}$$

  - Hilbert space $\mathbb{C}^{2S+1}$ spanned by $\{|-S\rangle, \cdots |S\rangle\}$
    - (Normalized) states $S^z|m\rangle = m|m\rangle, \quad m = -S, ..., S$

$$S^\pm|m\rangle = \sqrt{S(S+1) - m(m\pm 1)}|m \pm 1\rangle$$

- **Many spins**
  - Hilbert space $\mathcal{H} = (\mathbb{C}^{2S+1})^{\otimes N}$
  - Spin ops. $S_j^a = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes S^a \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j}, \quad a = x, y, z$

$$[S_i^a, S_j^b] = \mathrm{i}\delta_{ij} \sum_{c=x,y,z} \epsilon_{abc} S_i^c$$

  - States $m_j = -S, ..., S$

$$|m_1, m_2, ..., m_N\rangle = |m_1\rangle_1 |m_2\rangle_2 \cdots |m_N\rangle_N = |m_1\rangle \otimes |m_2\rangle \otimes \cdots \otimes |m_N\rangle$$

Addition of spins

- Total spin operators

$$S^a_{\text{tot}} := \sum_{j=1}^{N} S^a_j \quad a = x, y, z \qquad\qquad (\boldsymbol{S}_{\text{tot}})^2 = \sum_{a=x,y,z} (S^a_{\text{tot}})^2$$

- Two spin 1/2's
  - Singlet (S_{tot}=0): $|\psi_{0,0}\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ $\quad (\boldsymbol{S}_{\text{tot}})^2 = 0 \cdot 1 = 0$
  - Triplet (S_{tot}=1) $\begin{cases} |\psi_{1,1}\rangle = |\uparrow\uparrow\rangle \\ |\psi_{1,0}\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \\ |\psi_{1,-1}\rangle = |\downarrow\downarrow\rangle \end{cases}$ $\quad (\boldsymbol{S}_{\text{tot}})^2 = 1 \cdot 2 = 2$

- Two spin 1's
  - S_{tot}=0 $\quad |\psi_{0,0}\rangle = |+-\rangle - |00\rangle + |-+\rangle$
  - S_{tot}= 1

$$\begin{cases} |\psi_{1,1}\rangle = |+0\rangle - |0+\rangle \\ |\psi_{1,0}\rangle = |+-\rangle - |-+\rangle \\ |\psi_{1,-1}\rangle = |-0\rangle - |0-\rangle \end{cases}$$

  - S_{tot}= 2

$$\begin{cases} |\psi_{2,2}\rangle = |++\rangle \\ |\psi_{2,1}\rangle = |+0\rangle + |0+\rangle \\ |\psi_{2,0}\rangle = |+-\rangle + 2|00\rangle + |-+\rangle \\ |\psi_{2,-1}\rangle = |-0\rangle + |0-\rangle \\ |\psi_{2,-2}\rangle = |--\rangle \end{cases}$$

Anderson's paper in '50s

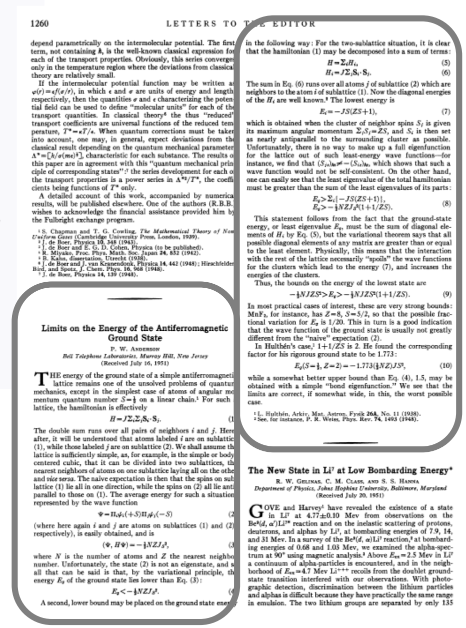
P. W. Anderson, *Phys. Rev*. **83** 1260 (1951)

Limits on the Energy of the Antiferromagnetic Ground State

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received July 16, 1951)

THE energy of the ground state of a simple antiferromagnetic lattice remains one of the unsolved problems of quantum mechanics, except in the simplest case of atoms of angular momentum quantum number $S=\frac{1}{2}$ on a linear chain.[1] For such a lattice, the hamiltonian is effectively

$$H=J\Sigma_i\Sigma_j\mathbf{S}_i\cdot\mathbf{S}_j. \qquad (1)$$

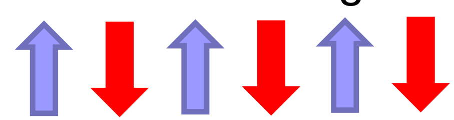


- Literally a half-page paper
- Upper and lower bounds on the ground-state energy of Heisenberg antiferromagnet

$$-\frac{1}{2}NzJS^2\left(1+\frac{1}{zS}\right)\leq E_0\leq -\frac{1}{2}NzJS^2$$

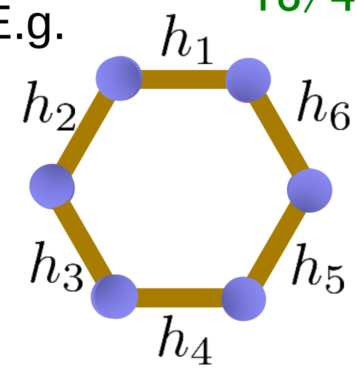
z: coordination number; *N*: # of spins

Neel state for large *zS*



Anderson's bound

E.g.



