A Note on Pricing Barrier Options under a Stochastic Volatility Model
-An Asymptotic Expansion with Static Hedging-

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Abstract

This short note proposes an approximation method of pricing barrier options under stochastic volatility environment by applying an asymptotic expansion approach combined with a static hedging method. In particular, through numerical examples it shows that the fifth-order normal approximation of an asymptotic expansion scheme (Shiraya-Takahashi-Toda[4],Takahashi-Takehara-Toda[7]) with a modification of a static hedging method by Fink[1] provides good approximations under the λ-SABR model.

Keywords: barrier option, knock-out option, static hedge, asymptotic expansion, stochastic volatility, λ-SABR model

1 Static Hedge

We will apply an asymptotic expansion method with a modification of a static hedging method by Fink[1] to approximate the value of barrier options. Especially, in addition to plain-vanilla options as described in Fink[1], digital options may be useful in static hedging for an in-the-money knock-out call option. We will briefly describe the method below.

• The payoff of an in-the-money knock-out call with maturity $T$, strike $K$ and barrier $B$:

$$(S_T - K)^+ 1_{\{M_T < B\}}$$

where $S_t$ denotes the underlying asset price at $t$, $M_t := \max\{S_u; 0 \leq u \leq t\}$ and $B$ is a constant such that $B > K$ (and $B > S_0$).

The payoff of an out-of-the-money knock-out call with maturity $T$, strike $K$ and barrier $B$:

$$(S_T - K)^+ 1_{\{Q_T > B\}}$$

where $Q_t := \min\{S_u; 0 \leq u \leq t\}$ and $B$ is a constant such that $B < K$ (and $B < S_0$).

• $C(t, T, K, v)$: the price of a plain-vanilla call option at $t$ with maturity $T$, strike $K$ and time-$t$ volatility $v$.

$D(t, T, K, v)$: the price of a digital option at $t$ with maturity $T$, strike $K$ and time-$t$ volatility $v$. The payoff is given by 1 if $S_T \geq K$ and 0 otherwise.

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• First of all, for replication of the value of the barrier option at maturity when the barrier is not hit until the maturity $T$, we long one unit of a plain-vanilla call option with maturity $T$ and strike $K$.

In addition, for an in-the-money knock-out call we may short $\alpha(B-K)$ (where $\alpha = 1$ or $\alpha = 2$) units of a digital option with maturity $T$ and strike $B$ to replicate the value when the barrier is hit just before the maturity.

• Example: in-the-money knock-out call with $K = 90$ and $B = 100$
  (a) long one unit of a plain-vanilla call option with $K = 90$
  (b) short 10 units when $\alpha = 1$ or 20 units when $\alpha = 2$ of a digital option with $K = 100$
  (c) a portfolio of call options with $K \geq 100$ (explained later)

Suppose that the barrier is hit just before the maturity.

The values of (a), (b), (c):
(a) about 10,
(b) about 5 when $\alpha = 1$, or 10 when $\alpha = 2$ (the value of a digital option at ATM just before the maturity is about a half of its payoff),
(c) about 0

When $\alpha = 1$, the replication error is reduced to about half of the error for the replication without digital options.

When $\alpha = 2$, the replication error is reduced to about 0. However, note that the error shows up when the barrier is hit at maturity.

• Next, fix $t_1(< T)$, $T_1(\in (t_1, T])$ and $v_1, v_2, \ldots, v_m$ (volatility at $t_1$).

Then, we consider the case when the barrier is hit at $t_1$.

We choose plain-vanilla call options with maturity $T_1$ so that the total value combined with $C(t_1, T, K, v_1) + \alpha(B-K)D(t_1, T, K, v_i)$ is 0 when the volatility at $t_1$ is $v_j (j = 1, \ldots, m)$. Their strikes are chosen above or equal to the barrier $B$ so that they expire out-of-the-money if the barrier is not hit until $T_1$.

Thus, at $t_1$ choose $x_{1j}$ ($j = 1, \ldots, m$) units of plain vanilla options with strikes $K = B + \gamma_j$ and maturity $T_1$ where $\gamma_j \geq 0$ ($j = 1, \ldots, m$) are given constants that are different each other and $\alpha$ is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.

In other words, solve the following system of linear equations with respect to $x_{1j}$ ($j = 1, \ldots, m$).

\[
\begin{align*}
C(t_1, T, K, v_1) + \alpha(B-K)D(t_1, T, K, v_1) + \sum_{j=1}^{m} x_{1j}C(t_1, T_1, B + \gamma_j, v_1) &= 0 \\
C(t_1, T, K, v_m) + \alpha(B-K)D(t_1, T, K, v_m) + \sum_{j=1}^{m} x_{1j}C(t_1, T_1, B + \gamma_j, v_m) &= 0
\end{align*}
\]

• Next, fix $t_2(< t_1)$, $T_2(\in (t_2, t_1])$ and $v_1, v_2, \ldots, v_m$ (volatility at $t_2$).

Then, we consider the case when the barrier is hit at $t_2$.

We choose plain-vanilla call options with maturity $T_2$ so that the total value combined with $C(t_1, T, K, v_1) + \alpha(B-K)D(t_1, T, K, v_i)+\sum_{j=1}^{m} x_{1j}C(t_1, T_1, B + \gamma_j, v_1)$ is 0 when the volatility at $t_2$ is $v_j (j = 1, \ldots, m)$. Their strikes are chosen above or equal to the barrier $B$ so that they expire out-of-the-money if the barrier is not hit until $T_2$.

