

Abelian quotients of the Y-filtration
on the homology cylinders via the LMO functor

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§1 Introduction

$$\Sigma_{g,1} := \overbrace{\sqcap \cdots \sqcap}^{g \geq 0}$$

$M_{g,1} := \text{Homeo}(\Sigma_{g,1} \text{ rel } \partial) / \text{isotopy}$ the mapping class group

$I_{g,1} := \text{Ker}(M_{g,1} \rightarrow \text{Aut } H_1(\Sigma_{g,1}; \mathbb{Z}))$ the Torelli group

$I_{g,1}(1) \supset \cdots \supset I_{g,1}(n) \supset \cdots$ the lower central series

$$\begin{matrix} \parallel & \parallel \\ I_{g,1} & [I_{g,1}(n-1), I_{g,1}] \end{matrix} \rightsquigarrow I_{g,1}(n)/I_{g,1}(n+1) \cong ??$$

↑
abelian groups

$$I_{g,1}(1)/I_{g,1}(2) = I_{g,1}^{\text{ab}} \quad \text{Johnson '85}$$

$$(I_{g,1}(n)/I_{g,1}(n+1)) \otimes \mathbb{Q} \quad \begin{matrix} \text{Hain '97} & \text{Morita '99} \\ \text{Morita - Sakasai - Suzuki '17} \end{matrix}$$

Prob 1 For $n \geq 2$, $\exists?$ torsion in $I_{g,1}(n)/I_{g,1}(n+1)$

Thm 1 (NSS) \exists torsion for $n = 3, 5$ & $g \geq 6$

$$I_{g,1} \supset \mathcal{K}_{g,1} \supset I_{g,1}(2)$$

$\text{Ker}(I_{g,1} \rightarrow \text{Aut}(\pi_1 \Sigma_{g,1} / \pi_1 \Sigma_{g,1}(3)))$ the Johnson kernel

Rem $M_g \supset I_g \supset \mathcal{K}_g$ are defined similarly.

Thm (MSS'17) For $g \geq 6$

$H_1(\mathcal{K}_g; \mathbb{Q}) \cong$ "Casson inv." \oplus "Morita's refinement $\tilde{\tau}_2$ of Johnson hom."

Prob 2 $\exists?$ torsion in $H_1(\mathcal{K}_{g(1)}; \mathbb{Z})$

Thm 2 (NSS) \exists torsion in $H_1(\mathcal{K}_g; \mathbb{Z})$ for $g \geq 6$.

Key tool : the LMO functor in 3-dim (quantum) topology

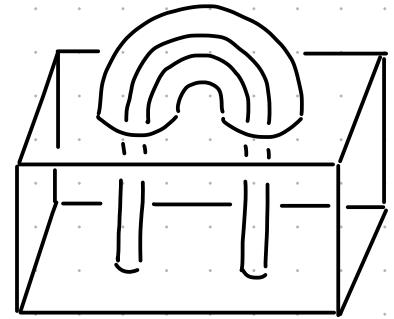
$$\tilde{\Sigma}: \mathcal{LCob}_g \longrightarrow {}^{\text{ts}}\mathcal{A}$$

the category of \checkmark \hookrightarrow the category of
Lagrangian g -cobordisms top-substantial Jacobi diagrams

M : connected oriented compact 3-mfd

$$m: \partial(\Sigma_{g,1} \times [-1,1]) \xrightarrow{\cong} \partial M$$

$$(M, m) \sim (M', m') \iff \begin{array}{c} M \xrightarrow{\exists \cong} M' \\ \text{def} \quad m \uparrow \quad \cup \quad \uparrow m' \\ \partial(\Sigma_{g,1} \times [-1,1]) \end{array}$$



