A Note on Pricing Barrier Options under a Stochastic Volatility Model -An Asymptotic Expansion with Static Hedging-

Kenichiro Shiraya*, Akihiko Takahashi † and Masashi Toda‡

First Version: July 14, 2009, Current Version: July 30, 2009

Abstract

This short note proposes an approximation method of pricing barrier options under stochastic volatility environment by applying an asymptotic expansion approach combined with a static hedging method. In particular, through numerical examples it shows that the fifth-order normal approximation of an asymptotic expansion scheme (Shiraya-Takahashi-Toda[4],Takahashi-Takehara-Toda[7]) with a modification of a static hedging method by Fink[1] provides good approximations under the λ-SABR model.

Keywords: barrier option, knock-out option, static hedge, asymptotic expansion, stochastic volatility, λ-SABR model

1 Static Hedge

We will apply an asymptotic expansion method with a modification of a static hedging method by Fink[1] to approximate the value of barrier options. Especially, in addition to plain-vanilla options as described in Fink[1], digital options may be useful in static hedging for an in-the-money knock-out call option. We will briefly describe the method below.

- The payoff of an in-the-money knock-out call with maturity T, strike K and barrier B:

  $$(S_T - K)^+ 1_{\{M_T < B\}}$$

  where S_t denotes the underlying asset price at t, $M_t := \max\{S_u; 0 \le u \le t\}$ and B is a constant such that $B > K$(and $B > S_0$).

  The payoff of an out-of-the-money knock-out call with maturity T, strike K and barrier B:

  $$(S_T - K)^+ 1_{\{Q_T > B\}}$$

  where $Q_t := \min\{S_u; 0 \le u \le t\}$ and B is a constant such that $B < K$ (and $B < S_0$).

- $C(t, T, K, v)$: the price of a plain-vanilla call option at t with maturity T, strike K and time-t volatility v.

  $D(t, T, K, v)$: the price of a digital option at t with maturity T, strike K and time-t volatility v. The payoff is given by 1 if $S_T \ge K$ and 0 otherwise.

*Mizuho-DL Financial Technology Co.,Ltd. The views expressed in this paper are those of the author and do not necessarily represent the views of Mizuho-DL Financial Technology Co., Ltd.
†Graduate School of Economics, the University of Tokyo
‡Graduate School of Economics, the University of Tokyo

- First of all, for replication of the value of the barrier option at maturity when the barrier is not hit until the maturity T, we long one unit of a plain-vanilla call option with maturity T and strike K.

  In addition, for an in-the-money knock-out call we may short $\alpha(B-K)$ (where $\alpha = 1$ or $\alpha = 2$) units of a digital option with maturity T and strike B to replicate the value when the barrier is hit just before the maturity.

- Example: in-the-money knock-out call with $K = 90$ and $B = 100$

  (a) long one unit of a plain-vanilla call option with $K = 90$

  (b) short 10 units when $\alpha = 1$ or 20 units when $\alpha = 2$ of a digital option with $K = 100$

  (c) a portfolio of call options with $K \geq 100$(explained later)

  Suppose that the barrier is hit just before the maturity.

  The values of (a), (b), (c):

  (a) about 10,

  (b) about 5 when $\alpha = 1$, or 10 when $\alpha = 2$ (the value of a digital option at ATM just before the maturity is about a half of its payoff.),

  (c) about 0

  When $\alpha = 1$, the replication error is reduced to about half of the error for the replication without digital options.

  When $\alpha = 2$, the replication error is reduced to about 0. However, note that the error shows up when the barrier is hit at maturity.

- Next, fix $t_1(< T)$, $T_1(\in (t_1, T])$ and $v_1, v_2, \cdots, v_m$(volatility at t_1).

  Then, we consider the case when the barrier is hit at t_1.

  We choose plain-vanilla call options with maturity T_1 so that the total value combined with $C(t_1, T, K, v_i)+\alpha(B-K)D(t_1, T, K, v_i)$ is 0 when the volatility at t_1 is $v_j (j = 1, \cdots, m)$. Their strikes are chosen above or equal to the barrier B so that they expire out-of-the-money if the barrier is not hit until T_1.

  Thus, at t_1 choose x_{1j} $(j = 1, \cdots, m)$ units of plain vanilla options with strikes $K = B + \gamma_j$ and maturity T_1 where $\gamma_j \geq 0$ $(j = 1, \cdots, m)$ are given constants that are different each other and α is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.

  In other words, solve the following system of linear equations with respect to x_{1j} $(j = 1, \cdots, m)$.

$$
\begin{cases}
C(t_1, T, K, v_1) + \alpha(B-K)D(t_1, T, K, v_1) + \sum_{j=1}^{m} x_{1j} C(t_1, T_1, B+\gamma_j, v_1) = 0 \\
\vdots \\
C(t_1, T, K, v_m) + \alpha(B-K)D(t_1, T, K, v_m) + \sum_{j=1}^{m} x_{1j} C(t_1, T_1, B+\gamma_j, v_m) = 0
\end{cases}
$$

- Next, fix $t_2(< t_1)$, $T_2(\in (t_2, t_1])$ and $v_1, v_2, \cdots, v_m$(volatility at t_2).

  Then, we consider the case when the barrier is hit at t_2.

  We choose plain-vanilla call options with maturity T_2 so that the total value combined with $C(t_1, T, K, v_i)+\alpha(B-K)D(t_1, T, K, v_i)+\sum_{j=1}^{m} x_{1j} C(t_1, T_1, B+\gamma_j, v_1)$ is 0 when the volatility at t_2 is $v_j (j = 1, \cdots, m)$. Their strikes are chosen above or equal to the barrier B so that they expire out-of-the-money if the barrier is not hit until T_2.

  In the same way as above, at t_2 choose x_{2j} $(j = 1, \cdots, m)$ units of plain vanilla call options with strikes $K = B + \gamma_j$ and maturity T_2 where $\gamma_j \geq 0$ $(j = 1, \cdots, m)$ are given constants and α is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.

In other words, solve the following system of linear equations with respect to x_{2j} $(j = 1, \cdots, m)$.

$$\begin{cases} C(t_2, T, K, v_1) + \alpha(B-K)D(t_2, T, K, v_1) \\ \qquad\qquad + \sum_{j=1}^{m} x_{1j}C(t_2, T_1, B+\gamma_j, v_1) + \sum_{j=1}^{m} x_{2j}C(t_2, T_2, B+\gamma_j, v_1) = 0 \\ \qquad\qquad\qquad\qquad \vdots \\ C(t_2, T, K, v_m) + \alpha(B-K)D(t_2, T, K, v_m) \\ \qquad\qquad + \sum_{j=1}^{m} x_{1j}C(t_2, T_1, B+\gamma_j, v_m) + \sum_{j=1}^{m} x_{2j}C(t_2, T_2, B+\gamma_j, v_m) = 0 \end{cases}$$

- In the same way, a portfolio of plain-vanilla call options for static hedging of a barrier option is recursively determined towards time 0 at prespecified time points $T = t_0 > t_1 > t_2 > \cdots > t_N = 0$.