In the same way as above, at $t_2$ choose $x_{2j}$ ($j = 1, \ldots, m$) units of plain vanilla call options with strikes $K = B + \gamma_j$ and maturity $T_2$ where $\gamma_j \geq 0$ ($j = 1, \ldots, m$) are given constants and $\alpha$ is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.
In other words, solve the following system of linear equations with respect to $x_{2j}$ ($j = 1, \cdots, m$).

$$
\begin{align*}
C(t_2, T, K, v_1) + \alpha(B - K)D(t_2, T, K, v_1) &+ \sum_{j=1}^{m} x_{1j}C(t_2, T_1, B + \gamma_j, v_1) + \sum_{j=1}^{m} x_{2j}C(t_2, T_2, B + \gamma_j, v_1) = 0 \\
\vdots \\
C(t_2, T, K, v_m) + \alpha(B - K)D(t_2, T, K, v_m) &+ \sum_{j=1}^{m} x_{1j}C(t_2, T_1, B + \gamma_j, v_m) + \sum_{j=1}^{m} x_{2j}C(t_2, T_2, B + \gamma_j, v_m) = 0
\end{align*}
$$

- In the same way, a portfolio of plain-vanilla call options for static hedging of a barrier option is recursively determined towards time 0 at prespecified time points $T = t_0 > t_1 > t_2 > \cdots > t_N = 0$.
  Hence, an approximate value at $t = 0$ of the barrier option is obtained by the value of the portfolio at $t = 0$.

## 2 Setup of Numerical Examples

This section shows setup of numerical examples that examine accuracy of our approximation method. We take $\lambda$-SABR Model for the underlying asset model, test both out-of-the-money and in-the-money knock-out call options, and compute benchmark prices by Monte Carlo simulations.

### 2.1 $\lambda$-SABR Model

For the underlying asset model, we take $\lambda$-SABR Model (e.g. Labordere[3]) where the dynamics of the underlying asset price $S$ is given as follows:

\[dS(t) = \mu S(t)dt + \sigma(t)S(t)\beta dW_1(t),\]
\[d\sigma(t) = \lambda(\theta - \sigma(t))dt + \nu_1 \sigma(t)dW_1(t) + \nu_2 \sigma(t)dW_2(t).\]

Here, $\mu$ is a constant, $\beta \in [0, 1]$ is a constant, $W = (W^1, W^2)$ is a two dimensional Brownian motion and $\nu_1 = \rho \nu$, $\nu_2 = (\sqrt{1 - \rho^2})\nu$ where $\nu$ is a positive constant and $\rho \in (-1, 1))$.

### 2.2 Parameters

- the initial price of the underlying asset: $S(0) = 100$
- the drift of the underlying asset price process: $\mu = 0$
- the interval of calendar spreads($\Delta t_i = t_{i-1} - t_i$($i = 1, \cdots, N$)): 0.1 ($T = 1, 2$), 0.01 ($T = 0.05$)
- $T_i = t_{i-1}$($i = 1, \cdots, N$)

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
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<tr>
<td>$\eta(t)$</td>
<td>$\eta(t)$</td>
<td>$\eta(t)$</td>
<td>$(1 + \nu)\eta(t)$</td>
<td>$(1 + 3\nu)\eta(t)$</td>
</tr>
</tbody>
</table>

The following three cases are tested for options in static hedging:

1. Only European Call Options (E)
2. European Call Options + [(B - K) units of a Digital Option] (E+D)
3. European Call Options + [2(B - K) units of a Digital Option] (E+DD)

3
Table 2: Strike Prices of Plain-Vanilla Options in Static Hedging (Strike Price = Barrier Price + $\gamma_i$)

\[
\begin{array}{cccccc}
T = 2 & 0 & 2.5 & 5 & 7.5 & 10 \\
T = 1 & 0 & 2 & 4 & 6 & 8 \\
T = 0.05 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

Table 3: Strike and Barrier Prices of Barrier Options (I-IV, IX: out-of-the-money knock-out, V-VIII, X: in-the-money knock-out)

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
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</tr>
<tr>
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<td>105</td>
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</tr>
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<td>III</td>
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<td>V</td>
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<td>VI</td>
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<td>VII</td>
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<td>110</td>
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<td>98</td>
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<tr>
<td>X</td>
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Table 4: Parameters

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<td>1</td>
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<td>0.3</td>
<td>-0.5</td>
<td>0.5</td>
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<td>iv</td>
<td>0.2</td>
<td>1.2</td>
<td>0.1</td>
<td>0.6</td>
<td>-0.5</td>
<td>0.5</td>
<td>1</td>
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<tr>
<td>v</td>
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<td>1.2</td>
<td>1</td>
<td>0.6</td>
<td>-0.5</td>
<td>0.5</td>
<td>1</td>
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<td>-0.5</td>
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<td>1</td>
</tr>
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<td>2</td>
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<td>1</td>
<td>0.6</td>
<td>-0.5</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>viii</td>
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<td>1.2</td>
<td>0.1</td>
<td>0.3</td>
<td>-0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>ix</td>
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<td>1</td>
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<td>-0.5</td>
<td>0.5</td>
<td>0.05</td>
</tr>
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<td>1.2</td>
<td>0.1</td>
<td>0.6</td>
<td>-0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>xi</td>
<td>2</td>
<td>1.2</td>
<td>1</td>
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<td>0.5</td>
<td>0.05</td>
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<tr>
<td>xii</td>
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<td>0.1</td>
<td>0.3</td>
<td>-0.5</td>
<td>1</td>
<td>0.05</td>
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</table>

Benchmark Prices are obtained by Monte Carlo simulations:

- Number of trials: 20,000,000
- Extrapolation method with 1000 and 2000 time steps for cases i, iii, vi, viii (small volatility cases)
- Extrapolation method with 2000 and 4000 time steps for cases ii, iv, v, vii (large volatility cases)
- Extrapolation method with 100 and 200 time steps for cases ix, xii (small volatility and short maturity cases)
- Extrapolation method with 200 and 400 time steps for cases x, xi (large volatility and short maturity cases)

(Extrapolation method: e.g. Gobet[2])
Finally, the approximate values of portfolios of plain-vanilla and digital options are computed by the fifth-order normal approximation of an asymptotic expansion scheme. ([4],[7]) See those papers for the detail of the method.