Def (M, m) is a **homology cylinder** over $\Sigma_{g,1}$

$$\iff (m_+)_* = (m_-)_*: H_1(\Sigma_{g,1}) \xrightarrow{\cong} H_1(M).$$

$\mathcal{C}_{g,1} := \{ \text{homology cylinders over } \Sigma_{g,1} \}$

is a monoid by $M \circ M' = \boxed{\begin{matrix} M' \\ M \end{matrix}}$

$$c: I_{g,1} \hookrightarrow \mathcal{C}_{g,1}$$

$$f \mapsto (\Sigma_{g,1} \times [-1,1], f \times 1 \cup \text{id} \times (-1))$$

$Y_1 \mathcal{C}_{g,1} \supset \dots \supset Y_n \mathcal{C}_{g,1} \supset \dots$: the "**Y-filtration**"

$\mathcal{C}_{g,1}$ submonoid satisfying $c(I_{g,1}(n)) \subset Y_n \mathcal{C}_{g,1}$

c induces a group hom $I_{g,1}(n)/I_{g,1}(n+1) \rightarrow Y_n \mathcal{C}_{g,1}/\sim_{Y_{n+1}}$

$Y_n \mathcal{C}_{g,1}/\sim_{Y_{n+1}}$

$Y_n \mathcal{C}_{g,1}$

Abelian quotients of the **Y-filtration**

on the **homology cylinders** via the **LMO functor**

$\mathcal{C}_{g,1}$

$\tilde{\Sigma}$

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§2. Preliminaries

④ Y-filtration (Goussarov'99, Habiro'00)

$$M = (M, m) \in \mathcal{C}_{g,1}$$

U

G : a **graph clasper**

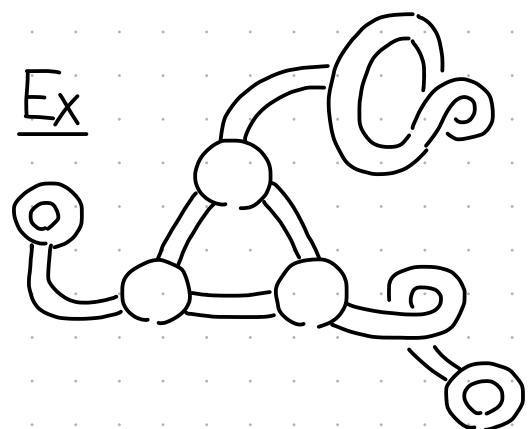
(a surface consisting of
disks, bands and leaves)

{

L_G : framed link in M

$$M_G = M|_{L_G} \in \mathcal{C}_{g,1}$$

Ex



Ex



Def M is **Y_n -equivalent** to M'

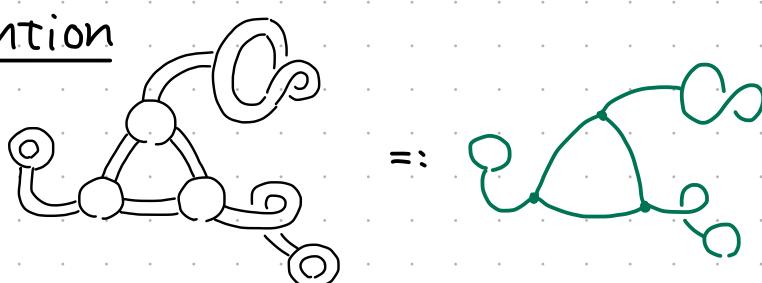
$\Leftrightarrow \exists G_1, \dots, G_r \subset M$ st. $M \underset{\text{def}}{\underset{j=1}{\overset{r}{\cup}}} G_j = M'$ & $\deg G_j = n$ \sim # of disks

$\text{Y}_n \mathcal{C}_{g,1} := \{ M \in \mathcal{C}_{g,1} \mid M \underset{\text{Y}_n}{\sim} \sum_{g,1} \times [-1,1] \}$ submonoid

$\mathcal{C}_{g,1} = \text{Y}_1 \mathcal{C}_{g,1} \supset \dots \supset \text{Y}_n \mathcal{C}_{g,1} \supset \dots$: the **Y-filtration**