- Setup
  - Total Hamiltonian $H = \sum_j h_j$
  - Sub-Hamiltonian h_j satisfies $h_j \geq E_j^{(0)} \mathbb{1}$

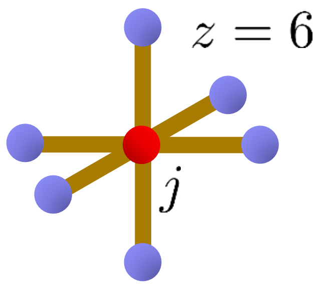
    $E_j^{(0)}$: the lowest eigenvalue of h_j
- Lower bound

  (The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

  - Proof) Let $|\Phi_0\rangle$ be a g.s. of *H*. Since $|\Phi_0\rangle$ is not necessarily a g.s. of h_j, we have $\langle \Phi_0 | h_j | \Phi_0 \rangle \geq E_j^{(0)}$.
- Application to Heisenberg AFM
  - $$h_j = \frac{J}{2} \sum_{i \text{ n.n. } j} \boldsymbol{S}_j \cdot \boldsymbol{S}_i = \frac{J}{4} \left\{ (\boldsymbol{S}_{\text{tot}})^2 - \Big(\sum_{i \text{ n.n. } j} \boldsymbol{S}_i \Big)^2 - S(S+1) \right\}$$
    $$\geq -\frac{J}{2} z S^2 \left(1 + \frac{1}{zS}\right) \mathbb{1}$$

    Minimum at $S_{\text{tot}} = (z\text{-}1)S$

    Maximum at $(\dots) = zS$



  - Proves the lower bound

Frustration-free systems

Setup

- Total Hamiltonian $H = \sum_j h_j$
- Sub-Hamiltonian h_j satisfies $h_j \geq E_j^{(0)} \mathbf{1}$

Frustration-free Hamiltonian

- Definition

> $H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j|\psi\rangle = E_j^{(0)}|\psi\rangle$ for all *j*.

- *ψ* saturates Anderson's bound
- NOTE) Depends on how you decompose *H*
- Many solvable models fall into this category
  - Majumdar-Ghosh model
  - Affleck-Kennedy-Lieb-Tasaki (AKLT) model
  - Kitaev's toric code

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Ferromagnetic Heisenberg model

- Hamiltonian (S=1/2, J > 0)

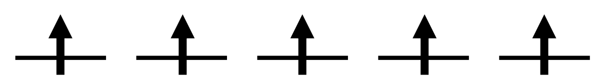
$$H = \sum_{j=1}^{N} h_j, \quad h_j = J\left(\frac{1}{4} - \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}\right) = J\left\{1 - \frac{1}{2}(\boldsymbol{S}_j + \boldsymbol{S}_{j+1})^2\right\}$$

Total spin

  - PBC imposed: $\boldsymbol{S}_{N+1} = \boldsymbol{S}_1$
  - SU(2) symmetry $[H, S^{\alpha}_{\mathrm{total}}] = 0, \quad S^{\alpha}_{\mathrm{tot}} = \sum_{j=1}^{N} S^{\alpha}_j, \quad (\alpha = z, +, -)$
  - $h_j = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes J P^{S=0} \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-2}$
    $P^{S=0} = |\psi_0\rangle\langle\psi_0|, \quad |\psi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$

⇒ $h_j \geq 0$ and $H \geq 0$

- H is frustration-free!
  - All-up state $|\Uparrow\rangle := |\uparrow\rangle_1|\uparrow\rangle_2 \cdots |\uparrow\rangle_N$
    is a zero-energy state of each h_j



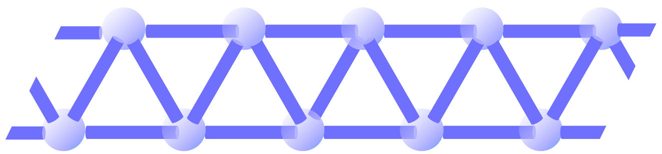
  - Other ground states: $(S^{-}_{\mathrm{tot}})^k|\Uparrow\rangle \quad (k = 0, 1, ..., N)$
  - Unique in each total S^z sector

ex.) Extension to higher-spin & dim.

Majumdar-Ghosh model (1)

■ Hamiltonian (S=1/2, J > 0) *J. Math. Phys.* **10**, 1388; 1399 (1969)

$$H = \sum_{j=1}^{N} h_j, \quad h_j = \frac{J}{2}\left(\boldsymbol{S}_{j-1}\cdot\boldsymbol{S}_j + \boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1} + \boldsymbol{S}_{j-1}\cdot\boldsymbol{S}_{j+1} + \frac{3}{4}\right)$$



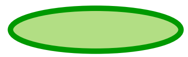
- N: even
- PBC imposed: $\boldsymbol{S}_{N+1} = \boldsymbol{S}_1$
- SU(2) symmetric
- Rewriting of h_j

$$h_j = \frac{J}{4}\left\{\underbrace{(\boldsymbol{S}_{j-1} + \boldsymbol{S}_j + \boldsymbol{S}_{j+1})}_{\text{Total spin}}{}^2 - \frac{3}{4}\right\}$$

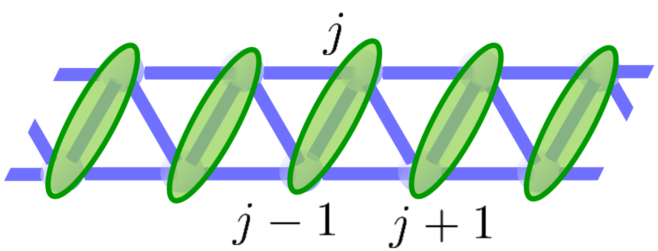
- Proportional to a projection operator
  - Addition of 3 spin 1/2's: $S_{j-1,j,j+1} = 1/2 \text{ or } 3/2$

