

Frustration-free Models and beyond

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Institute for
Physics of
Intelligence



Trans-Scale
Quantum Science
Institute

Outline of my lectures

- Day 1 (June 19)
 - Introduction to frustration-free systems
 - Systematic construction of models
- Day 2 (June 20)
 - Non-interacting Kitaev chain
 - Interacting Kitaev chain
- Day 3 (June 21)
 - Divergence-free conditions
 - Application to quantum many-body scars

Ground-state
Physics

Dynamics

Outline of today's lecture

1. Introduction & Motivation
 - Symmetries in Physics
 - Preliminaries
 - Inequalities
 - Spin operators
 - Anderson bound
 - Definition of frustration-free systems
2. Examples
3. Systematic construction of models
4. Summary

Symmetries in Physics

- Lead to conservation laws
 - Rotational sym. → Conservation of angular momentum
 - Allows for analytical treatments e.g. Kepler problem, Schrodinger eq. of Hydrogen atom, ...



Spherical cow
from *Wikipedia*

- Can be broken spontaneously
 - Result in various phases of matter
 - $SO(3)$ sym. breaking → (Ferro)magnetism
 - $U(1)$ sym. breaking → Superconductivity
- Can enrich symmetry unbroken phases
 - Time-reversal sym. → Topological insulators
 - Symmetry-protected topological phases beyond the Landau paradigm

Symmetries in Quantum Physics

■ Conserved quantities/charges

- H : Hamiltonian, A : Hermitian operator
- $[H, A] = 0 \Rightarrow$ They can be diagonalized simultaneously
 - H is block-diagonal w.r.t. the eigenvalues of A

$$H = \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \ddots \end{pmatrix}$$

- Can reduce the problem

- (Example) H : Hydrogen atom Hamiltonian
 A : one of angular momentum ops. $\mathbf{L} = (L_x, L_y, L_z)$

■ Infinitely many conserved charges

- Integrable models e.g. $S=1/2$ Heisenberg chain
- Strong constraints on their dynamics

Quiz

- A, B : Hermitian operator
- $[A, B] = 0 \Rightarrow A$ and B have a simultaneous eigenstate
- Q1: Does the converse hold?
- A1: No!

- Q2: Can you provide a counterexample?
- A2: *ns* wavefunction of hydrogen atom

➤ Angular momentum ops.: $[L_a, L_b] = i\hbar \sum_{c=x,y,z} \epsilon_{abc} L_c$

➤ s wavefunction $\psi_{0,0}$

➤ We have

$$L_x \psi_{0,0} = L_y \psi_{0,0} = L_z \psi_{0,0} = 0, \quad \mathbf{L}^2 \psi_{0,0} = 0$$

A baby example of a frustration-free system!

A crash course in inequalities

■ Positive semidefinite operators

Appendix of Tasaki, *Prog. Theor. Phys.* **99** (1998) or his book

\mathcal{H} : finite-dimensional Hilbert space

A, B : Hermitian operators on \mathcal{H}

- **Definition 1.** We write $A \geq 0$ and say A is **positive semidefinite (p.s.d.)** if $\langle \psi | A | \psi \rangle \geq 0, \forall |\psi\rangle \in \mathcal{H}$.
- **Definition 2.** We write $A \geq B$ if $A - B \geq 0$.

■ Important lemmas

- **Lemma 1.** $A \geq 0$ iff all the eigenvalues of A are nonnegative.
- **Lemma 2.** Let C be an arbitrary matrix on \mathcal{H} . Then $C^\dagger C \geq 0$.
Cor. A projection operator $P = P^\dagger$ is p.s.d.
- **Lemma 3.** If $A \geq 0$ and $B \geq 0$, we have $A + B \geq 0$.

Ex.) Prove them.

Spin-1/2 operators and states (1)

■ Single spin

- Pauli matrices on \mathbb{C}^2

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- States

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \sigma^z |\uparrow\rangle = |\uparrow\rangle, \quad \sigma^z |\downarrow\rangle = -|\downarrow\rangle$$

$$\alpha |\uparrow\rangle + \beta |\downarrow\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \quad (\alpha, \beta \in \mathbb{C})$$

- Dual (bra) states $\langle\uparrow| = (1, 0)$, $\langle\downarrow| = (0, 1)$

■ Tensor product of vectors

$$|v_1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \quad \Rightarrow \quad |v_1\rangle \otimes |v_2\rangle = \begin{pmatrix} \alpha \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \beta \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

- Often write it as $|v_1\rangle|v_2\rangle$.

Spin-1/2 operators and states (2)

■ Tensor product of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\Rightarrow A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix} = \begin{pmatrix} ae & af & * & * \\ ag & ah & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

• Matrix-vector multiplication

$$(A \otimes B)(|v_1\rangle \otimes |v_2\rangle) = (A|v_1\rangle) \otimes (B|v_2\rangle)$$

■ Many spins

• Spin operators acting on $(\mathbb{C}^2)^{\otimes N}$

$$S_j^a = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes \frac{\sigma^a}{2} \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-j}, \quad a = x, y, z$$

➤ ex.) $4S_1^x S_2^x + 2S_1^z = ?$

• States $\sigma_j = \uparrow, \downarrow$ ($j = 1, 2, \dots, N$)

$$|\sigma_1, \sigma_2, \dots, \sigma_N\rangle = |\sigma_1\rangle_1 |\sigma_2\rangle_2 \cdots |\sigma_N\rangle_N = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots \otimes |\sigma_N\rangle$$

Spin chain



Spin-S operators and states

■ Single spin $S = (S^x, S^y, S^z)$

- Commutation relations $[S^a, S^b] = i \sum_{c=x,y,z} \epsilon_{abc} S^c$

- Raising & lowering ops., total spin $c=x,y,z$

$$S^\pm := S^x \pm iS^y, \quad S^2 = (S^x)^2 + (S^y)^2 + (S^z)^2 = S(S+1)\mathbb{1}$$