3 Result

Tables 5 and 6 give the result. Generally, it shows that the method provides good approximations of barrier option prices. In particular, use of digital options seems effective for approximations of in-the-money knock-out call option prices.

<table>
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<tr>
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<th>E</th>
<th>Diff E</th>
<th>Plain Vanilla</th>
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<td>7.000</td>
<td>0.004</td>
<td>7.033</td>
</tr>
<tr>
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<td>8.728</td>
<td>8.729</td>
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<td>III</td>
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<td>8.291</td>
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<tr>
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<td>9.630</td>
<td>9.634</td>
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References


orders. Then, Here, where \( Y \) denotes the solution to the differential equation; \( dY_t = \partial V_0(X_t^{(0)}(0))dt + e V_0(X_t^{(0)}(0))dW_t \) for every \( \epsilon \geq 0 \). \( g_0T \) is the \( 0 \)-th element of \( A_1T \) and \( g_{\epsilon T} \) are smooth functions with bounded derivatives of all orders.

In the following appendix, we give a brief summary of an asymptotic expansion method following [7]. See the paper for the detail.

### A An Asymptotic Expansion in a General Markovian Setting

Let \((W, P)\) be the \( r\)-dimensional Wiener space. We consider a \( d\)-dimensional diffusion process \( X_t^{(c)} = (X_t^{(c)}(1), \ldots, X_t^{(c)}(d)) \) which is the solution to the following stochastic differential equation:

\[
dX_t^{(c),i} = V_0^j(X_t^{(c)})dt + eV_0^i(X_t^{(c)})dW_t \quad (i = 1, \ldots, d)
\]

\[
X_0^{(c)} = x_0 \in \mathbb{R}^d
\]

where \( W = (W^1, \ldots, W^r) \) is a \( r\)-dimensional standard Wiener process, and \( \epsilon \in (0, 1) \) is a known parameter. Let

Suppose that \( V_0 = (V_{0}^1, \ldots, V_{0}^d) : \mathbb{R}^d \times (0, 1] \mapsto \mathbb{R}^d \) and \( V = (V^1, \ldots, V^d) : \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^r \) satisfy some regularity conditions(e.g. \( V_0 \) and \( V \) are smooth functions with bounded derivatives of all orders.)

Next, suppose that a function \( g : \mathbb{R}^d \mapsto \mathbb{R} \) to be smooth and all derivatives have polynomial growth orders. Then, \( g(X_T^{(c)}) \) has its asymptotic expansion:

\[
g(X_T^{(c)}) \sim g_0T + \epsilon g_{\epsilon T} + \cdots
\]

in \( L^p \) for every \( p > 1 \)(or in \( D^\infty \)) as \( \epsilon \downarrow 0 \). \( g_{\epsilon T} \in D^\infty (n = 0, 1, \cdots) \), the coefficients in the expansion, can be obtained by Taylor’s formula and represented based on multiple Wiener-Ito integrals.

Let \( A_{kt} = \frac{\partial^k X_t^{(c)}}{\partial \epsilon^k} \bigg|_{\epsilon=0} \) and \( A_{ikt} \), \( i = 1, \ldots, d \) denote the \( i \)-th elements of \( A_{kt} \). In particular, \( A_{1t} \) is represented by

\[
A_{1t} = \int_0^t Y_s^{-1} \left( \partial V_0(X_u^{(0)})0du + V(X_u^{(0)})dW_u \right)
\]

where \( Y \) denotes the solution to the differential equation;

\[
dY_t = \partial V_0(X_t^{(0)}0)dt; \quad Y_0 = I_d.
\]

Here, \( \partial V_0 \) denotes the \( d \times d \) matrix whose \((j,k)\)-element is \( \partial_{ij} V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_j} \). \( V_0^j \) is the \( j \)-th element of \( V_0 \), and \( I_d \) denotes the \( d \times d \) identity matrix.

For \( k \geq 2 \), \( A_{ikt} \), \( i = 1, \ldots, d \) is recursively determined by the following:

\[
A_{ikt} = \int_0^t \frac{\partial^k V_0^j(X_s^{(0)}(0))ds}{\partial \epsilon^k} + \sum_{l=1}^k \frac{k!}{l!} \sum_{\beta=1} \sum_{\eta=1} \sum_{\beta=1} \int_0^t \sum_{\beta=1} \sum_{\eta=1} \sum_{\beta=1} \frac{1}{\beta!} \partial_{\beta_1,\ldots,\beta_d} \partial_{\beta_1,\ldots,\beta_d} V_0^j(X_s^{(0)}(0)) \prod_{j=1}^{\beta} A_{\beta_1,\ldots,\beta_d} \partial_{\beta_1,\ldots,\beta_d} dW_s \]

7
where \( \partial_i^\beta = \frac{\partial^\beta}{\partial x_1^{l_1} \cdots \partial x_d^{l_d}} \) and 
\[
L_{\beta,k} = \left\{ \tilde{l}_\beta = (l_1, \ldots, l_\beta); l_j \geq 0 (j = 1, \ldots, \beta), \sum_{j=1}^\beta l_j = k \right\}.
\]

Hence, Assumption 1 means that the distribution of \( f \) and \( g \) follow a normal distribution with variance \( \Sigma_T \) and make the following assumption:

For \( n \geq 2 \), \( g_{nT} \) is expressed as follows:

\[
g_{nT} = \sum_{n!} \frac{n!}{s_1! \cdots s_n!} \prod_{l=1}^n \left( \frac{1}{n!} \right)^{s_l} \sum_{p^1 \in P_s} \left( \frac{s_1!}{p_1^{s_1} \cdots p_d^{s_d}} \right) \partial_1^{s_{l_1}} \cdots \partial_d^{s_{l_d}} g(X_T^{(0)}) \prod_{i=1}^d (A_i^{(0)})^i (6)
\]

where

\[
S_n := \left\{ \tilde{s} = (s_1, \ldots, s_n); s_l \geq 0 (l = 1, \ldots, n), \sum_{l=1}^n l_{s_l} = n \right\},
\]

\[
P_s := \left\{ \tilde{p} = (p_1, \ldots, p_d); p_i \geq 0 (i = 1, \ldots, d), \sum_{i=1}^d p_i = s \right\}.
\]

Next, normalize \( g(X_T^{(0)}) \) to

\[
G^{(\epsilon)} = \frac{g(X_T^{(0)}) - g_{0T}}{\epsilon}
\]

for \( \epsilon \in (0, 1] \). Then,

\[
G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots
\]

in \( L^p \) for every \( p > 1 \)(or in \( D^\infty \)). Moreover, let

\[
\tilde{V}(x, t) = \left( \partial g(x) \right)'[Y_T Y_t^{-1} V(x)]
\]

and make the following assumption:

(Assumption 1) \( \Sigma_T = \int_0^T \tilde{V}(X_T^{(0)}, t) \tilde{V}(X_T^{(0)}, t)' dt > 0 \).

Note that \( g_{1T} \) follows a normal distribution with variance \( \Sigma_T \); the density function of \( g_{1T} \) denoted by \( f_{g_{1T}}(x) \) is given by

\[
f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi \Sigma_T}} \exp \left( -\frac{x^2}{2\Sigma_T} \right).
\]

Hence, Assumption 1 means that the distribution of \( g_{1T} \) does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, let \( S \) be the real Schwartz space of rapidly decreasing \( C^\infty \)-functions on \( \mathbb{R} \) and \( S' \) be its dual space that is the space of the Schwartz tempered distributions. Next, take \( \Phi \in S' \). Then, by Watanabe theory\([Watanabe 8], [Yoshida 9]\) \( \Phi(G^{(\epsilon)}) \) has an asymptotic expansion in \( D^\infty \)(a fortiori in \( D^\infty \)) as \( \epsilon \downarrow 0 \). In other words, the expectation of \( \Phi(G^{(\epsilon)}) \) is expanded around \( \epsilon = 0 \) as follows:

\[
E[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} E \left[ \Phi^{(m)}(g_{1T}) \left\{ \sum_{k \in K_j, m} C_{j,m,k} \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n} \right\} \right] + o(\epsilon^N)
\]
\[ \Phi(\Omega, F, P) \text{ be a probability space. Suppose that } \quad X \in L^2(\Omega, P) \quad \text{and } \quad Z \text{ is a random variable with Gaussian distribution with mean } 0 \quad \text{and variance } \Sigma. \quad \text{Then, the conditional expectation } \quad E[X | Z = x] \quad \text{has following expansion in } L^2(R, \mu) \quad \text{where } \mu \text{ is the Gaussian measure on } R \quad \text{with mean } 0 \quad \text{and variance } \Sigma: \]

\[ E[X | Z = x] = \sum_{n=0}^{\infty} a_n H_n(x; \Sigma) \quad \text{(8)} \]

where \( H_n(x; \Sigma) \) is the Hermite polynomial of degree \( n \) which is defined as

\[ H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma} \]

and coefficients \( a_n \) are given by

\[ a_n = \frac{1}{n!} \left( \frac{1}{\Sigma} \right)^n \frac{\partial^n}{\partial \xi^n} \left. \left\{ e^{\xi^2/\Sigma} E[e^{\xi Z} X] \right\} \right|_{\xi = 0} \quad \text{(9)} \]

- Recall \( \hat{g}_{1T} \) is defined as

\[ \hat{g}_{1T} = (\partial g(X_{0}^{(0)}))' \int_0^T [Y_t Y_t^{-1} V(X_{t}^{(0)})]dW_t = g_{1T} - C \]

where

\[ C = (\partial g(X_{T}^{(0)}))' \int_0^T Y_t Y_t^{-1} \partial_t V_0(X_{t}^{(0)}), 0)dt, \]

and define

\[ Z_{T}^{(N)} = \exp\{i\xi \hat{g}_{1T} + \frac{\xi^2}{2} \Sigma T\}. \]
Then, from Lemma 1 and (7), we have the following expression of $E[\Phi(G^{(\epsilon)})]$:

$$E[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbb{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^{m} \frac{\partial^{m}}{\partial x^{m}} \left( E \left[ X^{j,m,k} | g_{\epsilon}T = x - C \right] f_{g_{\epsilon}T}(x) \right) dx + o(\epsilon^{N})$$

$$= \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbb{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^{m} \frac{\partial^{m}}{\partial x^{m}} \left( \sum_{l=0}^{j+m} a_{l}^{j,m,k} H_{l}(x - C; \Sigma_{T}) f_{g_{\epsilon}T}(x) \right) dx + o(\epsilon^{N})$$

where

$$a_{l}^{j,m,k} = \frac{1}{l!} \frac{1}{(i\Sigma_{T})!} \frac{\partial^{l}}{\partial \xi^{l}} \left[ E \left[ X^{j,m,k} Z_{\Sigma_{T}}^{(\xi)} \right] \right].$$

In particular, let $\Phi$ be the delta function at $x \in \mathbb{R}$, $\delta_{x}$, we obtain the asymptotic expansion of density of $G^{(\epsilon)}$:

$$f_{G^{(\epsilon)}}(x) = E[\delta_{x}(G^{(\epsilon)})] = \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \sum_{k \in K_{j,m}} C^{j,m,k} \frac{(-1)^{m}}{m!} \frac{\partial^{m}}{\partial x^{m}} \left( H_{l}(x - C; \Sigma_{T}) f_{g_{\epsilon}T}(x) \right) + o(\epsilon^{N}).$$

Here it is noted that with this expression we now need to compute unconditional expectations $E[X^{j,m,k} Z_{\Sigma_{T}}^{(\xi)}]$ instead of the conditional expectations.