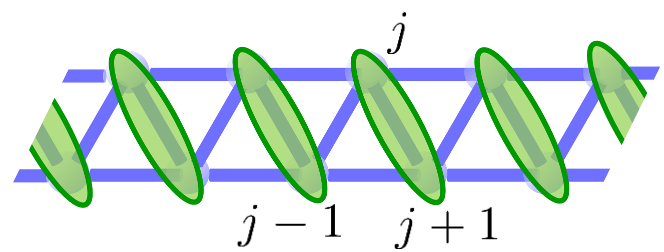
$$h_j = \frac{3}{4} J P_{j-1,j,j+1}^{S=3/2}$$ project out $S_{j-1,j,j+1} = 1/2$

Majumdar-Ghosh model (2)

- *H* is frustration-free!
  - Dimer i  $j = [i,j] := \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{i,j} - |\downarrow\uparrow\rangle_{i,j})$
  - Dimer states

$|\psi_1\rangle = [1,2][3,4]\cdots[N-1,N]$ $\qquad$ $|\psi_2\rangle = [2,3][4,5]\cdots[N,1]$





  - They are annihilated by h_j
    - Addition of a spin singlet and a spin 1/2 → $S_{j-1,j,j+1} = 1/2$
  - No other ground states
    - Proved by Caspers, Emmett, Magnus, *J. Phys. A* **17**, 2687 (1984)
- Generalizations
  - 2D model: Shastry, Sutherland, *Physica B+C* **108**, 1069 (1981)
  - Higher-spin model: Michaud *et al.*, *PRL* **108**, 127202 (2012)

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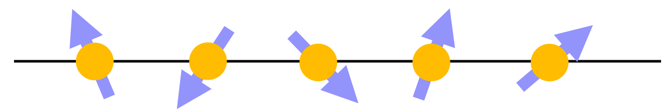
Haldane "conjecture" (early 80s)

■ Spin-*S* Heisenberg antiferromagnetic chain

- Hamiltonian (J > 0)

$$H_{\mathrm{Heis}} = J \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



- *S*=1/2, 3/2, 5/2, …
  Gapless, power-law decay of spin correlations
  NOTE) *S*=1/2 case is solvable (Bethe 1931)

Nobel Prize (2016)

- *S*=1, 2, 3, …
  a. Unique ground state
  b. Non-zero gap Δ (Haldane gap)
  c. Exponential decay of spin correlation

Established in many different ways!

$AgVP_2S_6$, NENP, …; ED, QMC, …

$$\Delta(S) = \begin{cases} 0.41048(6) & \text{for } S = 1 \\ 0.08917(4) & \text{for } S = 2 \\ 0.01002(3) & \text{for } S = 3 \end{cases}$$

Todo & Kato, *PRL* **87** (2001)

AKLT model (1)

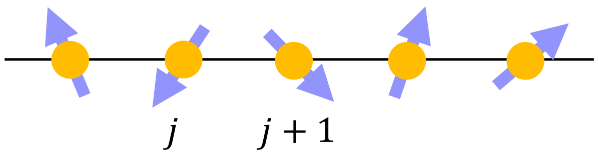
Affleck, Kennedy, Lieb & Tasaki, *PRL* **59** (1987), *CMP* **115** (1987)

■ Hamiltonian ($S=1$)

$$H = \sum_{j=1}^{N} h_j, \quad h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \frac{2}{3}$$

Bilinear　　Biquadratic

- PBC imposed: $\boldsymbol{S}_{N+1} = \boldsymbol{S}_1$
- SU(2) symmetric
- Rewriting of h_j



$$h_j = 2P^{S=2}_{j,j+1}, \qquad P^{S=2}_{j,j+1} = \frac{1}{24}(\boldsymbol{S}_j + \boldsymbol{S}_{j+1})^2\{(\boldsymbol{S}_j + \boldsymbol{S}_{j+1})^2 - 2\}$$

Total spin

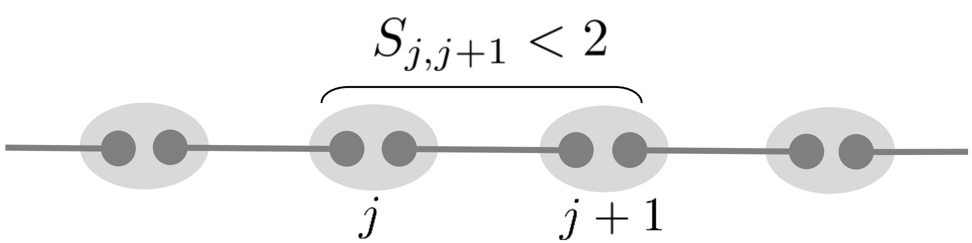
- Addition of 2 spin 1's:

$$S_{j,j+1} = 0, 1, \text{ or } 2$$

$h_j = 2P^{S=2}_{j,j+1}$ project out $S_{j,j+1} = 0, 1$ states

AKLT model (2)

- *H* is frustration-free!
  - Dimer = spin singlet:  $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$
  - Projection to spin-1 space:
  - Valence-bond-solid (VBS)/AKLT state



  - Each projector $P^{S=2}_{j,j+1}$ annihilates this state
    - Addition of a spin singlet and 2 spin 1/2's: $S_{j,j+1} = 0 \text{ or } 1$
  - AKLT proved
    - Uniqueness of the ground state
    - Nonzero gap above the ground state
    - Exponential decay of correlators

Partial support of Haldane's conjecture

VBS as a matrix product state

- Spin singlet  $\propto |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$
- Singlet product

$$\propto (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix} (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$$

