# *Semi static hedge of path dependent options*

Yuji Hishida & Kenji Yasutomi

yuji.hishida@mizuho-sc.com yasutomi@se.ritsumei.ac.jp

Fixed Income Group , Mizuho Securities Co., Ltd.

Dept. of Mathematical Sciences, Ritsumeikan University

## Contents

1. Method
2. Example
3. Principle
4. Complex details

# 1. Method

# Method

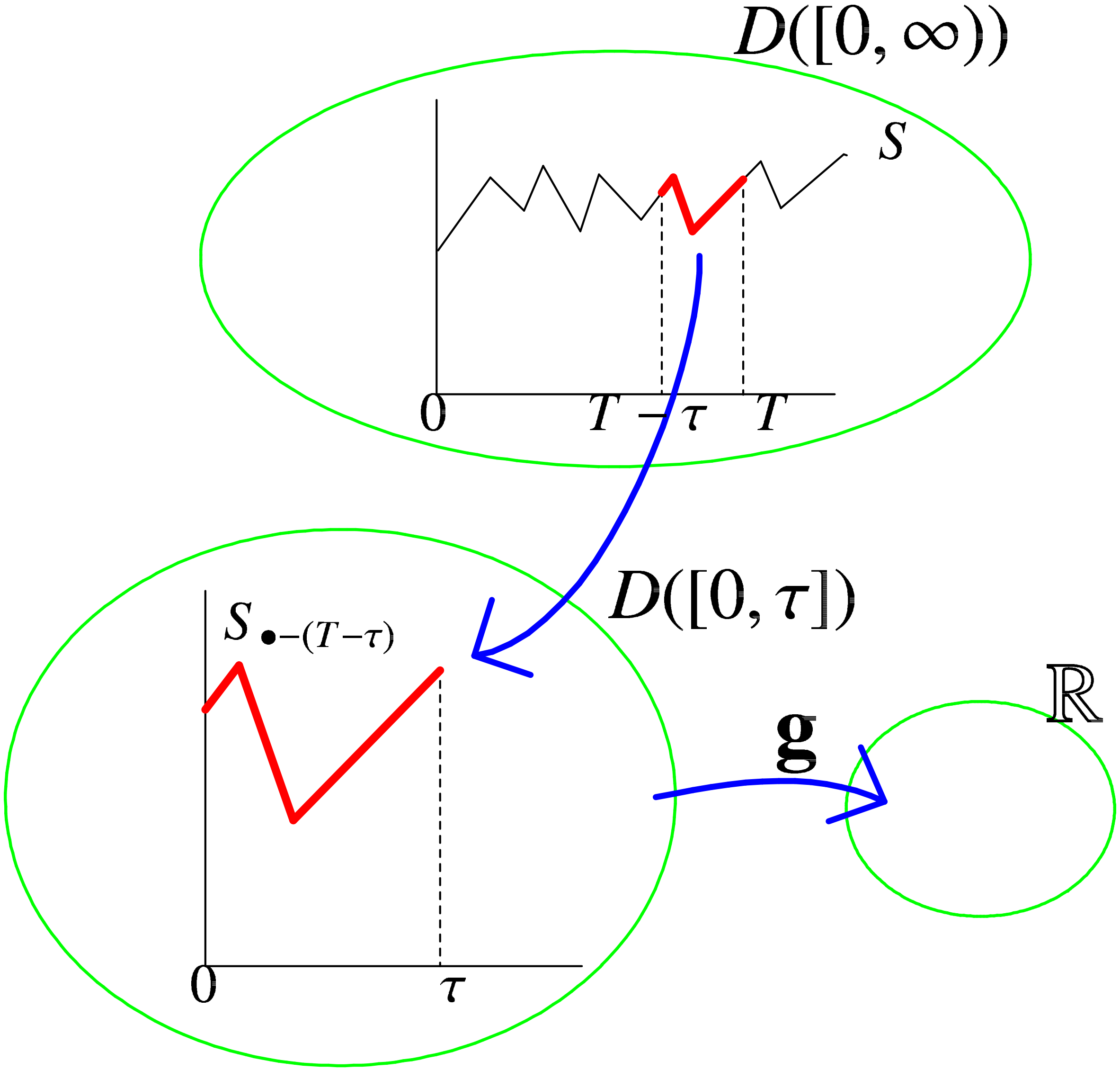
- $S = \{S_t\}$ : Markov process
- about a path dependent option
  $[T - \tau, T]$ : monitoring period
  where $T$: maturity time and $\tau > 0$ constant.