- Hilbert space \mathbb{C}^{2S+1} spanned by $\{|-S\rangle, \dots, |S\rangle\}$

- (Normalized) states $S^z|m\rangle = m|m\rangle, \quad m = -S, \dots, S$

$$S^\pm|m\rangle = \sqrt{S(S+1) - m(m \pm 1)}|m \pm 1\rangle$$

■ Many spins

- Hilbert space $\mathcal{H} = (\mathbb{C}^{2S+1})^{\otimes N}$

- Spin ops. $S_j^a = \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{j-1} \otimes S^a \otimes \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{N-j}, \quad a = x, y, z$

$$[S_i^a, S_j^b] = i\delta_{ij} \sum_{c=x,y,z} \epsilon_{abc} S_i^c$$

- States $m_j = -S, \dots, S$

$$|m_1, m_2, \dots, m_N\rangle = |m_1\rangle_1 |m_2\rangle_2 \cdots |m_N\rangle_N = |m_1\rangle \otimes |m_2\rangle \otimes \cdots \otimes |m_N\rangle$$

Addition of spins

■ Total spin operators

$$S_{\text{tot}}^a := \sum_{j=1}^N S_j^a \quad a = x, y, z \quad (\mathbf{S}_{\text{tot}})^2 = \sum_{a=x,y,z} (S_{\text{tot}}^a)^2$$

■ Two spin 1/2's

• Singlet ($\mathbf{S}_{\text{tot}}=0$): $|\psi_{0,0}\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \quad (\mathbf{S}_{\text{tot}})^2 = 0 \cdot 1 = 0$

• Triplet ($\mathbf{S}_{\text{tot}}=1$) $\left\{ \begin{array}{l} |\psi_{1,1}\rangle = |\uparrow\uparrow\rangle \\ |\psi_{1,0}\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \\ |\psi_{1,-1}\rangle = |\downarrow\downarrow\rangle \end{array} \right. \quad (\mathbf{S}_{\text{tot}})^2 = 1 \cdot 2 = 2$

■ Two spin 1's

• $\mathbf{S}_{\text{tot}}=0$ $|\psi_{0,0}\rangle = |+-\rangle - |00\rangle + |-+\rangle$

• $\mathbf{S}_{\text{tot}}=1$

$$\left\{ \begin{array}{l} |\psi_{1,1}\rangle = |+0\rangle - |0+\rangle \\ |\psi_{1,0}\rangle = |+-\rangle - |-+\rangle \\ |\psi_{1,-1}\rangle = |-0\rangle - |0-\rangle \end{array} \right.$$

• $\mathbf{S}_{\text{tot}}=2$

$$\left\{ \begin{array}{l} |\psi_{2,2}\rangle = |++\rangle \\ |\psi_{2,1}\rangle = |+0\rangle + |0+\rangle \\ |\psi_{2,0}\rangle = |+-\rangle + 2|00\rangle + |-+\rangle \\ |\psi_{2,-1}\rangle = |-0\rangle + |0-\rangle \\ |\psi_{2,-2}\rangle = |--\rangle \end{array} \right.$$

Anderson's paper in '50s

P. W. Anderson, *Phys. Rev.* **83** 1260 (1951)

Limits on the Energy of the Antiferromagnetic Ground State

P. W. ANDERSON

Bell Telephone Laboratories, Murray Hill, New Jersey

(Received July 16, 1951)

THE energy of the ground state of a simple antiferromagnetic lattice remains one of the unsolved problems of quantum mechanics, except in the simplest case of atoms of angular momentum quantum number $S = \frac{1}{2}$ on a linear chain.¹ For such a lattice, the hamiltonian is effectively

$$H = J \sum_i \sum_j \mathbf{S}_i \cdot \mathbf{S}_j \tag{1}$$

- Literally a half-page paper
- Upper and lower bounds on the ground-state energy of Heisenberg antiferromagnet

$$-\frac{1}{2} N z J S^2 \left(1 + \frac{1}{z S} \right) \leq E_0 \leq -\frac{1}{2} N z J S^2$$

z: coordination number; N: # of spins

1260 LETTERS TO THE EDITOR

depend parametrically on the intermolecular potential. The first term, not containing λ in the well-known classical expression for each of the transport properties. Obviously, this series converges only in the temperature region where the deviations from classical theory are relatively small.

If the intermolecular potential function may be written as $\phi(r) = \phi_0(r) + \lambda \phi_1(r)$, in which ϕ_0 and $\lambda \phi_1$ are units of energy and length respectively, then the quantities α and β characterizing the potential field can be used to define "molecular units" for each of the transport quantities. In classical theory the thus "reduced" transport coefficients are universal functions of the reduced temperature, $T^* = T/T_0$. When quantum corrections must be taken into account, one may, in general, expect deviations from the classical results depending on the quantum mechanical parameter $\lambda^2 \alpha^2 / (k_B T)$ characteristic for each substance. The results of this paper are in agreement with this "quantum mechanical principle of corresponding states": the series development for each of the transport properties is a power series in λ^2 / T^* , the coefficients being functions of T^* only.

A detailed account of this work, accompanied by numerical results, will be published elsewhere. One of the authors (R.B.B.) wishes to acknowledge the financial assistance provided him by the Fulbright exchange program.

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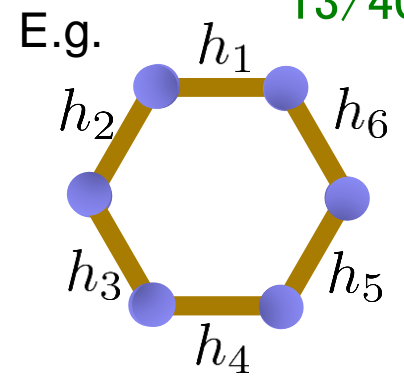
²³⁴ J. H. Van Vleck, *Phys. Rev.* **83**, 833 (1951).

²³⁵ J. H. Van Vleck, *Phys. Rev.* **83**, 833 (1951).

²³⁶ J. H. Van Vleck, *Phys. Rev.* **83**, 833 (1951).

²³⁷ J. H. Van Vleck, *Phys. Rev.* **83</**

Anderson's bound



■ Setup

- Total Hamiltonian $H = \sum_j h_j$
- Sub-Hamiltonian h_j satisfies $h_j \geq E_j^{(0)} \mathbb{1}$
 $E_j^{(0)}$: the lowest eigenvalue of h_j

■ Lower bound

(The g.s. energy of H) $=: E_0 \geq \sum_j E_j^{(0)}$

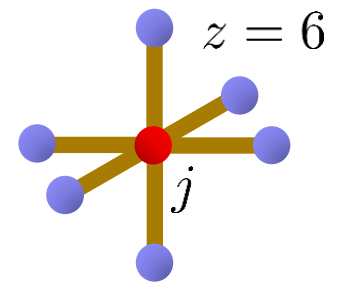
- Proof) Let $|\Phi_0\rangle$ be a g.s. of H . Since $|\Phi_0\rangle$ is not necessarily a g.s. of h_j , we have $\langle \Phi_0 | h_j | \Phi_0 \rangle \geq E_j^{(0)}$.

