Diagrammatic Algebra

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Tokyo 2019

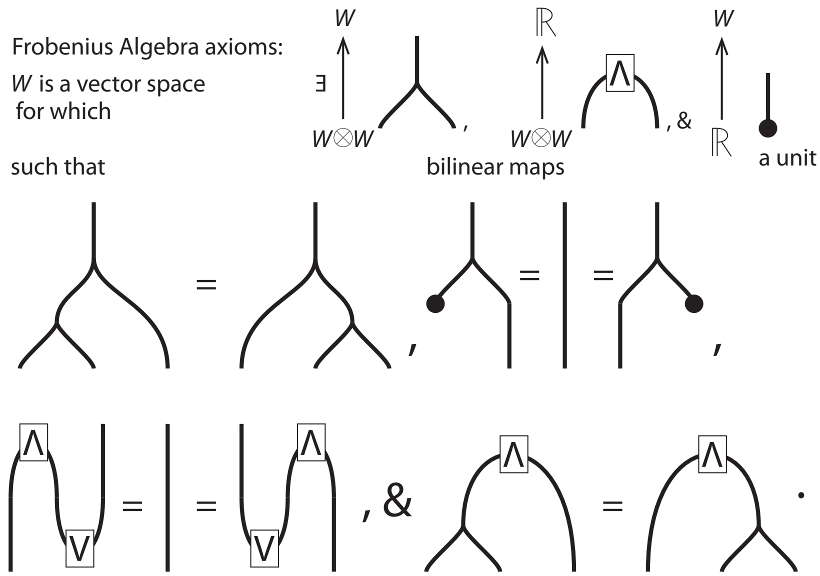
The bulk of this talk is based on Joint work with Seiichi Kamada. Other contributors include Victoria Lebed, Masahico Saito, and Seung Yeop Yang. This handout mostly contains drawings. Many of these will appear on the board. Others are too complicated to present in real time. Some will invite you to consider them at your leisure.

Outline

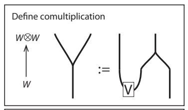
0. Metaphorical overview
1. Multi-categorical analogue of a Frobenius Algebra
   (a) $M(n, n)$
   (b) Frobenius Alg. axioms as diagrams
   (c) Objects, multiple arrows and exchangers
2. An associated category as a 4-category
3. Adding a braiding
   (a) Category of tangles.
   (b) Multi-category of 2-tangles
   (c) Qualgebras, knotted handle-bodies
   (d) Foams and foam moves
   (e) abstract tensors

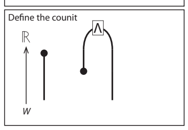
1. Frobenius Algebras

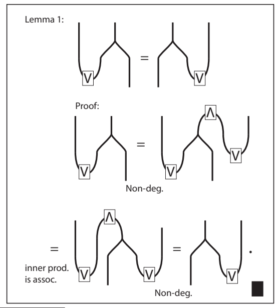
Example: $M(n,n)$.

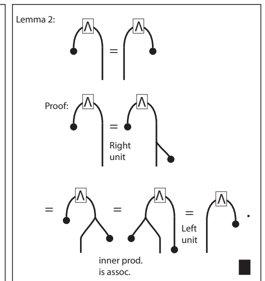


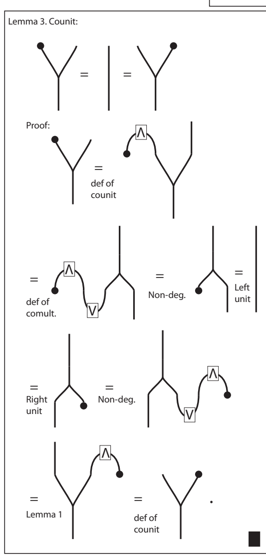
Thm: A Frobenius algebra is also a coalgebra.

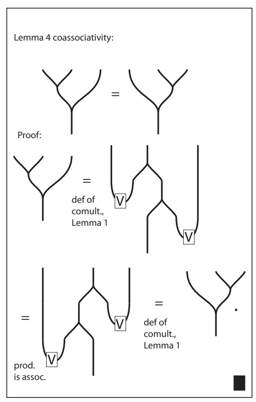


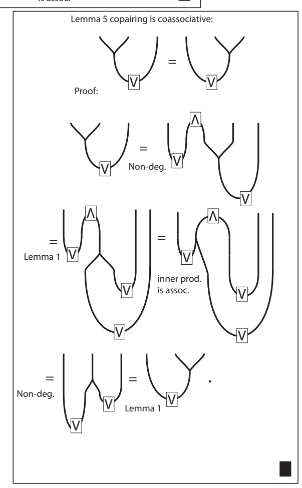






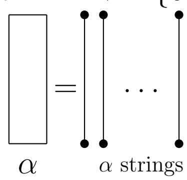


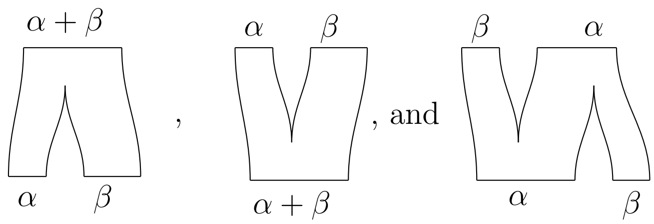


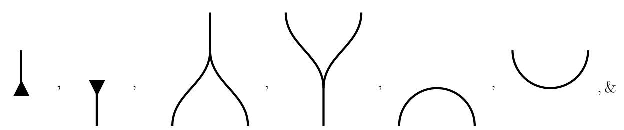


Categorification

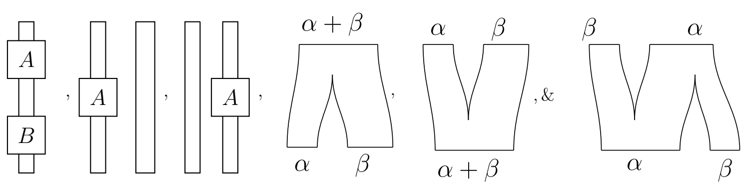
Objects of FA: $\mathbb{N} = \{0, 1, 2, \ldots\}$. Here $1 \leftrightharpoons -\bullet-$.

Id: on α 

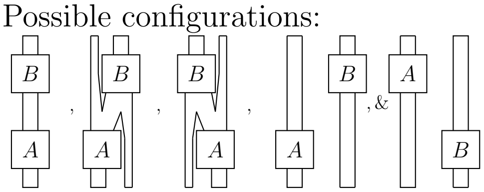


1. Generating arrows: 

2. If A and B are arrows, then these are arrows (careful about sources and targets)



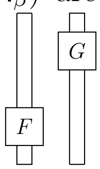
3. Possible configurations:



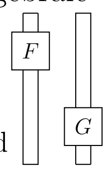
Exchanger axiom. *Suppose that* $\gamma \xleftarrow{F} \alpha$ *and* $\zeta \xleftarrow{G} \beta$ *are arrows. There is a natural family* X *of* 2*-arrows*

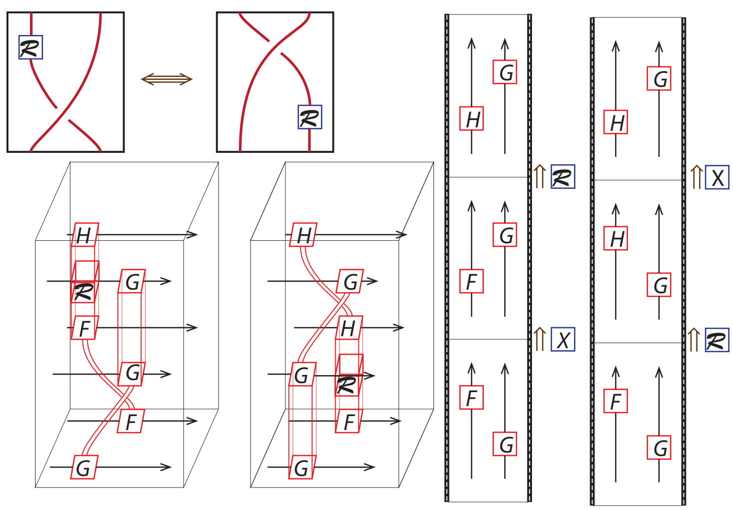
$$\mathsf{X} : (F \otimes \mathsf{I}_\zeta) \circ (\mathsf{I}_\alpha \otimes G) \Rightarrow (\mathsf{I}_\gamma \otimes G) \circ (F \otimes \mathsf{I}_\beta)$$

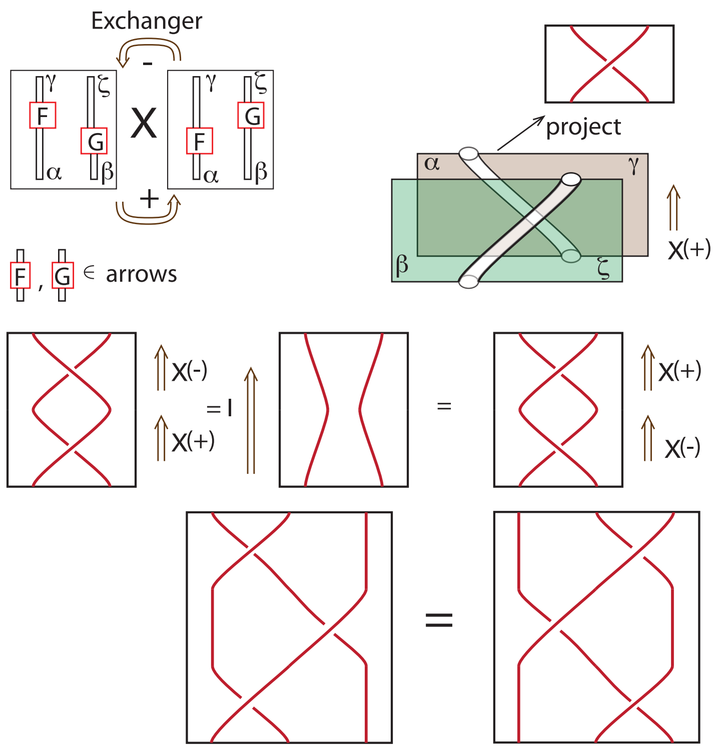
which are 2*-isomorphisms.* Here $(F \otimes \mathsf{I}_\zeta) \circ (\mathsf{I}_\alpha \otimes G)$ and $(\mathsf{I}_\gamma \otimes G) \circ (F \otimes \mathsf{I}_\beta)$ are algebraic expressions of the graphic (right and left respectively) :



and





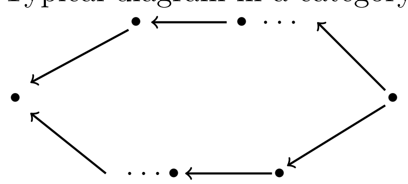


"Change followed by exchange" is the same thing as "exchange followed by change."

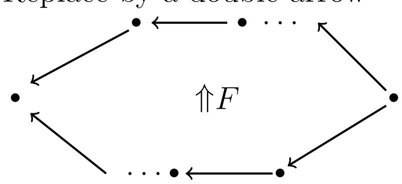
Three principles:

1. Different things are not equal.
2. Simultaneity does not occur.
3. Performing a process and undoing it should be compared to doing nothing.