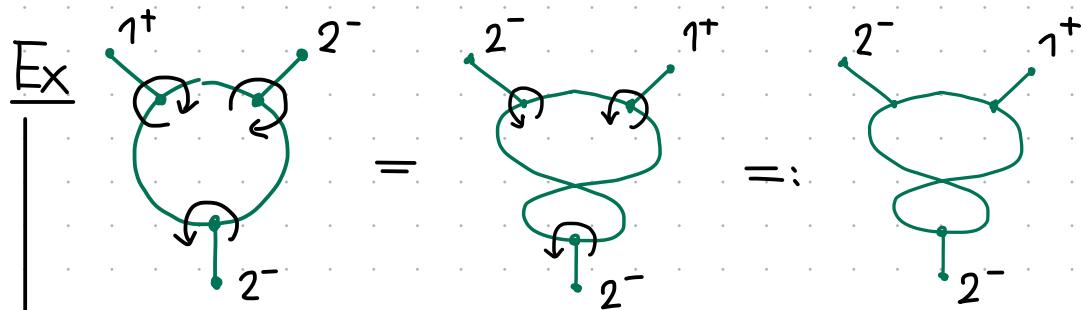
We want to study the abelian group $\text{Y}_n \mathcal{C}_{g,1} / \text{Y}_{n+1}$.

Convention



⊛ Jacobi diagrams

Def A **Jacobi diagram** colored with $\{1^+, \dots, g^+, 1^-, \dots, g^-\}$ is a uni-trivalent graph { univalent vertex is colored s.t. each trivalent vertex has a cyclic order }



$\mathbb{A}_n^C := \mathbb{Z} \{ \text{conn. Jacobi diagrams of } i\text{-deg} = n \} / \text{AS, IHX}$
 # of trivalent vertices
 (internal)

Ex $\mathbb{A}_1^C = \langle \begin{array}{c} i \\ \diagdown \\ \diagup \\ j \\ k \end{array} \text{'s} \rangle \cong \bigwedge_{\mathbb{Z}}^3 H_1(\Sigma_{g,1})$

self-loop

$$\left\{ \begin{array}{l} \text{---} = - \\ \text{---} - \text{---} + \text{---} = 0 \\ \text{---} = 0 \end{array} \right.$$

④ LMO functor & surgery map
 (Cheptea - Habiro - Massuyeau '08)

$$\log \tilde{\mathcal{Z}}(-) : \mathcal{C}_{g,1} \longrightarrow \widehat{\mathcal{A}}^c \otimes \mathbb{Q}, M \mapsto J_0 + J_1 + \cdots + J_n + \cdots$$

$$\begin{matrix} \cup \\ Y_n \mathcal{C}_{g,1} \end{matrix}$$

$$M \mapsto J_0 + 0 + \cdots + 0 + J_n + \cdots$$

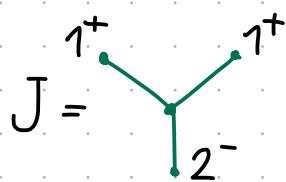
$$Y_n \mathcal{C}_{g,1} / Y_{n+1} \longrightarrow \mathcal{A}_n^c \otimes \mathbb{Q}, [M] \mapsto J_n$$

