An Asymptotic Expansion with Malliavin Weights: An Application to Pricing Discrete Barrier Options *

Akihiko Takahashi †and Toshihiro Yamada ‡

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Abstract

This paper proposes a new approximation method for pricing barrier options with discrete monitoring under stochastic volatility environment. In particular, the integration-by-parts formula in Malliavin calculus is effectively applied in an asymptotic expansion approach. First, the paper derives an expansion formula for generalized Wiener functionals. After it is applied to pricing path-dependent derivatives with discrete monitoring, the paper presents an analytic (approximation) formula for valuation of discrete barrier options under stochastic volatility environment. To our knowledge, this paper is the first one that shows an analytical formula for pricing discrete barrier options with stochastic volatility models.

Keywords: discrete barrier option, barrier option, knock-out option, Malliavin calculus, stochastic volatility, asymptotic expansion, Malliavin weight

1 Introduction

This paper develops an asymptotic expansion method for generalized Wiener functionals by applying the integration-by-parts formula providing divergence, that is so called *Malliavin weight* and *push down*(Malliavin(1997) and Malliavin-Thalmaier(2006)), that is the conditional expectation in Malliavin calculus. It also shows the equivalence between an asymptotic expansion developed by Watanabe(1987) and our expansion. Moreover, the paper applies the method to pricing path-dependent derivatives with discrete monitoring under stochastic volatility environment.

Further, it derives a concrete approximation formula for valuation of barrier options with discrete monitoring under stochastic volatility models. Fusai, Abrahams and Sgarra (2006) provided an analytical solution for pricing discrete barrier options in the Black-Scholes framework. Shiraya, Takahashi and Toda (2009) provided an analytic approximation formula for valuation of barrier options with continuous monitoring under stochastic volatility environment; however, their method cannot be applied to pricing discrete barrier options. Thus, to our knowledge, this paper is the first one that derives an analytic (approximation) formula for pricing discrete barrier options with stochastic volatility models. Also, our companion paper (Takahashi and Yamada[2009]) applies the method to deriving expansions of implied volatilities under stochastic volatility models and jump-diffusion models with stochastic volatilities.

The organization of the paper is as follows: After a brief summary of Malliavin calculus necessary for the remaining of the paper, the next section derives an asymptotic expansion formula for generalized Wiener functionals. Section 3 applies the general formula to pricing path-dependent derivatives with discrete monitoring. Section 4 provides an approximation formula for valuation of barrier options with discrete monitoring. Section 5 concludes. Appendix summarizes conditional expectation formulas used in the expansion.

2 Asymptotic Expansion

2.1 Preparation -Malliavin calculus-

This subsection summarizes basic facts on the Malliavin calculus which are necessary for the following discussion.

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†Graduate School of Economics, the University of Tokyo

‡Mitsubishi UFJ Trust Investment Technology Institute Co.,Ltd. (MTEC)

Let $(\mathcal{W}, \mu)$ be the d-dimensional Wiener space where

$$\mathcal{W} = \mathcal{W}^d = C_0([0,T] : \mathbf{R}^d) = \{w : [0,T] \to \mathbf{R}^d;\ \text{continuous},\ w(0) = 0\}$$

and μ is the Wiener measure. Next, let H be a Hilbert space such that

$$H = \left\{ h \in \mathcal{W}; h_i(t)(i = 1, \cdots, d) \text{ is absolute continuous with respect to } t \text{ and } \sum_{i=1}^{d} \int_0^T |\frac{dh_i(t)}{dt}|^2 dt < \infty \right\}$$

with an inner product $\langle h, h' \rangle_H = \sum_{i=1}^d \int_0^T \frac{dh_i(t)}{dt}\frac{dh_i'(t)}{dt} dt$. Then, H is called the Cameron-Martin space.

Define $L^{\infty-}(\mathcal{W})$ as $L^{\infty-}(\mathcal{W}) = \cap_{p<+\infty} L^p(\mathcal{W})$ and a distance on $L^{\infty-}(\mathcal{W})$ as $d_{L^{\infty-}(\mathcal{W})}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}(\min\{\|F_1 - F_2\|_{L^j}, 1\})$, where $\|\cdot\|_{L^p}$ denotes the L^p-norm in $(\mathcal{W}, \mu)$. Let $L^p(\mathcal{W} : H)$ denote the space of measurable maps from $\mathcal{W}$ to H such that $\|f\|_H \in L^p(\mathcal{W})$. The same definition is made for $L^{\infty-}(\mathcal{W} : H)$.

Then, consider the space

$$\mathbf{D}_1^p(\mathcal{W}) = \left\{ F \in L^p(\mathcal{W}) : \text{there exists } DF \in L^p(\mathcal{W} : H) \text{ such that for } h \in H,\ \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(w + \epsilon h) - F(w)] = \langle DF, h \rangle_H \right\}.$$

Here, DF is called the (Malliavin) derivative of F. Due to the identification between the Hirbert space $L^p(\mathcal{W} : H)$ and $L^p([0,T] \times \mathcal{W})$, DF is a stochastic process $\{D_t F = (D_{t,1}F, \cdots, D_{t,d}F) : t \in [0,T]\}$ such that $\langle DF, h \rangle_H = \sum_{i=1}^d \int_0^T (D_{t,i}F)(\frac{dh_i(t)}{dt})dt$.

A norm $\mathbf{D}_1^p(\mathcal{W})$ is given by $\|F\|_{\mathbf{D}_1^p} = \|F\|_{L^p(\mathcal{W})} + \|DF\|_{L^p(\mathcal{W}:H)}$. Also, $\mathbf{D}_1^\infty(\mathcal{W})$ is defined by $\mathbf{D}_1^\infty(\mathcal{W}) := \cap_{p<+\infty} \mathbf{D}_1^p(\mathcal{W})$, and a distance on $\mathbf{D}_1^\infty(\mathcal{W})$ is given by $d_{\mathbf{D}_1^\infty}(F_1, F_2) = \sum_{j=1}^\infty 2^{-j}(\min\{\|F_1 - F_2\|_{\mathbf{D}_1^j}, 1\})$.

For $r \geq 2 (r \in \mathbf{N})$, we introduce the spaces:

$$\mathbf{D}_r^p(\mathcal{W}) = \{F \in \mathbf{D}_{r-1}^p(\mathcal{W}) : DF \in \mathbf{D}_{r-1}^p(\mathcal{W} : H)\}$$

with $\|F\|_{\mathbf{D}_r^p} = \|F\|_{\mathbf{D}_{r-1}^p} + \|D^{n-1}F\|_{\mathbf{D}_1^p(H^{\otimes(n-1)})}$. We also define $\mathbf{D}_0^p(\mathcal{W})$ as $\mathbf{D}_0^p(\mathcal{W}) = L^p(\mathcal{W})$.

Some properties of these spaces are the following; $\mathbf{D}_{r'}^{p'}(\mathcal{W}) \subset \mathbf{D}_r^p(\mathcal{W})$, $r' \leq r$, and $p' \leq p$. The dual space of $(\mathbf{D}_r^q(\mathcal{W}))$, $(\mathbf{D}_r^q(\mathcal{W}))^*$ is given by $(\mathbf{D}_r^q(\mathcal{W}))^* = \mathbf{D}_{-r}^p(\mathcal{W})$, with $p^{-1} + q^{-1} = 1$.