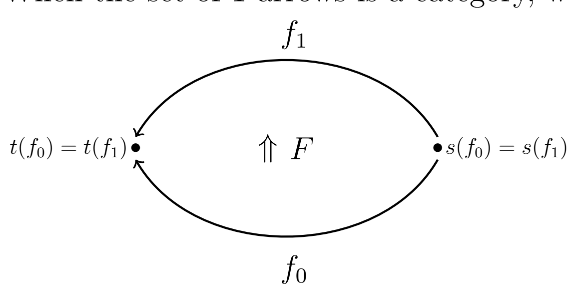
Typical diagram in a category:



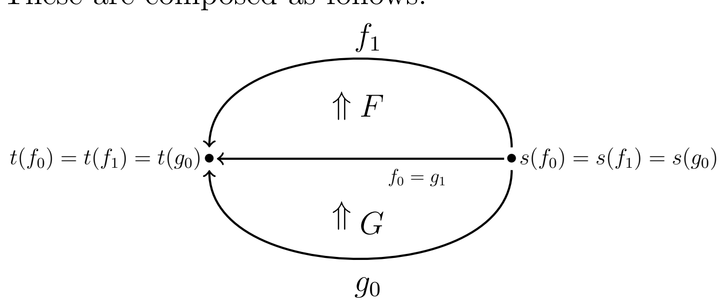
Replace by a double arrow



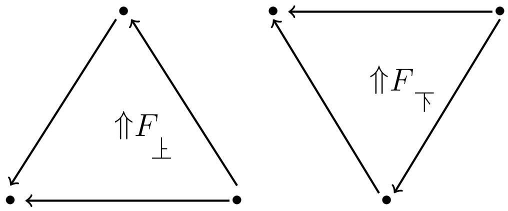
When the set of 1-arrows is a category, we have double arrows:



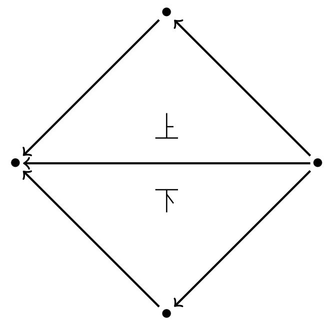
These are composed as follows:



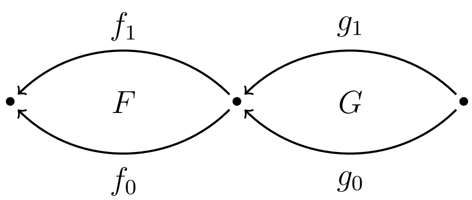
Compose all source arrows, but leave two target arrows (or vice versa):



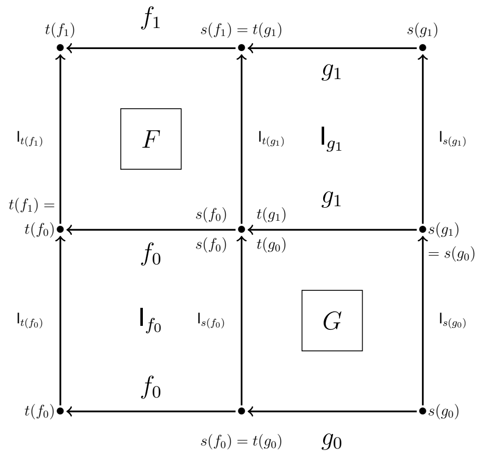
Globular compostion:



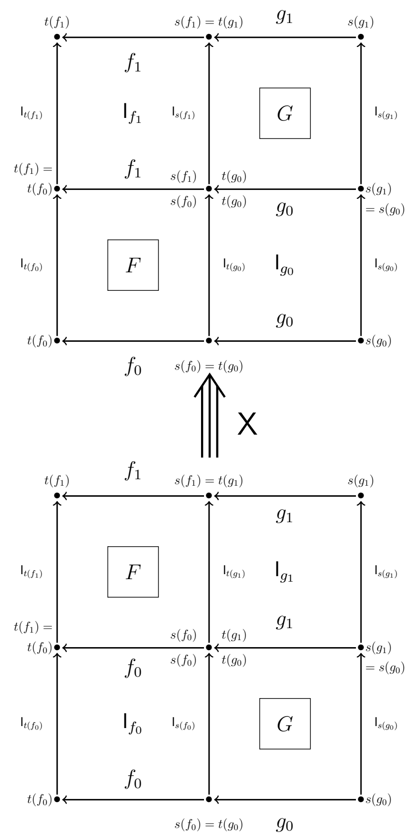
Don't allow:



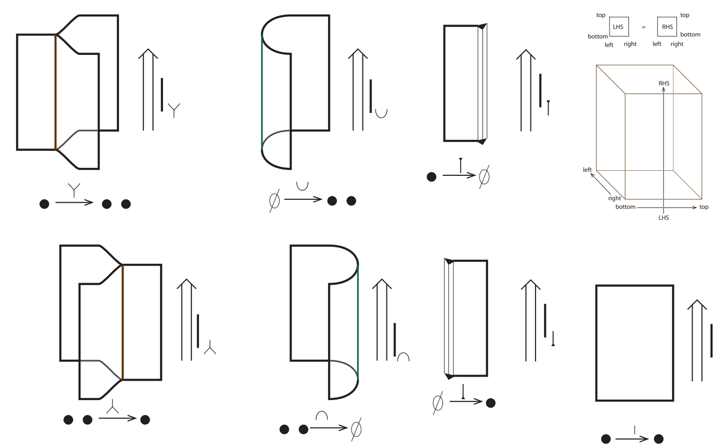
But replace with



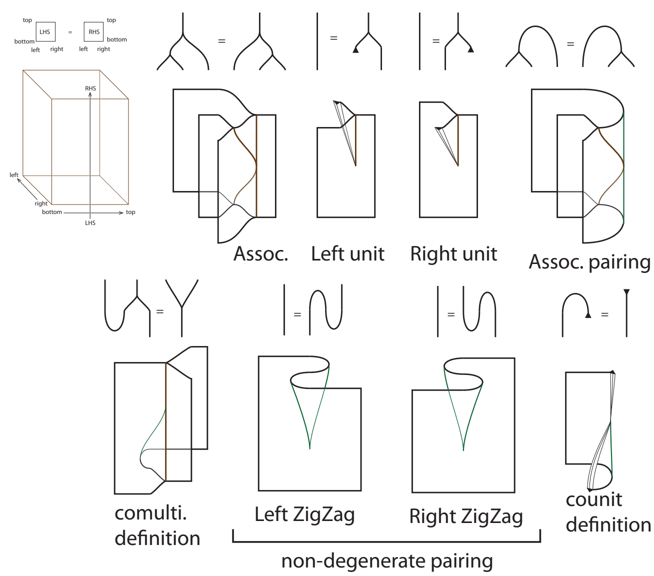
And assume a natural family of triple arrow isomorphisms:



Identity double arrows:



Generating double arrows:



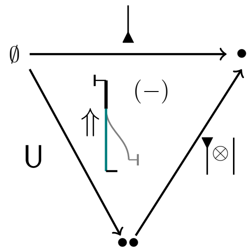
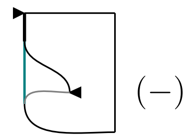
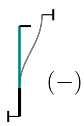
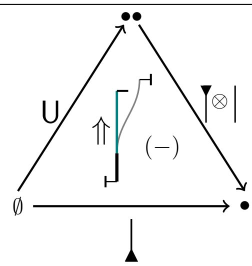
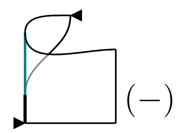
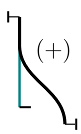
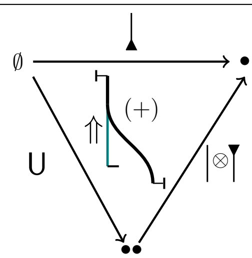
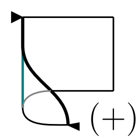
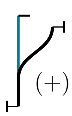
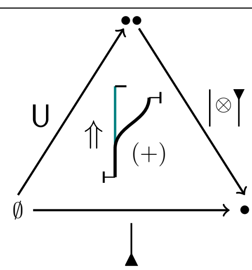
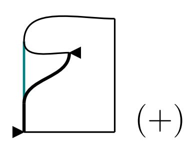
 unit via counit (left)		 Fold lower left corner back
 unit via counit inv. (left)		 Fold upper left corner back
 unit via counit (right)		 Fold lower left corner forward
 unit via counit inv. (right)		 Fold upper left corner forward

Table 1: Writing units in terms of counits

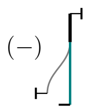
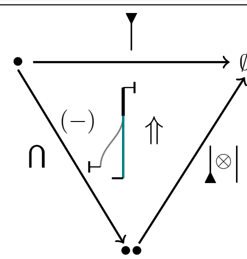
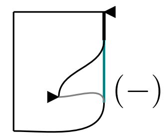
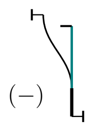
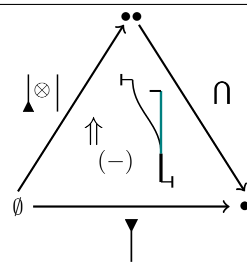
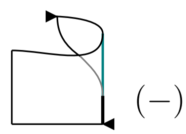
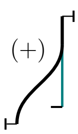
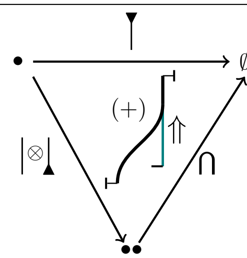
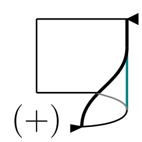
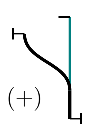
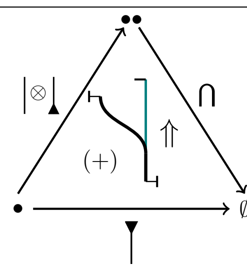
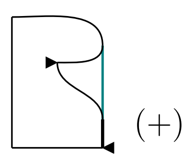
 counit via unit (left)		 Fold lower right corner back
 counit via unit inv. (left)		 Fold upper right corner back
 counit via unit (right)		 Fold lower right corner forward
 counit via unit inv. (right)		 Fold upper right corner forward

