Diagrammatic Algebra

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The bulk of this talk is based on Joint work with Seiichi Kamada. Other contributors include Victoria Lebed, Masahico Saito, and Seung Yeop Yang. This handout mostly contains drawings. Many of these will appear on the board. Others are too complicated to present in real time. Some will invite you to consider them at your leisure.

Outline

0. Metaphorical overview

1. Multi-categorical analogue of a Frobenius Algebra
   (a) $M(n,n)$
   (b) Frobenius Alg. axioms as diagrams
   (c) Objects, multiple arrows and exchangers

2. An associated category as a 4-category

3. Adding a braiding
   (a) Category of tangles.
   (b) Multi-category of 2-tangles
   (c) Qualgebras, knotted handle-bodies
   (d) Foams and foam moves
   (e) abstract tensors

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1. Frobenius Algebras

Example: $M(n, n)$.

Frobenius Algebra axioms:

$W$ is a vector space

for which

such that

$W \otimes W \rightarrow W$, bilinear maps

$\text{unit } A$, & $\text{unit } R$
Thm: A Frobenius algebra is also a coalgebra.

Lemma 1: \[V = V\]
Proof: \[V = V\]

Lemma 2: \[V = V\]
Proof: \[V = V\]

Lemma 3: Counit: \[V = V\]
Proof: \[V = V\]

Lemma 4: Coassociativity: \[V = V\]
Proof: \[V = V\]

Lemma 5: Copairing is coassociative: \[V = V\]
Proof: \[V = V\]
Categorification

Objects of $\mathbb{F}A$: $\mathbb{N} = \{0, 1, 2, \ldots\}$. Here $1 \cong \bullet$.

Id: on $\alpha$

Id strings

1. Generating arrows:
2. If $A$ and $B$ are arrows, then these are arrows (careful about sources and targets)

3. Possible configurations:
Exchanger axiom. Suppose that $\gamma \xrightarrow{F} \alpha$ and $\zeta \xrightarrow{G} \beta$ are arrows. There is a natural family $X$ of 2-arrows

$$X : (F \otimes l_\zeta) \circ (l_\alpha \otimes G) \Rightarrow (l_\gamma \otimes G) \circ (F \otimes l_\beta)$$

which are 2-isomorphisms. Here $(F \otimes l_\zeta) \circ (l_\alpha \otimes G)$ and $(l_\gamma \otimes G) \circ (F \otimes l_\beta)$ are algebraic expressions of the graphic (right and left respectively):
“Change followed by exchange” is the same thing as “exchange followed by change.”
Three principles:

1. Different things are not equal.
2. Simultaneity does not occur.
3. Performing a process and undoing it should be compared to doing nothing.

Typical diagram in a category:

Replace by a double arrow

When the set of 1-arrows is a category, we have double arrows:

These are composed as follows:
Compose all source arrows, but leave two target arrows (or vice versa):

Globular composition:

Don’t allow:

But replace with
And assume a natural family of triple arrow isomorphisms:

\[
\begin{align*}
&f_0 & g_1 & s(g_1) = s(g_0) \\
&f_1 & t(f_1) = t(f_0) & t(g_1) = t(g_0) \\
&I_f & t(f_0) = t(f_1) & t(g_0) = t(g_1) \\
&F & s(f_1) = s(f_0) & s(g_0) = s(g_1) \\
&G & s(f_0) = s(f_1) & s(g_1) = s(g_0) \\
\end{align*}
\]
Identity double arrows:

Generating double arrows:
Table 1: Writing units in terms of counits
Table 2: Writing counits in terms of units
<table>
<thead>
<tr>
<th>Left unit axiom</th>
<th>⊗</th>
<th>Left unit axiom inverse</th>
<th>[\Diamond]</th>
<th>upper left triangular flap back</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\triangleright] ((-))</td>
<td>[\downarrow]</td>
<td>[\Diamond] ((-))</td>
<td>[\downarrow]</td>
<td>lower left triangular flap back</td>
</tr>
<tr>
<td>Right unit axiom</td>
<td>[\uparrow] [\otimes]</td>
<td>[\rangle] ((+))</td>
<td>[\uparrow]</td>
<td>upper left triangular flap forward</td>
</tr>
<tr>
<td>Right unit axiom inverse</td>
<td>[\downarrow]</td>
<td>[\langle] ((+))</td>
<td>[\downarrow]</td>
<td>lower left triangular flap forward</td>
</tr>
</tbody>
</table>

Table 3: Units
Table 4: Counits
<table>
<thead>
<tr>
<th>Multi. via comult. left</th>
<th>Multi. via comult. inv. left</th>
<th>Multi. via comult. right</th>
<th>Multi. via comult. inv. right</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>I⊗Y</td>
<td><img src="image5" alt="Diagram" /></td>
<td>Y⊗I</td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
</tr>
<tr>
<td>Right, back, below fold ends on seam</td>
<td>Right, back, above fold ends on seam</td>
<td>Right, front, below fold ends on seam</td>
<td>Right, front, above fold ends on seam</td>
</tr>
</tbody>
</table>

**Table 5:** Writing multiplication in terms of comultiplication
Table 6: Writing comultiplication in terms of multiplication
<table>
<thead>
<tr>
<th>Diagram</th>
<th>Description</th>
<th>Diagram</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Associativity axiom" /></td>
<td>Associativity axiom</td>
<td><img src="image2" alt="Associativity axiom inverse" /></td>
<td>Associativity axiom inverse</td>
</tr>
<tr>
<td><img src="image3" alt="Coassociativity axiom" /></td>
<td>Coassociativity axiom</td>
<td><img src="image4" alt="Coassociativity axiom inverse" /></td>
<td>Coassociativity axiom inverse</td>
</tr>
</tbody>
</table>