■ Application to Heisenberg AFM

$$h_j = \frac{J}{2} \sum_{i \text{ n.n. } j} \mathbf{S}_j \cdot \mathbf{S}_i = \frac{J}{4} \left\{ (\mathbf{S}_{\text{tot}})^2 - \left(\sum_{i \text{ n.n. } j} \mathbf{S}_i \right)^2 - S(S+1) \right\}$$

$$\geq -\frac{J}{2} z S^2 \left(1 + \frac{1}{zS} \right) \mathbb{1}$$

↑ Minimum at $\mathbf{S}_{\text{tot}} = (z-1)S$
↑ Maximum at $(\dots) = zS$



- Proves the lower bound

Frustration-free systems

■ Setup

- Total Hamiltonian $H = \sum_j h_j$
- Sub-Hamiltonian h_j satisfies $h_j \geq E_j^{(0)} \mathbf{1}$

■ Frustration-free Hamiltonian

- Definition

$H = \sum_j h_j$ is said to be *frustration-free* if there exists a state $|\psi\rangle$ such that $h_j|\psi\rangle = E_j^{(0)}|\psi\rangle$ for all j .

- ψ saturates Anderson's bound
- NOTE) Depends on how you decompose H
- Many solvable models fall into this category
 - Majumdar-Ghosh model
 - Affleck-Kennedy-Lieb-Tasaki (AKLT) model
 - Kitaev's toric code

Outline of today's lecture

1. Introduction & Motivation
2. Examples
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 - AKLT model
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Ferromagnetic Heisenberg model

■ Hamiltonian ($S=1/2$, $J > 0$)

$$H = \sum_{j=1}^N h_j, \quad h_j = J \left(\frac{1}{4} - \mathbf{S}_j \cdot \mathbf{S}_{j+1} \right) = J \left\{ 1 - \frac{1}{2} \underbrace{(\mathbf{S}_j + \mathbf{S}_{j+1})^2}_{\text{Total spin}} \right\}$$


- PBC imposed: $\mathbf{S}_{N+1} = \mathbf{S}_1$

- SU(2) symmetry $[H, S_{\text{total}}^\alpha] = 0$, $S_{\text{tot}}^\alpha = \sum_{j=1}^N S_j^\alpha$, ($\alpha = z, +, -$)

$$h_j = \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{j-1} \otimes JP^{S=0} \otimes \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{N-2} \quad \Rightarrow \quad h_j \geq 0 \text{ and } H \geq 0$$

$$P^{S=0} = |\psi_0\rangle\langle\psi_0|, \quad |\psi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

■ H is frustration-free!

- All-up state $|\uparrow\uparrow\rangle := |\uparrow\rangle_1 |\uparrow\rangle_2 \cdots |\uparrow\rangle_N$  is a zero-energy state of each h_j
- Other ground states: $(S_{\text{tot}}^-)^k |\uparrow\uparrow\rangle$ ($k = 0, 1, \dots, N$)
- Unique in each total S^z sector

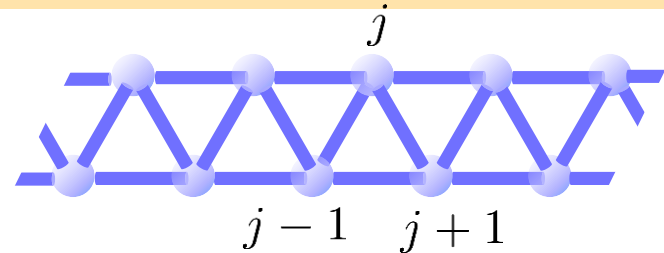
ex.) Extension to higher-spin & dim.

Majumdar-Ghosh model (1)

- Hamiltonian ($S=1/2$, $J > 0$) *J. Math. Phys.* **10**, 1388; 1399 (1969)

$$H = \sum_{j=1}^N h_j, \quad h_j = \frac{J}{2} \left(\mathbf{S}_{j-1} \cdot \mathbf{S}_j + \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \mathbf{S}_{j-1} \cdot \mathbf{S}_{j+1} + \frac{3}{4} \right)$$

- N : even
- PBC imposed: $\mathbf{S}_{N+1} = \mathbf{S}_1$
- SU(2) symmetric
- Rewriting of h_j




$$h_j = \frac{J}{4} \left\{ \underbrace{(\mathbf{S}_{j-1} + \mathbf{S}_j + \mathbf{S}_{j+1})^2}_{\text{Total spin}} - \frac{3}{4} \right\}$$

- Proportional to a projection operator
 - Addition of 3 spin 1/2's: $S_{j-1,j,j+1} = 1/2$ or $3/2$

$$h_j = \frac{3}{4} J P_{j-1,j,j+1}^{S=3/2} \quad \text{project out } S_{j-1,j,j+1} = 1/2$$

Majumdar-Ghosh model (2)

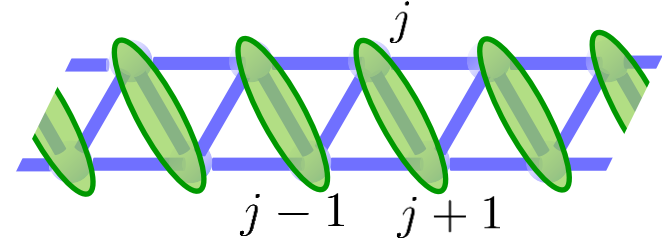
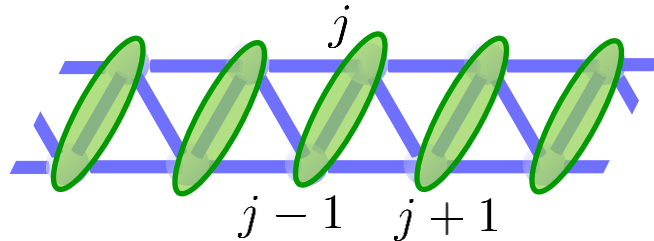
■ H is frustration-free!