Q. How to compute J_n from $[M]$?

A. When $[M] = S(J)$ for $J \in \mathcal{A}_n^c$, we have $J_n = \pm J$.

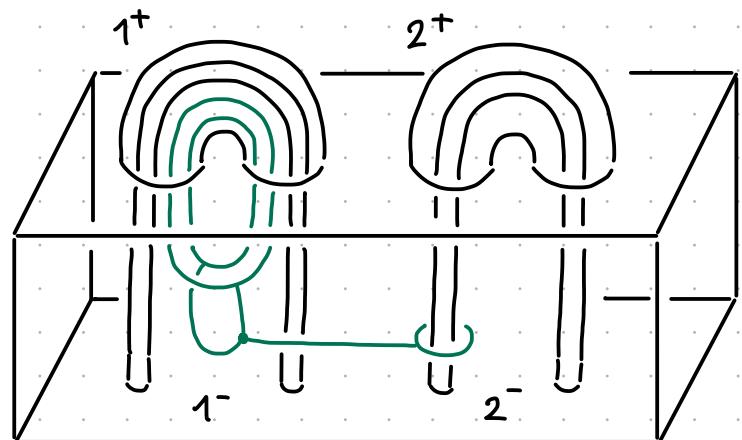
The surgery map $S : \mathcal{A}_n^c \xrightarrow{\text{hom}} Y_n \mathcal{C}_{g,1} / Y_{n+1}$ is defined by ...

Ex ($n=1, g=2$)



$$\sum_{2,1} \times [-1,1] \cup G_J$$

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$



$S(J) = (\sum_{2,1} \times [-1,1])_{G_J}$ is well-defined up to \sim_{Y_2}

§3. Main theorem

We expect that $\text{tor}(Y_n \mathcal{C}_{g,1} / Y_{n+1}) \xrightarrow{?} (\log \widehat{\Sigma}(-))_{n+1}$

$$[M] \in Y_n \mathcal{C}_{g,1} / Y_{n+1} \xrightarrow{\text{red}} \mathbb{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Q}$$

↓

well-defined $\sim \bar{\Sigma}_{n+1} \rightarrow \mathbb{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \ni (\log \widehat{\Sigma}(M))_{n+1} = J_{n+1}$

Rem $\bar{\Sigma}_2$ = "Birman - Craggs homomorphism"

Q. How to compute $\bar{\Sigma}_{n+1}([M]) = J_{n+1} \bmod \mathbb{Z}$?

A. When $[M] = s(J)$ for $J \in \mathbb{A}_n^c$, we obtain the formula:

Thm 3 (N.-Sato - Suzuki) For $n \geq 1$, $g \geq 0$.

$$\begin{array}{ccccc} \mathbb{A}_n^c & \xrightarrow{s} & Y_n \mathcal{C}_{g,1} / Y_{n+1} & \longrightarrow & Y_{n+1} \mathcal{C}_g / Y_{n+1} \\ \downarrow " \delta " & \cup & \downarrow \bar{\Sigma}_{n+1} & \cup & \downarrow " \widehat{\Sigma}_{n+1} " \\ \mathbb{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \xrightarrow{\text{id} \otimes \frac{1}{2}} & \mathbb{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} & \longrightarrow & (\mathbb{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})/\sim \end{array}$$

§4. The map δ and Proof of theorems

Def & Prop $U(J) = \{ \text{univalent vertices of } J \}$

$$\stackrel{\psi}{\sim} \quad l(v) \in \{1^+, \dots, q^+, 1^-, \dots, q^-\}$$

The map $\delta: \mathbb{A}_n^c \rightarrow \mathbb{A}_{n+1}^c \otimes \mathbb{Z}/2$ is well-defined.

$$J \mapsto \sum_{v \in U(J)} \delta_v(J) + \sum_{v \neq w \in U(J)} \delta_{vw}(J)$$

$l(v) = l(w)$

$$\delta_v \left(\begin{array}{c} l(v) \\ | \\ \dots \end{array} \right) := \dots \begin{array}{c} l(v) \quad l(v) \\ | \quad | \\ \dots \end{array} \dots + \dots \begin{array}{c} l(v) \quad l(v)^* \\ | \quad | \\ \dots \end{array} \dots$$

$(j^\pm)^* := j^\mp$

$$\delta_{vw} \left(\begin{array}{c} l(v) \quad l(w) \\ | \quad | \\ \dots \end{array} \right) := \begin{array}{c} l(v) \\ | \\ \dots \end{array}$$