Table 2: Writing counits in terms of units

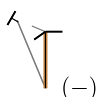
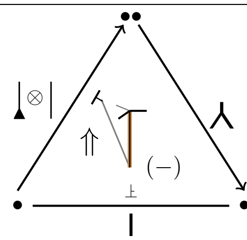
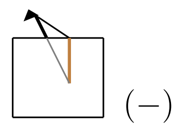
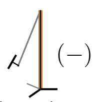
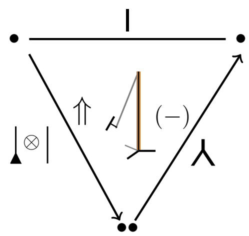
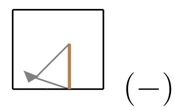
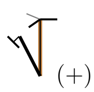
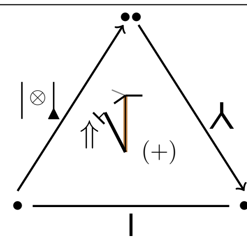
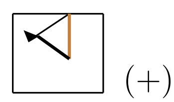
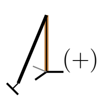
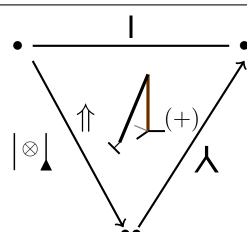
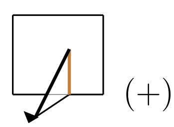
 Left unit axiom		 upper left triangular flap back
 Left unit axiom inverse		 lower left triangular flap back
 Right unit axiom		 upper left triangular flap forward
 Right unit axiom inverse		 lower left triangular flap forward

Table 3: Units

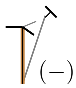
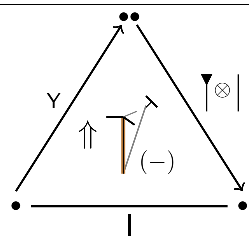
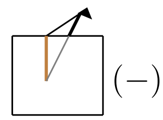
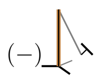
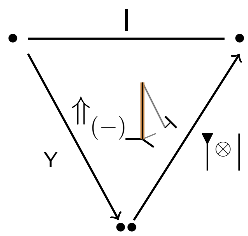
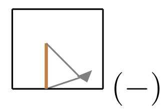
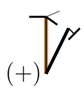
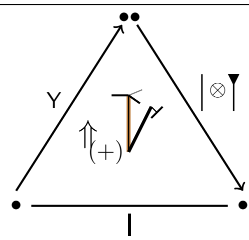
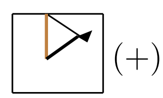
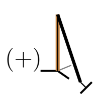
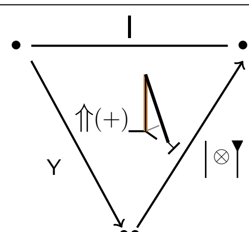
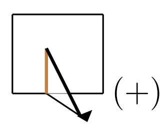
 Left counit axiom		 upper right triangular flap back
 Left counit axiom inverse		 lower right triangular flap back
 Right counit axiom		 upper right triangular flap forward
 Right counit axiom inverse		 lower right triangular flap forward

Table 4: Counits

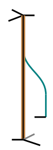
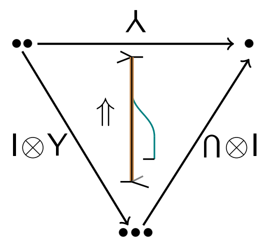
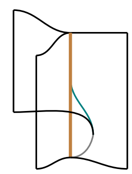
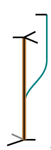
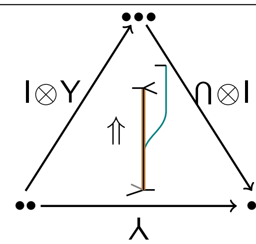
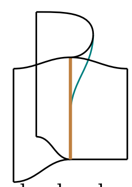
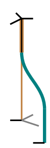
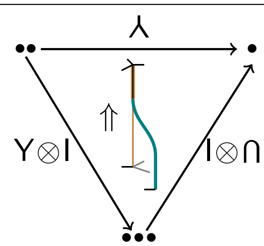
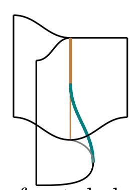
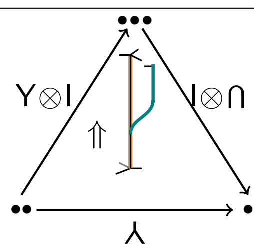
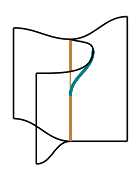
 Multi. via comult. left		 Right, back, below fold ends on seam
 Multi via comult. inv. left		 Right, back, above fold ends on seam
 Multi. via comult. right		 Right, front, below fold ends on seam
 Multi. via comult. inv. right		 Right, front, above fold ends on seam

Table 5: Writing multiplication in terms of comultiplication

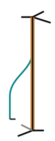
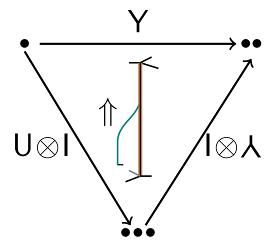
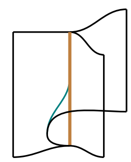
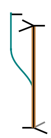
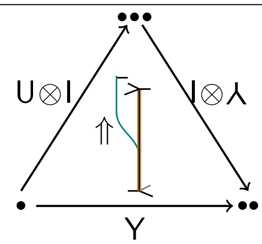
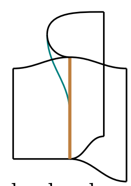
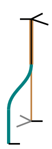
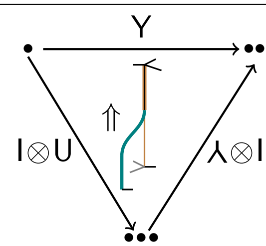
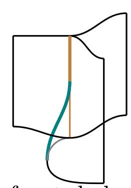
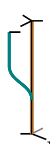
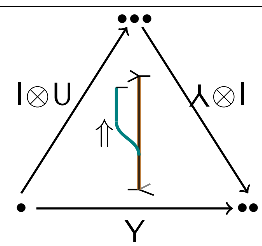
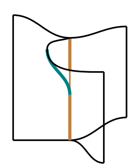
 Comult. via mult. left		 Left, back, below fold ends on seam
 Comulti via mult. inv. left		 Left, back, above fold ends on seam
 Comulti. via mult. right		 Left, front, below fold ends on seam
 Multi. via comult. inv. right		 left back above fold ends on seam

Table 6: Writing comultiplication in terms of multiplication

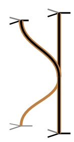
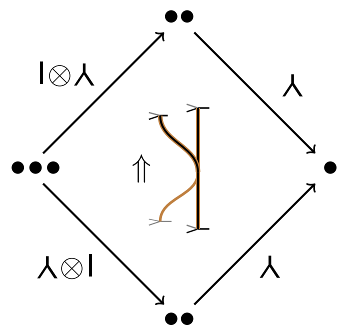
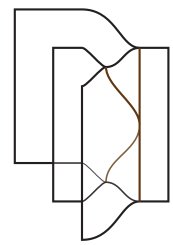
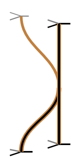
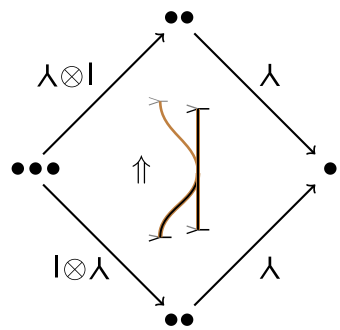
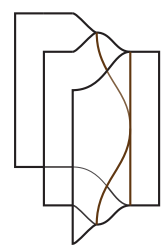
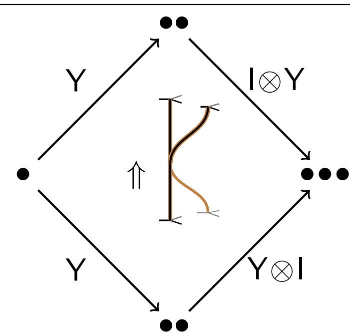
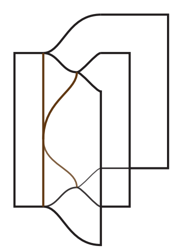
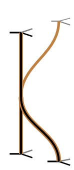
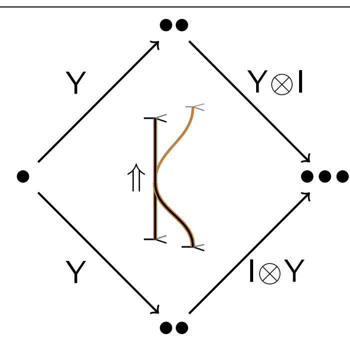
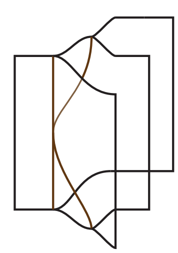
 Associativity axiom		 Left seam crosses top to bottom
 Associativity axiom inverse		 Left seam crosses bottom to top
 Coassociativity axiom		 Right seam crosses top to bottom
 Coassociativity axiom inverse		 Right seam crosses bottom to top

Table 7: Assocativity and coassociativity

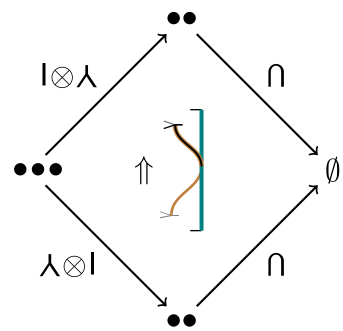
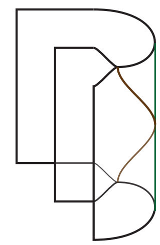
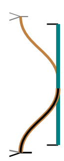
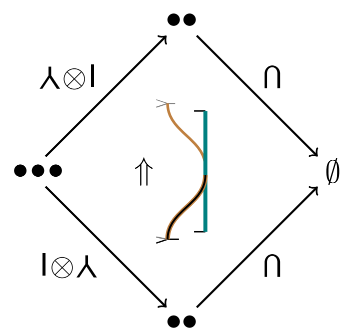
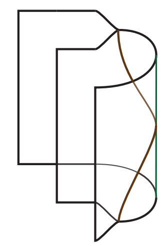
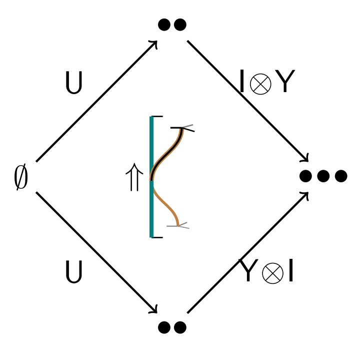
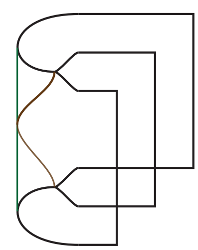
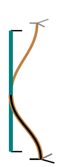
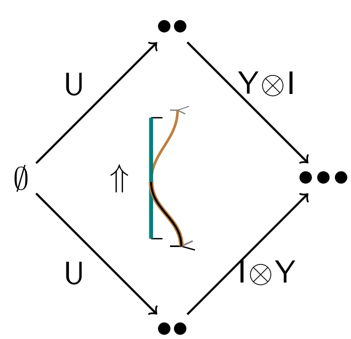
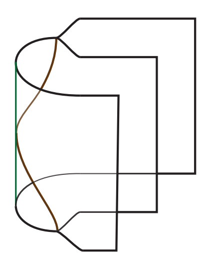
 Associative pairing		 Left seam crosses a fold on the right top to bottom
 Associativity pairing inv.		 Left seam crosses a fold on the right bottom to top
 Coassociative copairing		 Right seam crosses a fold on the left top to bottom
 Coassociativite copairing inv.		 Right seam crosses a fold on the right bottom to top