- Dimer i  $j = [i, j] := \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle_{i,j} - |\downarrow\uparrow\rangle_{i,j})$

- Dimer states

$$|\psi_1\rangle = [1, 2][3, 4] \cdots [N-1, N]$$

$$|\psi_2\rangle = [2, 3][4, 5] \cdots [N, 1]$$



- They are annihilated by h_j
 - Addition of a spin singlet and a spin 1/2 $\rightarrow S_{j-1,j,j+1} = 1/2$
- No other ground states
 - Proved by Caspers, Emmett, Magnus, *J. Phys. A* **17**, 2687 (1984)

■ Generalizations

- 2D model: Shastry, Sutherland, *Physica B+C* **108**, 1069 (1981)
- Higher-spin model: Michaud *et al.*, *PRL* **108**, 127202 (2012)

Outline of today's lecture

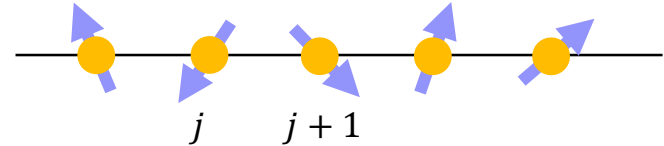
1. Introduction & Motivation
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Haldane “conjecture” (early 80s)

■ Spin- S Heisenberg antiferromagnetic chain

- Hamiltonian ($J > 0$)

$$H_{\text{Heis}} = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$



- $S=1/2, 3/2, 5/2, \dots$

Gapless, power-law decay of spin correlations

NOTE) $S=1/2$ case is solvable (Bethe 1931)

- $S=1, 2, 3, \dots$

- Unique ground state
- Non-zero gap Δ (Haldane gap)
- Exponential decay of spin correlation



Nobel Prize (2016)

Established in many different ways!

AgVP₂S₆, NENP, ...; ED, QMC, ...

$$\Delta(S) = \begin{cases} 0.41048(6) & \text{for } S = 1 \\ 0.08917(4) & \text{for } S = 2 \\ 0.01002(3) & \text{for } S = 3 \end{cases}$$

Todo & Kato, *PRL* **87** (2001)

AKLT model (1)

Affleck, Kennedy, Lieb & Tasaki,
PRL **59** (1987), *CMP* **115** (1987)

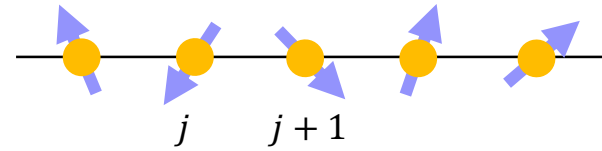
■ Hamiltonian ($S=1$)

$$H = \sum_{j=1}^N h_j, \quad h_j = \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{3}(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 + \frac{2}{3}$$

Bilinear

Biquadratic

- PBC imposed: $\mathbf{S}_{N+1} = \mathbf{S}_1$
- SU(2) symmetric
- Rewriting of h_j



$$h_j = 2P_{j,j+1}^{S=2}, \quad P_{j,j+1}^{S=2} = \frac{1}{24} \underbrace{(\mathbf{S}_j + \mathbf{S}_{j+1})^2}_{\text{Total spin}} \{ \underbrace{(\mathbf{S}_j + \mathbf{S}_{j+1})^2}_{\text{Total spin}} - 2 \}$$

Total spin


- Addition of 2 spin 1's:

$$S_{j,j+1} = 0, 1, \text{ or } 2$$

$$h_j = 2P_{j,j+1}^{S=2} \text{ project out } S_{j,j+1} = 0, 1 \text{ states}$$

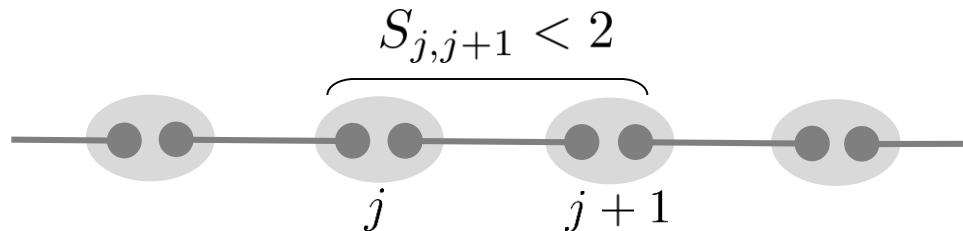
AKLT model (2)

■ H is frustration-free!

- Dimer = spin singlet:  $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

- Projection to spin-1 space: 


- Valence-bond-solid (VBS)/AKLT state




- Each projector $P_{j,j+1}^{S=2}$ annihilates this state
 - Addition of a spin singlet and 2 spin 1/2's: $S_{j,j+1} = 0$ or 1
- AKLT proved
 - Uniqueness of the ground state
 - Nonzero gap above the ground state
 - Exponential decay of correlators

Partial support of
Haldane's conjecture


VBS as a matrix product state

- Spin singlet  $\propto |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$

- Singlet product  $\propto (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix} (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$
 $= (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \\ -|\uparrow\uparrow\rangle & -|\uparrow\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$

- Projection to spin-1

$$P^{(1)}|\uparrow\uparrow\rangle = |+\rangle, \quad P^{(1)}|\uparrow\downarrow\rangle = P^{(1)}|\downarrow\uparrow\rangle = |0\rangle/\sqrt{2}, \quad P^{(1)}|\downarrow\downarrow\rangle = |-\rangle$$

-  $\propto (|\uparrow\rangle, |\downarrow\rangle) \begin{pmatrix} |0\rangle & \sqrt{2}|-\rangle \\ -\sqrt{2}|+\rangle & -|0\rangle \end{pmatrix} \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}$

- Repeat this procedure



$$\propto \text{Tr}[\mathcal{A}^{[1]} \dots \mathcal{A}^{[j]} \dots \mathcal{A}^{[L]}], \quad \mathcal{A}^{[j]} = \begin{pmatrix} |0\rangle_j & \sqrt{2}|-\rangle_j \\ -\sqrt{2}|+\rangle_j & -|0\rangle_j \end{pmatrix}$$

- Finitely correlated states Fannes, Nachtergaele & Werner, *CMP* **144**, 443 (1992)