We represent the path dependent option $A_T$ as

$$\mathbf{g} : D([0,\tau]) \to \mathbb{R}$$
$$A_T := E\left[\mathbf{g}(S_{\bullet-(T-\tau)})\right]$$

where $D([0,\tau]) := \{\omega : [0,\tau] \to \mathbb{R} \mid \omega \text{ is RCLL.}\}$.

# Method



# Method

For a given $A_T$,

- $\widetilde{A_T} := E\left[\widetilde{\mathbf{g}}(S_{\bullet-(T-\tau)})\right]$ : **easily calculated** ( or **quoted**) (ex. Plain Vanillas)

Then our study is

$$A_T - \widetilde{A_T} = E\left[\mathbf{h}(S_{\bullet-(T-\tau)})\right]$$

where $\mathbf{h} := \mathbf{g} - \widetilde{\mathbf{g}} : D([0,\tau]) \to \mathbb{R}$.

# Method

From markov property,

$$\begin{aligned} A_T - \widetilde{A_T} &= E\left[\mathbf{h}(S_{\bullet-(T-\tau)})\right] \\ &= E\left[h_S(S_{T-\tau})\right] \\ &= \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x) \end{aligned}$$

where $h_S : \mathbb{R} \to \mathbb{R}$ defined by

$$h_S(x) := E\left[\mathbf{h}(S_{\bullet-(T-\tau)}) | S_{T-\tau} = x\right]$$

# Method

$$A_T - \widetilde{A_T} = \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x)$$

# Method

$$A_T - \widetilde{A_T} \;=\; \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x)$$

- $A_T - \widetilde{A_T} \approx 0$ ?? How ??

# Method

$$A_T - \widetilde{A_T} = \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x)$$

- $A_T - \widetilde{A_T} \approx 0$ ?? How ??
- Only $P_{S_{T-\tau}}$ depends on $T$.

# Method

$$A_T - \widetilde{A_T} = \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x)$$

- $A_T - \widetilde{A_T} \approx 0$ ?? How ??
- Only $P_{S_{T-\tau}}$ depends on $T$.
- $P_{S_{T-\tau}}(M) \to 0 \quad (T \to \infty)$ for any compact set $M \in \mathbb{R}$.

# Method

$$A_T - \widetilde{A_T} = \int_0^\infty h_S(x)\, dP_{S_{T-\tau}}(x)$$

- $A_T - \widetilde{A_T} \approx 0$ ?? How ??
- Only $P_{S_{T-\tau}}$ depends on $T$.
- $P_{S_{T-\tau}}(M) \to 0 \quad (T \to \infty)$ for any compact set $M \in \mathbb{R}$.

Let us see the asymptotic behavior about $T$, i.e.,

$$\exists \alpha(T) \to 0 \text{ such that } \mu_T := \frac{P_{T-\tau}}{\alpha(T)} \to \mu \quad (T \to \infty)\,??$$

# Method

If $\exists \alpha(T) \to 0$ such that $\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$,

$$\begin{aligned} \frac{A_T - \widetilde{A_T}}{\alpha(T)} &= \int h_S(x) d\mu_T(x) \\ &\to \int h_S(x) d\mu =: C \ (T \to \infty) \end{aligned}$$

# Method

If $\exists \alpha(T) \to 0$ such that $\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$,

$$\frac{A_T - \widetilde{A_T}}{\alpha(T)} = \int h_S(x) d\mu_T(x)$$
$$\to \int h_S(x) d\mu =: C \ (T \to \infty)$$

**Method**

For large $T$,

$$A_T \approx \widetilde{A_T} + C\alpha(T)$$

(Path dependent option) $\approx$ (Plain vanillas) + (Remainder Term).