Furthermore, define the space $\mathbf{D}_\infty(\mathcal{W}) = \cap_{p,r} \mathbf{D}_r^p(\mathcal{W})$. Then, $\mathbf{D}_\infty(\mathcal{W})$ is a complete metric space under a metric, $d_{\mathbf{D}^\infty(\mathcal{W})}(F_1, F_2) = \sum_{p,r=1}^\infty \eta_{p,r}(\min\{\|F_1 - F_2\|_{\mathbf{D}_r^p}, 1\})$ where $\eta_{p,r} > 0$ such that $\sum_{p,r}^\infty \eta_{p,r} < \infty$. Note that this topology on $\mathbf{D}_\infty(\mathcal{W})$ is independent of the choice of the sequence $\{\eta_{p,r}\}$. We call $F \in \mathbf{D}_\infty(\mathcal{W})$ the smooth functional in the sense of Malliavin.

Given $Z = (Z_1(w), \cdots, Z_d(w)) \in \mathbf{D}_1^p(\mathcal{W} : H)$, there exists $D_i^*(Z_i) \in L^p(\mathcal{W})$, $i = 1, \cdots, d$ such that $E[\int_0^T D_{t,i}F(w)Z_i(w)dt] = E[F(w)D_i^*(Z_i(w))]$ for all $F \in \mathbf{D}_1^\infty(\mathcal{W})$. Then, define $D^*Z := \sum_{i=1}^d D_i^*(Z_i(w))$. So, there exists $C_p > 0$ such that $\|D^*Z\|_{L^p} \leq C_p \|Z\|_{\mathbf{D}_1^p(\mathcal{W}:H)}$. We call D^*Z the divergence of Z.

Definition 2.1 *Let $F = (F_1, \cdots, F_n) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be the n-dimensional smooth functional, we call F a non-degenerate in the sense of Malliavin if the Malliavin covariance matrix $\{\sigma_F^{ij}\}_{1 \leq i,j \leq n}$*

$$\sigma_F^{ij} := \langle DF_i, DF_j \rangle_H = \sum_{k=1}^d \int_0^T (D_{t,k}F_i(w))(D_{t,k}F_j(w))dt$$

is invertible and

$$(\det \sigma_F)^{-1} \in L^{\infty-}(\mathcal{W}).$$

Theorem 2.1 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a n-dimensional non-degenerate in the sense of Malliavin and $G \in \mathbf{D}_\infty(\mathcal{W})$. Then, for $\varphi \in C_b^1(\mathbf{R}^n)$,*

$$E[\partial_i \varphi(F)G] \quad = \quad E[\varphi(F)D^*(\sum_{j=1}^n G\gamma_{ij}^F DF^j)]$$

where $(\gamma_{ij}^F)_{1 \leq i,j \leq n}$ is the inverse matrix of Malliavin covariance of F.

(Proof) See Lemma Ⅲ.5.2. of Malliavin(1997). □

Theorem 2.2 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional. F has a smooth density $p^F \in \mathcal{S}(\mathbf{R}^n)$ where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of all infinitely differentiable functions $f : \mathbf{R}^n \mapsto \mathbf{R}$ such that for any $k \geq 1$, and for any multi-index $\beta \in \{1, \cdots, n\}^j$ one has $\sup_{x \in \mathbf{R}^n} |x|^k |\partial_\beta f(x)| < \infty$. (i.e. $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space and $\mathcal{S}'(\mathbf{R}^n)$ is its dual.)*

(Proof) See Theorem Ⅲ.5.1. of Malliavin(1997). □

Definition 2.2 *Consider the space $\mathbf{D}_{-\infty}(\mathcal{W}) = \cup_{p,r} \mathbf{D}^p_{-r}(\mathcal{W})$, that is, the dual of $\mathbf{D}_\infty$. We call $F \in \mathbf{D}_{-\infty}(\mathcal{W})$ a distribution on the Wiener space. We define the duality form on $\mathbf{D}_{-\infty} \times \mathbf{D}_\infty$, $(F, G) \mapsto \langle F, G \rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} = E[FG] \in \mathbf{R}$. We call this duality form the generalized expectation.*

2.2 Asymptotic Expansion for Generalized Wiener Functionals

First, let $O \subset \mathbf{R}^n$ and ν be the measure on O. Then, $\mathcal{S}(O, \nu)$ and $\mathcal{S}'(O, \nu)$ denote the Schwartz space of the rapidly decreasing functions and the set of the Schwartz distributions on the measure space (O, ν), respectively.

Watanabe (1983, 1984) introduced the distribution on Wiener space as composition of a non-degenerate map F by a Schwartz distribution T. The next theorem restates the result of Watanabe (1984) in terms of Malliavin(1997) and Malliavin-Thalmaier(2006).

For multi-index $\alpha^{(k)} = (\alpha_1, \cdots, \alpha_k)$, We define the iterated Malliavin weight. The Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows:

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)),$$

where

$$H_{(l)}(F, G) = D^*(\sum_{i=1}^n G \gamma^F_{li} DF_i).$$

Here, $\gamma^F = \{\gamma^F_{ij}\}_{1 \leq i,j \leq n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

By Malliavin(1997) and Malliavin-Thalmaier(2006) the conditional expectation on the Wiener space E^F gives a map:

$$E^F : L^p(\mathcal{W} : \mu) \to L^p(O : \nu).$$

where $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.

Theorem 2.3 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional. Let ν be the law of F and $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.*

1. *There exists a map*

$$(E^F)^* : \mathcal{S}'(O, \nu) \ni T \mapsto T \circ F \in \tilde{\mathbf{D}}_{-\infty} := \cup_s \cap_q \mathbf{D}^q_{-s} \subset \mathbf{D}_{-\infty}.$$

   $(E^F)^*$ is called the lifting up of T.

2. *The conditional expectation defines a map*

$$E^F : \mathbf{D}_\infty \ni G \mapsto E^F[G] \in \mathcal{S}(O, \nu).$$

   We call this map the push down of G.

3. *The following duality formula is obtained:*

$$\langle (E^F)^* T, G \rangle_{\mathbf{D}_{-\infty}(\mathcal{W}, \mu) \times \mathbf{D}_\infty(\mathcal{W}, \mu)} = \langle T, E^F[G] \rangle_{\mathcal{S}'(O, \nu) \times \mathcal{S}(O, \nu)}.$$

(Proof)

In this proof, we apply the discussions of Watanabe(1984), Malliavin(1997), Malliavin-Thalmaier(2006) and Nualart(2006).

1. Given $T \in \mathcal{S}'(\mathbf{R}^n)$, there exists $T_n \in \mathcal{S}(\mathbf{R}^n)$ such that $T_n \to T$ *in* $\mathcal{S}'(\mathbf{R}^n)$, *i.e*,

$$\|(1-\Delta+|x|^2)^{-m}T_n - (1-\Delta+|x|^2)^{-m}T\|_\infty \to 0, \qquad n \to \infty,$$

where $\|f\|_\infty = \sup_{x\in\mathbf{R}^n}|f(x)|$, $\Delta = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. By the Malliavin integration-by-parts formula, we can estimate as follows; for $p^{-1}+q^{-1}=1$,

$$\begin{aligned}
\|T_n(F)-T_{n'}(F)\|_{\mathbf{D}^q_{-2m}} &= \sup_{G\in\mathbf{D}^p_{2m},\|G\|_{\mathbf{D}^p_{2m}}\le 1} |E[T_n(F)G]-E[T_{n'}(F)G]| \\
&= \sup_{G\in\mathbf{D}^p_{2m},\|G\|_{\mathbf{D}^p_{2m}}\le 1} \left|E[(1-\Delta+|x|^2)^{-m}T_n(F)\vartheta^F_{(2m)}] - E[(1-\Delta+|x|^2)^{-m}T_{n'}(F)\vartheta^F_{(2m)}]\right| \\
&\le \sup_{G\in\mathbf{D}^p_{2m},\|G\|_{\mathbf{D}^p_{2m}}\le 1} \|(1-\Delta+|x|^2)^{-m}T_n - (1-\Delta+|x|^2)^{-m}T_{n'}\|_\infty \|\vartheta^F_{(2m)}\|_{L^1} \\
&\le C\|(1-\Delta+|x|^2)^{-m}T_n - (1-\Delta+|x|^2)^{-m}T_{n'}\|_\infty \to 0,
\end{aligned}$$

as $n, n' \to \infty$, where $\vartheta^F_{(2m)} \in \mathbf{D}_\infty$ and $C := \sup_{G\in\mathbf{D}^p_{2m},\|G\|_{\mathbf{D}^p_{2m}}\le 1}\|\vartheta^F_{(2m)}\|_{L^1} < \infty$. Then $(T_n(F))_{n\in\mathbf{N}}$ is a Cauchy sequence in $\mathbf{D}_{-\infty}$ and thus there exists $(E^F)^*T = T(F) \in \mathbf{D}_{-\infty}$: a composite functional $T(F)$ is uniquely determined.