  Hence, an approximate value at $t = 0$ of the barrier option is obtained by the value of the portfolio at $t = 0$.

2 Setup of Numerical Examples

This section shows setup of numerical examples that examine accuracy of our approximation method. We take λ-SABR Model for the underlying asset model, test both out-of-the-money and in-the-money knock-out call options. and compute benchmark prices by Monte Carlo simulations.

2.1 λ-SABR Model

For the underlying asset model, we take λ-SABR Model(e.g. Labordere[3]) where the dynamics of the underlying asset price S is given as follows:

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)^{\beta}dW^1(t), \tag{1}$$
$$d\sigma(t) = \lambda(\theta - \sigma(t))dt + \nu_1\sigma(t)dW^1(t) + \nu_2\sigma(t)dW^2(t). \tag{2}$$

Here, μ is a constant, $\beta \in [0, 1]$ is a constant, $W = (W^1, W^2)$ is a two dimensional Brownian motion and $\nu_1 = \rho\nu,\ \ \nu_2 = (\sqrt{1-\rho^2})\nu$ where ν is a positive constant and $\rho(\in (-1, 1))$.

2.2 Parameters

- the initial price of the underlying asset: $S(0) = 100$
- the drift of the underlying asset price process: $\mu = 0$
- the interval of calendar spreads($\Delta t_i = t_{i-1} - t_i (i = 1, \cdots, N)$): 0.1 ($T = 1, 2$), 0.01 ($T = 0.05$)
- $T_i = t_{i-1} (i = 1, \cdots, N)$

Table 1: Volatilities($\eta(t) = \left(\theta + (\sigma(0) - \theta)\, e^{-\lambda t}\right)$)

v_1	v_2	v_3	v_4	v_5
$\frac{\eta(t)}{(1+3\nu)}$	$\frac{\eta(t)}{(1+\nu)}$	$\eta(t)$	$(1+\nu)\eta(t)$	$(1+3\nu)\eta(t)$

The following three cases are tested for options in static hedging:

1. Only European Call Options(E)
2. European Call Options + [$(B-K)$ units of a Digital Option] (E+D)
3. European Call Options + [$2(B-K)$ units of a Digital Option] (E+DD)

Table 2: Strike Prices of Plain-Vanilla Options in Static Hedging (Strike Price = Barrier Price+γ_i)

	γ_1	γ_2	γ_3	γ_4	γ_5
$T = 2$	0	2.5	5	7.5	10
$T = 1$	0	2	4	6	8
$T = 0.05$	0	1	2	3	4

Table 3: Strike and Barrier Prices of Barrier Options (I-IV, IX:out-of-the-money knock-out, V-VIII, X:in-the-money knock-out)

	Strike	Barrier
I	95	85
II	105	85
III	95	90
IV	105	90
V	95	115
VI	105	115
VII	95	110
VIII	105	110
IX	100	98
X	100	102

Table 4: Parameters

	$\sigma(0)$	λ	θ	ν	ρ	β	T
i	1	1.2	1	0.3	-0.5	0.5	1
ii	0.2	1.2	0.1	0.6	-0.5	1	1
iii	1	1.2	1	0.3	-0.5	0.5	2
iv	0.2	1.2	0.1	0.6	-0.5	1	2
v	2	1.2	1	0.6	-0.5	0.5	1
vi	0.1	1.2	0.1	0.3	-0.5	1	1
vii	2	1.2	1	0.6	-0.5	0.5	2
viii	0.1	1.2	0.1	0.3	-0.5	1	2
ix	1	1.2	1	0.3	-0.5	0.5	0.05
x	0.2	1.2	0.1	0.6	-0.5	1	0.05
xi	2	1.2	1	0.6	-0.5	0.5	0.05
xii	0.1	1.2	0.1	0.3	-0.5	1	0.05

Benchmark Prices are obtained by Monte Carlo simulations:

- Number of trials: 20,000,000
- Extrapolation method with 1000 and 2000 time steps for cases i, iii, vi, viii(small volatility cases)

  Extrapolation method with 2000 and 4000 time steps for cases ii, iv, v, vii(large volatility cases)

  Extrapolation method with 100 and 200 time steps for cases ix, xii(small volatility and short maturity cases)

  Extrapolation method with 200 and 400 time steps for cases x, xi(large volatility and short maturity cases)

  (Extrapolation method: e.g. Gobet[2])

Finally, the approximate values of portfolios of plain-vanilla and digital options are computed by the fifth-order normal approximation of an asymptotic expansion scheme. ([4],[7]) See those papers for the detail of the method.

3 Result

Tables 5 and 6 give the result. Generally, it shows that the method provides good approximations of barrier option prices. In particular, use of digital options seems effective for approximations of in-the-money knock-out call option prices.

Table 5:

		MC	E	Diff E	Plain Vanilla			MC	E	Diff E	Plain Vanilla
I	i	6.997	7.000	0.004	7.033	III	i	6.690	6.694	0.004	7.033
	ii	8.728	8.729	0.001	9.247		ii	7.644	7.640	-0.004	9.247
	iii	8.288	8.291	0.004	8.543		iii	7.442	7.442	0.000	8.543
	iv	9.630	9.634	0.004	10.743		iv	8.050	8.061	0.011	10.743
	v	8.745	8.748	0.003	9.354		v	7.636	7.629	-0.007	9.354
	vi	6.954	6.960	0.005	6.984		vi	6.675	6.681	0.006	6.984
	vii	9.630	9.625	-0.005	10.893		vii	8.036	8.036	0.000	10.893
	viii	8.252	8.261	0.008	8.465		viii	7.455	7.448	-0.007	8.465
II	i	1.934	1.938	0.004	1.940	IV	i	1.909	1.908	-0.001	1.940
	ii	4.007	4.008	0.001	4.153		ii	3.660	3.661	0.001	4.153
	iii	3.429	3.435	0.005	3.473		iii	3.239	3.239	0.000	3.473
	iv	5.286	5.285	-0.001	5.705		iv	4.603	4.610	0.007	5.705
	v	3.935	3.937	0.003	4.107		v	3.587	3.584	-0.003	4.107
	vi	1.972	1.977	0.005	1.979		vi	1.950	1.954	0.005	1.979
	vii	5.190	5.181	-0.009	5.660		vii	4.508	4.505	-0.002	5.660
	viii	3.490	3.496	0.007	3.528		viii	3.313	3.310	-0.002	3.528
						IX	ix	0.858	0.858	-0.000	0.892
							x	1.321	1.317	-0.005	1.758
							xi	1.315	1.313	-0.002	1.759
							xii	0.859	0.858	-0.000	0.892