# 2. Example

2.1 model example

2.2 payoff example

# 2. Example

2.1 model example

2.2 payoff example

# Example of $\alpha(T)$

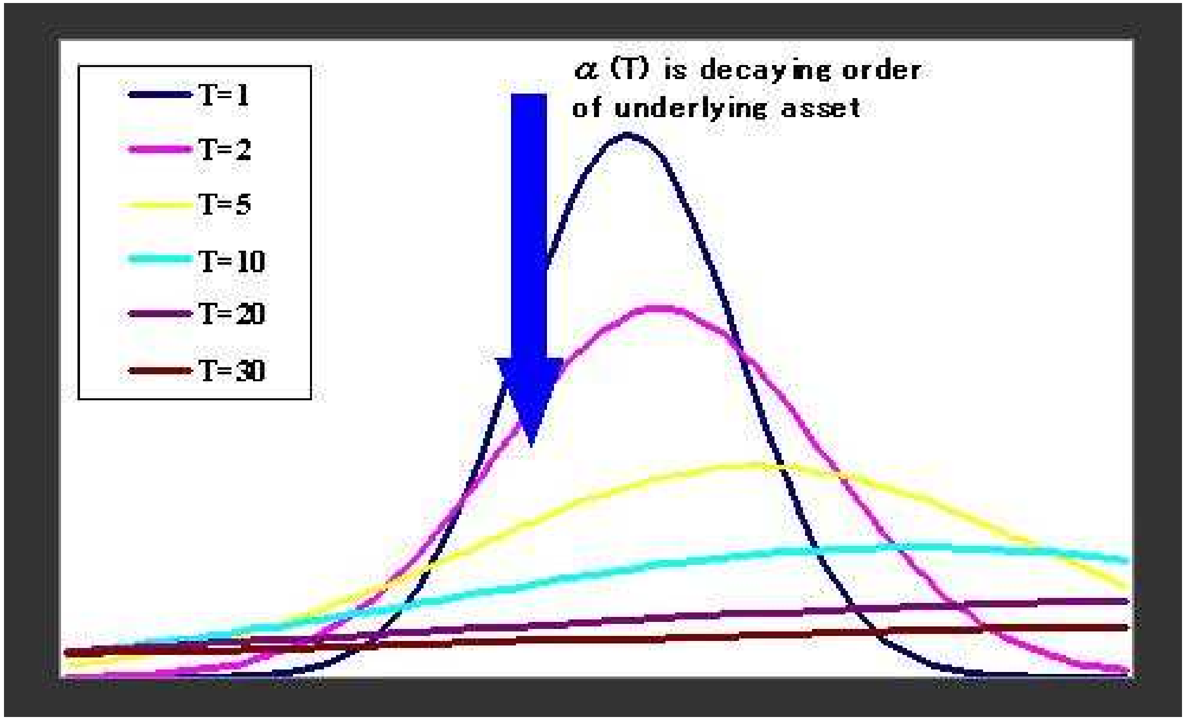
$$\alpha(T) \sim P(S_T \in M) \quad (T \to \infty)$$

for any compact set $M \in \mathbb{R}$.

# Example of $\alpha(T)$

$$\alpha(T) \sim P(S_T \in M) \quad (T \to \infty)$$

for any compact set $M \in \mathbb{R}$.



## Example of $\alpha(T)$

$\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$

Our method is $A_T \approx \widetilde{A_T} + C\alpha(T)$ . What is $\alpha(T)$ exactly given?

**Brownian Motion:**

$S_t = \exp(Z_t)$, $Z_t = aW_t + bt$ : Then,

$$\alpha(t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{b^2 t}{2a^2}\right)$$

$$\text{and} \quad \frac{d\mu}{dx}(x) = \frac{1}{\sqrt{2\pi a^2}} x^{\frac{b}{a^2}-1}$$

# Example of $\alpha(T)$

$\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$

Our principle is $A_T \approx \widetilde{A_T} + C\alpha(T)$. What is $\alpha(T)$ exactly given?

**Normal inverse Gaussian:**

$S_t = \exp(Z_t), \qquad Z_t = W_{\mathrm{IG}(t)} + \theta \mathrm{IG}(t) + bt$ , where

$\mathrm{IG}(t) = \inf\{s \mid B_s + \nu s > \delta t\}$. Then,

$$\alpha(t) = \frac{1}{\sqrt{t}} \exp\left(t\left(\nu\delta - \theta b - \sqrt{(b^2+\delta^2)(\theta^2+\nu^2)}\right)\right)$$

and $$\frac{d\mu}{dx}(x) = \frac{\delta}{\sqrt{2\pi(b^2+\delta^2)}} \left(\frac{\theta^2+\nu^2}{b^2+\delta^2}\right)^{\frac{1}{4}} x^{\left(\theta + b\sqrt{\frac{\theta^2+\nu^2}{b^2+\delta^2}}\right)-1}$$

# Example of $\alpha(T)$

$\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$

Our principle is $A_T \approx \widetilde{A_T} + C\alpha(T)$. What is $\alpha(T)$ exactly given?