Table 8: Assocativity and coassociativity of the pairing

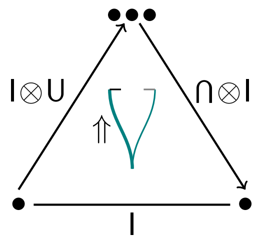
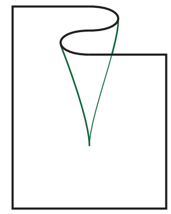
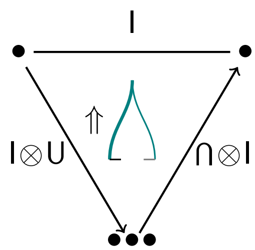
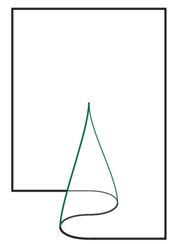
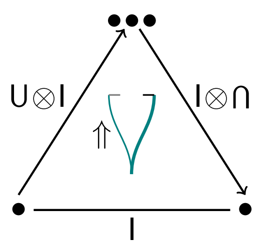
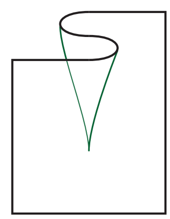
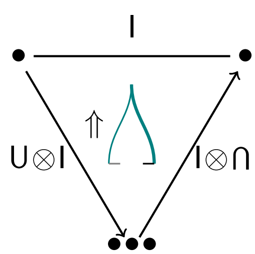
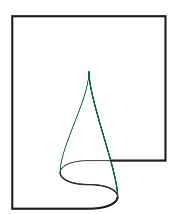
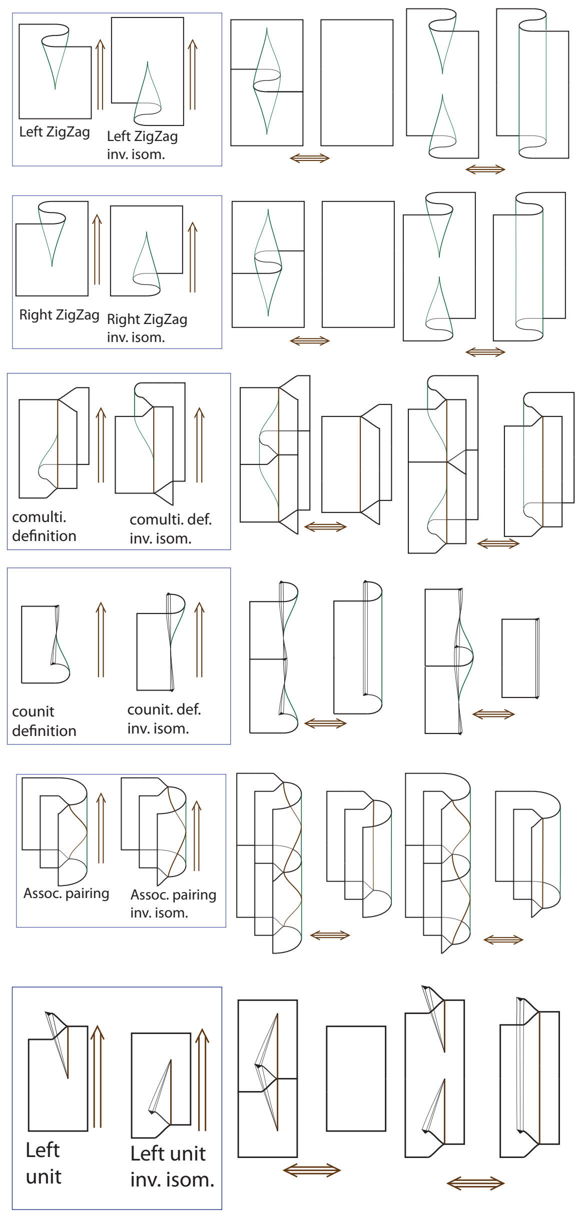
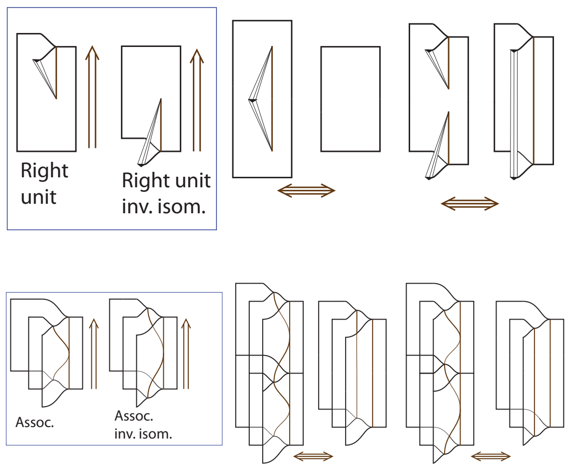
 Left cusp down		 left zig zag (top)
 Left cusp up		 left zig zag (bottom)
 Right cusp down		 Right zig zag (top)
 Right cusp up		 Right zig zag (bottom)

Table 9: Non-degeneracy of the pairing and copairing

Note: There are also triple arrows that can be seen as identities among relations. Here are some examples. These next few represent the third principle.

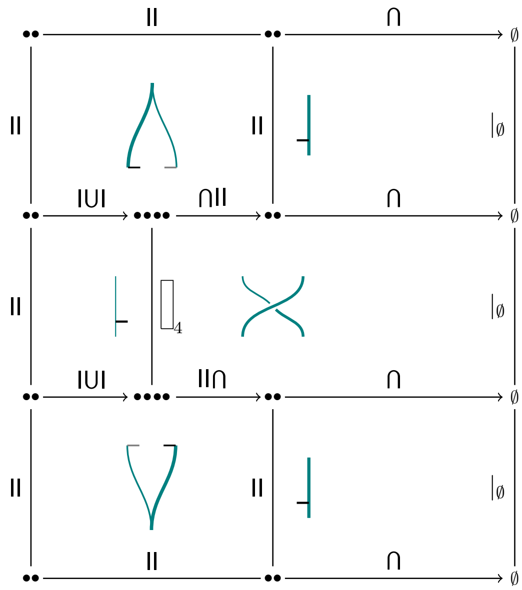




Let's consider this sequence of double arrows:

∩ ⇒ [∩∘(II∩)∘(IUI)] ⇒ [∩∘(∩II)∘(IUI)] ⇒ ∩

Rewrite as this:



Compare with the identity on:
$\bullet\bullet \xrightarrow{\cap} \emptyset$.
More on this later.

Additional double arrows. Consider the generating 1-arrows:

$\emptyset \xrightarrow{\blacktriangle} \bullet$	$\bullet \xrightarrow{\blacktriangledown} \emptyset$
$\bullet\bullet \xrightarrow{\curlywedge} \bullet$	$\bullet \xrightarrow{\curlyvee} \bullet\bullet$
$\bullet\bullet \xrightarrow{\cap} \emptyset$	$\emptyset \xrightarrow{\cup} \bullet\bullet$

Compare compositions in each row to identity arrows:

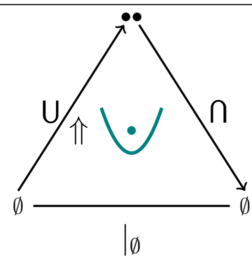
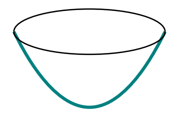
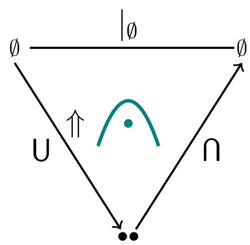
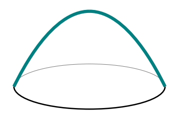
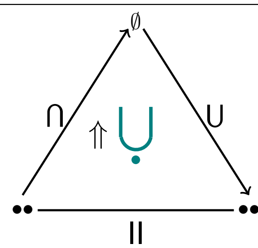
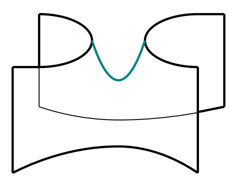
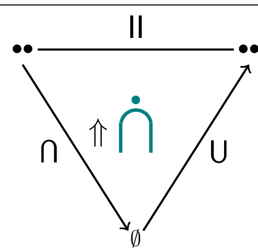
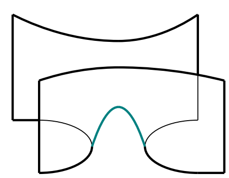
 birth glyph double arrow		 birth surface
 death glyph double arrow		 death surface
 saddle glyph double arrow		 saddle surface
 fork glyph double arrow		 fork surface

Table 10: Birth, death, saddle, and fork double arrows

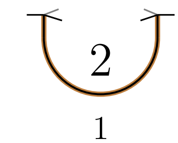
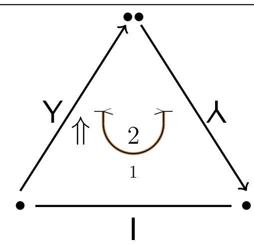
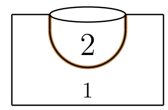
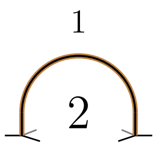
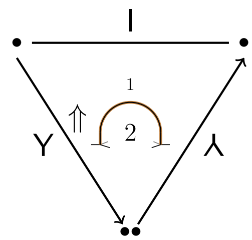
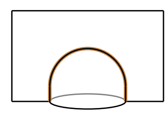
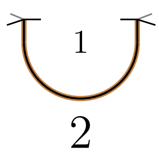
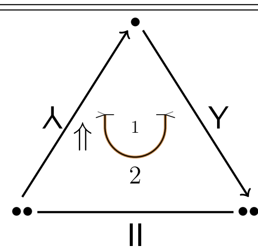
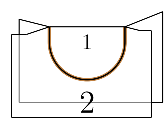
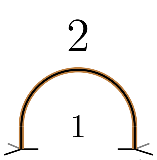
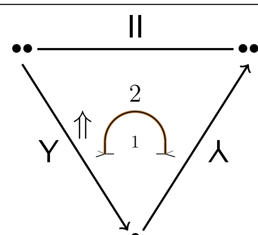
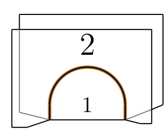
 zip close bottom, glyph		 zip close bottom, surface
 zip close top, glyph		 zip close top, surface
 zip open bottom, glyph		 zip open bottom, surface
 zip open top, glyph		 zip open top, surface

Table 11: Optimal points on seams

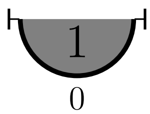
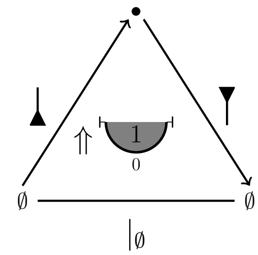
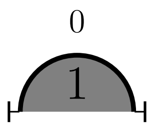
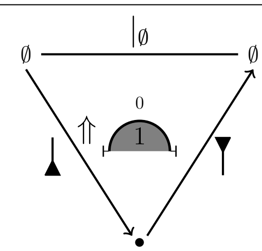
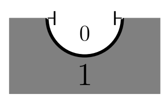
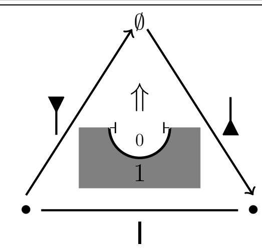
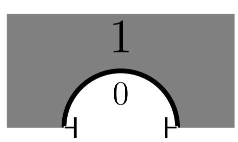
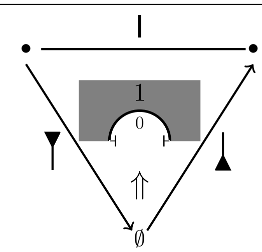
 Birth of unit/counit	
 death of unit/counit	
 canal between counit and unit	
 Arch from counit to unit	

Table 12: Birth, death, arch, and canal for unit/counit pairs

Summary Section 1: Frobenius Algebra axioms $\longrightarrow$ multiple category with obj $= \mathbb{N}$, arrows multi, comulti, unit, counit, pairing and copairing. Double arrows correspond to foams embedded in 3-space, and triple arrows correspond to isotopy of foams.