2. Given $G \in \mathbf{D}_\infty$. For any multi-index $s = (s_1, \cdots, s_k)$, For any $\varphi \in C_b^{|s|}(\mathbf{R}^n)$, by *push-down* and then the integration by parts formula on $\mathbf{R}^n$ or the integration by parts formula on $\mathcal{W}$ and then *push-down*, we obtain

$$E[\partial_s^{|s|}\varphi(F)G] = (-1)^{|s|}\int_{\mathbf{R}^n}\varphi(x)\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\}dx = \int_{\mathbf{R}^n}\varphi(x)E^{F=x}[\vartheta_s^F]p^F(x)dx,$$

where $\vartheta_s^F = H_s(F,G) \in \mathbf{D}_\infty$. It implies that

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[\vartheta_s^F]p^F(x),$$

where

$$E^{F=x}[\vartheta_s^F] \in L^p(O:\nu).$$

We define O_ϵ as $O_\epsilon = \{x \in \mathbf{R}^n : p^F(x) > \epsilon\}$. Therefore,

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \in L^p(O_\epsilon, dx),$$

for all s, which implies $E^{F=x}[G]p^F(x) \in C^\infty(O)$. As $p^F(x) \in C^\infty(O)$,

$$(p^F(x))^{-1}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[G] \in C^\infty(O).$$

Note that the conditional expectation has following expression:

$$E^{F=x}[G]p^F(x) = E[\mathbf{1}_{\{F>x\}}H_{(1,\cdots,n)}(F,G)].$$

For all $k \in \mathbf{N}$ and for all $j = 1, \cdots, n$, if $x_j > 0$,

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^d} x_j^{2k}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \\
=\ &\sup_{x\in\mathbf{R}^d} x_j^{2k}|E[\mathbf{1}_{F>x}H_s(F,H_{(1,\cdots,n)}(F,G))| \\
\le\ &E[|F_j|^{2k}|H_s(F,H_{(1,\cdots,n)}(F,G))|] \\
<\ &\infty,
\end{aligned}$$

if $x_j < 0$, we deduce a similar estimate. These facts imply $E^{F=x}[G] \in \mathcal{S}(O,\nu)$.

3. Therefore, for $T_n \in \mathcal{S}(\mathbf{R}^n)$, $n \in \mathbf{N}$, $T \in \mathcal{S}'(\mathbf{R}^n)$ such that $T_n \to T$ *in* $\mathcal{S}'(\mathbf{R}^n)$, we have

$$\langle T_n, E^F[G]\rangle_{\mathcal{S}'(O,\nu)\times\mathcal{S}(O,\nu)} = E[T_n(F)G] \to \langle (E^F)^*T, G\rangle_{\mathbf{D}_{-\infty}(\mathcal{W},\mu)\times\mathbf{D}_\infty(\mathcal{W},\mu)} = \langle T, E^F[G]\rangle_{\mathcal{S}'(O,\nu)\times\mathcal{S}(O,\nu)},$$

as $n \to \infty$.

□

The next theorem presents an asymptotic expansion formula for generalized Wiener functionals.

Theorem 2.4 *Consider a family of smooth Wiener functionals $F^\epsilon=(F_1^\epsilon,\cdots,F_n^\epsilon)\in\mathbf{D}_\infty(\mathcal{W}:\mathbf{R}^n)$ such that F^ϵ has an asymptotic expansion in $\mathbf{D}_\infty$ and satisfies the uniformly non-degenerate condition:*

$$\limsup_{\epsilon\downarrow 0}\|(\det\sigma_{F^\epsilon})^{-1}\|_{L^p}<\infty,\quad p<\infty. \tag{1}$$

Then, for a Schwartz distribution $T\in S'(\mathbf{R}^n)$, we have an asymptotic expansion in $\mathbf{R}$:

$$\left|E[T(F^\epsilon)]-\left\{\int_{\mathbf{R}^n}T(x)p^{F^0}(x)dx+\sum_{j=1}^N\frac{\epsilon^j}{j!}\int_{\mathbf{R}^n}T(x)E[\sum_k^{(j)}H_{\alpha^{(k)}}(F^0,\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l})|F^0=x]p^{F^0}(x)dx\right\}\right|=O(\epsilon^{N+1}), \tag{2}$$

or

$$\left|E[T(F^\epsilon)]-\left\{\int_{\mathbf{R}^n}T(x)p^{F^0}(x)dx+\sum_{j=1}^N\frac{\epsilon^j}{j!}\sum_k^{(j)}(-1)^k\int_{\mathbf{R}^n}T(x)\partial^k_{\alpha^{(k)}}\left\{E[\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l}|F^0=x]p^{F^0}(x)\right\}dx\right\}\right|=O(\epsilon^{N+1}), \tag{3}$$

where $F_i^{0,k}:=\frac{d^k}{d\epsilon^k}F_i^\epsilon|_{\epsilon=0}$, $k\in\mathbf{N}$, $i=1,\cdots,n$, $\alpha^{(k)}$ denotes a multi-index, $\alpha^{(k)}=(\alpha_1,\cdots,\alpha_k)$ and

$$\sum_k^{(j)}\equiv\sum_{k=1}^j\sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1}\sum_{\alpha^{(k)}\in\{1,\cdots,n\}^k}.$$

Also, Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows:

$$H_{\alpha^{(k)}}(F,G)=H_{(\alpha_k)}(F,H_{\alpha^{(k-1)}}(F,G)),$$

where

$$H_{(l)}(F,G)=D^*(\sum_{i=1}^n G\gamma^F_{li}DF_i).$$

Here, $\gamma^F=\{\gamma^F_{ij}\}_{1\leq i,j\leq n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

Remark 2.1 *The asymptotic expansion formula (3) is the formula developed by Watanabe (1987) and thus, this theorem shows the expansion (2) based on push down(conditional expectation) of Malliavin weight(divergence) is equivalent to the Watanabe's formula.*

(*Proof*)

We use α as an abbreviation of $\alpha^{(k)}$ in the proof. Under the uniformly non-degenerate condition of $F^\epsilon\in\mathbf{D}_\infty(\mathcal{W}:\mathbf{R}^n)$, the lifting up of $T\in\mathcal{S}'(\mathbf{R}^n)$, $(E^{F^\epsilon})^*T$, has the asymptotic expansion in distributions on the Wiener space $\mathbf{D}_{-\infty}$, i.e, for $N\in\mathbf{N}$, there exists $s\in\mathbf{N}$ s.t.

$$\|(E^{F^\epsilon})^*T-\{T\circ F^0+\sum_{j=1}^N\frac{\epsilon^j}{j!}\sum^{(j)}(\partial^k_\alpha T)\circ F^0\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l}\}\|_{\mathbf{D}^q_{-s}}=O(\epsilon^{N+1}),\quad \epsilon\in(0,1],q<\infty$$