**Variance Gamma model** :

$S_t = \exp(Z_t), \quad Z_t = \sigma W_{\gamma(t)} + \theta\gamma(t) + mt$ , where $\gamma(t)$ is gamma process with variance rate $\lambda$. Then,

$$\alpha(t) = \frac{1}{\sqrt{t}} \left( \frac{1+\eta}{2+\nu\theta^2/\sigma^2} \right)^{\frac{t}{\nu}} e^{(\frac{1-\eta}{\lambda} - \frac{\theta m}{\sigma^2})t}$$

and $$\frac{d\mu}{dx}(x) = \frac{1}{2}\sqrt{\frac{2+\lambda\theta^2/\sigma^2}{2\pi\eta(1+\eta)\sigma^2}}\, x^{\left((\frac{\theta}{\sigma^2} + \frac{\eta-1}{m\lambda})\right)-1}$$

where $\eta := \sqrt{1 + \frac{m^2\lambda^2}{\sigma^2}(\frac{2}{\lambda} + \frac{\theta^2}{\sigma^2})}$.

# 2. Example

2.1 model example

2.2 payoff example

# Payoff Example

| Path-dependent $A_T$ | Plain Vanillas $\widetilde{A_T}$ | Lemma |
|---|---|---|
| $\left(\frac{1}{N}\sum_{i=1}^{N} S_{T-\tau_i} - K\right)^+$ | $\frac{1}{N}\sum_{i=1}^{N}(S_{T-\tau_i} - K)^+$ | 1 |
| $(S_T - K)^+ \mathbf{1}_{\{\min_{i=1,2,\cdots N} S_{T-\tau_i} \geq L\}}$ | $(S_T - K)^+$ | 1 |
| $(\max_{i=1.2,\cdots N} S_{T-\tau_i} - K)^+$ | $M(S_T - K/M)^+$ | 2 |
| $\left(\frac{1}{N}\sum_{i=1}^{N} S_{T-\tau_i} - K\right)^+$ | $M(S_T - K/M)^+$ | 2 |

- $M$ : constant which depends only payoff type.
- For a given $A_T$, $\widetilde{A_T}$ is not unique.
- Both discrete and continuous are OK.

# 3. Principle

$$\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \rightarrow \mu \text{ ??}$$

# Principle

"Convergence in Law"

$$\forall f \in B \int f \, d\mu_T \to \int f \, d\mu$$

where $B := \{f : \mathbb{R} \to \mathbb{R} | f$ is bounded countinuous$\}$

# *Principle*

Recall in case of **BM**, $d\mu = \frac{1}{\sqrt{2\pi a^2}} x^{\frac{b}{a^2}-1} dx$

For example,

$$\int \sin(x)\ d\mu(x) \text{ is } \textbf{Not defined !!}$$

# Principle

$$\forall f \in B \int f\, d\mu_T \to \int f\, d\mu$$

where $B := \{f : \mathbb{R} \to \mathbb{R} | f$ is bounded countinuous$\}$

# Principle

$$\forall f \in E \int f \, d\mu_T \to \int f \, d\mu$$

where $E$: a set of function

# Principle

$$\forall f \in E \int f \, d\mu_T \to \int f \, d\mu$$

where $E$: a set of function

We denote it by

$$\mu_T \to_E \mu$$

# Principle

$\mu_T := \frac{P_{S_{T-\tau}}}{\alpha(T)} \to \mu$ ??

**Principle**

Find the set $E$ such that

1. $\mu_T \to_E \mu$
2. $E[\mathbf{h}(S_{\bullet-(T-\tau)})|S_{T-\tau} = x] =: h_S(x) \in E$

Then,

$$\frac{A_T - \widetilde{A_T}}{\alpha(T)} = \int h_S(x) d\mu_T(x) \to \int h_S(x) d\mu =: C \ (T \to \infty)$$

# 4. Complex details

# Complex details

$$E_S := \left\{ h \middle| \int h(x) P(\sup_{0\le t\le\tau} S_t > \frac{1}{x}) P(\sup_{0\le t\le\tau} S_t < \frac{1}{x})\ dx < \infty \right\}$$

For $E_s$,

1. Can we find $h_S \in E_S$ ?? $\Longrightarrow$ **Lemma1, Lemma2**.
2. $\mu_T \to_{E_S} \mu$ ?? $\Longrightarrow$ **Lemma3**.