Thm. There is a multi-functor b/2 the multi-category gen by

$\emptyset \xrightarrow{\blacktriangle} \bullet$	$\bullet \xrightarrow{\blacktriangledown} \emptyset$
$\bullet\bullet \xrightarrow{\curlywedge} \bullet$	$\bullet \xrightarrow{Y} \bullet\bullet$
$\bullet\bullet \xrightarrow{\cap} \emptyset$	$\emptyset \xrightarrow{\cup} \bullet\bullet$

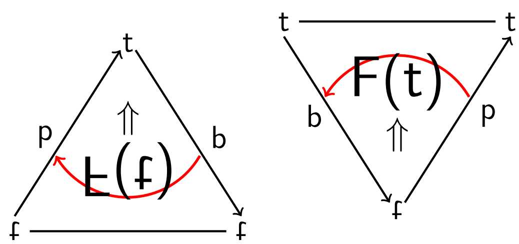
and isotopy classes of embedded foams.

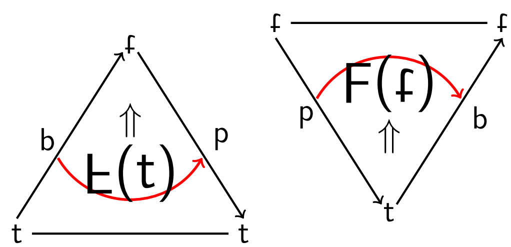
2. Promote to a multi-category with quadruple arrows

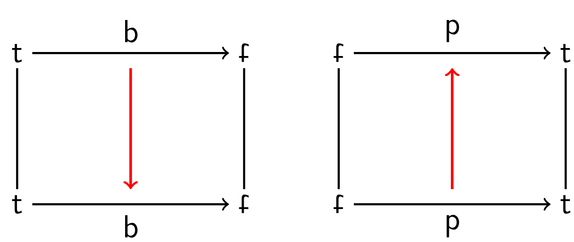
Obj: t and ɟ.
Generating 1-arrows:
ɟ —— ɟ
t —— t
p : ɟ → t
b : t → ɟ

In general, a reduced non-identity arrow is a finite sequence pbpb . . . or bpbp . . . .

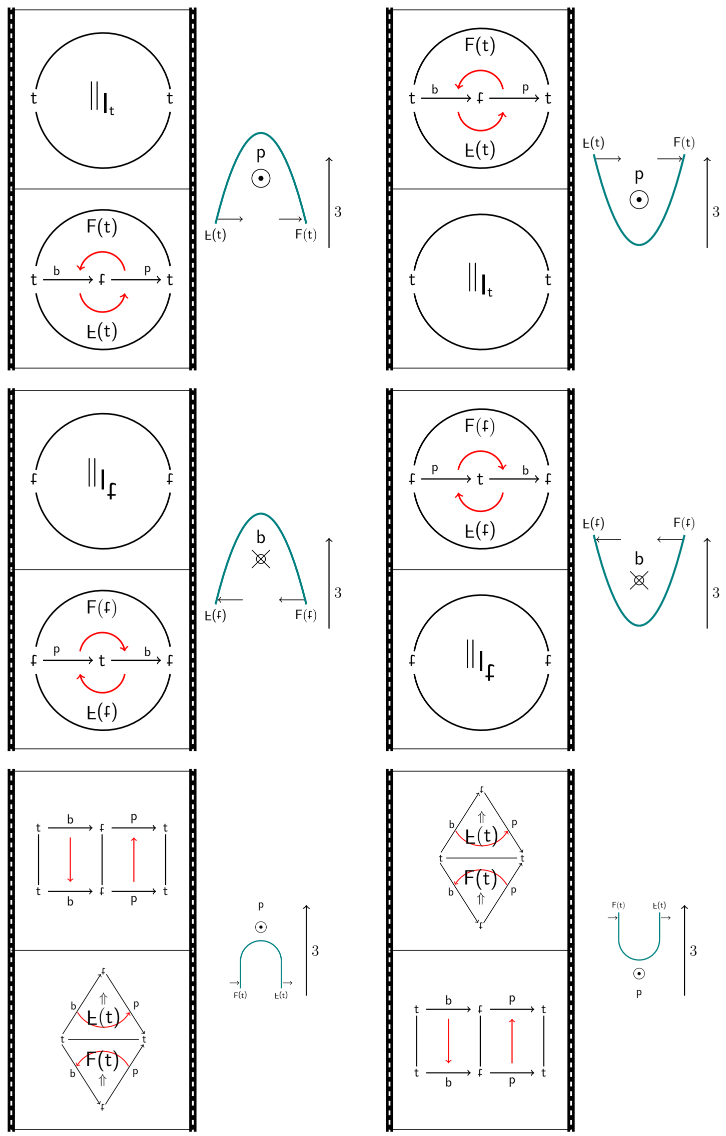
Generating double arrows:

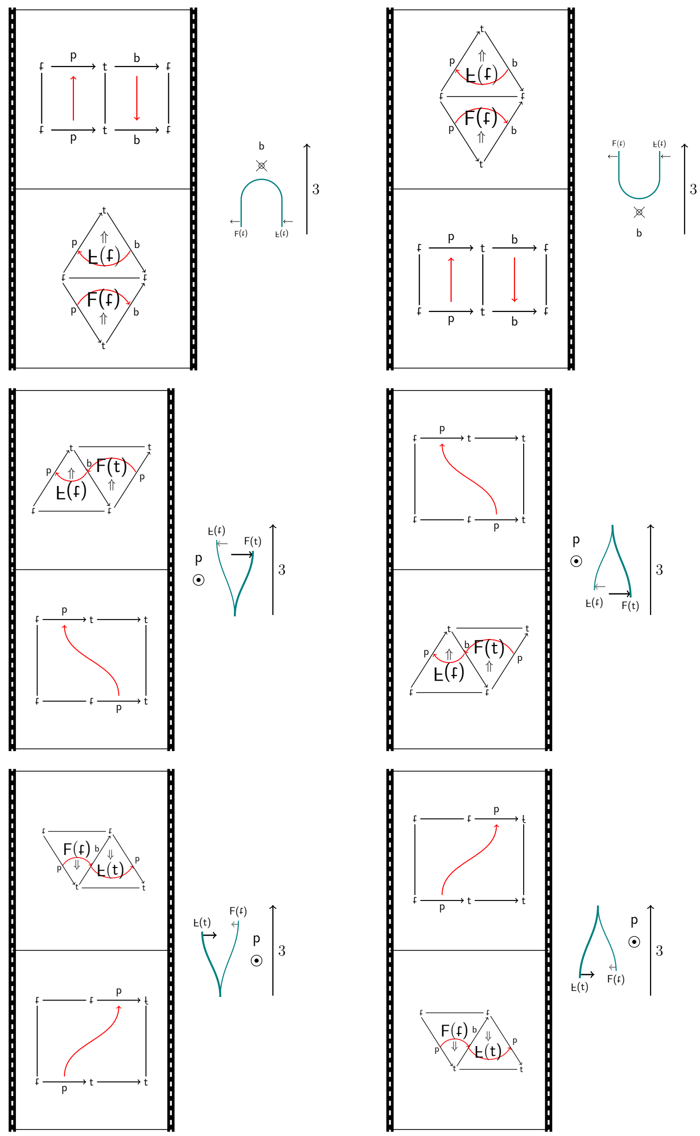


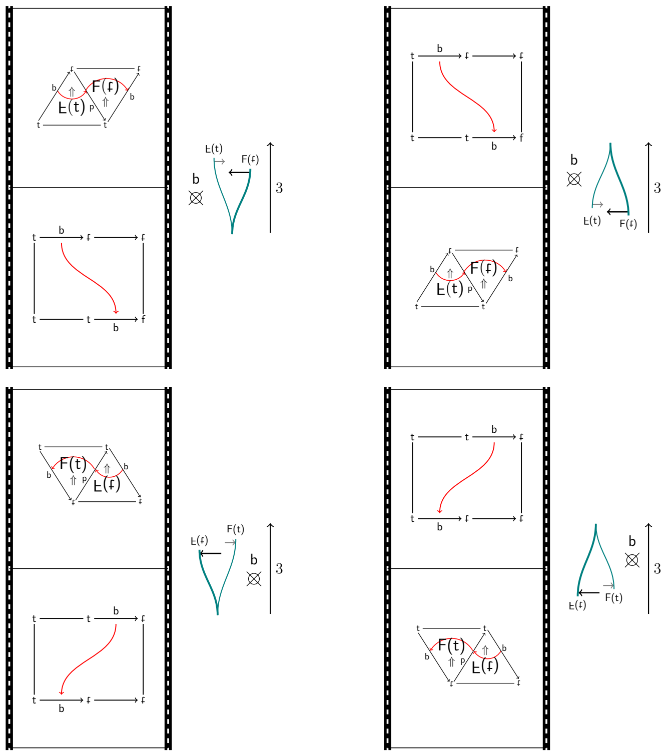




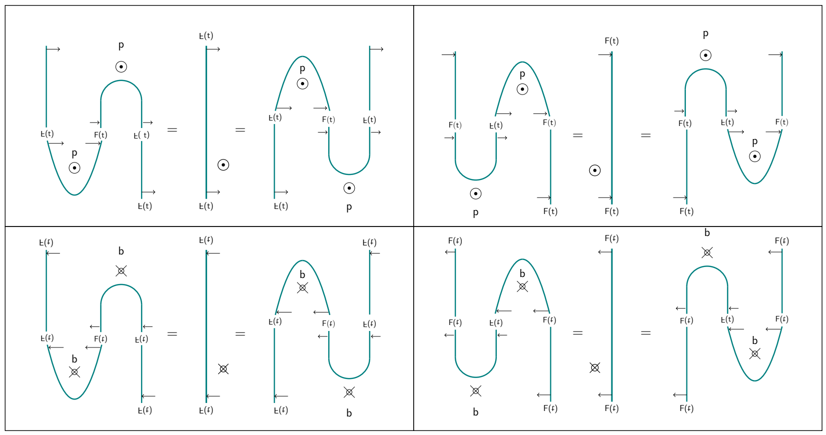
Generating triple arrows:

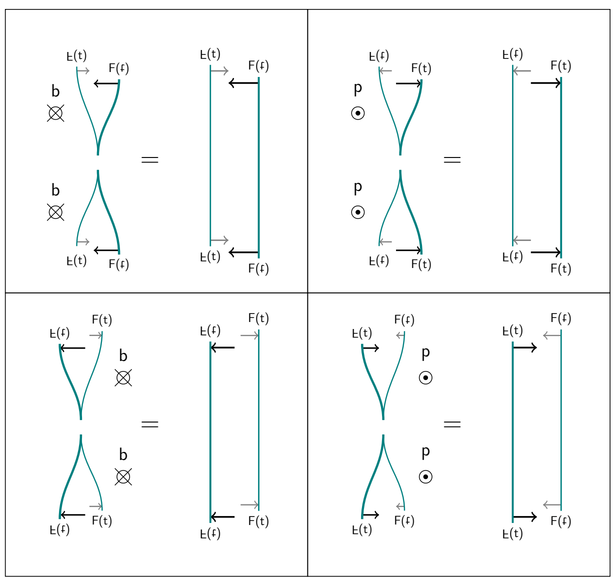


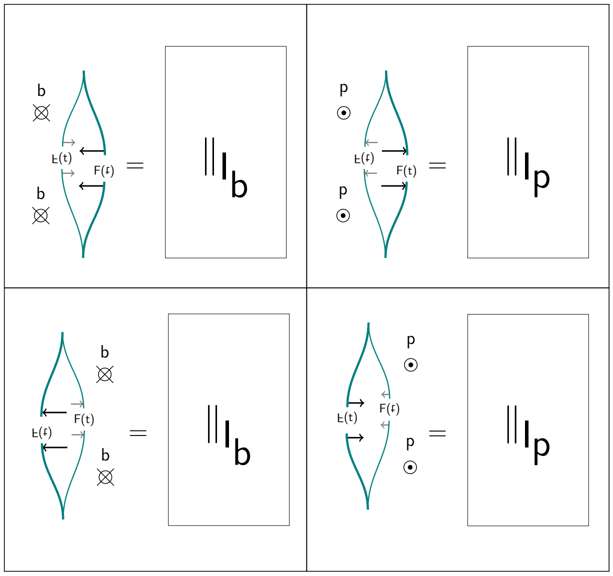


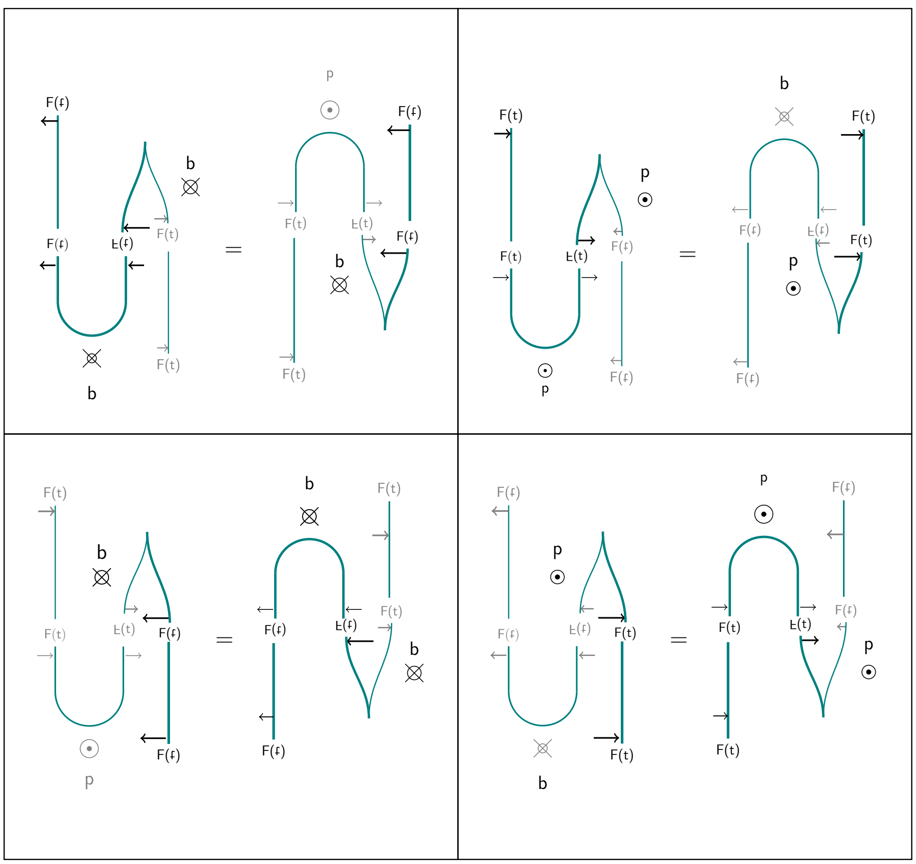


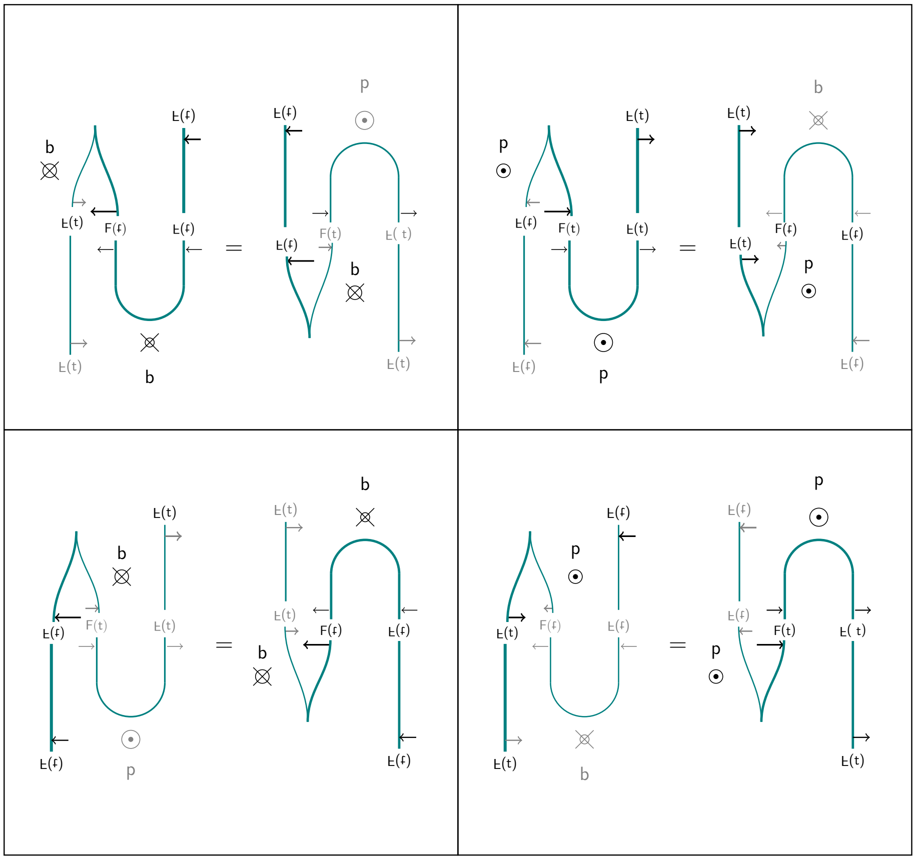
Quadruple arrows are equalities:

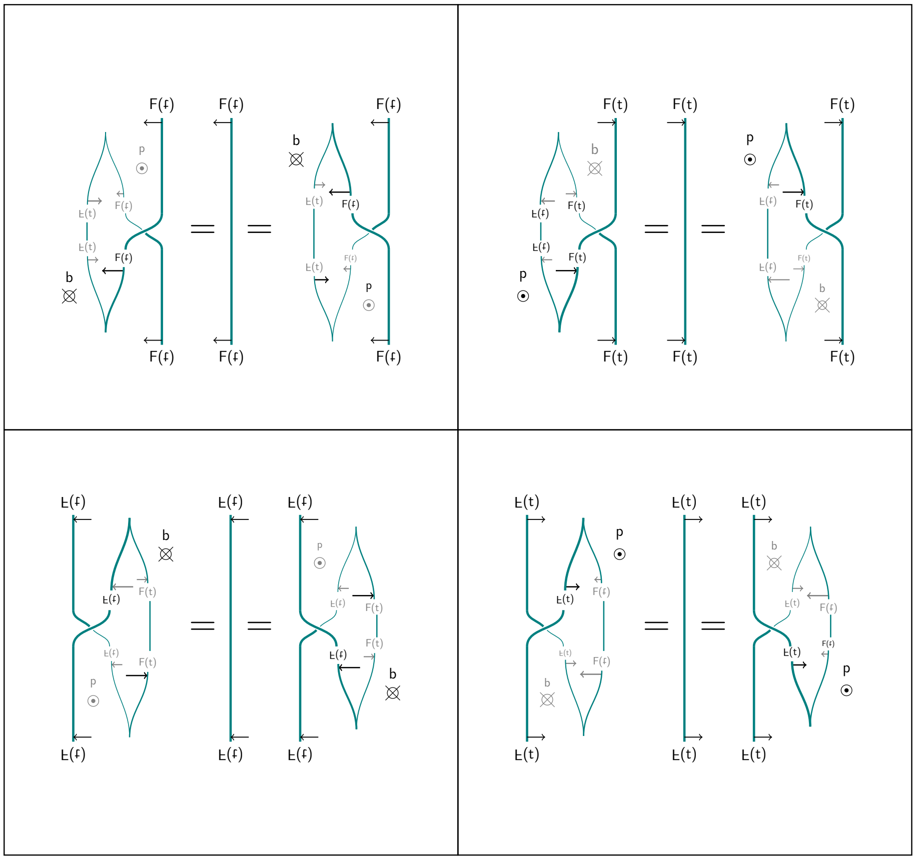












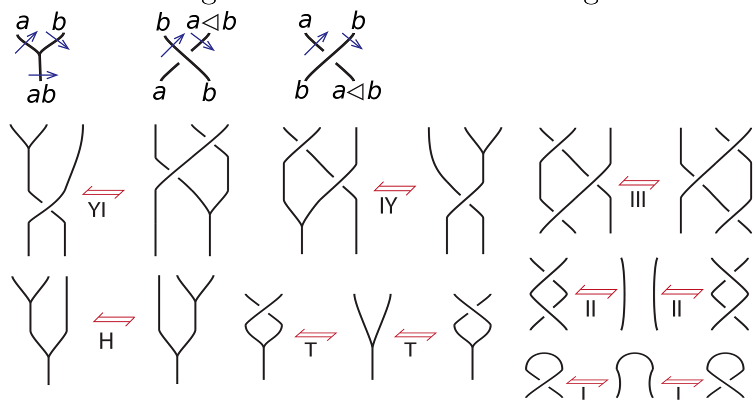
Add exchanger axioms.