Table 7: Assocativity and coassociativity
Table 8: Assocativity and coassociativity of the pairing

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative pairing</td>
<td>Left seam crosses a fold on the right top to bottom</td>
</tr>
<tr>
<td>Associativity pairing inv.</td>
<td>Left seam crosses a fold on the right bottom to top</td>
</tr>
<tr>
<td>Coassociative copairing</td>
<td>Right seam crosses a fold on the left top to bottom</td>
</tr>
<tr>
<td>Coassociativity copairing inv.</td>
<td>Right seam crosses a fold on the right bottom to top</td>
</tr>
</tbody>
</table>
Table 9: Non-degeneracy of the pairing and copairing
Note: There are also triple arrows that can be seen as identities among relations. Here are some examples. These next few represent the third principle.
Let’s consider this sequence of double arrows:

\[
\require{AMScd}
\begin{CD}
\n @>>> [n \circ (\Pi \Pi) \circ (IUI)] \\
\end{CD}
\]

Rewrite as this:
Compare with the identity on:
\[ \bullet \mapsto \emptyset. \]
More on this later.

Additional double arrows. Consider the generating 1-arrows:

Compare compositions in each row to identity arrows:
<table>
<thead>
<tr>
<th>Birth Glyph Double Arrow</th>
<th>Birth Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Death Glyph Double Arrow</td>
<td>Death Surface</td>
</tr>
<tr>
<td>Saddle Glyph Double Arrow</td>
<td>Saddle Surface</td>
</tr>
<tr>
<td>Fork Glyph Double Arrow</td>
<td>Fork Surface</td>
</tr>
</tbody>
</table>

Table 10: Birth, Death, Saddle, and Fork Double Arrows
Table 11: Optimal points on seams

<table>
<thead>
<tr>
<th>Zip Type</th>
<th>Glyph</th>
<th>Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Close Bottom</td>
<td><img src="zip_close_bottom_glyph.png" alt="Diagram" /></td>
<td><img src="zip_close_bottom_surface.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Close Top</td>
<td><img src="zip_close_top_glyph.png" alt="Diagram" /></td>
<td><img src="zip_close_top_surface.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Open Bottom</td>
<td><img src="zip_open_bottom_glyph.png" alt="Diagram" /></td>
<td><img src="zip_open_bottom_surface.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Open Top</td>
<td><img src="zip_open_top_glyph.png" alt="Diagram" /></td>
<td><img src="zip_open_top_surface.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Table 12: Birth, death, arch, and canal for unit/counit pairs
**Summary Section 1:** Frobenius Algebra axioms $\to$ multiple category with obj $= \mathbb{N}$, arrows multi, comulti, unit, counit, pairing and copairing. Double arrows correspond to foams embedded in 3-space, and triple arrows correspond to isotopy of foams.

**Thm.** There is a multi-functor $b/2$ the multi-category gen by

and isotopy classes of embedded foams.
2. Promote to a multi-category with quadruple arrows

Obj: $t$ and $t$.
Generating 1-arrows:

\[
\begin{array}{c}
\text{f} \\
\text{t} \\
p : \text{f} \rightarrow \text{t} \\
b : \text{t} \rightarrow \text{f}
\end{array}
\]

In general, a reduced non-identity arrow is a finite sequence $\text{p}_1\text{p}_2\text{p}_3\ldots$ or $\text{b}_1\text{b}_2\text{b}_3\ldots$.

Generating double arrows:

Generating triple arrows:
Quadruple arrows are equalities:
Add exchanger axioms.

**Thm.** There is a multi-functor from the multi-category that is described above to the multi-category of isotopy classes of embedded oriented surfaces in 3-space.

Such a surface separates space into regions. Checkerboard color these. The two objects correspond to the black and white regions. The arrows correspond to arcs that intersect the surface transversely. The double arrows are embedded disks, and the triple arrows are chunks of space.
3. Adding a braiding

Start with one object and only the identity arrow defined upon it. Then create a generating non-identity double arrow from the identity arrow to itself. The exchanger axiom allows for braiding.

Qualgebra Axioms:

\[ H \quad (ab)c = a(bc) \quad \text{(associativity)}; \]
\[ YI \quad (ab) \triangleright c = (a \triangleleft c)(b \triangleleft c) \quad \text{(distributivity)}; \]
\[ IY \quad (a \triangleleft b) \triangleright c = a \triangleleft (bc) \quad \text{(exponential law)}; \]
\[ III \quad (a \triangleleft b) \triangleright c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad \text{(self-distributivity)}; \]
\[ II \quad \forall a, b \exists c \text{ such that } c \triangleleft b = a \quad \text{(right invertibility)}; \]
\[ I \quad a \triangleleft a = a \quad \text{(idempotence)}; \]
\[ T \quad a \cdot b = b \cdot (a \triangleleft b) \quad \text{(twisted commutativity)}. \]
<table>
<thead>
<tr>
<th>Partition</th>
<th>Foam move name</th>
<th>Associated chain</th>
<th>Prismatic structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3, 4)</td>
<td>YV</td>
<td>(a, b, c, d)</td>
<td>$\Delta^4$</td>
</tr>
<tr>
<td>(1, 2, 3)</td>
<td>4</td>
<td>HI</td>
<td>(a, b, c)</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>(3, 4)</td>
<td>YY</td>
<td>(a, b)</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>3</td>
<td>4</td>
<td>YII</td>
</tr>
<tr>
<td>1</td>
<td>(2, 3, 4)</td>
<td>IH</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>(2, 3)</td>
<td>4</td>
<td>IYI</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(3, 4)</td>
<td>IIY</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 13: The foam moves, corresponding chains, partitions, and prismatic sets
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