Table 6:

		MC	E	E+D	E+DD	Diff E	Diff E+D	Diff E+DD	Plain Vanilla
V	i	4.572	4.559	4.562	4.566	-0.013	-0.010	-0.006	7.033
	ii	2.523	2.534	2.525	2.516	0.010	0.001	-0.008	9.247
	iii	2.880	2.885	2.882	2.878	0.005	0.002	-0.001	8.543
	iv	1.659	1.668	1.659	1.650	0.010	0.001	-0.008	10.743
	v	2.721	2.738	2.728	2.718	0.017	0.007	-0.004	9.354
	vi	4.338	4.335	4.337	4.338	-0.003	-0.002	-0.000	6.984
	vii	1.806	1.817	1.807	1.797	0.011	0.001	-0.009	10.893
	viii	2.669	2.681	2.676	2.671	0.011	0.007	0.002	8.465
VI	i	0.689	0.684	0.686	0.687	-0.005	-0.003	-0.002	1.940
	ii	0.383	0.395	0.391	0.386	0.012	0.007	0.003	4.153
	iii	0.434	0.436	0.434	0.432	0.002	0.001	-0.001	3.473
	iv	0.246	0.256	0.251	0.247	0.010	0.005	0.001	5.705
	v	0.433	0.446	0.441	0.436	0.013	0.008	0.003	4.107
	vi	0.629	0.629	0.629	0.630	-0.000	0.000	0.001	1.979
	vii	0.281	0.291	0.286	0.281	0.010	0.005	0.000	5.660
	viii	0.383	0.388	0.386	0.383	0.005	0.003	0.000	3.528
VII	i	2.368	2.370	2.368	2.367	0.002	0.000	-0.001	7.033
	ii	1.018	1.028	1.022	1.016	0.009	0.003	-0.003	9.247
	iii	1.177	1.188	1.185	1.183	0.010	0.008	0.006	8.543
	iv	0.618	0.627	0.622	0.616	0.009	0.004	-0.001	10.743
	v	1.075	1.085	1.078	1.072	0.009	0.003	-0.004	9.354
	vi	2.257	2.249	2.254	2.260	-0.008	-0.002	0.003	6.984
	vii	0.655	0.662	0.657	0.651	0.008	0.002	-0.004	10.893
	viii	1.117	1.125	1.123	1.120	0.009	0.006	0.004	8.465
VIII	i	0.114	0.116	0.115	0.115	0.001	0.001	0.000	1.940
	ii	0.046	0.052	0.050	0.048	0.005	0.003	0.002	4.153
	iii	0.053	0.055	0.054	0.053	0.002	0.002	0.001	3.473
	iv	0.028	0.032	0.030	0.028	0.004	0.002	0.000	5.705
	v	0.052	0.057	0.055	0.053	0.005	0.003	0.001	4.107
	vi	0.104	0.101	0.103	0.105	-0.003	-0.001	0.001	1.979
	vii	0.031	0.035	0.033	0.031	0.004	0.002	0.000	5.660
	viii	0.047	0.050	0.049	0.048	0.002	0.002	0.001	3.528
X	ix	0.121	0.138	0.130	0.122	0.017	0.009	0.001	0.892
	x	0.023	0.030	0.030	0.029	0.007	0.007	0.006	1.758
	xi	0.023	0.025	0.024	0.024	0.002	0.002	0.001	1.759
	xii	0.119	0.133	0.127	0.120	0.014	0.007	0.000	0.892

References

[1] Fink, J. *An examination of the effectiveness of static hedging in the presence of stochastic volatility*, Journal of Futures Markets, 23(9):859-890,2003.

[2] Gobet, E. *Advanced Monte Carlo methods for barrier and related exotic options*, 2008, to appear in Mathematical Modelling and Numerical Methods in Finance , A. Bensoussan(editor).

[3] P.H.Labordere, *Analysis, Geometry and Modeling in Finance : Advanced Methods in Options Pricing*, Chapman and Hall, 2008.

[4] Shiraya, K. Takahashi, A. and Toda, M., *Pricing Average Options under Stochastic Volatility Models*, FSA Research Review 5, 179-214, 2009.

[5] Takahashi, A. [1995], "Essays on the Valuation Problems of Contingent Claims," Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[6] Takahashi, A. [1999], "An Asymptotic Expansion Approach to Pricing Contingent Claims," *Asia-Pacific Financial Markets*, Vol. 6, 115-151.

[7] Takahashi, A., Takehara, K. and Toda, M., *Computation in an Asymptotic Expansion Method*, Preprint, May, 2009.

[8] Watanabe, S. [1987], "Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels," *The Annals of Probability*, Vol. 15, 1-39.

[9] Yoshida, N. [1992a], "Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe," *Probability Theory and Related Fields*, Vol. 92, 275-311.

In the following appendix, we give a brief summary of an asymptotic expansion method following [7]. See the paper for the detail.

A An Asymptotic Expansion in a General Markovian Setting

Let (W, P) be the r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^{(\epsilon),i} &= V_0^i(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^i(X_t^{(\epsilon)})dW_t \quad (i = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d \end{aligned} \tag{3}$$

where $W = (W^1, \cdots, W^r)$ is a r-dimensional standard Wiener process, and $\epsilon \in (0, 1]$ is a known parameter. Let

Suppose that $V_0 = (V_0^1, \cdots, V_0^d) : \mathbf{R}^d \times (0, 1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions.(e.g. V_0 and V are smooth functions with bounded derivatives of all orders.)

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \cdots$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$. $g_{nT} \in \mathbf{D}^\infty(n = 0, 1, \cdots)$, the coefficients in the expansion, can be obtained by Taylor's formula and represented based on multiple Wiener-Ito integrals.