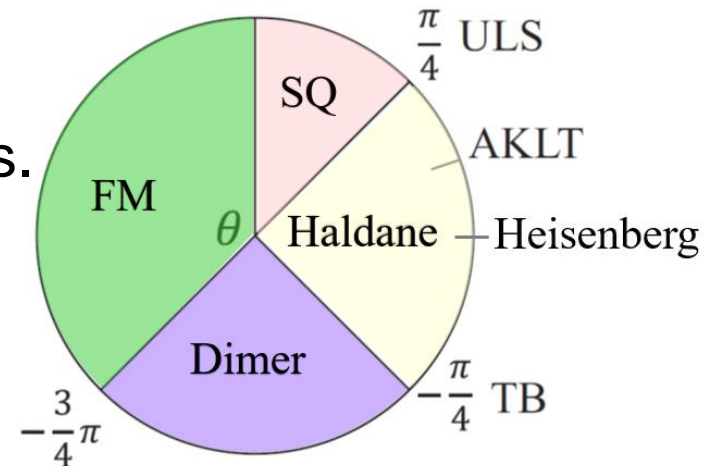
Quantum spin-1 chain with SU(2)

■ Bilinear-biquadratic (BLBQ) model

$$H_{\text{BLBQ}}(\theta) = \sum_{j=1}^L [\cos \theta (\mathbf{S}_j \cdot \mathbf{S}_{j+1}) + \sin \theta (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2]$$

■ Phase diagram Lauchli, Schmid & Trebst, *PRB* **74**, 144426 (2006)

- Spin-quadrupolar (SQ): gapless, dominant nematic corr.
- Ferromagnetic (FM)
- Dimer: gapped, 2-fold degenerate g.s.
- Haldane phase
 - ✓ Gapped unique g.s.
 - ✓ Edge states
 - ✓ Hidden AFM order (string order)

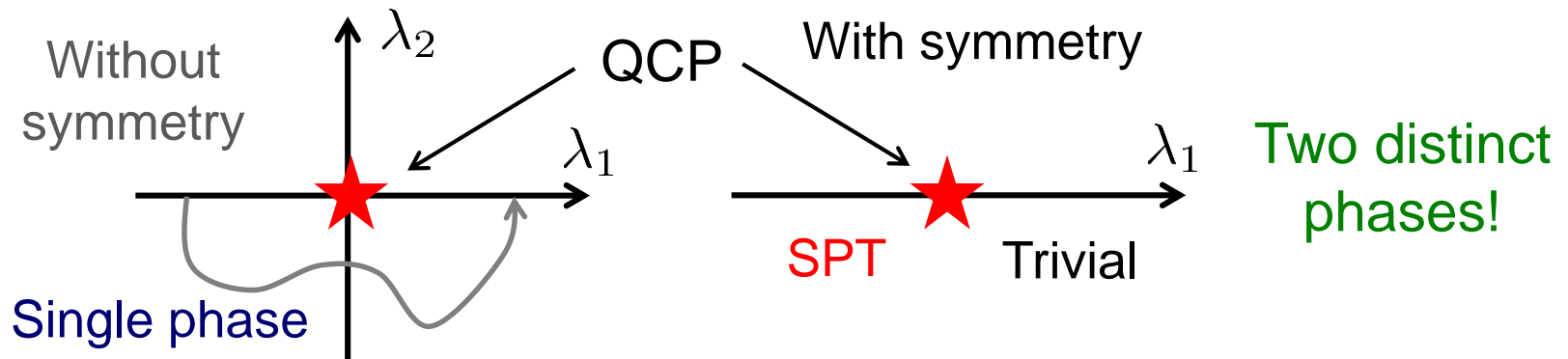


Prototype of Symmetry-protected topological (SPT) phase!

Haldane phase as SPT phase

■ What is SPT?

Gu & Wen, *PRB* **80** (2009). Pollmann, Berg, Turner & Oshikawa, *PRB* **81** (2010); **85** (2012)



■ Symmetry protection

$S=1$ Haldane phase is protected by ANY one of three symmetries:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (ii) time-reversal, (iii) bond centered inversion

$$Z_\pi = X_\pi Y_\pi = \exp(-i\pi \sum_j S_j^z)$$

$$Y_\pi = \exp(-i\pi \sum_j S_j^y)$$

$$X_\pi = \exp(-i\pi \sum_j S_j^x)$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, X_\pi, Y_\pi, Z_\pi\}$$

BLBQ has symmetries (i), (ii) & (iii)

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Kitaev's toric code

Kitaev, Ann. Phys. **303**, 2 (2003)

■ Setup

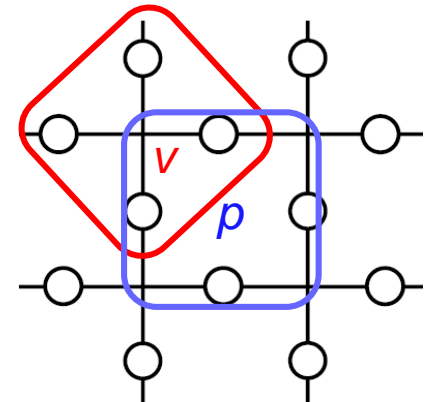
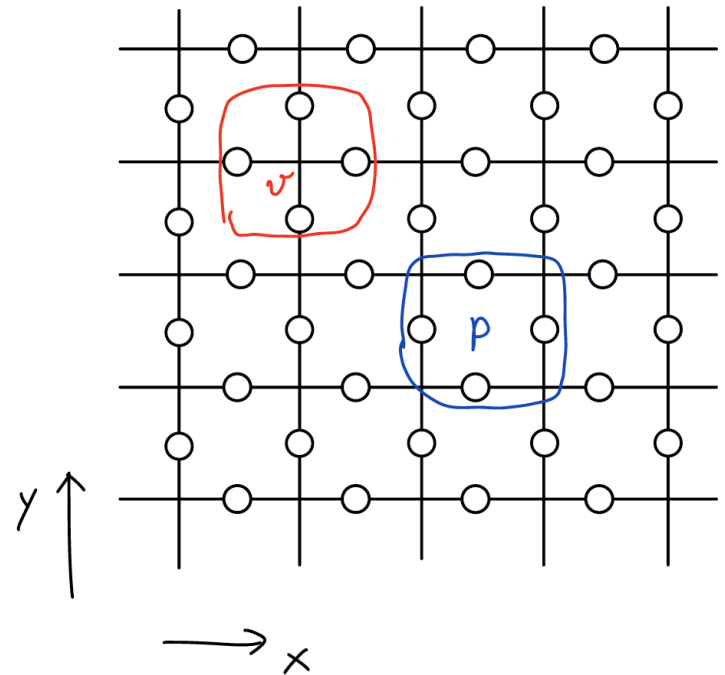
- 2D square lattice
- PBC in x & y directions
- Spin $1/2$'s on edges
- N : number of vertices