Then, there exists an asymptotic expansion of $\langle(E^{F^\epsilon})^*T,\mathbf{1}\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty}$. Let ν be the law of F^0. The push-down of the divergence are computed as follows:

$$\begin{aligned}\left\langle\partial^k_\alpha T(F^0),\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l}\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} &= \left\langle T(F^0),H_\alpha(F^0,\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l})\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty}\\ &= \left\langle T,E^{F^0}[H_\alpha(F^0,\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l})]\right\rangle_{\mathcal{S}'(\nu(dx))\times\mathcal{S}(\nu(dx))}\\ &= \int_{\mathbf{R}^n}T(x)E[H_\alpha(F^0,\prod_{l=1}^k F^{0,\beta_l}_{\alpha_l})|F^0=x]\nu(dx).\end{aligned}$$

On the other hands,

$$\begin{aligned}
\left\langle \partial_\alpha^k T(F^0), \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l} \right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} &= \left\langle \partial_\alpha^k T(x), E^{F^0}[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}] \right\rangle_{\mathcal{S}'(\nu(dx))\times\mathcal{S}(\nu(dx))} \\
&= \left\langle T(x), (\partial^*)_\alpha^k E^{F^0}[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}] \right\rangle_{\mathcal{S}'(\nu(dx))\times\mathcal{S}(\nu(dx))} \\
&= (-1)^k \int_{\mathbf{R}^n} T(x)\partial_\alpha^k \left\{ E[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F^0 = x]p^{F^0}(x) \right\} dx,
\end{aligned}$$

where $(\partial^*)_\alpha^k = \partial_\alpha^* \cdots \partial_\alpha^*$($k$ times) and ∂_α^* denotes the divergence(creation) operator on the space $(\mathbf{R}^n, \nu(dx))$.
□

Corollary 2.1 *The density $p^{F^\epsilon}(y)$ is expressed as following asymptotic expansion with the push-down of Malliavin weights:*

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^m \frac{\epsilon^j}{j!} E[\sum_k^{(j)} H_{\alpha^{(k)}}(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l})|F^0 = y]p^{F^0}(y) + O(\epsilon^{m+1}),$$

where $p^{F^0}(y)$ is the density of F^0. Alternative expression is given by:

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^m \frac{\epsilon^j}{j!} \sum_k^{(j)} (-1)^k \partial_{\alpha^{(k)}}^k \left\{ E[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F^0 = y]p^{F^0}(y) \right\} + O(\epsilon^{m+1}).$$

(*Proof*)
Take a delta function $\delta_y \in \mathcal{S}'$ in the theorem above. □

3 Pricing Path-dependent Derivatives with Discrete Monitoring

This section presents an approximation formula for pricing a path-dependent derivative whose payoff is determined by the underlying asset's value at finite number of time points during the contract period, as an application of Theorem 2.4 in the previous section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space and $(W_{1,t}, W_{2,t})_{t\in[0,T]}$ be a two dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. We consider the following stochastic volatility model;

$$\begin{aligned}
dS_t^{(\epsilon)} &= V(\sigma_t^{(\epsilon)}, t)S_t^{(\epsilon)}dW_{1,t}, \\
d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)}, t)dt + \epsilon A_1(\sigma_t^{(\epsilon)}, t)(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \\
S_0^{(\epsilon)} &= S_0^{(0)} = s,
\end{aligned} \tag{4}$$

where $\rho \in [-1, 1]$ and $\epsilon \in [0, 1]$. $V, A_0, A_1 : \mathbf{R} \times [0, T] \mapsto \mathbf{R}$ are continuous and C^∞ for each $t \in [0, T]$ with bounded derivatives of any orders in the first argument.

Under this stochastic volatility model, we consider a derivative whose payoff depends on the underlying asset price S at monitoring time points, $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$. More specifically, let $\varphi : \mathbf{R}^N \mapsto \mathbf{R}$ be the payoff function of a path-dependent derivative with discrete monitoring. First, we impose the following assumption.

Assumption 3.1 *For all $t \in (0, T]$,*

$$\int_0^t V(\sigma_t^{(0)}, t)^2 dt > 0.$$

Denote $X_{t_i}^{(\epsilon)}$ by the logarithmic process of $S_{t_i}^{(\epsilon)}$;

$$X_{t_i}^{(\epsilon)} := \log S_{t_i}^{(\epsilon)}, \qquad i = 1, \cdots, N.$$

Then, regarding the valuation of the path-dependent derivative with discrete monitoring, the following theorem is obtained.

Theorem 3.1 *Let $\varphi : \mathbf{R}^N \mapsto \mathbf{R}$ be the payoff function of a path-dependent derivative with discrete monitoring. Then, an asymptotic expansion formula for valuation of the derivative under the stochastic volatility model (4);*

$$
\begin{aligned}
& E[\varphi(S_{t_1}^{(\epsilon)}, \cdots, S_{t_N}^{(\epsilon)})] \\
= & \int_{\mathbf{R}^N} \varphi(e^{x_1}, \cdots, e^{x_N}) p^{X^{(0)}}(x_1, \cdots, x_N) dx_1 \cdots dx_N \qquad (5) \\
& + \sum_{j=1}^{m} \frac{\epsilon^j}{j!} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \cdots, e^{x_N}) \sum_{k}^{(j)} E[H_{\alpha^{(k)}}(X^{(0)}, \prod_{l=1}^{k} X_{\beta_l}^{0,\alpha_l}) | X^{(0)} = (x_1, \cdots, x_N)] p^{X^{(0)}}(x_1, \cdots, x_N) dx_1, \cdots, dx_N \\
& + O(\epsilon^{m+1}).
\end{aligned}
$$

where $X_k^{0,i} := \frac{d^k}{d\epsilon^k} X_{t_i}^{(\epsilon)}|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \cdots, N$ *and* $p^{X^0}(x_1, \cdots, x_N)$ *is the density function of* $X^{(0)} = (X_{t_1}^{(0)}, \cdots, X_{t_N}^{(0)})$.

(*Proof*)

The Malliavin covariance matrix $\{\sigma_{X^{(0)}}\}_{ij}$ is given by

$$
\begin{aligned}
\sigma_{X^0} &= \begin{bmatrix} \langle DX_{t_1}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \cdots & \langle DX_{t_1}^{(0)}, DX_{t_N}^{(0)} \rangle_H \\ \vdots & \ddots & \vdots \\ \langle DX_{t_N}^{(0)}, DX_{t_1}^{(0)} \rangle_H & \cdots & \langle DX_{t_N}^{(0)}, DX_{t_N}^{(0)} \rangle_H \end{bmatrix} \\
&= \begin{bmatrix} \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_1} \mathbf{1}_{t \le t_N} dt \\ \vdots & \ddots & \vdots \\ \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_N} \mathbf{1}_{t \le t_1} dt & \cdots & \int_0^T V(\sigma_t^{(0)}, t)^2 \mathbf{1}_{t \le t_N} dt \end{bmatrix} \\
&= \begin{bmatrix} \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt \\ \vdots & \ddots & \vdots \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_N} V(\sigma_t^{(0)}, t)^2 dt \end{bmatrix}.
\end{aligned}
$$

For $n = 1, \cdots, N$, define

$$
\Sigma_n = \begin{bmatrix} \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt \\ \vdots & \ddots & \vdots \\ \int_0^{t_1} V(\sigma_t^{(0)}, t)^2 dt & \cdots & \int_0^{t_n} V(\sigma_t^{(0)}, t)^2 dt \end{bmatrix}.
$$