Let $A_{kt} = \frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and A_{kt}^i, $i = 1, \cdots, d$ denote the i-th elements of A_{kt}. In particular, A_{1t} is represented by

$$A_{1t} = \int_0^t Y_t Y_u^{-1} \left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u \right) \tag{4}$$

where Y denotes the solution to the differential equation;

$$dY_t = \partial V_0(X_t^{(0)}, 0)Y_t dt;\ Y_0 = I_d.$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j, k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

For $k \geq 2$, A_{kt}^i, $i = 1, \cdots, d$ is recursively determined by the following:

$$\begin{aligned} A_{kt}^i &= \int_0^t \partial_\epsilon^k V_0^i(X_s^{(0)}, 0)ds \\ &+ \sum_{l=1}^k \frac{k!}{l!(k-l)!} \sum_{\beta=1}^l \sum_{\vec{l}_\beta \in L_{\beta,l}} \int_0^t \frac{1}{\beta!} \sum_{d_1,\cdots,d_\beta=1}^d \partial_{d_1,\cdots,d_\beta}^\beta \partial_\epsilon^{k-l} V_0^i(X_s^{(0)}, 0) \prod_{j=1}^\beta A_{l_j,s}^{d_j} ds \\ &+ \sum_{\beta=1}^k \sum_{\vec{l}_\beta \in L_{\beta,k-1}} \int_0^t \frac{1}{\beta!} \sum_{\alpha=1}^r \sum_{d_1,\cdots,d_\beta=1}^d \partial_{d_1,\cdots,d_\beta}^\beta V_\alpha^i(X_s^{(0)}) \prod_{j=1}^\beta A_{l_j,s}^{d_j} dW_s^\alpha, \end{aligned} \tag{5}$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial \epsilon^l}$, $\partial_{d_1,\cdots,d_\beta}^\beta = \frac{\partial^\beta}{\partial x_{d_1}\cdots \partial x_{d_\beta}}$ and

$$L_{\beta,k} = \left\{ \vec{l}_\beta = (l_1,\cdots,l_\beta); l_j \geq 0 (j=1,\cdots,\beta), \sum_{j=1}^{\beta} l_j = k \right\}.$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) A_{1T}^i. \end{aligned}$$

For $n \geq 2$, g_{nT} is expressed as follows:

$$g_{nT} = \sum_{\vec{s}\in S_n} \left(\frac{n!}{s_1!\cdots s_n!} \right) \prod_{l=1}^{n} \left(\frac{1}{l!} \right)^{s_l} \sum_{\vec{p}^{s_l}\in P_{s_l}} \left(\frac{s_l!}{p_1^{s_l}!\cdots p_d^{s_l}!} \right) \partial_1^{p_1^{s_l}} \cdots \partial_d^{p_d^{s_l}} g(X_T^{(0)}) \prod_{i=1}^{d} \left(A_{lT}^i \right)^{p_i^{s_l}} \tag{6}$$

where

$$\begin{aligned} S_n &:= \left\{ \vec{s} = (s_1,\cdots,s_n);\ s_l \geq 0 (l=1,\cdots,n),\ \sum_{l=1}^{n} l s_l = n \right\}, \\ P_s &:= \left\{ \vec{p}^s = (p_1^s,\cdots,p_d^s);\ p_i^s \geq 0 (i=1,\cdots,d),\ \sum_{i=1}^{d} p_i^s = s \right\}. \end{aligned}$$

Next, normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0,1]$. Then,

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$). Moreover, let

$$\hat{V}(x,t) = (\partial g(x))' [Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)},t) \hat{V}(X_t^{(0)},t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{x^2}{2\Sigma_T} \right).$$

Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory(Watanabe [8], Yoshida [9]) $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\tilde{\mathbf{D}}^{-\infty}$(a fortiori in $\mathbf{D}^{-\infty}$) as $\epsilon \downarrow 0$. In other words, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $N = 0,1,2,\cdots$,

$$\mathbf{E}[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \left\{ \sum_{k\in K_{j,m}} C^{j,m,k} \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n} \right\} \right] + o(\epsilon^N)$$

$$
\begin{aligned}
&= \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\mathbf{E}\left[\Phi^{(m)}(g_{1T})\sum_{k\in K_{j,m}} C^{j,m,k}\mathbf{E}\left[X^{j,m,k}\middle| g_{1T}\right]\right] + o(\epsilon^N)\\
&= \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi^{(m)}(x)\sum_{k\in K_{j,m}} C^{j,m,k}\mathbf{E}\left[X^{j,m,k}\middle| g_{1T}=x\right] f_{g_{1T}}(x)dx + o(\epsilon^N)\\
&= \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}} C^{j,m,k}(-1)^m\frac{\partial^m}{\partial x^m}\{\mathbf{E}\left[X^{j,m,k}\middle| g_{1T}=x\right] f_{g_{1T}}(x)\}dx + o(\epsilon^N)
\end{aligned}
\tag{7}
$$

where $\Phi^{(m)}(g_{1T}) = \left.\frac{\partial^m\Phi(x)}{\partial x^m}\right|_{x=g_{1T}}$,

$$
K_{j,m} = \left\{(k_1,\cdots,k_{j-m+1}); k_n \geq 0, \sum_{n=1}^{j-m+1} k_n = m, \sum_{n=1}^{j-m+1} nk_n = j\right\},
$$

$$
\begin{aligned}
X^{j,m,k} &= \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n},\\
C^{j,m,k} &= \prod_{n=1}^{j-m+1} \frac{m!}{k_1!\cdots k_{j-m+1}!}.
\end{aligned}
$$

B Computation Scheme

- To compute conditional expectations in the right hand side of (7), we use the following lemma which can be derived from the property of Hermite polynomials and leads us to compute the unconditional expectations instead of the conditional ones.

  Lemma 1 *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega, P)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z=x]$ has following expansion in $L^2(\mathbf{R},\mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance Σ:*

  $$E[X|Z=x] = \sum_{n=0}^{\infty} a_n H_n(x;\Sigma) \tag{8}$$

  where $H_n(x;\Sigma)$ is the Hermite polynomial of degree n which is defined as

  $$H_n(x;\Sigma) = (-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma}$$

  and coefficients a_n are given by

  $$a_n = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\left.\frac{\partial^n}{\partial\xi^n}\right|_{\xi=0}\left\{e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]\right\}. \tag{9}$$

- Recall $\hat{g}_{1T}$ is defined as

  $$\hat{g}_{1T} = (\partial g(X_T^{(0)}))'\int_0^T [Y_T Y_t^{-1} V(X_t^{(0)})]dW_t = g_{1T} - C$$

  where

  $$C = (\partial g(X_T^{(0)}))'\int_0^T Y_T Y_t^{-1}\partial_\epsilon V_0(X_t^{(0)},0)dt,$$

  and define

  $$Z_T^{\langle\xi\rangle} = \exp\{i\xi\hat{g}_{1T} + \frac{\xi^2}{2}\Sigma_T\}.$$