■ Hamiltonian $(J_e > 0, J_m > 0)$

$$H = -J_e \sum_v A_v - J_m \sum_p B_p$$

- Vertex ops. $A_v = \prod_{j \sim v} \sigma_j^z$
- Plaquette ops. $B_p = \prod_{j \sim p} \sigma_j^x$
- They all commute

A_v and B_p share 0 or 2 spins. $\{\sigma_j^z, \sigma_j^x\} = 0$



Toric code is frustration-free

■ Properties of local Hamiltonians

- A_v and B_p square to 1
- Their eigenvalues are ± 1
- Anderson bound: $H \geq -N(J_e + J_m)$

■ Ground state

- All-up state $|\uparrow\uparrow\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$
 - is a ground state of $-A_v$ for all v
 - But is not even an eigenstate of $-B_p$

- $|\Psi\rangle = \left\{ \prod_p (1 + B_p) \right\} |\uparrow\uparrow\rangle$ is a nonzero state

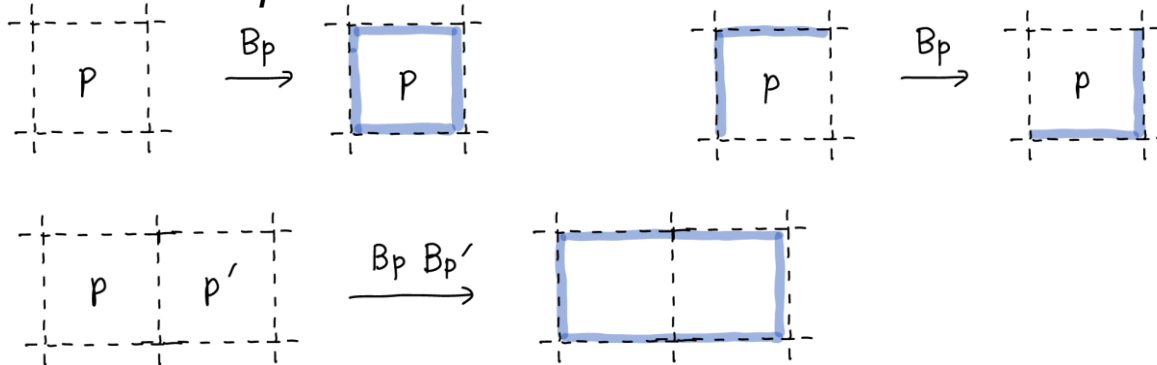
- Since $[A_v, B_p] = 0$, $-A_v|\Psi\rangle = -|\Psi\rangle$
- Since $B_p(1 + B_p) = (1 + B_p)$ & $[B_p, B_{p'}] = 0$, $-B_p|\Psi\rangle = -|\Psi\rangle$
- $|\Psi\rangle$ saturates the lower bound $E = -N(J_e + J_m)$

Graphical representation of Ψ

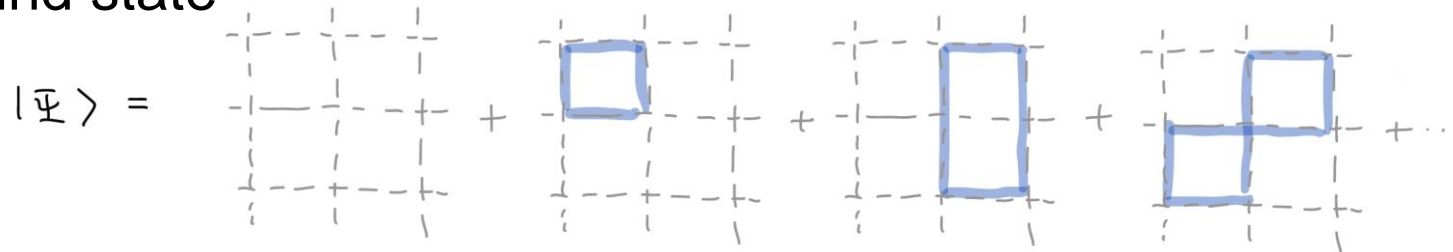
- Spin \leftrightarrow line segment



- Action of B_p



- Ground state



- 0, 2, or 4 lines emanating from each vertex in each config.
- $|\Psi\rangle$ = superposition of all such loop configurations
- Same diagrams appear in high- T expansion of Ising
- Another view: projection of cluster state

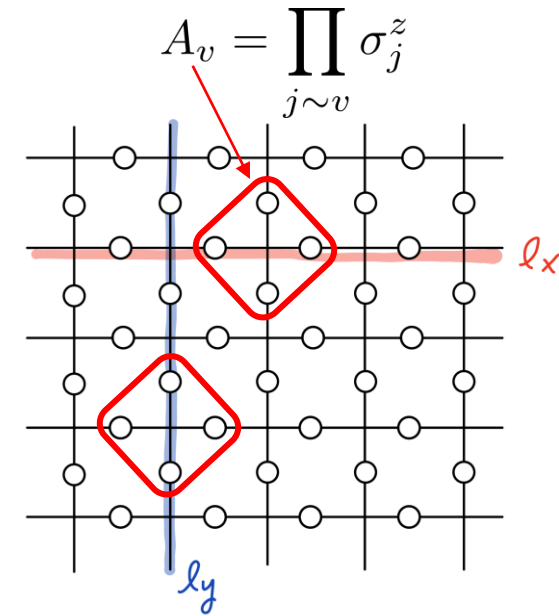
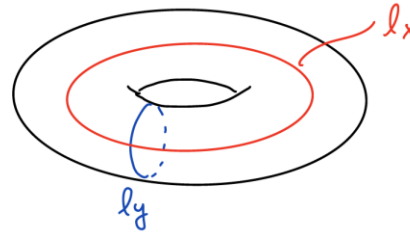
Topological degeneracy

■ Electric path ops

- Model is on a torus
- Closed paths l_x & l_y

$$X_\alpha := \prod_{j \in l_\alpha} \sigma_j^x \quad (\alpha = x, y)$$

- Commute with H



■ Degenerate ground states

- $|\Psi\rangle, X_x|\Psi\rangle, X_y|\Psi\rangle, X_x X_y|\Psi\rangle$
- Graphical rep.

$$X_y|\Psi\rangle = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \dots$$

Loops cross this line an odd number of times

The graphical representation shows the action of X_y on the ground state $|\Psi\rangle$. It is shown as a sum of configurations. The first configuration is a vertical blue line l_y crossing a horizontal red line l_x . The second configuration is the same as the first, but with a blue loop added that crosses the red line an odd number of times. The third configuration is the same as the first, but with a blue loop added that crosses the red line an even number of times. The red line is dashed in the diagrams.

