The Asymptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims *

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August 1999
(Fourth Revision)

Abstract

We propose a new methodology for the valuation problem of financial contingent claims when the underlying asset prices follow a general class of continuous Itô processes. Our method can be applied to a wide range of valuation problems including complicated contingent claims associated with the term structure of interest rates. We illustrate our method by giving two examples: the valuation problems of swaptions and average (Asian) options for interest rates. Our method gives some explicit formulae for solutions, which are suffciently numerically accurate for practical purposes in most cases. The continuous stochastic processes for spot interest rates and forward interest rates are not necessarily Markovian nor diffusion processes in the usual sense; nevertheless our approach can be rigorously justified by the Malliavin-Watanabe Calculus in stochastic analysis.

Key Words

Derivatives, Term Structure of Interest Rates, Asymptotic Expansion, Small Disturbance Asymptotics, Malliavin-Watanabe Calculus

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*This paper is a revised version of Discussion Paper No.95-F-19, Faculty of Economics, University of Tokyo, which was presented at the Japan-Russia Symposium on Probability Theory and Mathematical Statistics held at Tokyo in July 1995. We thank N.Yoshida and S.Kusuoka for some discussions on the related issues. Also we thank the referee and an associate editor of this journal for their helpful comments on the previous versions. However, we are responsible for all possible errors in this paper.

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1 Introduction

In the past decade various contingent claims have been introduced and actively traded in financial markets. In particular, various types of interest rates based contingent claims have appeared and attracted attention in financial markets. This paper presents a new methodology which is applicable to the valuation problem of financial contingent claims such as options, swaps, and other derivative securities when the underlying asset prices follow the general class of continuous It$\hat{o}$ processes. Our method is especially useful for the pricing problem of interest rate based derivatives when the underlying forward rates follow a general class of continuous It$\hat{o}$ processes.

In the valuation problem of financial contingent claims, it has been known that we can rarely obtain explicit formulae on solutions when the underlying assets follow the general class of continuous It$\hat{o}$ processes. This is particularly evident for contingent claims based on the term structure of interest rates because their payoff functions are usually complicated functionals of the underlying asset prices and the term structure of interest rates must satisfy the strong restrictions implied by fundamental economic theory. In order to cope with these problems, two methods, called the partial differential equation (PDE) approach and the Monte Carlo (MC) approach have been widely known and used for practical valuation problems. (See Duffie (1992) or Hull (1993) for the details of these methods.) The asymptotic expansion approach we are proposing in this paper is different from these conventional methods. As we shall show later, our method has several advantages compared to those existing methods.

The asymptotic expansion approach is based on the key empirical observation on many asset prices, including interest rates, that the observed and estimated volatilities for financial asset prices may vary over time, but they are not very large in comparison with the observed levels of asset prices. This observation has even been true for the stock prices whose volatilities are relatively large in comparison with other financial prices. It was using this key observation that Kunitomo and Takahashi (1992) had developed the asymptotic theory called *Small Disturbance Asymptotics* [1] for solving the valuation problem of average (or Asian) options for foreign exchange rates when the volatility parameter goes to zero. They have proposed to use the limiting distribution as the first order approximation to the exact distribution of the payoff function of average options

[1] The small disturbance asymptotic theory had originally been developed for the analysis of simultaneous equation systems in econometrics. See Anderson (1977) or Kunitomo et. al. (1983) for the details.

when the underlying asset prices follow geometric Brownian motion. Although the approximations they proposed have given relatively accurate numerical values, they are not completely satisfactory in some cases for practical purposes. For the same valuation problem in Kunitomo and Takahashi (1992), Yoshida (1992a) has obtained further results on average options when the underlying asset prices follow geometric Brownian motion by using an asymptotic expansion technique originally developed for an application in statistics.