By Assumption 3.1, each principal minor's determinant of the Malliavin covariance matrix is positive;

$$
\det \Sigma_n > 0, \quad n = 1, \cdots, N,
$$

then the Malliavin covariance matrix is positive definite. Thus, the uniformly non-degenerate condition is satisfied by the similar argument to Takahashi and Yoshida (2004). For the payoff function $\varphi \in \mathcal{S}'$, Theorem 2.4 can be applied and hence the following asymptotic expansion formula is obtained:

$$
\begin{aligned}
& E[\varphi(S_{t_1}^{(\epsilon)}, \cdots, S_{t_N}^{(\epsilon)})] \\
= & \int_{\mathbf{R}^N} \varphi(e^{x_1}, \cdots, e^{x_N}) p^{X^0}(x_1, \cdots, x_N) dx_1 \cdots dx_N \\
& + \sum_{j=1}^{m} \frac{\epsilon^j}{j!} \int_{\mathbf{R}^N} \varphi(e^{x_1}, \cdots, e^{x_N}) \sum^{(j)} E[H_{\alpha}(X^0, \prod_{l=1}^{k} X_{\beta_l}^{0,\alpha_l}) | X^0 = (x_1, \cdots, x_N)] p^{X^0}(x_1, \cdots, x_N) dx_1, \cdots, dx_N \\
& + O(\epsilon^{m+1}).
\end{aligned}
$$

□

4 Pricing Barrier Options with Discrete Monitoring

This section provides an approximation formula for valuation of barrier options with discrete monitoring as a concrete example of the previous section. Let $B \subset \mathbf{R}$ be the barrier. For example, $B = [L, \infty)$, $B = (-\infty, H]$ and $[L, H]$ for some constants, $-\infty < L < H < \infty$. Also the same stochastic volatility model (4) as in the previous section is applied:

$$\begin{aligned} dS_t^{(\epsilon)} &= V(\sigma_t^{(\epsilon)}, t) S_t^{(\epsilon)} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)}, t)dt + \epsilon A_1(\sigma_t^{(\epsilon)}, t)(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}). \end{aligned}$$

Hereafter, the following notations are used:

$$\begin{aligned} Y_{t_i} &:= \int_0^{t_i} V(\sigma_t^{(0)}, t) dW_{1t}, \\ \xi_i &:= \int_0^{t_i} V(\sigma_t^{(0)}, t)^2 dt, \\ X_{t_i} &:= \log S_0 + Y_{t_i} - \frac{1}{2}\xi_i, \\ v_{1t} &:= \partial_x V(\sigma_t^{(0)}, t)\sigma_t^{(1)}, \ \partial_x V(\sigma_t^{(0)}, t) := \frac{\partial V(x,t)}{\partial x}|_{x=\sigma_t^{(0)}} \\ \sigma_t^{(1)} &:= \frac{\partial \sigma_t^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0} = \eta_t \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)})(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}), \\ \eta_t &:= \exp\left\{\int_0^t \partial_x A_0(\sigma_u^{(0)}, u) du\right\}, \\ \Psi_i &:= \int_0^{t_i} v_{1t} dW_{1t} - \int_0^{t_i} V(\sigma_t^{(0)}, t) v_{1t} dt, \\ \Psi_{i-1,i} &:= \int_{t_{i-1}}^{t_i} v_{1t} dW_{1t} - \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)}, t) v_{1t} dt, \\ \Sigma_{(i-1,i)} &:= \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)}, t)^2 dt. \end{aligned}$$

$Barrier_N^{SV}$ denotes the price at time 0 of a discrete barrier option with strike K and maturity T under the stochastic volatility (4). Also, $Barrier_N^{BS}$ denotes the price of this discrete barrier option under the Black-Scholes model;

$$Barrier_N^{BS} = \int_{\mathbf{R}^N} \psi(y_1, \cdots, y_N) p(y_1, \cdots, y_N) dy_1 \cdots dy_N,$$

where

$$\begin{aligned} \psi(y_1, \cdots, y_N) &= \mathbf{1}_{\{se^{y_1 - \frac{1}{2}\xi_1} \in B\}} \cdots \mathbf{1}_{\{se^{y_N - \frac{1}{2}\xi_N} \in B\}} (se^{y_N - \frac{1}{2}\xi_N} - K)^+, \\ p(y_1, \cdots, y_N) &= \frac{1}{(2\pi)^{N/2} \Sigma_{(0,1)}^{1/2} \cdots \Sigma_{(N-1,N)}^{1/2}} e^{-\sum_{i=1}^N \frac{(y_i - y_{i-1})^2}{2\Sigma_{i-1,i}}}, \ (y_0 = 0). \end{aligned}$$

Then, the following result is obtained.

Proposition 4.1 *An asymptotic expansion of $Barrier_N^{SV}$, the price at time 0 of a discrete barrier option with strike K and maturity T under the stochastic volatility (4) is given by:*

$$Barrier_N^{SV} = Barrier_N^{BS} + \epsilon \int_{\mathbf{R}^N} \psi(y_1, \cdots, y_N) \vartheta(y_1, \cdots, y_N) p(y_1, \cdots, y_N) dy_1 \cdots dy_N + O(\epsilon^2),$$

where

$$\begin{aligned} \vartheta(y_1, \cdots, y_N) &= \sum_{k=1}^N \zeta_{k-1,k} \left(\frac{(y_k - y_{k-1})^3}{\Sigma_{k-1,k}^3} - \frac{3(y_k - y_{k-1})}{\Sigma_{k-1,k}} - \frac{(y_k - y_{k-1})^2}{\Sigma_{k-1,k}^2} + \frac{1}{\Sigma_{k-1,k}} \right) \\ &\quad + \sum_{k=2}^N \sum_{l=1}^{k-1} \zeta_{k-1,k}^{(l-1,l)} \left(\frac{y_l - y_{l-1}}{\Sigma_{l-1,l}} \right) \left(\frac{(y_k - y_{k-1})^2}{\Sigma_{k-1,k}^2} - \frac{1}{\Sigma_{k-1,k}} - \frac{y_k - y_{k-1}}{\Sigma_{k-1,k}} \right), \end{aligned}$$

with

$$\zeta_{k-1,k} = \rho \int_{t_{k-1}}^{t_k} \partial V(\sigma_t^{(0)}, t) \eta_t V(\sigma_t^{(0)}, t) \int_{t_{k-1}}^t \eta_s^{-1} A_1(\sigma_s^{(0)}, s) V(\sigma_s^{(0)}, s) ds dt,$$

and

$$\zeta_{k-1,k}^{(l-1,l)} \quad = \quad \rho \int_{t_{k-1}}^{t_k} \partial V(\sigma_t^{(0)},t)\eta_t V(\sigma_t^{(0)},t)dt \int_{t_{l-1}}^{t_l} \eta_s^{-1} A_1(\sigma_s^{(0)},s)V(\sigma_s^{(0)},s)ds.$$

(*Proof*)

[**IBP on $\mathbf{D}_{-\infty} \times \mathbf{D}_{\infty} \rightarrow$ Push down**]

Let $\varphi : (x_1, \cdots, x_N) \mapsto \mathbf{1}_{e^{x_1} \in B} \cdots \mathbf{1}_{e^{x_N} \in B}(e^{x_N} - K)^+$.

$$\begin{aligned}
Barrier_N^{SV} &= E[\varphi(X)] + \epsilon \sum_{i=1}^N E\left[\frac{\partial}{\partial x_i}\varphi(X)\Psi_i\right] + O(\epsilon^2) \\
&= E[\varphi(X)] + \epsilon \sum_{i=1}^N \sum_{j=1}^N E[\varphi(X) D_1^*(\Psi_i \gamma_{ij}^{X^{(0)}} D_1 X_j^{(0)})] + O(\epsilon^2) \\
&= \int_{\mathbf{R}^N} \psi(y_1, \cdots, y_N) p(y_1, \cdots, y_N) dy_1 \cdots dy_N \\
&\quad + \epsilon \sum_{i=1}^N \sum_{j=1}^N \int_{\mathbf{R}^N} \psi(y_1, \cdots, y_N) E^{Y=y}[D_1^*(\Psi_i \gamma_{ij}^{X^{(0)}} D_1 X_j^{(0)})] p(y_1, \cdots, y_N) dy_1 \cdots dy_N + O(\epsilon^2).
\end{aligned}$$