- Then, from Lemma 1 and (7), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$
\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}}C^{j,m,k}(-1)^m\frac{\partial^m}{\partial x^m}\{\mathbf{E}\left[X^{j,m,k}\middle|\hat{g}_{1T}=x-C\right]f_{g_{1T}}(x)\}dx+o(\epsilon^N)\\
&= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}}C^{j,m,k}(-1)^m\frac{\partial^m}{\partial x^m}\{\sum_{l=0}^{j+m}a_l^{j,m,k}H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}dx+o(\epsilon^N)
\end{aligned}
$$

where

$$
a_l^{j,m,k}=\frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[X^{j,m,k}Z_T^{\langle\xi\rangle}]\right\}.
$$

- In particular, let Φ be the delta function at $x\in\mathbf{R}$, δ_x, we obtain the asymptotic expansion of density of $G^{(\epsilon)}$:

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})]\\
&= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\sum_{k\in K_{j,m}}\sum_{l=0}^{j+m}\frac{a_l^{j,m,k}C^{j,m,k}}{m!}(-1)^m\frac{\partial^m}{\partial x^m}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}+o(\epsilon^N).
\end{aligned}
\tag{10}
$$

- Here it is noted that with this expression we now need to compute unconditional expectations $\mathbf{E}[X^{j,m,k}Z_T^{\langle\xi\rangle}]$ instead of the conditional expectations.

- **Remark**
We remark the relation between our method and an approach presented by Takahashi[5],[6] in which the density function of $G^{(\epsilon)}$ is derived by Fourier inversion of its formally expanded characteristic function. In fact, [5],[6] formally expanded $\Psi_{G^{(\epsilon)}}(\xi)=\mathbf{E}[e^{i\xi G^{(\epsilon)}}]$ as

$$
\begin{aligned}
\Psi_{G^{(\epsilon)}}(\xi)=\mathbf{E}\left[e^{i\xi G^{(\epsilon)}}\right] &= e^{-\frac{\xi^2}{2}\Sigma_T+i\xi C}\sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\sum_{k\in K_{j,m}}C^{j,m,k}(i\xi)^m\mathbf{E}\left[X^{j,m,k}Z_T^{\langle\xi\rangle}\right]+o(\epsilon^N) \qquad (11)\\
&= e^{-\frac{\xi^2}{2}\Sigma_T+i\xi C}\sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\sum_{k\in K_{j,m}}C^{j,m,k}(i\xi)^m\mathbf{E}\left[\mathbf{E}\left[X^{j,m,k}Z_T^{\langle\xi\rangle}|g_{1T}\right]\right]+o(\epsilon^N)
\end{aligned}
$$

and computed the conditional expectations in this expansion. Then, $f_{G^{(\epsilon)}}(x)$ was derived by Fourier inversion of $\Psi_{G^{(\epsilon)}}(\xi)$;

$$
f_{G^{(\epsilon)}}(x)=\mathcal{F}^{-1}\left(\Psi_{G^{(\epsilon)}}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ix\xi}\Psi_{G^{(\epsilon)}}(\xi)d\xi. \tag{12}
$$

- This approach is completely equivalent to our method. From (10) we obtain

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\sum_{k\in K_{j,m}}\sum_{l=0}^{j+m}\frac{a_l^{j,m,k}C^{j,m,k}}{m!}(-1)^m\frac{\partial^m}{\partial x^m}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}+o(\epsilon^N)\\
&= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\sum_{k\in K_{j,m}}\sum_{l=0}^{j+m}\frac{a_l^{j,m,k}C^{j,m,k}}{m!}\mathcal{F}^{-1}\left((i\xi)^m(i\xi\Sigma_T)^l e^{-\frac{\xi^2}{2}\Sigma_T+i\xi C}\right)+o(\epsilon^N)\\
&= \mathcal{F}^{-1}\left(e^{-\frac{\xi^2}{2}\Sigma_T+i\xi C}\sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\sum_{k\in K_{j,m}}C^{j,m,k}(i\xi)^m\sum_{l=0}^{j+m}a_l^{j,m,k}(i\Sigma_T)^l\xi^l\right)+o(\epsilon^N)\\
&= \mathcal{F}^{-1}\left(e^{-\frac{\xi^2}{2}\Sigma_T+i\xi C}\sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\frac{1}{m!}\sum_{k\in K_{j,m}}C^{j,m,k}(i\xi)^m\mathbf{E}\left[X^{j,m,k}Z_T^{\langle\xi\rangle}\right]\right)+o(\epsilon^N).
\end{aligned}
$$

Then it is obvious that the inversion of the characteristic function expanded up to ϵ^N-order (11) coincides with the density function obtained by our approach.

B.1 Asymptotic Expansion of Density Function

- In this subsection, we propose a new computational method for the asymptotic expansion of the density function (10). In particular, we show that coefficients in the expansion is obtained through a system of ordinary differential equations that is solved easily, and derive an expression of the expansion up to ϵ^3-order.

  First, we write down the equation (10) more explicitly up to ϵ^3-order:

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= a_0^{0,0,(0)} H_0(x-C;\Sigma_T) f_{g_{1T}}(x) \\
&+\epsilon\left\{\sum_{l=0}^{2} a_l^{1,1,(1)}(-1)\frac{\partial}{\partial x}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right\} \\
&+\epsilon^2\left\{\sum_{l=0}^{3} a_l^{2,1,(0,1)}(-1)\frac{\partial}{\partial x}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right. \\
&\left.+\frac{1}{2}\sum_{l=0}^{4} a_l^{2,2,(2,0)}\frac{\partial^2}{\partial x^2}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right\} \\
&+\epsilon^3\left\{\sum_{l=0}^{4} a_l^{3,1,(0,0,1)}(-1)\frac{\partial}{\partial x}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right. \\
&+\frac{1}{2}\sum_{l=0}^{5} a_l^{3,2,(1,1,0)}\frac{\partial^2}{\partial x^2}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\} \\
&\left.+\frac{1}{6}\sum_{l=0}^{6} a_l^{3,3,(3,0,0)}(-1)\frac{\partial^3}{\partial x^3}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right\} \\
&+o(\epsilon^3),
\end{aligned}
$$

  where coefficients $a_l^{j,m,k}$ are given by

$$
\begin{aligned}
a_l^{0,0,(0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{1,1,(1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,1,(0,1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,2,(2,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,1,(0,0,1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{4T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,2,(1,1,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,3,(3,0,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}^3Z_T^{\langle\xi\rangle}]\right\}.
\end{aligned}
\tag{13}
$$