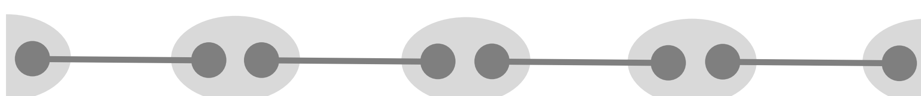
$$= (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \\ -|\uparrow\uparrow\rangle & -|\uparrow\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$$

- Projection to spin-1

$$P^{(1)}|\uparrow\uparrow\rangle = |+\rangle, \quad P^{(1)}|\uparrow\downarrow\rangle = P^{(1)}|\downarrow\uparrow\rangle = |0\rangle/\sqrt{2}, \quad P^{(1)}|\downarrow\downarrow\rangle = |-\rangle$$

$$\propto (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |0\rangle & \sqrt{2}|-\rangle \\ -\sqrt{2}|+\rangle & -|0\rangle \end{pmatrix} \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$$

- Repeat this procedure



$$\propto \mathrm{Tr}[\mathcal{A}^{[1]} \cdots \mathcal{A}^{[j]} \cdots \mathcal{A}^{[L]}], \quad \mathcal{A}^{[j]} = \begin{pmatrix} |0\rangle_j & \sqrt{2}|-\rangle_j \\ -\sqrt{2}|+\rangle_j & -|0\rangle_j \end{pmatrix}$$

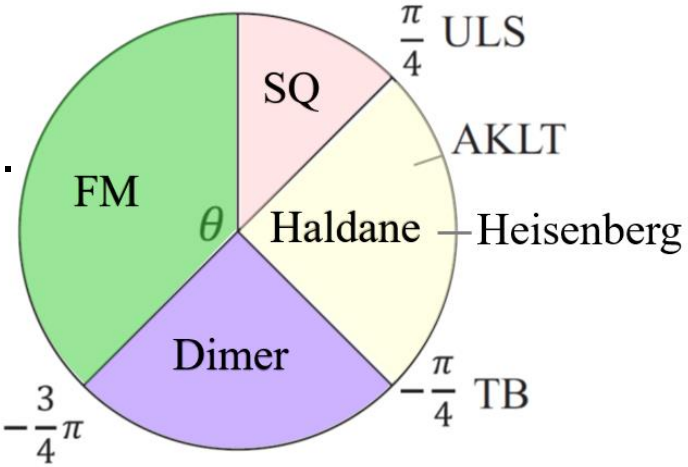
- Finitely correlated states — Fannes, Nachtergaele & Werner, *CMP* **144**, 443 (1992)

Quantum spin-1 chain with SU(2)

- Bilinear-biquadratic (BLBQ) model

$$H_{\mathrm{BLBQ}}(\theta) = \sum_{j=1}^{L} \left[\cos\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \sin\theta(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\right]$$

- Phase diagram Lauchli, Schmid & Trebst, *PRB* **74**, 144426 (2006)
  - Spin-quadrupolar (SQ): gapless, dominant nematic corr.
  - Ferromagnetic (FM)
  - Dimer: gapped, 2-fold degenerate g.s.
  - Haldane phase
    - ✓ Gapped unique g.s.
    - ✓ Edge states
    - ✓ Hidden AFM order (string order)

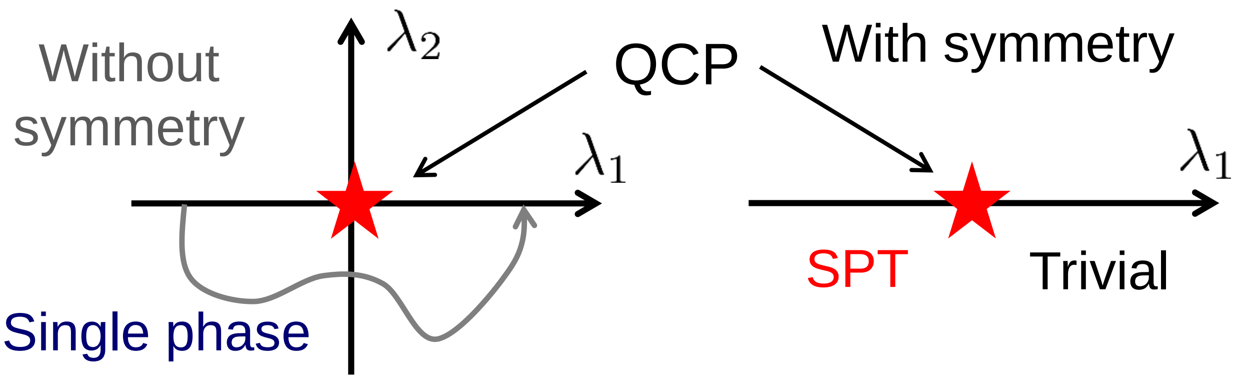


Prototype of Symmetry-protected topological (SPT) phase!

Haldane phase as SPT phase

■ What is SPT?

Gu & Wen, *PRB* **80** (2009). Pollmann, Berg, Turner & Oshikawa, *PRB* **81** (2010); **85** (2012)

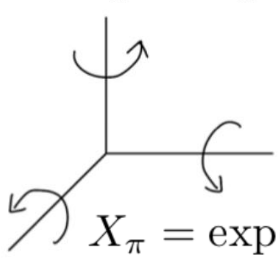


Two distinct phases!