The main purpose of the present paper is to show that the asymptotic expansion method in small disturbance asymptotics can be effectively applicable to various valuation problems of contingent claims appearing when the underlying asset prices follow the general class of continuous Itô processes. In particular, we shall show that the asymptotic expansion approach is very simple, yet gives a unified approach to the problem of interest rate based contingent claims valuation. However, we shall point out that some economic considerations of theoretical restrictions on the structure of stochastic processes should be indispensable when we apply the asymptotic expansion method. In the term structure of interest rates, for instance, we require strong conditions on the form of their drift functions because of no-arbitrage theory. This implies that the continuous stochastic processes for spot interest rates and forward rates are not necessarily Markovian nor diffusion processes in the usual sense.

In a companion of the present paper, Takahashi (1997) [2] has systematically investigated the valuation problem of various contingent claims when the asset price $S(t)$ follows a diffusion process:

$$S(t) = S(0) + \int_0^t \mu(S(v), v)dv + \int_0^t \sigma^*(S(v), v)dB(v) , \tag{1.1}$$

where $\mu(S(v), v)$ and $\sigma^*(S(v), v)$ are the instantaneous mean and volatility functions, and $B(v)$ is the standard Brownian motion. It is evident that the Black-Scholes economy and the Cox-Ingersol-Ross model on the spot interest rate are special cases of this framework. In the simplest Black-Scholes economy, for instance, let $S^{(\epsilon)}(t)$ satisfy the integral equation of a diffusion process:

$$S^{(\epsilon)}(t) = S(0) + \int_0^t rS^{(\epsilon)}(v)dv + \epsilon \int_0^t \sigma(S^{(\epsilon)}(v), v)dB(v) , \tag{1.2}$$

[2] This is essentially based on Chapter 1 of an unpublished Ph.D. Dissertation (Takahashi (1995)) at Berkeley. It gives many numerical examples and discusses the validity of the *small disturbance* asymptotic expansion method when the asset price follows a general non-homogeneous diffusion process.

where $\sigma(S^{(\epsilon)}(v), v)$ is the volatility term with $0 < \epsilon \leq 1$ and r is the risk free (constant) rate. Then a *small disturbance* asymptotic theory can be constructed by considering the situation when $\epsilon \to 0$ and we can develop the valuation method of contingent claims based on $\{S^{(\epsilon)}(t)\}$ with the no-arbitrage theory.

As we shall explain in the Appendix, our method is not an ad-hoc approximation method because it can be rigorously justified by the Malliavin-Watanabe theory in stochastic analysis. The Malliavin-Watanabe Calculus has been developed as an infinite dimensional analysis of Wiener functional by several probablists in the last two decades. We intend to apply this powerful calculus on continuous stochastic processes to the valuation problem of financial contingent claims along the line developed by Watanabe (1987) and Yoshida (1992a). However, we should mention that the spot and forward interest rates in the term structure model are not necessarily Markovian in the usual sense while the existing asymptotic expansion methods initiated by Watanabe (1987) and refined by Yoshida (1992a) in stochastic analysis have been developed for the time homogeneous Markovian processes. Hence, we need to extend the existing results on the validity of the asymptotic expansion approach to the solution of certain stochastic integral equations for the interest rate processes. Since this extension to a class of non-Markovian continuous processes is not a trivial task, the asymptotic expansion approach developed in this paper would be also interesting to researchers in stochastic analysis.

Furthermore, as we shall illustrate in Section 4, the resulting formulae we shall derive for complicated contingent claims are numerically accurate in many practical situations. Thus the asymptotic expansion approach would be not only theoretically interesting, but also quite useful for researchers in financial economics and practitioners in financial markets.

In Section 2, we formulate the valuation problem of contingent claims based on the term structure of interest rates. In Section 3, we shall explain the asymptotic expansion approach for this problem and give some theoretical results. Then, in Section 4, we shall show some numerical results on interest rate derivatives as illustrative examples. Section 5 will summarize our results and provide concluding comments. Some mathematical details including useful formulae and the mathematical validity of our method via the Malliavin-Watanabe theory will be presented in Section 6.

