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# New examples of elements in the kernel of the Magnus representation of the Torelli group

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**Abstract.**

From our previous paper, it is known that the Magnus representation of the Torelli group is not faithful. In this paper, we show a certain kind of elements in the kernel of the representation.

## §1. Introduction

Let $\Sigma_g$ be an oriented closed surface and $\Sigma_{g,1}$ a compact surface removed an open disk from $\Sigma_g$. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, which is the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act trivially on the first homology group of $\Sigma_{g,1}$.

From our previous paper [6], the Magnus representation of the Torelli group

$$r_1 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}[H])$$

is not faithful for $g \geq 2$, where $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. Then this representation cannot determine the linearity of the Torelli group. In [9], we characterize the kernel of $r_1$ for the commutator of two BSCC maps, where the Dehn twist along a bounding simple closed curve is called BSCC map.

In Section 2, we review the definition and the irreducible decomposition of the Magnus representation of the Torelli group.

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## §2. Definition and irreducible decomposition of the Magnus representation of the Torelli group

In this section, we recall the definition and the irreducible decomposition of the Magnus representation of the Torelli group from [4], [6] and [7].

**Definition 2.1.** We call the mapping

$$\begin{array}{rccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g;\mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

the Magnus representation of the mapping class group, where $\frac{\partial}{\partial \gamma_i}$ : $\mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the Fox derivation of the integral group ring $\mathbb{Z}[\Gamma_0]$ and $^- : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the antiautomorphism induced by the mapping $\gamma \mapsto \gamma^{-1}$.

The mapping $r$ is not a homomorphism but a crossed homomorphism. Namely, the Magnus representation of the mapping class group is not a genuine representation.

**Proposition 2.2** (Morita [4]). *For arbitrary two elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

*where ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ on each entry.*

*Proof.* It follows from the chain rule of the Fox derivation. Q.E.D.

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