■ Symmetry protection

S=1 Haldane phase is protected by ANY one of three symmetries: (i) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (ii) time-reversal, (iii) bond centered inversion

$$Z_\pi = X_\pi Y_\pi = \exp(-i\pi \textstyle\sum_j S_j^z)$$



$$Y_\pi = \exp(-i\pi \textstyle\sum_j S_j^y)$$

$$X_\pi = \exp(-i\pi \textstyle\sum_j S_j^x)$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, X_\pi, Y_\pi, Z_\pi\}$$

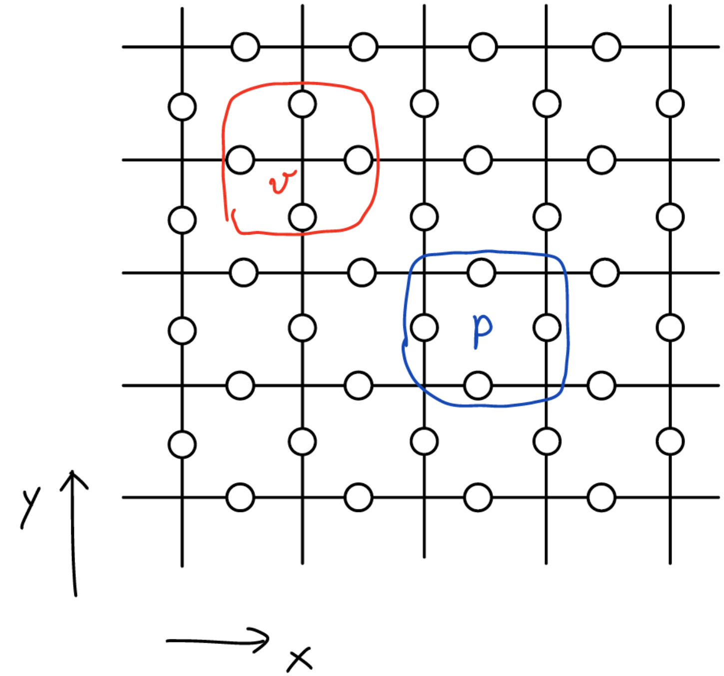
BLBQ has symmetries (i), (ii) & (iii)

Outline of today's lecture

1. Introduction & Motivation
2. Examples
   - Ferromagnetic Heisenberg model
   - Majumdar-Ghosh model
   - AKLT model
   - Kitaev's toric code
3. Systematic construction of models
4. Summary

Kitaev's toric code

Kitaev, Ann. Phys. **303**, 2 (2003)

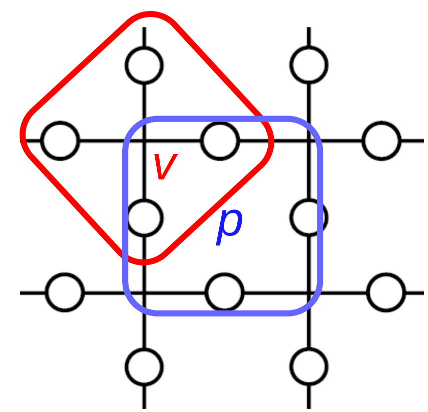


- Setup
  - 2D square lattice
  - PBC in *x* & *y* directions
  - Spin 1/2's on edges
  - *N*: number of vertices
- Hamiltonian $(J_{\mathrm{e}} > 0, J_{\mathrm{m}} > 0)$

$$H = -J_{\mathrm{e}} \sum_{v} A_v - J_{\mathrm{m}} \sum_{p} B_p$$

  - Vertex ops. $A_v = \prod_{j \sim v} \sigma_j^z$
  - Plaquette ops. $B_p = \prod_{j \sim p} \sigma_j^x$
  - They all commute

    A_v and B_p share 0 or 2 spins. $\{\sigma_j^z, \sigma_j^x\} = 0$



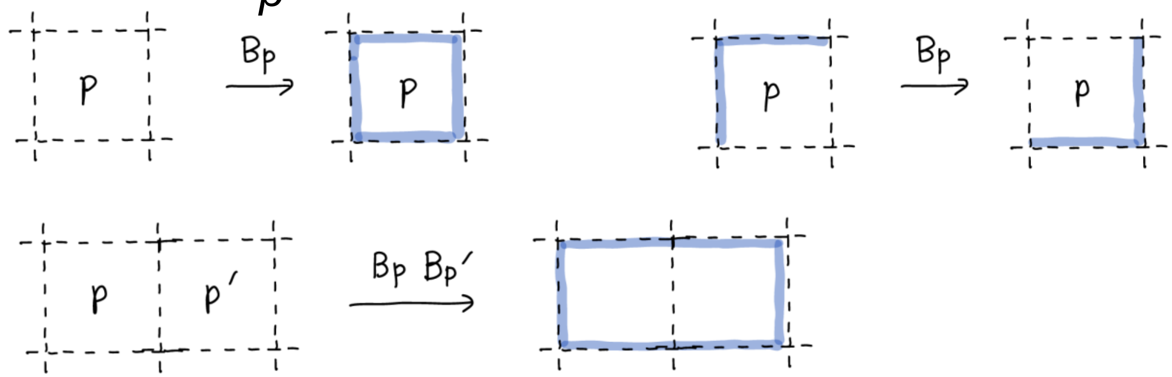
Toric code is frustration-free

- Properties of local Hamiltonians
  - A_v and B_p square to 1
  - Their eigenvalues are ± 1
  - → Anderson bound: $H \geq -N(J_{\mathrm{e}} + J_{\mathrm{m}})$
- Ground state
  - All-up state $|\Uparrow\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$
    - is a ground state of $-A_v$ for all v
    - But is not even an eigenstate of $-B_p$
  - $$|\Psi\rangle = \left\{\prod_p (1 + B_p)\right\} |\Uparrow\rangle$$ is a nonzero state
  - Since $[A_v, B_p] = 0$, $-A_v|\Psi\rangle = -|\Psi\rangle$
  - Since $B_p(1 + B_p) = (1 + B_p)$ & $[B_p, B_{p'}] = 0$, $-B_p|\Psi\rangle = -|\Psi\rangle$
  - $|\Psi\rangle$ saturates the lower bound $E = -N(J_{\mathrm{e}} + J_{\mathrm{m}})$