Ex ($n=1$) $J = \begin{array}{c} 1^+ \\ | \\ 2^- \end{array}$ $(J \neq 0, 2J=0)$

$$\begin{aligned} \delta(J) &= \underbrace{\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^- \\ | \quad | \\ 2^- \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^- \\ | \quad | \\ 2^- \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^+ \end{array}}_{=} + \underbrace{\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \end{array}}_{=} \\ &= \begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \quad 2^- \end{array} + \begin{array}{c} 1^+ \\ | \\ 2^- \end{array} \end{aligned}$$

$\left(\begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^+ \quad 2^- \end{array} \right) \xrightarrow{\text{IHX}} \left(- \begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^+ \quad 2^- \end{array} \right) = 0$

Note We gave some examples of Σ_{n+1} by the computer, and found the map δ .

Thm 3 (N.-Sato - Suzuki)

$$\begin{array}{ccccc}
 A_n^c & \xrightarrow{s} & Y_n C_{g,1} / Y_{n+1} & \longrightarrow & Y_{n+1} C_g / Y_{n+1} \\
 \downarrow " \delta " & \cup & \downarrow \bar{z}_{n+1} & \cup & \downarrow " \hat{z}_{n+1} " \\
 A_{n+1}^c \otimes \mathbb{Z}/2 & \xrightarrow{\text{id} \otimes \frac{1}{2}} & A_{n+1}^c \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & (A_{n+1}^c \otimes \mathbb{Q}/\mathbb{Z})/\sim
 \end{array}$$

Ex ($n=1$) $J = \begin{array}{c} 1^+ \\ \diagup \quad \diagdown \\ \text{---} \\ 2^- \end{array}$ ($2s(J) = 0, s(J) \neq 0$)

$$\bar{\Sigma}_2(s(J)) = \frac{1}{2} \delta(J) = \frac{1}{2} \begin{array}{c} 1^+ \quad 1^+ \\ | \quad | \\ 2^- \quad 2^- \end{array} + \frac{1}{2} \begin{array}{c} 1^+ \\ \text{---} \\ 2^- \end{array} \neq 0 \quad (\therefore s(J) \neq 0)$$

Proof of Thm 3

Step 1: Decompose $s(J)$ into elementary pieces.

Step 2: Compute $\tilde{\Sigma}_{\leq 2}$ for these pieces.

Step 3: Compute $\tilde{\Sigma}_{\leq 2}(s(J))$ by the functoriality of $\tilde{\Sigma}$.

Then $\frac{1}{2} \delta(J)$ appears. \square

Thm 2 (NSS) $\text{tor } H_1(\mathcal{K}_g; \mathbb{Z}) \neq 0$ for $g \geq 6$

Proof Step 1: Check that ζ is a homomorphism.

$$\begin{array}{ccc}
 (\log \tilde{\Sigma} \circ C(-))_4 =: \zeta & & \\
 \mathcal{K}_g \xrightarrow{\quad} (\mathbb{A}_4^C \oplus \mathbb{Q}/\mathbb{Z})/\sim & & \\
 \cup & & \uparrow \cup \\
 I_g(2) \xrightarrow{\quad} I_g(2)/I_g(4) \xrightarrow{c} Y_2 \mathcal{C}_g / Y_4 & & \\
 \Downarrow f \mapsto & & \downarrow \cup \quad \downarrow \cup \quad \uparrow \zeta_4 \\
 Y_3 \mathcal{C}_g / Y_4 & \xrightarrow{\quad} (\mathbb{A}_4^C \oplus \mathbb{Q}/\mathbb{Z})/\sim & \\
 \Downarrow s(J) & &
 \end{array}$$

Step 2: For $J = \begin{array}{ccccc} & b & c & b & \\ a & | & | & | & a \end{array}$, find $f \in I_g(2)$ s.t. $f \mapsto s(J)$

Step 3: Show $\zeta(f) \neq 0$ by Thm 3 ($\therefore [f] \neq 0 \in H_1(\mathcal{K}_g)$)

Step 4: Prove that Casson inv. $\oplus \tilde{T}_2$ send f to 0
 (Key: $2J = 0$)