**Remark**

We remark the relation between our method and an approach presented by Takahashi[5],[6] in which the density function of $G^{(\epsilon)}$ is derived by Fourier inversion of its formally expanded characteristic function. In fact, [5],[6] formally expanded $\Psi_{G^{(\epsilon)}}(\xi) = E[e^{i\xi G^{(\epsilon)}}]$ as

$$\Psi_{G^{(\epsilon)}}(\xi) = E \left[ e^{i\xi G^{(\epsilon)}} \right] = e^{-\frac{\xi^{2}}{2} \Sigma_{T} + i\xi C} \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \sum_{k \in K_{j,m}} C^{j,m,k} (i\xi)^{m} E \left[ X^{j,m,k} Z_{\Sigma_{T}}^{(\xi)} \right] + o(\epsilon^{N})$$

and computed the conditional expectations in this expansion. Then, $f_{G^{(\epsilon)}}(x)$ was derived by Fourier inversion of $\Psi_{G^{(\epsilon)}}(\xi)$;

$$f_{G^{(\epsilon)}}(x) = \mathcal{F}^{-1}(\Psi_{G^{(\epsilon)}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \Psi_{G^{(\epsilon)}}(\xi) d\xi.$$

This approach is completely equivalent to our method. From (10) we obtain

$$f_{G^{(\epsilon)}}(x) = E^{J} \left[ e^{i\xi G^{(\epsilon)}} \right] = \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \sum_{k \in K_{j,m}} a_{l}^{j,m,k} \frac{C^{j,m,k}}{m!} \left( -1 \right)^{m} \frac{\partial^{m}}{\partial x^{m}} \left( H_{l}(x - C; \Sigma_{T}) f_{g_{\epsilon}T}(x) \right) + o(\epsilon^{N})$$

$$= \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \sum_{k \in K_{j,m}} a_{l}^{j,m,k} \frac{C^{j,m,k}}{m!} \mathcal{F}^{-1} \left( (i\xi)^{m} (i\xi \Sigma_{T})^{l} e^{-\frac{\xi^{2}}{2} \Sigma_{T} + i\xi C} \right) + o(\epsilon^{N})$$

$$= \mathcal{F}^{-1} \left( e^{-\frac{\xi^{2}}{2} \Sigma_{T} + i\xi C} \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \sum_{k \in K_{j,m}} C^{j,m,k} (i\xi)^{m} \sum_{l=0}^{j+m} a_{l}^{j,m,k} (i\xi)^{l} \right) + o(\epsilon^{N})$$

$$= \mathcal{F}^{-1} \left( e^{-\frac{\xi^{2}}{2} \Sigma_{T} + i\xi C} \sum_{j=0}^{N} e^{j} \sum_{m=0}^{j} \frac{1}{m!} \sum_{k \in K_{j,m}} C^{j,m,k} (i\xi)^{m} E \left[ X^{j,m,k} Z_{\Sigma_{T}}^{(\xi)} \right] \right) + o(\epsilon^{N}).$$
In this subsection, we propose a new computational method for the asymptotic expansion of the

B.1 Asymptotic Expansion of Density Function

First, we write down the equation (10) more explicitly up to expansion up to a system of ordinary differential equations that is solved easily, and derive an expression of the expansion of its solution in terms of $T_a$. We have

For example,

where coefficients $a_i^{j,m,k}$ are given by

$a_i^{0,0,0} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[Z_T^{(\xi)}] \right\}$

$a_i^{1,1,1} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{2,T} Z_T^{(\xi)}] \right\}$

$a_i^{2,1,0} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{3,T} Z_T^{(\xi)}] \right\}$

$a_i^{2,2,2} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{2,T}^2 Z_T^{(\xi)}] \right\}$

$a_i^{3,1,0,0} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{4,T} Z_T^{(\xi)}] \right\}$

$a_i^{3,2,1,0} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{2,T} g_{3,T} Z_T^{(\xi)}] \right\}$

$a_i^{3,3,0,0} = \frac{1}{i!} \left( \frac{\partial}{\partial x} \right)^i \left|_{x=0} \right. \left\{ E[g_{2,T}^3 Z_T^{(\xi)}] \right\}.

(13)

Since $E[Z_T^{(\xi)}] = 1$, we have $a_i^{0,0,0} = 1$ and $a_i^{0,0,0} = 0$ for $l \geq 1$. The other expectations above are expressed in terms of $A_{k,T}$ and $Z_T^{(\xi)}$. For example,
\[
E[g_{2T}Z_T^{(g)}] = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) E[A_{1T}A_{1T}^* Z_T^{(g)}] + \frac{1}{2} \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) E[A_{1T}^* Z_T^{(g)}]
\]

\[
E[g_{3T}Z_T^{(g)}] = \frac{1}{6} \sum_{i,j,k,l=1}^{d} \partial_i \partial_j \partial_k \partial_l g(X_T^{(0)}) E[A_{1T}A_{1T}^* A_{1T}^* Z_T^{(g)}] + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) E[A_{1T}^* A_{1T}^* Z_T^{(g)}]
\]

\[
E[g_{2T}^2 Z_T^{(g)}] = \frac{1}{4} \sum_{i,j,k,l=1}^{d} \partial_i \partial_j g(X_T^{(0)}) \partial_k \partial_l g(X_T^{(0)}) E[A_{1T}A_{1T}^* A_{1T}^* A_{1T}^* Z_T^{(g)}]
\]

where \( A_{1T} \) is given by (4), and \( A_{2T} \) and \( A_{3T} \) are expressed as