The Malliavin covariance matrix is given by

$$\sigma_{X^{(0)}} = \begin{bmatrix}
\int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \cdots & \int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt \\
\int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)},t)^2 dt & \cdots & \int_0^{t_2} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)},t)^2 dt \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)},t)^2 dt & \cdots & \int_0^{t_{N-1}} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_{N-1}} V(\sigma_t^{(0)},t)^2 dt \\
\int_0^{t_1} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_2} V(\sigma_t^{(0)},t)^2 dt & \cdots & \int_0^{t_{N-1}} V(\sigma_t^{(0)},t)^2 dt & \int_0^{t_N} V(\sigma_t^{(0)},t)^2 dt
\end{bmatrix},$$

and its determinant is given by

$$\det(\sigma_{X^{(0)}}) = \Sigma_{(0,1)}\Sigma_{(1,2)} \cdots \Sigma_{(N-1,N)},$$

where

$$\Sigma_{(i-1,i)} = \int_{t_{i-1}}^{t_i} V(\sigma_t^{(0)},t)^2 dt.$$

The inverse matrix is given by

$$\gamma^{X^{(0)}} =$$

$$\begin{bmatrix}
\frac{1}{\Sigma_{(0,1)}} + \frac{1}{\Sigma_{(1,2)}} & -\frac{1}{\Sigma_{(1,2)}} & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{\Sigma_{(1,2)}} & \frac{1}{\Sigma_{(1,2)}} + \frac{1}{\Sigma_{(2,3)}} & -\frac{1}{\Sigma_{(2,3)}} & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{\Sigma_{(2,3)}} & \frac{1}{\Sigma_{(2,3)}} + \frac{1}{\Sigma_{(3,4)}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & \frac{1}{\Sigma_{(N-3,N-2)}} + \frac{1}{\Sigma_{(N-2,N-1)}} & -\frac{1}{\Sigma_{(N-2,N-1)}} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\Sigma_{(N-2,N-1)}} & \frac{1}{\Sigma_{(N-2,N-1)}} + \frac{1}{\Sigma_{(N-1,N)}} & -\frac{1}{\Sigma_{(N-1,N)}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{\Sigma_{(N-1,N)}} & \frac{1}{\Sigma_{(N-1,N)}}
\end{bmatrix}.$$

$$\Psi_k := \int_0^{t_k} A_t \int_0^t B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) dW_{1t} - \int_0^{t_k} C_t \int_0^t B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) dt,$$

where $A_t = \partial_x V(\sigma_t^{(0)},t) \cdot \eta_t$, $B_s = \eta_s^{-1} A_1(\sigma_s^{(0)},s)$ and $C_t = V(\sigma_t^{(0)},t) \cdot \partial_x V(\sigma_t^{(0)},t) \cdot \eta_t = V(\sigma_t^{(0)},t) A_t$.

$$\Psi_k = \Psi_{k-1} + \Psi_{k-1,k} + \Psi_{k-1,k}^{(0,k-1)},$$

where $\Psi_0 = 0$, $\Psi_{0,1}^{(0,0)} = 0$ and

$$
\begin{aligned}
\Psi_{k-1,k} &:= \int_{t_{k-1}}^{t_k} A_t \left(\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) dW_{1t} - \int_{t_{k-1}}^{t_k} C_t \left(\int_{t_{k-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) dt, \\
\Psi_{k-1,k}^{(0,k-1)} &:= \left(\int_0^{t_{k-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} A_t dW_{1t} \right) - \left(\int_0^{t_{k-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} C_t dt \right), \\
\Psi_{k-1,k}^{(l-1,l)} &:= \left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} A_t dW_{1t} \right) - \left(\int_{t_{l-1}}^{t_l} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} C_t dt \right),
\end{aligned}
$$

$1 \le l \le k-1$.

Using the integration by parts formula, we have

$$
\begin{aligned}
\pi_{i,j} &= D_1^* \left(\Psi_i \gamma_{ij}^{X^{(0)}} D_1 \int_0^{t_j} V(\sigma_t^{(0)}, t) dW_{1,t} \right) \\
&= \gamma_{ij}^{X^{(0)}} \left[\Psi_i \int_0^T V(\sigma_u^{(0)}, u) \mathbf{1}_{u \le t_j} dW_{1,u} - \int_0^T D_{u,1} \Psi_i \mathbf{1}_{u \le t_i} V(\sigma_u^{(0)}, u) \mathbf{1}_{t \le t_j} du \right]
\end{aligned}
$$

Then the Malliavin weight is given by

$$
\begin{aligned}
\pi &= \sum_{i,j=1}^{N} \pi_{ij} \\
&= \sum_{k=1}^{N} \left(\frac{1}{\Sigma_{k-1,k}} \right) \left\{ \Psi_{k-1,k} \int_{t_{k-1}}^{t_k} V(\sigma_{0u}) dW_{1,u} - \int_{t_{k-1}}^{t_k} D_{u,1} \Psi_{k-1,k} V(\sigma_{0u}) du \right\} \\
&\quad + \sum_{k=2}^{N} \sum_{l=1}^{k-1} \left(\frac{1}{\Sigma_{k-1,k}} \right) \left\{ \Psi_{k-1,k}^{(l-1,l)} \int_{t_{k-1}}^{t_k} V(\sigma_{0u}) dW_{1,u} - \int_{t_{k-1}}^{t_k} D_{u,1} \Psi_{k-1,k}^{(l-1,l)} V(\sigma_{0u}) du \right\}.
\end{aligned}
$$

For $t_{i-1} < u \le t_i$,

$$
\begin{aligned}
D_{u,1}\Psi_i &= D_{u,1}\Psi_{i-1,i} + D_{u,1}\Psi_{i-1,i}^{(0,i-1)} \\
&= D_{u,1} \int_{t_{i-1}}^{t_i} A_t \int_{t_{i-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) dW_{1,t} - D_{u,1} \int_{t_{i-1}}^{t_i} C_t \int_{t_{i-1}}^{t} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) dt \\
&\quad + D_{u,1} \left(\int_0^{t_{i-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{i-1}}^{t_i} A_t dW_{1,t} \right) \\
&\quad - D_{u,1} \left(\int_0^{t_{i-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{i-1}}^{t_i} C_t dt \right) \\
&= \rho A_u \int_{t_{i-1}}^{u} B_s dW_{1,s} + \rho B_u \int_u^{t_i} A_t dW_{1,t} - \rho B_u \int_u^{t_i} C_t dt + A_u \left(\int_0^{t_{i-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k} V(\sigma_{0u}) du &= \rho \int_{t_{k-1}}^{t_k} A_u \int_{t_{i-1}}^{u} B_s dW_{1,s} du + \rho \int_{t_{k-1}}^{t_k} B_u \int_u^{t_i} A_t dW_{1,t} du - \rho \int_{t_{k-1}}^{t_k} B_u \int_u^{t_i} C_t dt du \\
\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k}^{(0,k-1)} V(\sigma_{0u}) du &= \left(\int_0^{t_{i-1}} B_s(\rho dW_{1,s} + \sqrt{1-\rho^2} dW_{2,s}) \right) \left(\int_{t_{k-1}}^{t_k} A_u du \right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
E[\Psi_{k-1,k} \int_{t_{k-1}}^{t_k} V(\sigma_{0u}) dW_{1,u} | Y = y] &= \zeta_{k-1,k} \left(\frac{(y_k - y_{k-1})^2}{\Sigma_{k-1,k}^2} - \frac{(y_k - y_{k-1})}{\Sigma_{k-1,k}} \right) (y_k - y_{k-1}), \\
E[\int_{t_{k-1}}^{t_k} D_{u,1}\Psi_{k-1,k} V(\sigma_{0u}) du | Y = y] &= \zeta_{k-1,k} \left(\frac{2(y_k - y_{k-1})}{\Sigma_{k-1,k}} - 1 \right),
\end{aligned}
$$