2 The Valuation Problem of Interest Rate Based Contingent Claims

We consider a continuous time economy with a trading interval $[0, \bar{T}]$, where $\bar{T} < +\infty$; it is also complete in the proper economic sense. Let $P(t,T)$ denote the price of the discount bond at t with maturity date T $(0 \leq t \leq T \leq \bar{T} < +\infty)$. We use the notational convention that $P(T,T) = 1$ at maturity date $t = T$ for normalization. Let also $P(t,T)$ be continuously differentiable with respect to T and $P(t,T) > 0$ for $0 \leq t \leq T \leq \bar{T}$. Then the instantaneous forward rate at s for future date t $(0 \leq s \leq t \leq T)$ is defined by

$$
f(s,t) = -\frac{\partial \log P(s,t)}{\partial t} \quad . \tag{2.1}
$$

In the term structure model of interest rates we assume that a family of forward rate processes [3] $\{f(s,t)\}$ for $0 \leq s \leq t \leq T$ follow the stochastic integral equation

$$
\begin{aligned}
f(s,t) = f(0,t) \quad &+ \int_0^s \left[\sum_{i=1}^n \sigma_i^*(f(v,t),v,t) \int_v^t \sigma_i^*(f(v,y),v,y)dy \right] dv \\
&+ \int_0^s \sum_{i=1}^n \sigma_i^*(f(v,t),v,t) dB_i(v) \ ,
\end{aligned} \tag{2.2}
$$

where $f(0,t)$ are non-random initial forward rates, $\{B_i(v), i = 1, \cdots, n\}$ are n independent Brownian motions, and $\{\sigma_i^*(f(v,t),v,t), i = 1, \cdots, n\}$ are the volatility functions. We assume that the initial forward rates are observable and fixed. When $f(s,t)$ is continuous at $s = t$ for $0 \leq s \leq t \leq T$, the spot interest rate at t can be defined by $r(t) = f(t,t)$.

Consider the contingent claims based on the term structure of interest rates. There have been many interest rate based contingent claims developed and traded in financial markets. Most of those contingent claims can be regarded as functionals of bond prices with different maturities. Let $\{c_j, j = 1, \cdots, m\}$ be a sequence of non-negative coupon payments and $\{T_j, j = 1, \cdots, m\}$ be a sequence of payment periods satisfying the condition $0 \leq t \leq T_1 \leq \cdots \leq T_m \leq \bar{T}$. Then the price of the coupon bond with coupon payments $\{c_j, j = 1, \cdots, m\}$ at t should be given by [4]

[3] The restrictions in (2.2) we are imposing in this arbitrage-free formulation have been derived by Heath, Jarrow, and Morton (1992). There have been other approaches to the problem of the term structure of interest rates as discussed by Duffie (1992) or Hull (1993).

[4] We implicitly assume that there does not exist any default risk associated with bonds or any transaction costs.

$$P_{m,\{T_j\},\{c_j\}}(t) = \sum_{j=1}^{m} c_j P(t, T_j), \tag{2.3}$$

where $\{P(t, T_j), j = 1, \cdots, m\}$ are the prices of zero-coupon bonds with different maturities. For illustrations we give two examples of interest rate based contingent claims, which are important for practice in financial markets.

Example 1 : The payoff functions of options on the coupon bond with coupon payments $\{c_j, j = 1, \cdots, m\}$ at $\{T_j, j = 1, \cdots, m\}$ and swaptions expiring on date T $(0 < T \leq T_m)$ can be written as

$$V^{(1)}(T) = \left[P_{m,\{T_j\},\{c_j\}}(T) - K\right]^+, \tag{2.4}$$

and

$$V^{(2)}(T) = \left[K - P_{m,\{T_j\},\{c_j\}}(T)\right]^+, \tag{2.5}$$

where K is a fixed strike price and the max function is defined by $[X]^+ = \max\ (X, 0)$. $V^{(1)}(T)$ and $V^{(2)}(T)$ are the payoffs of the call options and put options on the coupon bond, respectively.