- Since $E[Z_T^{\langle\xi\rangle}]=1$, we have $a_0^{0,0,(0)}=1$ and $a_l^{0,0,(0)}=0$ for $l\geq 1$. The other expectations above are expressed in terms of A_{kT} and $Z_T^{\langle\xi\rangle}$. For example,

$$
\begin{aligned}
\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}] &= \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jZ_T^{\langle\xi\rangle}]+\frac{1}{2}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{2T}^iZ_T^{\langle\xi\rangle}]\\
\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}] &= \frac{1}{6}\sum_{i,j,k=1}^{d}\partial_i\partial_j\partial_k g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kZ_T^{\langle\xi\rangle}]+\frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{1T}^jZ_T^{\langle\xi\rangle}]\\
&\quad+\frac{1}{6}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{3T}^iZ_T^{\langle\xi\rangle}],\\
\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}] &= \frac{1}{4}\sum_{i,j,k,l=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lZ_T^{\langle\xi\rangle}]\\
&\quad+\frac{1}{2}\sum_{i,j,k=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_i g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{2T}^kZ_T^{\langle\xi\rangle}]\\
&\quad+\frac{1}{4}\sum_{i,j=1}^{d}\partial_i g(X_T^{(0)})\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{2T}^jZ_T^{\langle\xi\rangle}]
\end{aligned}
$$

where A_{1t} is given by (4), and A_{2t} and A_{3t} are expressed as

$$
\begin{aligned}
A_{2t} &= \int_0^t Y_tY_u^{-1}\Bigg(\sum_{j,k=1}^{d}\partial_j\partial_k V_0(X_u^{(0)},0)A_{1u}^jA_{1u}^k du+2\sum_{j=1}^{d}\partial_\epsilon\partial_j V_0(X_u^{(0)},0)A_{1u}^j du\\
&\quad+\partial_\epsilon^2 V_0(X_u^{(0)},0)du+2\sum_{j=1}^{d}\partial_j V(X_u^{(0)})A_{1u}^j dW_u\Bigg),\\
A_{3t} &= \int_0^t Y_tY_u^{-1}\Bigg(\sum_{j,k,l=1}^{d}\partial_j\partial_k\partial_l V_0(X_u^{(0)},0)A_{1u}^jA_{1u}^kA_{1u}^l du+3\sum_{j,k=1}^{d}\partial_j\partial_k V_0(X_u^{(0)},0)A_{1u}^jA_{2u}^k du\\
&\quad+3\sum_{j,k=1}^{d}\partial_j\partial_k\partial_\epsilon V_0(X_u^{(0)},0)A_{1u}^jA_{1u}^k du+3\sum_{j=1}^{d}\partial_j\partial_\epsilon V_0(X_u^{(0)},0)A_{2u}^j du\\
&\quad+3\sum_{j=1}^{d}\partial_j\partial_\epsilon^2 V_0(X_u^{(0)},0)A_{1u}^j du+\partial_\epsilon^3 V_0(X_u^{(0)},0)du\\
&\quad+3\sum_{j,k=1}^{d}\partial_j\partial_k V(X_u^{(0)})A_{1u}^jA_{1u}^k dW_u+3\sum_{j=1}^{d}\partial_j V(X_u^{(0)})A_{2u}^j dW_u\Bigg).
\end{aligned}
$$

- Here, we redefine $\hat{g}_1=\{\hat{g}_{1t};t\in\mathbf{R}^+\}$ and $Z^{\langle\xi\rangle}=\{Z_t^{\langle\xi\rangle};t\in\mathbf{R}^+\}$ as a stochastic processes

  $$\hat{g}_{1t}=\int_0^t\hat{V}(X_u^{(0)},u)dW_u$$

  and

  $$Z_t^{\langle\xi\rangle}=\exp\{i\xi\hat{g}_{1t}+\frac{\xi^2}{2}\Sigma_t\},$$

  respectively where

  $$\hat{V}(x,t)=(\partial g(X_T^{(0)}))'[Y_TY_t^{-1}V(x)].$$

- We define $\eta_{1,1}^i$, $\eta_{2,1}^i$, $\eta_{2,2}^{i,k}$, $\eta_{3,1}^i$, $\eta_{3,2}^{i,k}$, $\eta_{3,3}^{i,k}$, $\eta_{4,1}^i$, $\eta_{4,2,1}^{i,k}$, $\eta_{4,2,2}^{i,k}$, $\eta_{5,2}^{i,k}$, $\eta_{5,3}^{i,k}$, and $\eta_{6,3}^{i,k,l}$ as

  $$\eta_{1,1}^i(t):=E[A_{1t}^iZ_t^{\langle\xi\rangle}],\quad \eta_{2,1}^i(t):=E[A_{2t}^iZ_t^{\langle\xi\rangle}],\quad \eta_{2,2}^{i,k}(t):=E[A_{1t}^iA_{1t}^kZ_t^{\langle\xi\rangle}],$$

$$
\begin{aligned}
&\eta^i_{3,1}(t) := E[A^i_{3t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k}_{3,2}(t) := E[A^i_{1t}A^k_{2t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l}_{3,3}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}Z^{\langle\xi\rangle}_t],\\
&\eta^i_{4,1}(t) := E[A^i_{4t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k}_{4,2,1}(t) := E[A^i_{1t}A^k_{3t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k}_{4,2,2}(t) := E[A^i_{2t}A^k_{2t}Z^{\langle\xi\rangle}_t],\\
&\eta^{i,k,l}_{4,3}(t) := E[A^i_{1t}A^k_{1t}A^l_{2t}Z_t], \quad \eta^{i,k,l,m}_{4,4}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}A^m_{1t}Z^{\langle\xi\rangle}_t],\\
&\eta^{i,k}_{5,2}(t) := E[A^i_{2t}A^k_{3t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l}_{5,3,1}(t) := E[A^i_{1t}A^k_{1t}A^l_{3t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l}_{5,3,2}(t) := E[A^i_{1t}A^k_{2t}A^l_{2t}Z^{\langle\xi\rangle}_t],\\
&\eta^{i,k,l,m}_{5,4}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}A^m_{2t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l,m,n}_{5,5}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}A^m_{1t}A^n_{1t}Z^{\langle\xi\rangle}_t],\\
&\eta^{i,k,l}_{6,3}(t) := E[A^i_{2t}A^k_{2t}A^l_{2t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l,m}_{6,4}(t) := E[A^i_{1t}A^k_{1t}A^l_{2t}A^m_{2t}Z^{\langle\xi\rangle}_t],\\
&\eta^{i,k,l,m,n}_{6,5}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}A^m_{1t}A^n_{2t}Z^{\langle\xi\rangle}_t], \quad \eta^{i,k,l,m,n,o}_{6,5}(t) := E[A^i_{1t}A^k_{1t}A^l_{1t}A^m_{1t}A^n_{1t}A^o_{1t}Z^{\langle\xi\rangle}_t].
\end{aligned} \tag{14}
$$