- Degeneracy is robust against local perturbations
- ex.) Prove that the four states are orthogonal

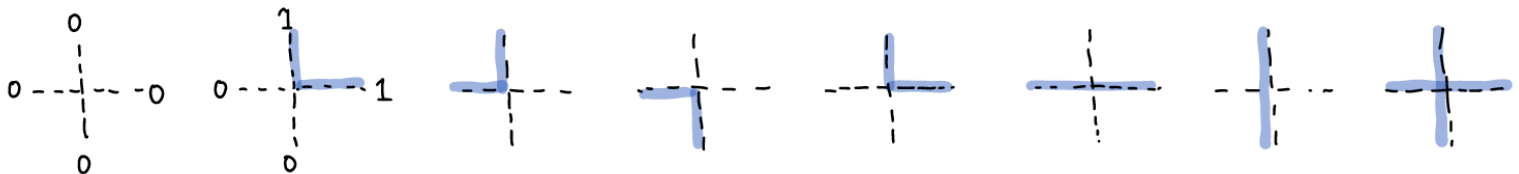
Ψ as a tensor network

■ Rokhsar-Kivelson (RK) states

- RK state is equal-weight superposition of classical configurations (with constraints)
- Originally discussed in the context of quantum dimer model *PRL* **61**, 2376 (1988)
- Can be expressed as a tensor network
- Ψ is an example of an RK state!

■ Building blocks

- Local tensor



- The weight of each local config. is 1
- Can be made anisotropic \rightarrow Quantum 8-vertex model
 - Ardonne, Fendley & Fradkin, *Ann. Phys.* **310**, 493 (2004)

Outline of today's lecture

1. Introduction & Motivation
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3. Systematic construction of models
 - Unitary transformation
 - Witten's conjugation
 - Sandwiching method
 - Wouters, Katsura, Schuricht,
SciPost Phys. Core **4**, 027 (2021)
arXiv:2005.12825
4. Summary

Can we cook up new models?

■ Trivial model

- Hamiltonian $H = \sum_{j=1}^N (1 - \sigma_j^x) \geq 0$ (p.s.d)
- Ground state

$$|\Psi_+\rangle = |+\rangle_1 |+\rangle_2 \cdots |+\rangle_N$$

$$\text{Local basis: } \sigma^x |\pm\rangle = \pm |\pm\rangle$$

➤ No entanglement...

■ Cluster model

- Hamiltonian $\tilde{H} = \sum_{j=1}^N (1 - \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z) = U H U^\dagger$

➤ PBC imposed

- Unitarily equivalent to H

➤ Unitary tr. $U = (CZ)_{1,2} (CZ)_{2,3} \cdots (CZ)_{N,1}$

$$(CZ)|a, b\rangle = (-1)^{ab} |a, b\rangle, \quad (a, b = 0, 1)$$

- Ground state $|\tilde{\Psi}_+\rangle = U |\Psi_+\rangle$

➤ Entangled!

Anything beyond unitary transformation?

Yes! **Witten's conjugation** is a key!

$N=2$ supersymmetric (SUSY) QM

■ Algebra

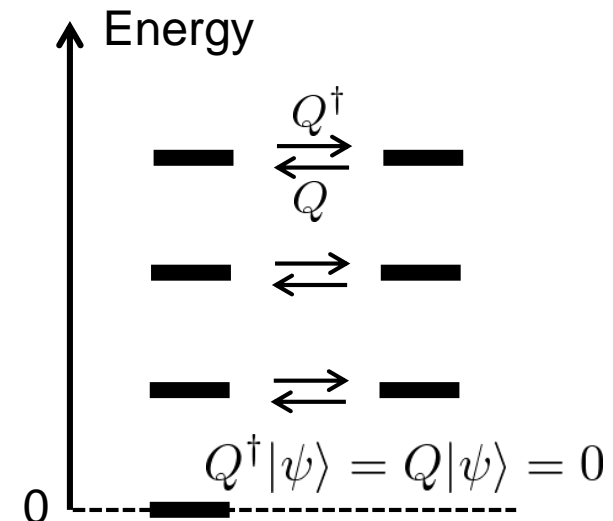
- Supercharges: $Q, Q^\dagger, Q^2 = 0, (Q^\dagger)^2 = 0$
- Fermionic parity: $(-1)^F, \{Q, (-1)^F\} = \{Q^\dagger, (-1)^F\} = 0$
- Hamiltonian: $H = \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q$
- Symmetry: $[H, Q] = [H, Q^\dagger] = [H, (-1)^F] = 0$

■ Spectrum of H

- $E \geq 0$ for all states, as H is p.s.d
- $E > 0$ states **come in pairs** $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
- $E = 0$ iff a state is a SUSY singlet

Ground-state energy = 0 \rightarrow SUSY **unbroken**

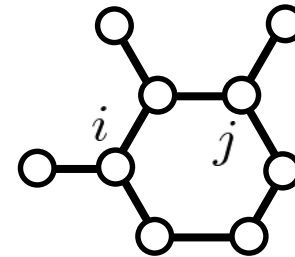
Ground-state energy > 0 \rightarrow SUSY **broken**



Elementary example

■ Lattice bosons and fermions

- Lattice sites: $i, j = 1, 2, \dots, N$
- Creation, annihilation ops.



$$[b_i, b_j^\dagger] = \delta_{i,j}, \quad \{c_i, c_j^\dagger\} = \delta_{i,j}, \quad [b_i, b_j] = \{c_i, c_j\} = 0.$$

➤ bosons and fermions are mutually commuting.