$$
\begin{aligned}
E[\Psi^{(l-1,l)}_{k-1,k}\int_{t_{k-1}}^{t_k} V(\sigma_{0u})dW_{1,u}|Y=y] &= \zeta^{(l-1,l)}_{k-1,k}\left(\frac{(y_l-y_{l-1})}{\Sigma_{l-1,l}}\right)\left(\frac{(y_k-y_{k-1})}{\Sigma_{k-1,k}}-1\right)(y_k-y_{k-1}),\\
E[\int_{t_{k-1}}^{t_k} D_{u,1}\Psi^{(l-1,l)}_{k-1,k}V(\sigma_{0u})du|Y=y] &= \zeta^{(l-1,l)}_{k-1,k}\left(\frac{(y_l-y_{l-1})}{\Sigma_{l-1,l}}\right)(y_k-y_{k-1}).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\vartheta(y_1,\cdots,y_N)=E[\pi|Y=y] &= \sum_{k=1}^{N}\zeta_{k-1,k}\left(\frac{(y_k-y_{k-1})^3}{\Sigma^3_{k-1,k}}-\frac{3(y_k-y_{k-1})}{\Sigma^2_{k-1,k}}-\frac{(y_k-y_{k-1})^2}{\Sigma^2_{k-1,k}}+\frac{1}{\Sigma_{k-1,k}}\right)\\
&+\sum_{k=2}^{N}\sum_{l=1}^{k-1}\zeta^{(l-1,l)}_{k-1,k}\left(\frac{y_l-y_{l-1}}{\Sigma_{l-1,l}}\right)\left(\frac{(y_k-y_{k-1})^2}{\Sigma^2_{k-1,k}}-\frac{1}{\Sigma_{k-1,k}}-\frac{y_k-y_{k-1}}{\Sigma_{k-1,k}}\right).
\end{aligned}
$$

□

Remark 4.1 *Proposition 4.1 can be proved in the following way, too.*
[**Push down → IBP on $\mathcal{S}'\times\mathcal{S}$**]

$$
\begin{aligned}
\vartheta(y_1,\cdots,y_N)p(y_1,\cdots,y_N) &= \sum_{k=1}^{N}\left(-\frac{\partial}{\partial y_k}\left\{E^{Y^0=y}[\Psi_k]p(y_1,\cdots,y_N)\right\}\right)\\
&= -\sum_{k=1}^{N}\left(\frac{\partial}{\partial y_k}\left\{E^{Y^0=y}[\Psi_{k-1,k}]p(y_1,\cdots,y_N)\right\}\right)\\
&\quad -\sum_{k=2}^{N}\left(\frac{\partial}{\partial y_k}\left\{E^{Y^0=y}[\Psi^{(0,k-1)}_{k-1,k}]p(y_1,\cdots,y_N)\right\}\right)\\
&= -\sum_{k=1}^{N}\left(\frac{\partial}{\partial y_k}\left\{E^{Y^0=y}[\Psi_{k-1,k}]\right\}p(y_1,\cdots,y_N)+E^{Y^0=y}[\Psi_{k-1,k}]\frac{\partial}{\partial y_k}p(y_1,\cdots,y_N)\right)\\
&\quad -\sum_{k=2}^{N}\left(\frac{\partial}{\partial y_k}\left\{E^{Y^0=y}[\Psi^{(0,k-1)}_{k-1,k}]\right\}p(y_1,\cdots,y_N)+E^{Y^0=y}[\Psi^{(0,k-1)}_{k-1,k}]\frac{\partial}{\partial y_k}p(y_1,\cdots,y_N)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
E^{Y^0=y}[\Psi_{k-1,k}] &= \frac{1}{\Sigma^2_{k-1,k}}\zeta_{k-1,k}\left\{(y_k-y_{k-1})^2-\Sigma_{k-1,k}(y_k-y_{k-1})-\Sigma_{k-1,k}\right\},\\
E^{Y^0=y}[\Psi^{(0,k-1)}_{k-1,k}] &= \left(\int_{t_{k-1}}^{t_k}C_t dt\right)\left\{\frac{(y_k-y_{k-1})}{\Sigma_{k-1,k}}-1\right\}\left(\sum_{i=1}^{k-1}\frac{\rho}{\Sigma_{i-1,i}}\left(\int_{t_{i-1}}^{t_i}B_sV(s,\sigma^{(0)}_s)ds\right)(y_i-y_{i-1})\right),\\
\left(\frac{-\partial}{\partial y_k}\right)\left\{E^{Y^0=y}[\Psi_{k-1,k}]\right\} &= \frac{\zeta_{k-1,k}}{\Sigma^2_{k-1,k}}(-2(y_k-y_{k-1})+\Sigma_{k-1,k}),\\
\left(\frac{-\partial}{\partial y_k}\right)\left\{E^{Y^0=y}[\Psi^{(0,k-1)}_{k-1,k}]\right\} &= \frac{-1}{\Sigma_{k-1,k}}\left(\int_{t_{k-1}}^{t_k}C_t dt\right)\left(\sum_{i=1}^{k-1}\frac{\rho}{\Sigma_{i-1,i}}\left(\int_{t_{i-1}}^{t_i}B_sV(s,\sigma^{(0)}_s)ds\right)(y_i-y_{i-1})\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\vartheta(y_1,\cdots,y_N)p(y_1,\cdots,y_N) &= \sum_{k=1}^{N}\zeta_{k-1,k}\left(\frac{(y_k-y_{k-1})^3}{\Sigma^3_{k-1,k}}-\frac{3(y_k-y_{k-1})}{\Sigma^2_{k-1,k}}-\frac{(y_k-y_{k-1})^2}{\Sigma^2_{k-1,k}}+\frac{1}{\Sigma_{k-1,k}}\right)p(y_1,\cdots,y_N)\\
&+\sum_{k=2}^{N}\sum_{l=1}^{k-1}\zeta^{(l-1,l)}_{k-1,k}\left(\frac{y_l-y_{l-1}}{\Sigma_{l-1,l}}\right)\left(\frac{(y_k-y_{k-1})^2}{\Sigma^2_{k-1,k}}-\frac{1}{\Sigma_{k-1,k}}-\frac{y_k-y_{k-1}}{\Sigma_{k-1,k}}\right)p(y_1,\cdots,y_N).
\end{aligned}
$$

5 Conclusion

This paper developed an asymptotic expansion method for generalized Wiener functionals by applying the integration-by-parts formula providing so called *Malliavin weight* and *push down*(Malliavin(1997) and Malliavin-Thalmaier(2006)) in Malliavin calculus. It also showed the equivalence between an asymptotic expansion developed by Watanabe(1987) and our expansion. Moreover, the paper applied the method to pricing path-dependent derivatives with discrete monitoring under stochastic volatility environment. Further, it derived a concrete approximation formula for valuation of barrier options with discrete monitoring with stochastic volatility models. To our knowledge, this paper is the first one that shows an analytical approximation formula for pricing barrier options with discrete monitoring under stochastic volatility environment. Numerical experiments and higher order expansions are our next research topic.