\[
A_{2T} = \int_0^t Y_t Y_u^{-1} \left( \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)},0) A_{1u}^j A_{1u}^k du + \sum_{j=1}^{d} \partial_j V_0(X_u^{(0)},0) A_{1u}^j du \right)
\]

\[
A_{3T} = \int_0^t Y_t Y_u^{-1} \left( \sum_{j,k,l=1}^{d} \partial_j \partial_k \partial_l V_0(X_u^{(0)},0) A_{1u}^j A_{1u}^k A_{1u}^l du + \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)},0) A_{1u}^j A_{1u}^k du \right)
\]

\[
+ 3 \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)},0) A_{1u}^j A_{1u}^k du + \sum_{j=1}^{d} \partial_j V_0(X_u^{(0)},0) A_{1u}^j du
\]

\[
+ 3 \sum_{j,k=1}^{d} \partial_j \partial_l V_0(X_u^{(0)},0) A_{1u}^j A_{1u}^l du + 3 \sum_{j=1}^{d} \partial_j V_0(X_u^{(0)},0) A_{1u}^j du
\]

\[
+ 3 \sum_{j,k=1}^{d} \partial_j \partial_l V(X_u^{(0)}) A_{1u}^j A_{1u}^l du + 3 \sum_{j=1}^{d} \partial_j V(X_u^{(0)}) A_{1u}^j du
\]

• Here, we redefine \( \hat{g}_t = \{ \hat{g}_t ; t \in \mathbb{R}^+ \} \) and \( Z_t^{(g)} = \{ Z_t^{(g)} ; t \in \mathbb{R}^+ \} \) as a stochastic processes

\[
\hat{g}_t = \int_0^t \hat{V}(X_u^{(0)}, u) du
\]

and

\[
Z_t^{(g)} = \exp \{ i \xi \hat{g}_t + \frac{\xi^2}{2} \Sigma_t \},
\]

respectively where

\[
\hat{V}(x, t) = (\partial g(X_T^{(0)})) \left[ Y_t Y_t^{-1} V(x) \right].
\]

• We define \( \eta_{i,1}^i, \eta_{i,2}^i, \eta_{i,1}, \eta_{i,2}, \eta_{i,3}^i, \eta_{i,3}^k, \eta_{i,1}^k, \eta_{i,2}^k, \eta_{i,3}^i, \eta_{i,3}^k, \eta_{i,1}, \eta_{i,2}, \eta_{i,3}^i, \eta_{i,3}^k, \eta_{i,1}, \eta_{i,2}, \eta_{i,3}^i, \eta_{i,3}^k, \eta_{i,1}, \eta_{i,2}, \eta_{i,3}^i, \eta_{i,3}^k, \eta_{i,1}, \eta_{i,2}, \eta_{i,3}^i, \eta_{i,3}^k \) as

\[
\eta_{i,1}^i(t) := E[A_{1T}^i Z_t^{(g)}], \quad \eta_{i,2}^i(t) := E[A_{1T}^i Z_t^{(g)}], \quad \eta_{i,3}^i(t) := E[A_{1T}^i A_{1T}^i Z_t^{(g)}],
\]

12
We derive a system of ordinary differential equations of \( \eta \). Here, we have the following ordinary differential equation of \( V_t \):

\[
V_t = A_t^i A_{t}^{i,k} Z_t^{(\xi)} \]

\( \eta_{5,1}(t) := E[\eta_{5,1}^{(0)}], \quad \eta_{5,2}^{i,k}(t) := E[A_1^i A_2^k Z_t^{(\xi)}], \quad \eta_{5,3}^{i,k,l}(t) := E[A_1^i A_1^k A_2^l Z_t^{(\xi)}], \quad \eta_{5,4}^{i,k,l,m}(t) := E[A_2^i A_2^k A_2^l A_2^m Z_t^{(\xi)}], \\
\eta_{5,5}^{i,k,l,m,n}(t) := E[A_1^i A_1^k A_1^l A_2^m A_2^n Z_t^{(\xi)}], \quad \eta_{5,5}^{i,k,l,m,n,o}(t) := E[A_2^i A_2^k A_2^l A_2^m A_2^n A_2^o Z_t^{(\xi)}]. \tag{14}
\]

- We derive a system of ordinary differential equations of \( \eta \). In the followings, for simplicity, we assume that \( V_0 \) doesn’t depend on \( \epsilon \), and write \( V_0(x, \epsilon) \) as \( V_0(x) \).

- Consider the evaluation of \( \eta_{5,1}^{(1)}(t) = E[A_2^i A_2^k Z_t^{(\xi)}] \) which appears in the \( \epsilon \)-order. Applying Ito’s formula to \( A_2^i Z_t^{(\xi)} \), we have:

\[
d(A_2^j Z_t^{(\xi)}) = A_2^j dZ_t^{(\xi)} + Z_t^{(\xi)} dA_2^j + dA_2^j dZ_t^{(\xi)}
\]

\[
= \left\{ 2(i\xi) \sum_{j' = 1}^d A_{i,j'} Z_t^{(\xi)} \frac{\partial}{\partial j'} \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) + \sum_{j' = 1}^d A_{i,j'} Z_t^{(\xi)} \frac{\partial}{\partial j'} V_0^j(X_t^{(0)}) \right\} dt
\]

\[
+ \left\{ (i\xi) A_2^j Z_t^{(\xi)} \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) + 2 \sum_{j' = 1}^d A_{i,j'} Z_t^{(\xi)} \frac{\partial}{\partial j'} V_0^j(X_t^{(0)}) \right\} dW_t.
\]