Thm. There is a multi-functor from the multi-category that is described above to the multi-category of isotopy classes of embedded oriented surfaces in 3-space.

Such a surface separates space into regions. Checkerboard color these. The two objects correspond to the black and white regions. The arrows correspond to arcs that intersect the surface transversely. The double arrows are embedded disks, and the triple arrows are chunks of space.

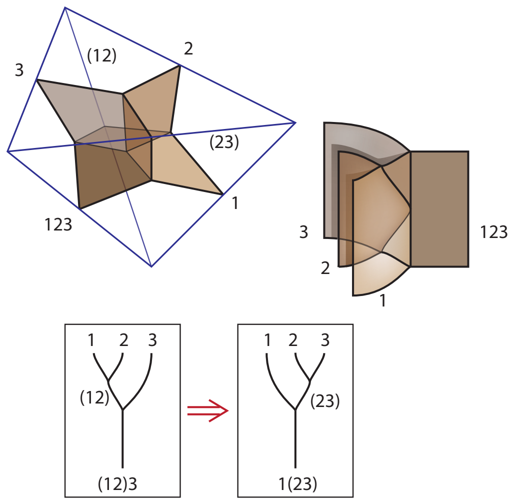
3. Adding a braiding

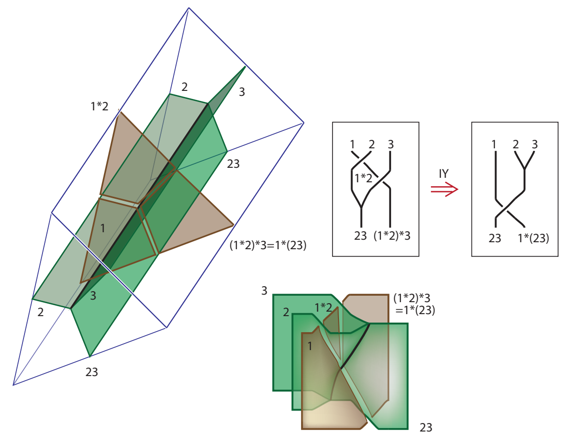
Start with one object and only the identity arrow defined upon it. Then create a generating non-identity double arrow from the identity arrow to itself. The exchanger axiom allows for braiding.

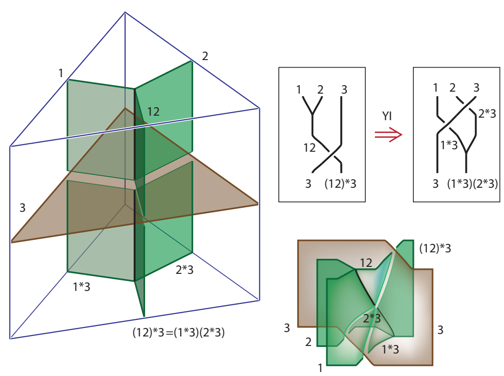


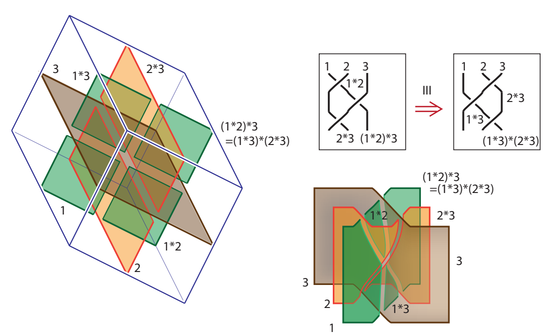
Qualgebra Axioms:

H	$(ab)c = a(bc)$	(associativity);
YI	$(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$	(distributivity);
IY	$(a \triangleleft b) \triangleleft c = a \triangleleft (bc)$	(exponential law);
III	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(self-distributivity);
II	$\forall a, b\ \exists! c$ such that $c \triangleleft b = a$	(right invertibility);
I	$a \triangleleft a = a$	(idempotence);
T	$a \cdot b = b \cdot (a \triangleleft b)$	(twisted commutativity).





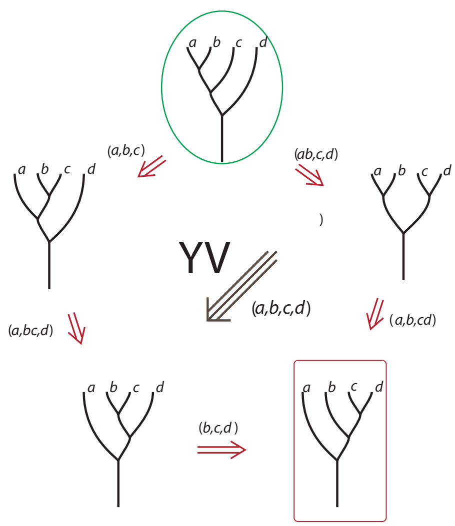


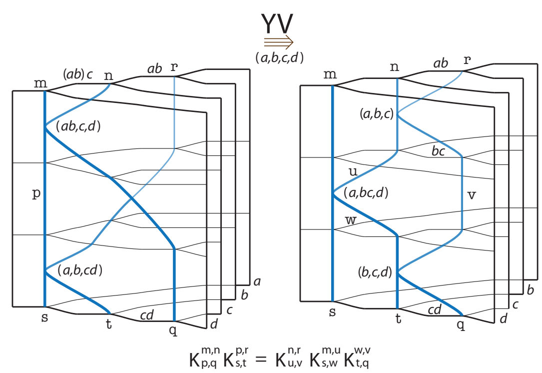


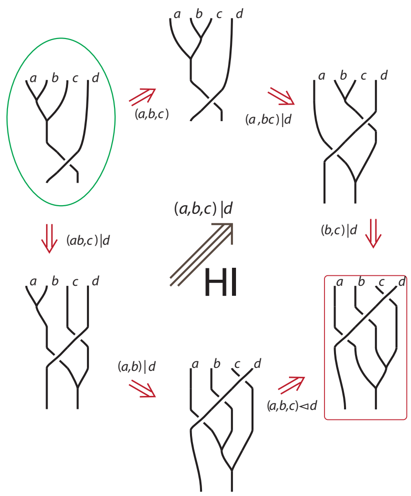
H	$\mathsf{Y}^{\ell,n}_{j}\mathsf{Y}^{j,k}_{i} = \mathsf{Y}^{n,k}_{p}\mathsf{Y}^{\ell,p}_{i}.$
YI	$\mathsf{Y}^{a,b}_{d}\mathsf{X}^{d,c}_{e,f} = \mathsf{X}^{b,c}_{d,e}\mathsf{X}^{a,d}_{e,g}\mathsf{Y}^{g,e}_{f}.$
IY	$\mathsf{X}^{a,b}_{d,g}\mathsf{X}^{g,c}_{h,f}\mathsf{Y}^{d,h}_{e} = \mathsf{Y}^{b,c}_{i}\mathsf{X}^{a,i}_{e,f}.$
III	$\mathsf{X}^{a,b}_{d,e}\mathsf{X}^{e,c}_{i,h}\mathsf{X}^{d,i}_{f,g} = \mathsf{X}^{b,c}_{j,k}\mathsf{X}^{a,j}_{f,e}\mathsf{X}^{e,k}_{g,h}$

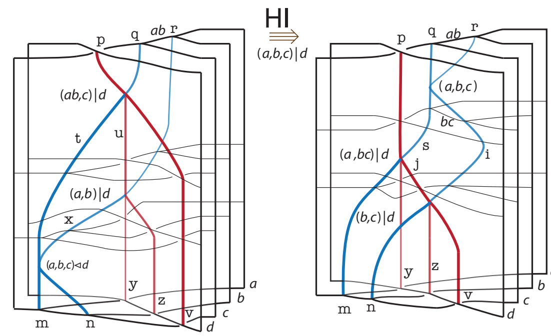
Partition	Foam move name	Associated chain	Prismatic structure
$(1,2,3,4)$	YV	(a,b,c,d)	Δ^4
$(1,2,3)\|4$	HI	$(a,b,c)\|d$	$\Delta^3\times\Delta^1$
$(1,2)\|(3,4)$	YY	$(a,b)\|(c,d)$	$\Delta^2\times\Delta^2$
$(1,2)\|3\|4$	YII	$(a,b)\|c\|d$	$\Delta^2\times\Delta^1\times\Delta^1$
$1\|(2,3,4)$	IH	$a\|(b,c,d)$	$\Delta^1\times\Delta^3$
$1\|(2,3)\|4$	IYI	$a\|(b,c)\|d$	$\Delta^1\times\Delta^2\times\Delta^1$
$1\|2\|(3,4)$	IIY	$a\|b\|(c,d)$	$\Delta^1\times\Delta^1\times\Delta^2$
$1\|2\|3\|4$	IIII	$a\|b\|c\|d$	$\Delta^1\times\Delta^1\times\Delta^1\times\Delta^1$

Table 13: The foam moves, corresponding chains, partitions, and prismatic sets

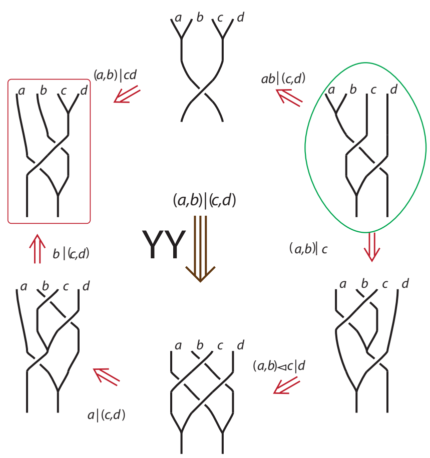


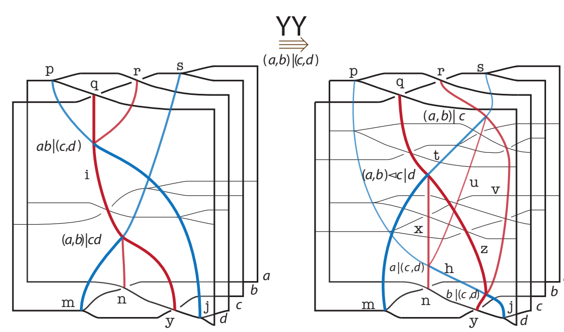




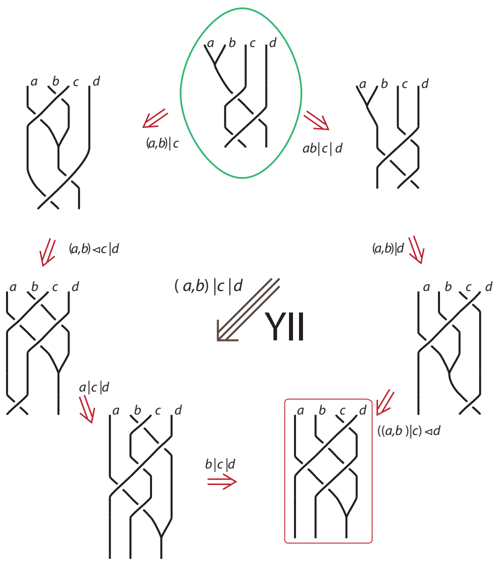


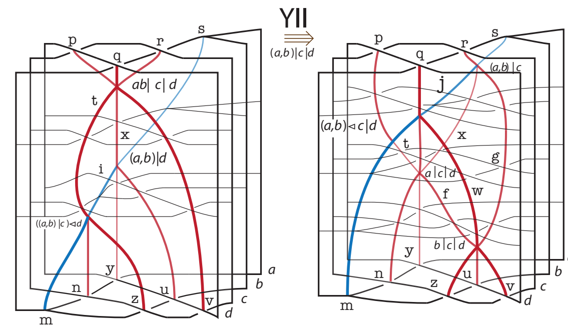
$$M^{p,q}_{t,u,v}\, M^{u,r}_{x,y,z}\, K^{t,x}_{m,n} = K^{q,r}_{s,i}\, M^{p,s}_{m,y,j}\, M^{j,i}_{n,z,v}$$