A Formulas for the conditional expectations of the Wiener-Itô integrals

This appendix summarizes conditional expectation formulas useful for explicit computation of the asymptotic expansion in pricing barrier options with discrete monitoring.

1.
$$\mathbf{E}\left[\int_0^T q_{2t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] = \left(\int_0^T q_{2t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

2.
$$\mathbf{E}\left[\int_0^T \int_0^t q_{2u}^{'} dW_u q_{3t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T \int_0^t q_{2u}^{'} q_{1u} du q_{3t}^{'} q_{1t} dt\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$

3.
$$\mathbf{E}\left[\left(\int_0^T q_{2u}^{'} dW_u\right)\left(\int_0^T q_{3s}^{'} dW_s\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{2u}^{'} q_{1u} du\right)\left(\int_0^T q_{3s}^{'} q_{1s} ds\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+ \int_0^T q_{2t}^{'} q_{3t} dt$$

4.
$$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2u}^{'} dW_u q_{3s}^{'} dW_s q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{4t}^{'} q_{1t} \int_0^t q_{3s}^{'} q_{1s} \int_0^s q_{2u}^{'} q_{1u} du ds dt\right) \frac{H_3(x;\Sigma)}{\Sigma^3}$$

5.
$$\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2u}^{'} dW_u\right)\left(\int_0^t q_{3s}^{'} dW_s\right) q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left\{\int_0^T \left(\int_0^t q_{2u}^{'} q_{1u} du\right)\left(\int_0^t q_{3s}^{'} q_{1s} ds\right) q_{4t}^{'} q_{1t} dt\right\} \frac{H_3(x;\Sigma)}{\Sigma^3}$$
$$+ \left(\int_0^T \int_0^t q_{2u}^{'} q_{3u} du q_{4t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

6.
$$\mathbf{E}\left[\left(\int_0^T \int_0^t q_{2s}^{'} dW_s q_{3t}^{'} dW_t\right)\left(\int_0^T q_{4u}^{'} dW_u\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\left(\int_0^T q_{4u}^{'} q_{1u} du\right) \frac{H_3(x;\Sigma)}{\Sigma^3}$$
$$+ \left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{4s} ds dt + \int_0^T q_{3t}^{'} q_{4t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

7.

$$\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T\int_0^r q'_{4u}dW_u q'_{5r}dW_r\right) \mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{1s}dsdt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\int_0^T q'_{3t}q_{1t}\int_0^t q'_{5r}q_{1r}\int_0^r q'_{2u}q_{4u}dudrdt + \int_0^T q'_{5t}q_{1t}\int_0^t q'_{3r}q_{1r}\int_0^r q'_{2u}q_{4u}dudrdt\right.$$

$$+\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2r}q_{5r}\int_0^r q'_{4u}q_{1u}dudrdt + \int_0^T q'_{3t}q_{5t}\left(\int_0^t q'_{2s}q_{1s}ds\right)\left(\int_0^t q'_{4u}q_{1u}du\right)dt$$

$$\left.+\int_0^T q'_{5r}q_{1r}\int_0^r q'_{3u}q_{4u}\int_0^u q'_{2s}q_{1s}dsdudr\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T\int_0^t q'_{2u}q_{4u}du q'_{3t}q_{5t}dt$$

8.

$$\mathbf{E}\left[\left(\int_0^T q'_{2t}dW_t\right)\left(\int_0^T q'_{3s}dW_s\right)\left(\int_0^T\int_0^r q'_{4u}dW_u q'_{5r}dW_r\right) \mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{3s}q_{1s}ds\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\left(\int_0^T q'_{3s}q_{1s}ds\right)\left(\int_0^T q'_{2r}q_{5r}\int_0^r q'_{4u}q_{1u}dudr\right) + \left(\int_0^T q'_{2t}q_{3t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\right.$$

$$+\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{3s}q_{4s}dsdr\right) + \left(\int_0^T q'_{3t}q_{1t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{2u}q_{4u}dudr\right)$$

$$\left.+\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{3r}q_{5r}\int_0^r q'_{4u}q_{1u}dudr\right)\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T q'_{2t}q_{5t}\int_0^t q'_{3s}q_{4s}dsdt + \int_0^T q'_{3r}q_{5r}\int_0^r q'_{2u}q_{4u}dudr$$

References

[1] Bismut, J.-M. : Large deviations and the Malliavin calculus, Progress in Mathematics, vol. 45, Birkhauser. (1984)

[2] Fouque, J-P., Papanicolaou, G., Sircar, K.R. : Derivatives in financial markets with stochastic volatility, Cambridge University Press, Cambridge. (2000)

[3] Fourni*é*, E., Lasry, J.-M., Lebuchoux, J., Lions, P.-L., Touzi, N. : Applications of Malliavin calculus to Monte-Carlo methods in Finance, Finance Stoch. **3** (2001), no. 4, 391-412.

[4] Fusai, G., Abrahams, D., Sgarra, C. : An exact analytical solution for discrete barrier options, Finance Stoch. **10** (2006), no. 1, 1-26.

[5] Ikeda, N., Watanabe, S. : Stochastic differential equations and diffusion processes, 2nd edition, North-Holland/ Kodansha. (1989)

[6] Kunitomo, N., Takahashi, A.: On validity of the asymptotic expansion approach in contingent claim analysis, Ann. Appl. Probab. **13** (2003), no. 3, 914-952.

[7] Leandre, R.: Quantitative and geometric applications of the Malliavin calculus, Contemporary mathematics **73** (1988) , 173-196.

[8] L*ü*tkebomert, E.: An asymptotic expansion for a Black-Scholes type model, Bull. Sci. Math. **128** (2004), no. 8, 661-685.

[9] Malliavin, P., Thalmaier, A.: Stochastic calculus of variations in mathematical finance, Springer. (2006)

[10] Malliavin, P. : Stochastic Analysis, Springer. (1997)

[11] Nualart, D. : The Malliavin calculus and related topics, Springer. (2006)

[12] Nualart, D., Ustunel. A.S., and Zakai, M. : On the moments of a multiple Wiener Ito integral and the space induced by the polynomials of the integrals. Stoch. **25** (1988) 233-240.

[13] Shiraya, K., Takahashi, A., Toda, M. : Pricing Barrier and Average Options under Stochastic Volatility Environment. Preprint, CARF-F-176, Graduate School of Economics, University of Tokyo. (2009)

[14] Siopacha, M., Teichmann, J. : Weak and strong Taylor methods for numerical solutions of stochastic differential equations, arXiv.org.(2007)

[15] Takahashi, A.: An asymptotic expansion approach to pricing financial contingent claims , Asia-Pacific Financial Markets **6** (1999), 115-151

[16] Takahashi, A.: An asymptotic expansion approach in finance , Working Paper CARF-F-102. (2007)

[17] Takahashi, A., Yamada, T.: An Asymptotic Expansion with Push-Down of Malliavin Weights (2009), Preprint.

[18] Takahashi, A., Yoshida, N.: An asymptotic expansion scheme for optimal investment problems, Statistical Inference for Stochastic Processes **7** (2004), 153-188.

[19] Watanabe, S.: Malliavin's calculus in terms of generalized Wiener functionals, Lecture Notes in Control and Inform. Sci., **49**, Springer. (1983)

[20] Watanabe, S.: Lectures on Stochastic Differential Equations and Malliavin Calculus. (1984)

[21] Watanabe, S.: Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, Ann. Probab. **15** (1987), no. 1, 1-39.

[22] Yoshida, N.: Asymptotic expansion for statistics related to small diffusions, J.Japan Statist. Soc. **22** (1992), no.2, 139-159.