- Fermion number $F = \sum_j c_j^\dagger c_j$
- Vacuum state $b_i |\text{vac}\rangle \stackrel{j}{=} c_i |\text{vac}\rangle = 0, \forall i$

■ Supercharges and Hamiltonian

$$Q = \sum_j b_j^\dagger c_j, \quad Q^\dagger = \sum_j c_j^\dagger b_j$$

➔ $\{Q, Q^\dagger\} = \sum_j b_j^\dagger b_j + \sum_j c_j^\dagger c_j$

Just the total particle number!
 $|\text{vac}\rangle$ is a SUSY singlet.

Zero-energy states (ZES)

■ Cohomology

- Zero-energy states (ZESs) are in 1-to-1 correspondence with nontrivial cohomology classes of Q .

Proof) Any ZES $|\psi\rangle$ is annihilated by both Q and Q^\dagger .
 But $|\psi\rangle$ cannot be written as $|\psi\rangle = Q|\phi\rangle$ for any $|\phi\rangle$
 since this would imply that $\langle\psi|\psi\rangle = \langle\phi|Q^\dagger|\psi\rangle = 0$.

$$\# (\text{ZES of } H) = \dim (\text{Ker } Q / \text{Im } Q)$$

■ Witten's conjugation (*Nucl. Phys. B* **202**, 253 (1982).)

- Invertible operator M
- New supercharge & Hamiltonian

$$\tilde{Q} := MQM^{-1}, \quad \tilde{H} := \{\tilde{Q}, \tilde{Q}^\dagger\}$$

- $\dim (\text{Ker } \tilde{Q} / \text{Im } \tilde{Q}) = \dim (\text{Ker } Q / \text{Im } Q)$

The deformation preserves the number of zero-energy states.

Conjugation argument (non-SUSY ver.)

■ Universal form of frustration-free systems

- Set the ground state (g.s.) energy to zero
- Hamiltonian (p.s.d.)
- Zero-energy ground state

$$H = \sum_j L_j^\dagger L_j$$

$$|\psi\rangle \text{ s.t. } L_j|\psi\rangle = 0 \quad \forall j$$

■ Recipe for the construction

- New L operators $\tilde{L}_j := M L_j M^{-1}$ (M : invertible)
- New Hamiltonian
- Zero-energy ground state

$$\tilde{H} = \sum_j \tilde{L}_j^\dagger \tilde{L}_j$$

$$|\tilde{\psi}\rangle = M|\psi\rangle, \quad \tilde{L}_j|\tilde{\psi}\rangle = 0 \quad \forall j$$

- \tilde{H} is inequivalent to H unless M is unitary
- \tilde{H} and H have the same number of g.s.
- Similar to the idea of “Doob transform”

Sandwiching method

■ A slight generalization

- Positive definite operator C_j
- Sandwich it between \tilde{L}_j^\dagger and \tilde{L}_j
 - Does not change the g.s. manifold

• Newer Hamiltonian

$$\tilde{H} = \sum_j \tilde{L}_j^\dagger C_j \tilde{L}_j \quad \tilde{L}_j = M L_j M^{-1}$$

• Zero-energy g.s.

$$|\tilde{\psi}\rangle = M|\psi\rangle,$$

■ Theorem

Let $|\Psi_k\rangle$ ($k = 1, \dots, n$) be linearly independent zero-energy g.s. of H . The ground-state manifold of \tilde{H} is given by

$$\tilde{G} = \text{span}\{M|\Psi_1\rangle, \dots, M|\Psi_n\rangle\}.$$

Thus, the g.s. degeneracies of H and \tilde{H} are identical.

- Proof) Just follows from $\tilde{G} \subseteq \bigcap_j \ker(\tilde{L}_j)$ and $\tilde{G} \supseteq \bigcap_j \ker(\tilde{L}_j)$.

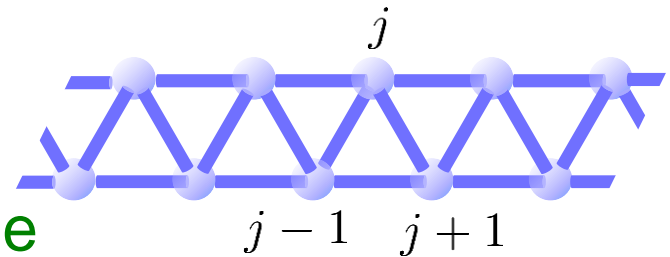
Exercise

Shastry, Sutherland,
Physica B+C **108**, 1069 (1981)

- Deformed Majumdar-Ghosh model
 - Hamiltonian ($S=1/2$, N : even, PBC imposed)

$$H = \sum_{j=1}^N h_j, \quad h_j = \sum_{a=x,y,z} \frac{J_a}{2} \left(S_{j-1}^a S_j^a + S_j^a S_{j+1}^a + S_{j-1}^a S_{j+1}^a + \frac{1}{4} \right)$$

- $J_a > 0$ ($a = x, y, z$) are distinct
- SU(2) symmetry is broken



- Q. Prove that H is frustration-free
 - Hint: h_j is a sum of positive semidefinite ops.

Further extensions

- 3-coloring condition: $J_x J_y + J_y J_z + J_z J_x = 0$
 - Changlani *et al.*, *PRL* **120**, 117202 (2018)
 - Palle & Benton, *PRB* **103**, 214428 (2021)

Summary

■ Frustration-free systems

- The Hamiltonian $H = \sum_j h_j$ is said to be frustration-free if there exists a simultaneous eigenstate of h_j with their lowest eigenvalues for all j
- h_j do not have to commute with each other
- Examples
 - Ferro-Heisenberg, Majumdar-Ghosh, AKLT, Toric code, ...

■ Construction of frustration-free models

- Witten's conjugation
- Sandwiching method

$$H = \sum_j L_j^\dagger L_j \quad \longrightarrow \quad \tilde{H} = \sum_j \tilde{L}_j^\dagger C_j \tilde{L}_j \quad C_j > 0, \quad \tilde{L}_j = M L_j M^{-1} \\ (M: \text{invertible})$$

- Allow for constructing new models
 - Wouters, Katsura, Schuricht, *SciPost Phys. Core* **4**, 027 (2021)