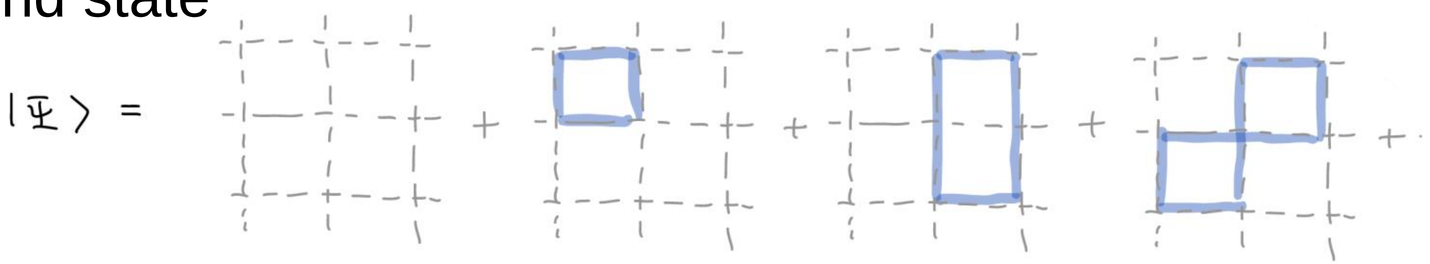
Graphical representation of Ψ

- Spin ←→ line segment

- Action of B_p



- Ground state

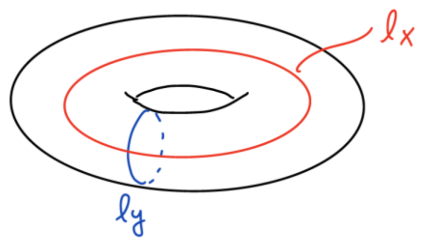


  - 0, 2, or 4 lines emanating from each vertex in each config.
  - $|\Psi\rangle$ = superposition of all such loop configurations
  - Same diagrams appear in high-*T* expansion of Ising

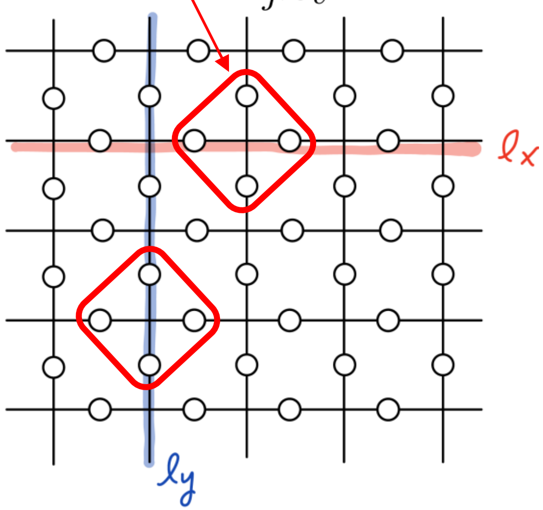
- Another view: projection of cluster state

Topological degeneracy

- Electric path ops
  - Model is on a torus
  - Closed paths ℓ_x & ℓ_y
  - $X_\alpha := \prod_{j \in \ell_\alpha} \sigma_j^x \quad (\alpha = x, y)$
    - Commute with *H*

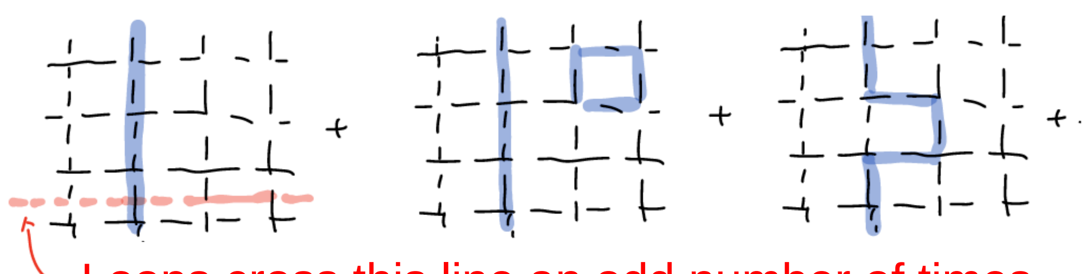


$$A_v = \prod_{j \sim v} \sigma_j^z$$



- Degenerate ground states
  - $|\Psi\rangle,\ X_x|\Psi\rangle,\ X_y|\Psi\rangle,\ X_xX_y|\Psi\rangle$
  - Graphical rep.

$X_y|\Psi\rangle =$

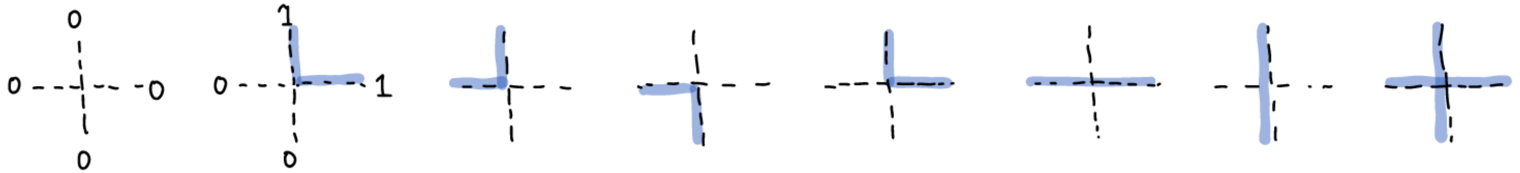


Loops cross this line an odd number of times

  - Degeneracy is robust against local perturbations
  - ex.) Prove that the four states are orthogonal