# Complex details

**Lemma 1.** *If there exist $M_U$, $M_L$ and $M$ such that*

$$\inf_{0\le s\le\tau} \omega(s) > M_L \Rightarrow \mathbf{h}(\omega) = 0,$$
$$\sup_{0\le s\le\tau} \omega(s) < M_U \Rightarrow \mathbf{h}(\omega) = 0,$$
$$|\mathbf{h}(\omega)| \le \sup_{0\le s\le\tau} \omega(s) + M.$$

*Then,* $E\left[\mathbf{h}(S_{\bullet-(T-\tau)})|S_{T-\tau} = x\right] =: h_S \in E_S$

**(ex., Asian Option or Barrier Option)**

# Complex details

**Lemma 2.** $\mathbf{g} : D([0,\tau]) \to \mathbb{R}$ *satisfys*

$$\inf_{0\le s\le\tau} \omega(s) \le \mathbf{g}(\omega) \le \sup_{0\le s\le\tau} \omega(s),$$
$$\mathbf{g}(a\omega) = a\mathbf{g}(\omega),$$

*for any* $\omega \in D([0,\tau])$ *and* $a \in \mathbb{R}$, *and let* $\mathbf{h}$ *represent*

$$\mathbf{h}(\omega) = (\mathbf{g}(\omega) - K)^+ - M(\omega(0) - K/M)^+,$$

*where* $M := E[\mathbf{g}(\omega)]$. *Then,* $E\left[\mathbf{h}(S_{\bullet-(T-\tau)})|S_{T-\tau} = x\right] =: h_S \in E_S$

**(ex., Asian Option or Look back Option)**

# Complex details

$$E_S := \left\{ h \middle| \int h(x) P(\sup_{0\le t\le\tau} S_t > \frac{1}{x}) P(\sup_{0\le t\le\tau} S_t < \frac{1}{x}) \; dx < \infty \right\}$$

For $E_s$,

1. Can we find $h_S \in E_S$ ?? $\Longrightarrow$ **Lemma1, Lemma2**.
2. $\mu_T \rightarrow_{E_S} \mu$ ?? $\Longrightarrow$ **Lemma3**.

# Complex details

**Lemma 3.** *$S_t = e^{Z_t}$ : $Z_t$: "good" Levy process then,*

$$\mu_T \to_{E_S} \mu$$

Remark that "good" means the good condition about the tail order!!

**(ex., BM, NIG, VG)**

# Summary and Conclusion

## *Summary and Conclusion*

- The basic idea for our method is to approximate the path dependent option by combinations of plain vanillas.

# Summary and Conclusion

- The basic idea for our method is to approximate the path dependent option by combinations of plain vanillas.
- For the large maturity $T$, we obtain the asymptotic behavior $A_T \approx \widetilde{A_T} + C\alpha(T)$,

# Summary and Conclusion

- The basic idea for our method is to approximate the path dependent option by combinations of plain vanillas.
- For the large maturity $T$, we obtain the asymptotic behavior $A_T \approx \widetilde{A_T} + C\alpha(T)$,
- Since $\alpha(T)$ is the decaying order of $S_T$, $\alpha(T) \to 0$, furthormore, it can be expressed by the elementary function in some cases.

## Summary and Conclusion

- The basic idea for our method is to approximate the path dependent option by combinations of plain vanillas.
- For the large maturity $T$, we obtain the asymptotic behavior $A_T \approx \widetilde{A_T} + C\alpha(T)$,
- Since $\alpha(T)$ is the decaying order of $S_T$, $\alpha(T) \to 0$, furthormore, it can be expressed by the elementary function in some cases.
- Therefore, we can price several path dependent option (Asian option, barrier option, look back option, etc) by elementary function, and give the semi static hedge strategy.

## Summary and Conclusion

- By giving "Principle", Whether our method can be applied or not reduces the argument about the convergence of some measures.

# Summary and Conclusion

- By giving "Principle", Whether our method can be applied or not reduces the argument about the convergence of some measures.
- By using "Principle", it might be possible to find the more models or many payoff.

## Summary and Conclusion

- By giving "Principle", Whether our method can be applied or not reduces the argument about the convergence of some measures.
- By using "Principle", it might be possible to find the more models or many payoff.
- Some stochastic volatility models (e.x Heston model) are now left for future study.

Thank you for your attention!!