- We derive a system of ordinary differential equations of η. In the followings, for simplicity, we assume that V_0 doesn't depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$.
- Consider the evaluation of $\eta^j_{2,1}(T) = E[A^j_{2T}Z^{\langle\xi\rangle}_T]$ which appears in the ϵ-order. Applying It$\hat{o}$'s formula to $A^j_{2t}Z^{\langle\xi\rangle}_t$, we have

$$
\begin{aligned}
d(A^j_{2t}Z^{\langle\xi\rangle}_t) &= A^j_{2t}dZ^{\langle\xi\rangle}_t + Z^{\langle\xi\rangle}_t dA^j_{2t} + dA^j_{2t}dZ^{\langle\xi\rangle}_t\\
&= \Bigg\{2(i\xi)\sum_{j'=1}^d A^{j'}_{1t}Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t)\partial_{j'}V^j(X^{(0)}_t)' + \sum_{j'=1}^d A^{j'}_{2t}Z^{\langle\xi\rangle}_t\partial_{j'}V^j_0(X^{(0)}_t)\\
&\quad + \sum_{j'=1}^d\sum_{k'=1}^d A^{j'}_{1t}A^{k'}_{1t}Z^{\langle\xi\rangle}_t\partial_{j'}\partial_{k'}V^j_0(X^{(0)}_t)\Bigg\}dt\\
&\quad + \Bigg\{(i\xi)A^j_{2t}Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t) + 2\sum_{j'=1}^d A^{j'}_{1t}Z^{\langle\xi\rangle}_t\partial_{j'}V^j(X^{(0)}_t)\Bigg\}dW_t.
\end{aligned}
$$

- Note that the second and third terms are martingales. Thus, taking the expectation on both sides, we have the following ordinary differential equation of $\eta^j_{2,1}$:

$$
\begin{aligned}
\frac{d}{dt}\eta^j_{2,1}(t) &= 2(i\xi)\sum_{j'=1}^d \eta^{j'}_{1,1}(t)\hat{V}(X^{(0)}_t,t)\partial_{j'}V^j(X^{(0)}_t)'\\
&\quad + \sum_{j'=1}^d \eta^{j'}_{2,1}(t)\partial_{j'}V^j_0(X^{(0)}_t) + \sum_{j'=1}^d\sum_{k'=1}^d \eta^{j',k'}_{2,2}(t)\partial_{j'}\partial_{k'}V^j_0(X^{(0)}_t).
\end{aligned}
$$

- Here, $\eta^j_{1,1}(j=1,\cdots,d)$ appearing in the right hand side of above ODE are evaluated in the similar manner:

$$
\begin{aligned}
d(A^j_{1t}Z^{\langle\xi\rangle}_t) &= A^j_{1t}dZ^{\langle\xi\rangle}_t + Z^{\langle\xi\rangle}_t dA^j_{1t} + dA^j_{1t}dZ^{\langle\xi\rangle}_t\\
&= \Bigg\{(i\xi)Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)' + \sum_{j'=1}^d A^{j'}_{1t}Z^{\langle\xi\rangle}_t\partial_{j'}V^j_0(X^{(0)}_t)\Bigg\}dt\\
&\quad + \Big\{(i\xi)A^j_{1t}Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t) + Z^{\langle\xi\rangle}_t V^j(X^{(0)}_t)\Big\}dW_t,
\end{aligned}
$$

  hence, we have

$$
\frac{d}{dt}\eta^j_{1,1}(t) = (i\xi)\hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)' + \sum_{j'=1}^d \eta^{j'}_{1,1}(t)\partial_{j'}V^j_0(X^{(0)}_t).
$$

- $\eta_{2,2}^{j,k}$ is evaluated in the same way.

$$\begin{aligned}\frac{d}{dt}\eta_{2,2}^{j,k}(t) &= (i\xi)\left\{\eta_{1,1}^{k}(t)\hat{V}(X_t^{(0)},t)V^j(X_t^{(0)})' + \eta_{1,1}^{j}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})'\right\} \\ &\quad + V^j(X_t^{(0)})V^k(X_t^{(0)})' + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{i',k'}(t)\partial_{i'}V_0^j(X_t^{(0)})\partial_{k'}V_0^k(X_t^{(0)})\end{aligned}$$

- Note that $\eta_{1,1}^{j}$, $\eta_{2,2}^{j,k}$ and $\eta_{2,1}^{j}$ are evaluated in the following order:

$$\eta_{1,1}^{j} \rightarrow \eta_{2,2}^{j,k} \rightarrow \eta_{2,1}^{j}.$$

- The key observation is that each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. Hence, one can easily solve (analytically or numerically) the system of ODEs and evaluate expectations.

Proposition 1 *For $\eta_{j,m,k}$ defined in (14), the following system of ordinary differential equations holds:*

$$\frac{d}{dt}\eta_{1,1}^{i}(t) = (i\xi)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \sum_{i'=1}^{d}\eta_{1,1}^{i'}(t)\partial_{i'}V_0^i(X_t^{(0)})$$