Ψ as a tensor network

- Rokhsar-Kivelson (RK) states
  - RK state is equal-weight superposition of classical configrations (with constraints)
  - Originally discussed in the context of quantum dimer model *PRL* **61**, 2376 (1988)
  - Can be expressed as a tensor network
  - *Ψ* is an example of an RK state!
- Building blocks
  - Local tensor



  - The weight of each local config. is 1
  - Can be made anisotropic → Quantum 8-vertex model
    - Ardonne, Fendley & Fradkin, Ann. Phys. **310**, 493 (2004)

Outline of today's lecture

1. Introduction & Motivation
2. Examples
3. Systematic construction of models
   - Unitary transformation
   - Witten's conjugation
   - Sandwiching method
     - ➢ Wouters, Katsura, Schuricht, *SciPost Phys. Core* **4**, 027 (2021) arXiv:2005.12825
4. Summary

Can we cook up new models?

- **Trivial model**
  - Hamiltonian $$H = \sum_{j=1}^{N} (1 - \sigma_j^x) \quad \geq 0 \quad \text{(p.s.d)}$$
  - Ground state

$$|\Psi_+\rangle = |+\rangle_1 |+\rangle_2 \cdots |+\rangle_N$$

Local basis: $\sigma^x |\pm\rangle = \pm|\pm\rangle$

  - ➢ No entanglement…

- **Cluster model**
  - Hamiltonian $$\widetilde{H} = \sum_{j=1}^{N} (1 - \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z) \ = UHU^\dagger$$
    - ➢ PBC imposed
  - Unitarily equivalent to *H*
    - ➢ Unitary tr. $$U = (CZ)_{1,2}(CZ)_{2,3} \cdots (CZ)_{N,1}$$

$$(CZ)|a, b\rangle = (-1)^{ab}|a, b\rangle, \quad (a, b = 0, 1)$$

  - Ground state $$|\widetilde{\Psi}_+\rangle = U|\Psi_+\rangle$$
    - ➢ Entangled!

Anything beyond unitary transformation?

Yes! Witten's conjugation is a key!

N=2 supersymmetric (SUSY) QM

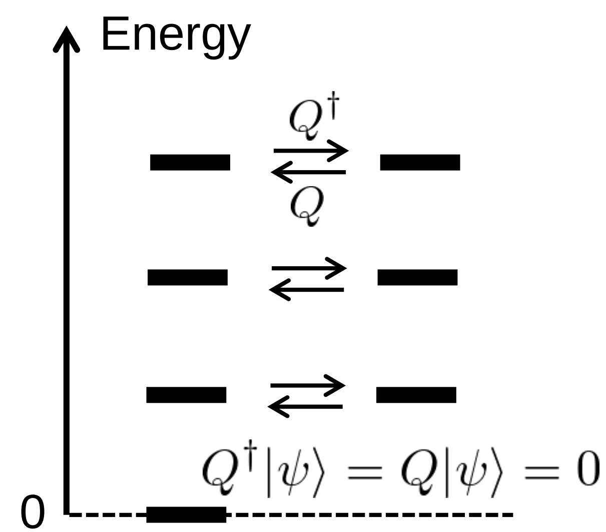
■ Algebra

- Supercharges: $Q,\ Q^\dagger, \quad Q^2 = 0,\ (Q^\dagger)^2 = 0$
- Fermionic parity: $(-1)^F, \quad \{Q, (-1)^F\} = \{Q^\dagger, (-1)^F\} = 0$
- Hamiltonian: $H = \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q$
- Symmetry: $[H, Q] = [H, Q^\dagger] = [H, (-1)^F] = 0$

■ Spectrum of *H*

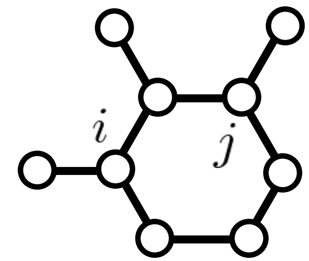
- $E \geqq 0$ for all states, as *H* is p.s.d
- $E > 0$ states come in pairs $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
- $E = 0$ iff a state is a SUSY singlet

Ground-state energy = 0 → SUSY ***unbroken***
Ground-state energy > 0 → SUSY ***broken***



Elementary example

■ Lattice bosons and fermions



- Lattice sites: *i*, *j* = 1,2, …, *N*
- Creation, annihilation ops.

$$[b_i, b_j^\dagger] = \delta_{i,j}, \quad \{c_i, c_j^\dagger\} = \delta_{i,j}, \quad [b_i, b_j] = \{c_i, c_j\} = 0.$$

  - ➢ bosons and fermions are mutually commuting.

- Fermion number $F = \sum_j c_j^\dagger c_j$
- Vacuum state $b_i|\mathrm{vac}\rangle \stackrel{?}{=} c_i|\mathrm{vac}\rangle = 0, \forall i$

■ Supercharges and Hamiltonian

$$Q = \sum_j b_j^\dagger c_j, \quad Q^\dagger = \sum_j c_j^\dagger b_j$$

$$\Rightarrow \{Q, Q^\dagger\} = \sum_j b_j^\dagger b_j + \sum_j c_j^\dagger c_j$$

Just the total particle number!
|vac> is a SUSY singlet.

Zero-energy states (ZES)

■ Cohomology

- Zero-energy states (ZESs) are in 1-to-1 correspondence with nontrivial cohomology classes of Q.