Therefore, [MSS'17] implies $[f] \neq 0 \in \text{tor } H_1(\mathcal{K}_g)$ \square

Similarly, we can prove the following:

Thm 1 (NSS)

$\text{tor } (I_{g,1}(n)/I_{g,1}(n+1)) \neq 0$ for $n=3,5$ & $g \geq 6$

④ Future perspective

We would like to

- determine $I_{g,1}(n)/I_{g,1}(n+1)$.
- determine $H_1(K_g; \mathbb{Z})$.
- reveal a topological/geometric meaning of $\bar{\Sigma}_{n+1}$.
- consider " $\bar{\Sigma}_{n+2}$ ".

($Y_n C_{g,1}/Y_{n+2}$ is also an abelian group for $n \geq 2$)

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④ $Y_n C_{g,1}/Y_{n+1}$ ($n=1, 2$ Massuyeau-Melhan '03 '13)

Thm (NSS)

$$0 \rightarrow (\overset{3}{\wedge} H \oplus \overset{2}{\wedge} H) \otimes \mathbb{Z}/2 \rightarrow A_3^c \xrightarrow{S} Y_3 C_{g,1}/Y_4 \rightarrow 0 \text{ (exact)}$$

$$\begin{aligned} a \wedge b \wedge c &\mapsto \underset{a}{\cancel{b}} \underset{c}{\cancel{b}} \underset{a}{\cancel{b}} + \underset{b}{\cancel{c}} \underset{a}{\cancel{c}} \underset{c}{\cancel{b}} + \underset{c}{\cancel{a}} \underset{b}{\cancel{a}} \underset{a}{\cancel{c}} \\ a \wedge b &\mapsto \begin{array}{c} a \\ \diagup \quad \diagdown \\ \text{---} \end{array} + \begin{array}{c} b \\ \diagup \quad \diagdown \\ \text{---} \end{array} \end{aligned}$$

④ Goussarov-Habiro Conjecture for the Y_4 -equivalence

Prop (NSS)

$$M \sim_{Y_4} M' \text{ iff } f(M) = f(M') \text{ for all finite-type inv. of } \deg \leq 3$$

④ $\mathcal{H}_{g,1} = \mathcal{C}_{g,1}/\sim$ the homology cobordism group
of homology cylinders

$(M, m) \sim (N, n) \Leftrightarrow \exists W^4: \text{cpt ori smooth st.}$

$$\partial W = M \cup (-N) \quad \& \quad H_*(M) \xrightarrow[m \circ n^{-1}]{} H_*(W) \xleftarrow{\cong} H_*(N)$$

$$\begin{array}{c} \mathcal{C}_{g,1} \supset \cdots \supset Y_n \mathcal{C}_{g,1} \supset \cdots \\ \downarrow \qquad \qquad \downarrow \\ \mathcal{H}_{g,1} \supset \cdots \supset Y_n \mathcal{H}_{g,1} \supset \cdots \end{array}$$

Johnson hom & Birman-Craggs hom factor through $Y_n \mathcal{H}_{g,1}$

Prop (NSS) $\bar{\Sigma}_{n+1}$ ($n > 1$ odd) does NOT factor through $Y_n \mathcal{H}_{g,1}$

$$④ \bar{\Sigma}_{n+1} = \bigoplus_{l=0}^{\infty} \bar{\Sigma}_{n+1,l} : Y_n \mathcal{C}_{g,1} / Y_{n+1} \longrightarrow \bigoplus_{l=0}^{\infty} \mathcal{A}_{n+1,l}^c$$

Thm (NSS) The $\bar{\Sigma}_{n+1,l}$ is non-trivial for n odd & $l = 0, 1$.
 $(g \geq \frac{n+5}{2})$

Recently, we showed that $\bar{\Sigma}_{4l-2, l}$ is non-trivial for $\forall l > 0$
(tool: weight system) $(g \geq 2)$