- Note that the second and third terms are martingales. Thus, taking the expectation on both sides, we have the following ordinary differential equation of \( \eta_{5,1}^{(1)} \):

\[
\frac{d}{dt} \eta_{5,1}^{(1)}(t) = 2(i\xi) \sum_{j' = 1}^d \eta_{5,1}^{(1)}(t) \frac{\partial}{\partial j'} \frac{\partial}{\partial j} V_0^j(X_t^{(0)})
\]

\[
+ \sum_{j' = 1}^d \eta_{5,1}^{(1)}(t) \frac{\partial}{\partial j'} V_0^j(X_t^{(0)}) + \sum_{j' = 1}^d \sum_{k' = 1}^d \eta_{5,2}^{(1)}(t) \frac{\partial}{\partial j'} \frac{\partial}{\partial k'} V_0^j(X_t^{(0)}).
\]

- Here, \( \eta_{5,1}^{(j)}(j = 1, \ldots, d) \) appearing in the right hand side of above ODE are evaluated in the similar manner:

\[
d(A_2^i Z_t^{(\xi)}) = A_2^i dZ_t^{(\xi)} + Z_t^{(\xi)} dA_2^i + A_2^i dZ_t^{(\xi)}
\]

\[
= \left\{ (i\xi) Z_t^{(\xi)} \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) + \sum_{j' = 1}^d A_{i,j'} Z_t^{(\xi)} \frac{\partial}{\partial j'} V_0^j(X_t^{(0)}) \right\} dt
\]

\[
+ \left\{ (i\xi) A_2^i Z_t^{(\xi)} \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) + Z_t^{(\xi)} \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) \right\} dW_t,
\]

hence, we have

\[
\frac{d}{dt} \eta_{5,1}^{(1)}(t) = (i\xi) \frac{\partial}{\partial j} V_0^j(X_t^{(0)}) + \sum_{j' = 1}^d \eta_{5,1}^{(1)}(t) \frac{\partial}{\partial j'} V_0^j(X_t^{(0)}).
\]
\( \eta_{2,2} \) is evaluated in the same way.

\[
\frac{d}{dt} \eta_{2,2}^j(t) = (i\xi) \{ \eta_{1,1}^j(t) \tilde{V}(X_t^{(0)}, t)V^j(X_t^{(0)})' + \eta_{1,1}^j(t)\tilde{V}(X_t^{(0)}, t)V^k(X_t^{(0)})' \}
+ V^j(X_t^{(0)})V^k(X_t^{(0)})' + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{2,2}^{i',k'}(t)\partial_{i'}V_0^j(t)\partial_{k'}V_0^k(t)
\]

- Note that \( \eta_{1,1}^j, \eta_{2,2}^{i,k} \) and \( \eta_{2,1}^j \) are evaluated in the following order:

\[
\eta_{1,1}^j \rightarrow \eta_{2,2}^{i,k} \rightarrow \eta_{2,1}^j.
\]

- The key observation is that each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. Hence, one can easily solve (analytically or numerically) the system of ODEs and evaluate expectations.

**Proposition 1** For \( \eta_{i,m,k} \) defined in (14), the following system of ordinary differential equations holds:

\[
\frac{d}{dt} \eta_{1,1}^i(t) = (i\xi)\tilde{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + \sum_{i'=1}^{d} \eta_{1,1}^{i'}(t)\partial_{i'}V_0^i(t)
\]

\[
\frac{d}{dt} \eta_{2,1}^i(t) = 2(i\xi) \sum_{i'=1}^{d} \eta_{2,1}^{i'}(t)\tilde{V}(X_t^{(0)}, t)\partial_{i'}V^i(X_t^{(0)})'
+ \sum_{i'=1}^{d} \eta_{2,1}^{i'}(t)\partial_{i'}V_0^i(t) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{2,2}^{i',k'}(t)\partial_{i'}\partial_{k'}V_0^i(t)
\]

\[
\frac{d}{dt} \eta_{2,2}^{i,k}(t) = (i\xi) \{ \eta_{1,1}^{k}(t)\tilde{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + \eta_{1,1}^{i}(t)\tilde{V}(X_t^{(0)}, t)V^k(X_t^{(0)})' \}
+ V^i(X_t^{(0)})V^k(X_t^{(0)})' + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{2,2}^{i',k'}(t)\partial_{i'}V_0^i(t)\partial_{k'}V_0^k(t)
\]

\[
\frac{d}{dt} \eta_{3,1}^i(t) = (i\xi) \left\{ 3 \sum_{i'=1}^{d} \eta_{2,1}^{i'}(t)\tilde{V}(X_t^{(0)}, t)\partial_{i'}V^i(X_t^{(0)})'
+ 3 \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{2,2}^{i',k'}(t)\tilde{V}(X_t^{(0)}, t)\partial_{i'}\partial_{k'}V^i(X_t^{(0)})' \right\}
+ \eta_{3,1}^{i'}(t)\partial_{i'}V_0^i(t) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{2,2}^{i',k'}(t)\partial_{i'}\partial_{k'}V_0^i(t)
+ \sum_{i'=1}^{d} \sum_{k'=1}^{d} \sum_{l'=1}^{d} \eta_{3,3}^{i',k',l'}(t)\partial_{i'}\partial_{k'}\partial_{l'}V_0^i(t)
\]

\[
\frac{d}{dt} \eta_{3,2}^{i,k}(t) = (i\xi) \left\{ \eta_{2,1}^{k}(t)\tilde{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + 2 \sum_{i'=1}^{d} \eta_{2,2}^{i',k}(t)\tilde{V}(X_t^{(0)}, t)\partial_{i'}V^k(X_t^{(0)})' \right\}
+ 2 \sum_{i'=1}^{d} \eta_{2,2}^{i',k}(t)\partial_{i'}V_0^k(t) + \sum_{k'=1}^{d} \eta_{3,2}^{i,k}(t)\partial_{k'}V_0^k(t)
+ \sum_{k'=1}^{d} \eta_{3,2}^{i,k}(t)\partial_{k'}V_0^k(t)
\]
We summarize the discussion above as the following theorem:

The asymptotic expansion of density of \( G^{(t)} \) up to \( c^3 \)-order is given by

\[
f_{G^{(t)}}(x) = f_{g_{1T}}(x) + \epsilon \left\{ \sum_{l=1}^{3} C_{1l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x) + \epsilon^2 \left\{ \sum_{l=1}^{6} C_{2l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x) + \epsilon^3 \left\{ \sum_{l=1}^{9} C_{3l} H_l(x; \Sigma_T) \right\} f_{g_{1T}}(x) + o(\epsilon^3).
\]

where

\[
C_{1l} = \Sigma_T a_{l-1,1}^{1,1},
\]
\[
C_{21} = \Sigma_T a_{1}^{2,1,0,1},
\]
\[
C_{2l} = \Sigma_T a_{l-1}^{2,1,0,1} + \frac{1}{2} \Sigma_T a_{l-2}^{2,2,2,2}(l \geq 2),
\]
\[
C_{31} = \Sigma_T a_{1}^{3,1,0,0,1},
\]
\[
C_{32} = \Sigma_T a_{1}^{3,1,0,0,1} + \frac{1}{2} \Sigma_T a_{1}^{3,2,2,2,2}(l \geq 2),
\]
\[
C_{3l} = \Sigma_T a_{l-1}^{3,1,0,0,1} + \frac{1}{2} \Sigma_T a_{l-2}^{3,2,2,2,2} + \frac{1}{6} \Sigma_T a_{l-3}^{3,3,2,2,2}(l \geq 3).
\]

Here, \( a_{l}^{i,m,k} \) are given by (13):

\[
a_{l}^{0,0,0} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[Z_T^0] \},
\]
\[
a_{l}^{1,1,1} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{1T}Z_T^1] \},
\]
\[
a_{l}^{2,1,0,1} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{3T}Z_T^1] \},
\]
\[
a_{l}^{2,2,2,0} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{3T}Z_T^2] \},
\]
\[
a_{l}^{3,1,0,0,1} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{4T}Z_T^1] \},
\]
\[
a_{l}^{3,2,2,1,0} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{2T}g_{3T}Z_T^1] \},
\]
\[
a_{l}^{3,3,3,0,0} = \frac{1}{l!} \frac{1}{(\Sigma_T)^l} \frac{\partial^l}{\partial \xi^l} \left|_{\xi=0} \right\} \{ \mathbb{E}[g_{2T}g_{3T}Z_T^0] \}.
\]
The expectations in (13) are obtained as the solutions to the system of ordinary differential equations given in Proposition 1.

B.2 Asymptotic Expansion of Option Prices

- We apply the asymptotic expansion to option pricing. We consider the plain vanilla option on the underlying asset \( g(X_T^{(0)}) \) where the dynamics of \( X_T^{(0)} \) is given by (3).
- For example, an asymptotic expansion up to \( \epsilon^{N+1} \) of a call option price at time 0 with maturity \( T \) and strike price \( K \) where \( K = X_T^{(0)} - \epsilon y \) for arbitrary \( y \in \mathbb{R} \) is given by

\[
C(K, T) = \epsilon P(0, T) \int_{-y}^{\infty} (x + y) f_{g^{(1)}, N}(x) \, dx + o(\epsilon^{N+1}).
\]

Here, \( P(0, T) \) denotes the price at time 0 of a zero coupon bond with maturity \( T \) and \( f_{g^{(1)}, N} \) is the normal asymptotic expansion of density of \( G^{(1)}(\epsilon) \) up to \( \epsilon^N \)-th order given by (10):

\[
f_{g^{(1)}, N}(x) = \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \sum_{k \in \mathbb{K}} \sum_{l=0}^{j+m} a_{j,m,k,l} \frac{(-1)^m}{m!} \partial_x^m \{ H_l(x - C; \Sigma_T) f_{g^{(1)}}(x) \}
\]

- In particular, using Theorem 1, an asymptotic expansion up to \( \epsilon^4 \) of a call option price at time 0 with maturity \( T \) and strike price \( K \) where \( K = X_T^{(0)} - \epsilon y \) for arbitrary \( y \in \mathbb{R} \) is expressed as

\[
C(K, T) = \epsilon P(0, T) \int_{-y}^{\infty} (x + y) f_{g^{(1)}}(x) \, dx
+ \epsilon^2 P(0, T) \int_{-y}^{\infty} (x + y) \left\{ \sum_{l=1}^{3} C_{1l} H_l(x; \Sigma_T) \right\} f_{g^{(1)}}(x) \, dx
+ \epsilon^3 P(0, T) \int_{-y}^{\infty} (x + y) \left\{ \sum_{l=1}^{6} C_{2l} H_l(x; \Sigma_T) \right\} f_{g^{(1)}}(x) \, dx
+ \epsilon^4 P(0, T) \int_{-y}^{\infty} (x + y) \left\{ \sum_{l=1}^{9} C_{3l} H_l(x; \Sigma_T) \right\} f_{g^{(1)}}(x) \, dx
+ o(\epsilon^4).
\]

- Integrals appeared in the right hand side can be calculated by following formulas related to the Hermite polynomial

\[
\int_{-y}^{\infty} H_k(x; \Sigma) f_{g^{(1)}}(x) \, dx = \Sigma H_{k-1}(-y; \Sigma) f_{g^{(1)}}(y) \ (k \geq 1),
\]
\[
\int_{-y}^{\infty} x H_k(x; \Sigma) f_{g^{(1)}}(x) \, dx = -\Sigma y H_{k-1}(-y; \Sigma) f_{g^{(1)}}(y)
+ \Sigma^2 H_{k-2}(-y; \Sigma) f_{g^{(1)}}(y) \ (k \geq 2).
\]