  Proof) Any ZES $|\psi\rangle$ is annihilated by both Q and $Q^\dagger$.
  But $|\psi\rangle$ cannot be written as $|\psi\rangle = Q|\phi\rangle$ for any $|\phi\rangle$
  since this would imply that $\langle\psi|\psi\rangle = \langle\phi|Q^\dagger|\psi\rangle = 0$.

(ZES of H) = $\dim\left(\mathrm{Ker}\, Q/\mathrm{Im}\, Q\right)$

■ Witten's conjugation (*Nucl. Phys*. B **202**, 253 (1982).)

- Invertible operator M
- New supercharge & Hamiltonian

$$\widetilde{Q} := MQM^{-1}, \qquad \widetilde{H} := \{\widetilde{Q}, \widetilde{Q}^\dagger\}$$

- $\dim\left(\mathrm{Ker}\,\widetilde{Q}/\mathrm{Im}\,\widetilde{Q}\right) = \dim\left(\mathrm{Ker}\, Q/\mathrm{Im}\, Q\right)$

The deformation preserves the number of zero-energy states.

Conjugation argument (non-SUSY ver.)

- Universal form of frustration-free systems
  - Set the ground state (g.s.) energy to zero
  - Hamiltonian (p.s.d.)

$$H = \sum_j L_j^\dagger L_j$$

  - Zero-energy ground state

$$|\psi\rangle \text{ s.t. } L_j|\psi\rangle = 0 \ \forall j$$

- Recipe for the construction
  - New *L* operators $\widetilde{L}_j := M L_j M^{-1} \quad (M\text{: invertible})$
  - New Hamiltonian

$$\widetilde{H} = \sum_j \widetilde{L}_j^\dagger \widetilde{L}_j$$

  - Zero-energy ground state

$$|\widetilde{\psi}\rangle = M|\psi\rangle, \quad \widetilde{L}_j|\widetilde{\psi}\rangle = 0 \ \forall j$$

  - $\widetilde{H}$ is inequivalent to H unless *M* is unitary
  - $\widetilde{H}$ and H have the same number of g.s.
  - Similar to the idea of “Doob transform”

Sandwiching method

- A slight generalization
  - Positive definite operator C_j
  - Sandwich it between $\widetilde{L}_j^\dagger$ and $\widetilde{L}_j$
    - Does not change the g.s. manifold
  - Newer Hamiltonian

$$\widetilde{H} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j \qquad \widetilde{L}_j = M L_j M^{-1}$$

  - Zero-energy g.s.

$$|\widetilde{\psi}\rangle = M|\psi\rangle,$$

- Theorem

Let $|\Psi_k\rangle\ (k=1,\ldots,n)$ be linearly independent zero-energy g.s. of H. The ground-state manifold of $\widetilde{H}$ is given by

$$\widetilde{G} = \mathrm{span}\{M|\Psi_1\rangle, \ldots, M|\Psi_n\rangle\}.$$

Thus, the g.s. degeneracies of H and $\widetilde{H}$ are identical.

  - Proof) Just follows from $\widetilde{G} \subseteq \bigcap_j \ker(\widetilde{L}_j)$ and $\widetilde{G} \supseteq \bigcap_j \ker(\widetilde{L}_j)$.

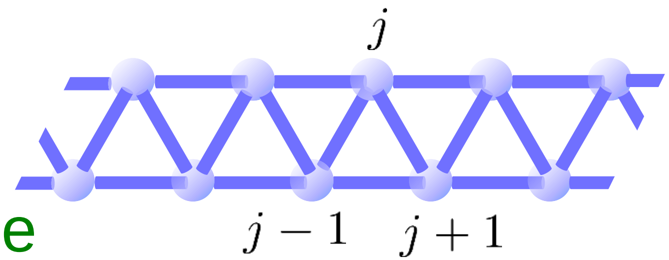
Exercise

Shastry, Sutherland, *Physica B+C* **108**, 1069 (1981)

- Deformed Majumdar-Ghosh model
  - Hamiltonian (*S*=1/2, *N*: even, PBC imposed)

$$H=\sum_{j=1}^{N} h_j, \quad h_j=\sum_{a=x,y,z}\frac{J_a}{2}\left(S_{j-1}^aS_j^a+S_j^aS_{j+1}^a+S_{j-1}^aS_{j+1}^a+\frac{1}{4}\right)$$

  - $J_a>0\ (a=x,y,z)$ are distinct
  - SU(2) symmetry is broken



- Q. Prove that *H* is frustration-free
  - Hint: h_j is a sum of positive semidefinite ops.

- Further extensions
  - 3-coloring condition: $J_xJ_y+J_yJ_z+J_zJ_x=0$
    - Changlani *et al.*, *PRL* **120**, 117202 (2018)
    - Palle & Benton, *PRB* **103**, 214428 (2021)

Summary

■ Frustration-free systems

- The Hamiltonian $H = \sum_j h_j$ is said to be frustration-free if there exists a simultaneous eigenstate of h_j with their lowest eigenvalues for all *j*
- h_j do not have to commute with each other
- Examples
  - Ferro-Heisenberg, Majumdar-Ghosh, AKLT, Toric code, …

■ Construction of frustration-free models

- Witten's conjugation
- Sandwiching method

$$H = \sum_j L_j^\dagger L_j \quad \Rightarrow \quad \widetilde{H} = \sum_j \widetilde{L}_j^\dagger C_j \widetilde{L}_j$$

$$C_j > 0, \ \widetilde{L}_j = M L_j M^{-1} \quad (M\text{: invertible})$$

- Allow for constructing new models
  - Wouters, Katsura, Schuricht, *SciPost Phys. Core* **4**, 027 (2021)