Example 2 : The yield of a zero coupon bond at t with time to maturity of τ $(0 < t < t + \tau < T_m)$ years is given by

$$L^{\tau}(t) = \left[\frac{1}{P(t, t+\tau)} - 1\right] \frac{1}{\tau}\ . \tag{2.6}$$

The payoffs of the options on average interest rates can be written as

$$V^{(3)}(T) = \left[\frac{1}{T}\int_0^T L^{\tau}(t)dt - K\right]^+ \tag{2.7}$$

and

$$V^{(4)}(T) = \left[K - \frac{1}{T}\int_0^T L^{\tau}(t)dt\right]^+, \tag{2.8}$$

where K is a fixed strike price. $V^{(3)}(T)$ and $V^{(4)}(T)$ are the payoffs of the call options and put options of average options on interest rates, respectively. Note that the options are on the average of L^{τ}, where L^{τ} is a constant duration, for example 6 month LIBOR rates.

The valuation problem of a contingent claim in a complete market can be simply defined as the determination of its "fair" value at financial markets. Let

$V(T)$ be the payoff of a contingent claim at the terminal period T. Then the standard martingale theory in financial economics predicts that the fair price of $V(T)$ at time t $(0 \leq t < T)$ should be given by

$$V_t(T) = \mathbf{E}_t\left[e^{-\int_t^T r(s)ds} V(T)\right] , \tag{2.9}$$

where $\mathbf{E}_t[\cdot]$ stands for the conditional expectation operator given the information available at t with respect to the equivalent martingale measure. When we do not impose the drift restrictions given by (2.2) for forward rate processes, we have to change the underlying probability measure into the equivalent martingale measure by the no-arbitrage theory and we can obtain the same results reported in this paper. Since this complicates our notations as well as explanations, we have directly imposed the restrictions given by (2.2).

3 The Asymptotic Expansion Approach

There are mainly two difficulties in the valuation problem of interest rate based contingent claims. First, the payoff functions and the discount factors are usually non-linear functionals of bonds with different maturities and the spot interest rate. More importantly, the coupon bond prices are also complicated functionals of instantaneous forward rate processes. Therefore except some special cases we cannot obtain explicit formulae for the solution in the valuation problems of interest rate based contingent claims.

In order to develop a new asymptotic expansion approach, we first re-formulate (2.2) and we assume that a family of instantaneous forward rate processes obey the stochastic integral equation:

(3.1)

$$\begin{aligned} f^{(\varepsilon)}(s,t) = f(0,t) \quad &+ \quad \varepsilon^2 \int_0^s \left[\sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy\right] dv \\ &+ \quad \varepsilon \int_0^s \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t)dB_i(v), \end{aligned}$$

where $0 < \varepsilon \leq 1$ and $0 \leq s \leq t \leq T \leq \bar{T}$. The volatility function $\sigma_i(f^{(\varepsilon)}(s,t),s,t)$ depends not only on s and t, but also on $f^{(\varepsilon)}(s,t)$ in the general case. Let $f^{(\varepsilon)}(s,t)$ be continuous at $s = t$ for $0 \leq s \leq t \leq T \leq \bar{T}$. Then the spot interest rate process can be defined by

$$r^{(\varepsilon)}(t) = f^{(\varepsilon)}(t,t) . \tag{3.2}$$

We note that these equations for $\{r^{(\varepsilon)}(t)\}$ and $\{f^{(\varepsilon)}(s,t)\}$ can be obtained simply by substituting $\varepsilon\sigma_i(f^{(\varepsilon)}(v,t),v,t)$ for $\sigma_i^*(f(v,t),v,t)$ in (2.2).