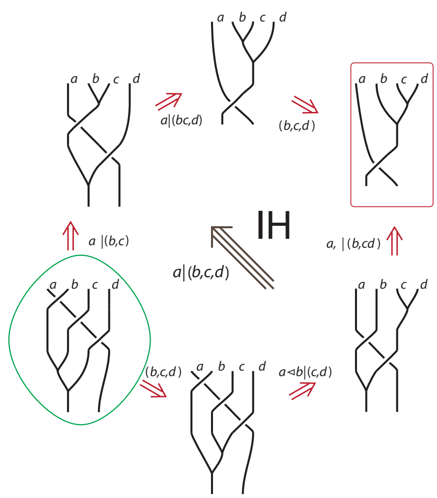


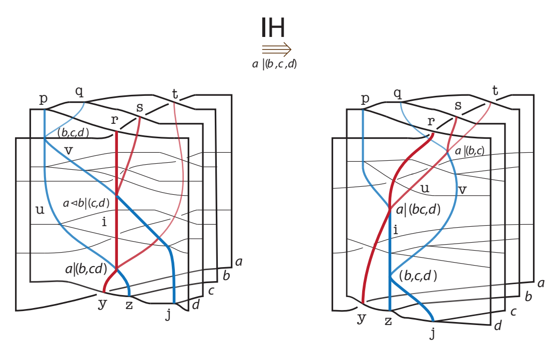
$$W^{p,q,r}_{i,j}\, M^{i,s}_{m,n,y} = M^{r,s}_{t,u,v}\, M^{q,t}_{m,x,z}\, W^{p,x,u}_{n,h}\, W^{h,z,v}_{y,j}$$



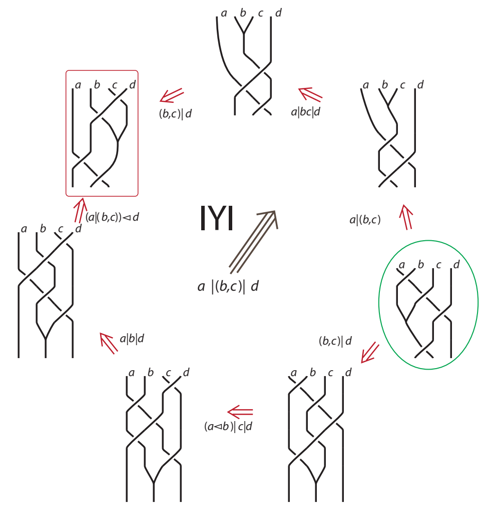


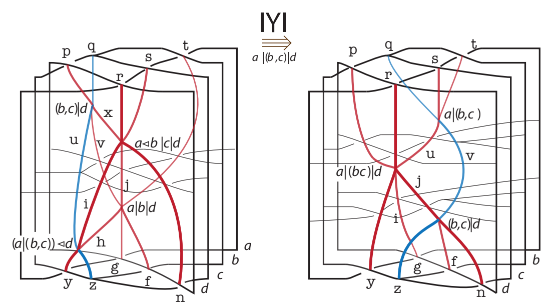
$$X^{p,q,r}_{t,x,v}\ M^{x,s}_{i,y,u}\ M^{t,i}_{m,n,z} = M^{r,s}_{j,x,g}\ M^{q,j}_{m,t,w} X^{p,t,x}_{n,y,f} X^{f,w,g}_{z,u,v}$$



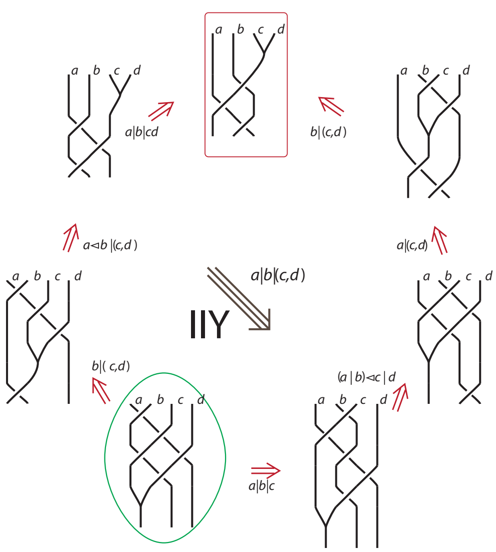


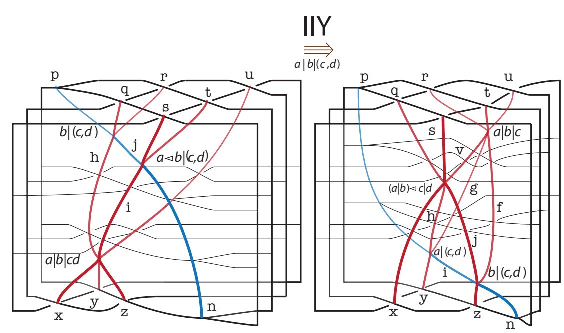
$$K^{p,q}_{u,v} W^{v,r,s}_{i,j}\ W^{u,i,t}_{y,z} = W^{q,s,t}_{u,v} W^{p,r,u}_{y,i} K^{i,v}_{z,j}$$



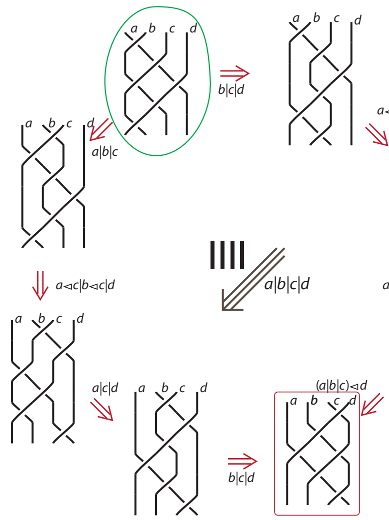


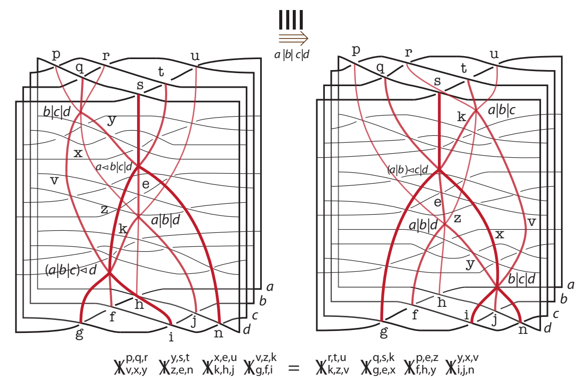
$$M^{p,q}_{u,v,x}\, X^{x,r,s}_{i,j,n}\, X^{v,j,t}_{h,g,f}\, W^{u,i,h}_{y,z} = W^{q,s,t}_{u,v}\, X^{p,r,u}_{y,i,j}\, M^{j,v}_{z,f,n}$$





$$W^{p,q,r}_{h,j}\, W^{j,s,t}_{i,n}\, X^{h,i,u}_{x,y,z} = X^{r,t,u}_{v,g,f}\, X^{q,s,v}_{x,h,j}\, W^{p,h,g}_{y,i}\, W^{i,j,f}_{z,n}$$





YV	$\mathsf{K}^{m,n}_{p,q}\mathsf{K}^{p,r}_{s,t} = \mathsf{K}^{n,r}_{u,v}\mathsf{K}^{m,u}_{s,w}\mathsf{K}^{w,v}_{t,q}$
HI	$\mathsf{M}^{p,q}_{t,u,v}\mathsf{M}^{u,r}_{x,y,z}\mathsf{K}^{t,x}_{m,n} = \mathsf{K}^{q,r}_{s,i}\mathsf{M}^{p,s}_{m,y,j}\mathsf{M}^{j,i}_{n,z,v}$
YY	$\mathsf{W}^{p,q,r}_{i,j}\mathsf{M}^{i,s}_{m,n,y} = \mathsf{M}^{r,s}_{t,u,v}\mathsf{M}^{q,t}_{m,x,z}\mathsf{W}^{p,x,u}_{n,h}\mathsf{W}^{h,z,v}_{y,i}$
YII	$\text{Ж}^{p,q,r}_{t,x,v}\mathsf{M}^{x,s}_{i,y,u}\mathsf{M}^{t,i}_{m,n,z} = \mathsf{M}^{r,s}_{j,x,g}\mathsf{M}^{q,j}_{m,t,x}\text{Ж}^{p,t,x}_{n,y,f}\text{Ж}^{f,w,g}_{z,u,v}$
IH	$\mathsf{K}^{p,q}_{u,v}\mathsf{W}^{v,r,s}_{i,j}\mathsf{W}^{u,i,t}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\mathsf{W}^{p,r,u}_{y,i}\mathsf{K}^{i,v}_{z,j}$
IYI	$\mathsf{M}^{p,q}_{u,v,x}\text{Ж}^{x,r,s}_{i,j,n}\mathsf{X}^{v,j,t}_{h,g,f}\mathsf{W}^{u,i,h}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\text{Ж}^{p,r,u}_{y,i,j}\mathsf{M}^{j,v}_{z,f,n}$
IIY	$\mathsf{W}^{p,q,r}_{h,j}\mathsf{W}^{j,s,t}_{i,n}\text{Ж}^{h,i,u}_{x,y,z} = \text{Ж}^{r,t,u}_{v,g,f}\text{Ж}^{q,s,v}_{x,h,j}\mathsf{W}^{p,h,g}_{y,i}\mathsf{W}^{i,j,f}_{z,n}$
IIII	$\text{Ж}^{p,q,r}_{v,x,y}\text{Ж}^{y,s,t}_{z,e,n}\text{Ж}^{x,e,u}_{k,h,j}\text{Ж}^{v,z,k}_{g,f,i} = \text{Ж}^{r,t,u}_{k,z,v}\text{Ж}^{q,s,k}_{g,e,x}\text{Ж}^{p,e,z}_{f,h,y}\text{Ж}^{y,x,v}_{i,j,n}$