$$\begin{aligned}\frac{d}{dt}\eta_{2,1}^{i}(t) &= 2(i\xi)\sum_{i'=1}^{d}\eta_{1,1}^{i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^i(X_t^{(0)})' \\ &\quad + \sum_{i'=1}^{d}\eta_{2,1}^{i'}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{i',k'}(t)\partial_{i'}\partial_{k'}V_0^i(X_t^{(0)}) \\ \frac{d}{dt}\eta_{2,2}^{i,k}(t) &= (i\xi)\left\{\eta_{1,1}^{k}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{1,1}^{i}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})'\right\} \\ &\quad + V^i(X_t^{(0)})V^k(X_t^{(0)})' + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{i',k'}(t)\partial_{i'}V_0^i(X_t^{(0)})\partial_{k'}V_0^k(X_t^{(0)})\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\eta_{3,1}^{i}(t) &= (i\xi)\Big\{3\sum_{i'=1}^{d}\eta_{2,1}^{i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^i(X_t^{(0)})' \\ &\quad + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{i',k'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}\partial_{k'}V^i(X_t^{(0)})'\Big\} \\ &\quad + \sum_{i'=1}^{d}\eta_{3,1}^{i'}(t)\partial_{i'}V_0^i(X_t^{(0)}) + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{3,2}^{i',k'}(t)\partial_{i'}\partial_{k'}V_0^i(X_t^{(0)}) \\ &\quad + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta_{3,3}^{i',k',l'}(t)\partial_{i'}\partial_{k'}\partial_{l'}V_0^i(X_t^{(0)}) \\ \frac{d}{dt}\eta_{3,2}^{i,k}(t) &= (i\xi)\Big\{\eta_{2,1}^{k}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + 2\sum_{i'=1}^{d}\eta_{2,2}^{i,i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^k(X_t^{(0)})'\Big\} \\ &\quad + 2\sum_{i'=1}^{d}\eta_{1,1}^{i'}(t)V^i(X_t^{(0)})\partial_{i'}V^k(X_t^{(0)})' \\ &\quad + \sum_{i'=1}^{d}\eta_{3,2}^{i',k}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{3,2}^{i,k'}(t)\partial_{k'}V_0^k(X_t^{(0)})\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{3,3}^{i,i',k'}(t)\partial_{i'}\partial_{k'}V_0^k(X_t^{(0)}) \\
\frac{d}{dt}\eta_{3,3}^{i,k,l}(t) \quad = \quad &(i\xi)\Big\{\eta_{2,2}^{k,l}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})'+\eta_{2,2}^{i,l}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})'+\eta_{2,2}^{i,k}(t)\hat{V}(X_t^{(0)},t)V^l(X_t^{(0)})'\Big\} \\
&+\eta_{1,1}^{i}(t)V^k(X_t^{(0)})V^l(X_t^{(0)})'+\eta_{1,1}^{k}(t)V^i(X_t^{(0)})V^l(X_t^{(0)})'+\eta_{1,1}^{l}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{3,3}^{i',k,l}(t)\partial_{i'}V_0^i(X_t^{(0)})+\sum_{k'=1}^{d}\eta_{3,3}^{i,k',l}(t)\partial_{k'}V_0^k(X_t^{(0)})+\sum_{l'=1}^{d}\eta_{3,3}^{i,k,l'}(t)\partial_{l'}V_0^l(X_t^{(0)})
\end{aligned}
$$

(The remaining equations are omitted.)

- We summarize the discussion above as the following theorem:

Theorem 1 *The asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^3-order is given by*

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) \quad = \quad & f_{g_{1T}}(x) \\
&+\epsilon\left\{\sum_{l=1}^{3}C_{1l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x) \\
&+\epsilon^2\left\{\sum_{l=1}^{6}C_{2l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x) \\
&+\epsilon^3\left\{\sum_{l=1}^{9}C_{3l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x) \\
&+o(\epsilon^3).
\end{aligned}
$$

where

$$
\begin{aligned}
&C_{1l}=\Sigma_T a_{l-1}^{1,1,(1)}, \\
&C_{21}=\Sigma_T a_0^{2,1,(0,1)}, \quad C_{2l}=\Sigma_T a_{l-1}^{2,1,(0,1)}+\frac{1}{2}\Sigma_T^2 a_{l-2}^{2,2,(2,0)}(l\geq 2), \\
&C_{31}=\Sigma_T a_0^{3,1,(0,0,1)}, \quad C_{32}=\Sigma_T a_1^{3,1,(0,0,1)}+\frac{1}{2}\Sigma_T^2 a_0^{3,2,(1,2,0)}, \\
&C_{3l}=\Sigma_T a_{l-1}^{3,1,(0,0,1)}+\frac{1}{2}\Sigma_T^2 a_{l-2}^{3,2,(1,2,0)}+\frac{1}{6}\Sigma_T^3 a_{l-3}^{3,3,(3,0,0)}(l\geq 3).
\end{aligned}
$$

Here, $a_l^{j,m,k}$ are given by (13):

$$
\begin{aligned}
a_l^{0,0,(0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{1,1,(1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,1,(0,1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,2,(2,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,1,(0,0,1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{4T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,2,(1,1,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,3,(3,0,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}^3Z_T^{\langle\xi\rangle}]\right\}.
\end{aligned}
$$

The expectations in (13) are obtained as the solutions to the system of ordinary differential equations given in Proposition 1.

B.2 Asymptotic Expansion of Option Prices

- We apply the asymptotic expansion to option pricing. We consider the plain vanilla option on the underlying asset $g(X_T^{(\epsilon)})$ where the dynamics of $X_T^{(\epsilon)}$ is given by (3).
- For example, an asymptotic expansion upto $\epsilon^{(N+1)}$ of a call option price at time 0 with maturity T and strike price K where $K = X_T^{(0)} - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given by

$$C(K,T) \;=\; \epsilon P(0,T) \int_{-y}^{\infty} (x+y) f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}).$$

Here, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T and $f_{G^{(\epsilon)},N}$ is the normal asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order given by (10):

$$f_{G^{(\epsilon)},N}(x) \;=\; \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \sum_{k \in K_{j,m}} \sum_{l=0}^{j+m} \frac{a_l^{j,m,k} C^{j,m,k}}{m!} (-1)^m \frac{\partial^m}{\partial x^m} \{H_l(x - C; \Sigma_T) f_{g_{1T}}(x)\}$$

- In particular, using Theorem 1, an asymptotic expansion upto ϵ^4 of a call option price at time 0 with maturity T and strike price K where $K = X_T^{(0)} - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is expressed as

$$\begin{aligned} C(K,T) \;=\; & \epsilon P(0,T) \int_{-y}^{\infty} (x+y) f_{g_{1T}}(x)dx \\ & + \epsilon^2 P(0,T) \int_{-y}^{\infty} (x+y) \left\{ \sum_{l=1}^{3} C_{1l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x)dx \\ & + \epsilon^3 P(0,T) \int_{-y}^{\infty} (x+y) \left\{ \sum_{l=1}^{6} C_{2l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x)dx \\ & + \epsilon^4 P(0,T) \int_{-y}^{\infty} (x+y) \left\{ \sum_{l=1}^{9} C_{3l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x)dx \\ & + o(\epsilon^4). \end{aligned}$$

- Integrals appeared in the right hand side can be calculated by following formulas related to the Hermite polynomial

$$\begin{aligned} \int_{-y}^{\infty} H_k(x;\Sigma) f_{g_{1T}}(x)dx \;&=\; \Sigma H_{k-1}(-y;\Sigma) f_{g_{1T}}(y) \ (k \geq 1), \\ \int_{-y}^{\infty} x H_k(x;\Sigma) f_{g_{1T}}(x)dx \;&=\; -\Sigma y H_{k-1}(-y;\Sigma) f_{g_{1T}}(y) \\ & \quad + \Sigma^2 H_{k-2}(-y;\Sigma) f_{g_{1T}}(y) \ (k \geq 2). \end{aligned}$$