The asymptotic expansion approach we are proposing in this paper consists of the following three steps. First, given the future forward rate process, we consider, first, a functional of the entire forward rate process $\{f^{(\varepsilon)}(s,t)\}$ with a parameter $\varepsilon(0<\varepsilon\leq 1)$ as

$$U_T^{(\varepsilon)} = U(\{f^{(\varepsilon)}(s,t)\}), \tag{3.3}$$

where $U(\cdot)$ is a smooth functional of the process $\{f^{(\varepsilon)}(s,t)\}$ as we shall illustrate later in this section. Since we do not know the distribution of (3.3), we consider, first, its stochastic expansion around the deterministic process

$$U_T^{(0)} = U(\{f^{(0)}(s,t)\}) \tag{3.4}$$

when the volatility parameter ε goes to zero. Second, the formal stochastic expansions of the discounted payoff function can be taken with respect to the polynomial order of the volatility coefficients ε^k $(k=1,2,\cdots)$. Then the asymptotic expansion for the corresponding density function can be derived with respect to the parameter ε. Finally, we truncate the resulting stochastic expansion and take the expectation in (2.9) with respect to the density function given the information available at time t.

In order to implement this procedure, we first need to obtain the stochastic expansions of the stochastic processes $\{f^{(\varepsilon)}(s,t)\}$ and $\{r^{(\varepsilon)}(t)\}$. We shall make the following assumptions [5] :

Assumption I : (i) The volatility functions $\{\sigma_i(f^{(\varepsilon)}(s,t),s,t)\}$ are non-negative, bounded, and smooth in their first argument, where given ε $(0<\varepsilon\leq 1)$ $\{\sigma_i(f^{(\varepsilon)}(s,t),s,t)\}$ is a real-valued function defined in $\{0<s\leq t\leq T\}$. (ii) The volatility functions and all derivatives in their first argument are bounded and Lipschitz continuous. (iii) The initial (non-random) forward rates $f(0,t)$ are Lipschitz continuous with respect to t.

Assumption II : For any $0<s\leq t\leq T$,

$$\Sigma(s,t) = \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v,t)^2 dv > 0, \tag{3.5}$$

where $\{\sigma_i^{(0)}(v,t)\}$ are non-random functions given by

$$\sigma_i^{(0)}(v,t) = \sigma_i(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0} \tag{3.6}$$

[5] We also implicitly assume the measurability conditions on functions in Assumption I, which are necessary for rigorous mathematical arguments in Section 6.3.

and $\{\sigma_i(f^{(0)}(s,t),s,t)\}$ is a real-valued function defined in $\{0 < s \le t \le T\}$.

The conditions we have made in the first part of Assumption I exclude the possible explosion of the solution for (3.1). They are quite strong and could be relaxed considerably, which may be interesting from the view of stochastic analysis. For practical purposes, however, we can often use the truncation argument as in an example given by Heath, Jarrow, and Morton (1992). (See (4.1) in Section 4 for an example.) Assumption II and the second part of Assumption I ensure the key conditions for the truncated version for the non-degeneracy of the Malliavin-covariance in our problem; which, as we shall see in the following derivations, is essential for the validity of the asymptotic expansion approach . Under these assumptions, we can derive the stochastic expansions of the forward and spot interest rate processes. The outline of these derivations and their mathematical validity are given in Section 6.

Theorem 3.1 *: Under Assumption I, the stochastic expansion of the instantaneous forward rate $\{f^{(\varepsilon)}(s,t)\}$ in (3.1) is given by*

$$f^{(\varepsilon)}(s,t) = f(0,t) + \varepsilon A(s,t) + \varepsilon^2 C(s,t) + o_p(\varepsilon^2) \tag{3.7}$$

as $\varepsilon \to 0$. In particular, the spot rate process can be expanded as

$$r^{(\varepsilon)}(t) = f(0,t) + \varepsilon A(t,t) + \varepsilon^2 C(t,t) + o_p(\varepsilon^2). \tag{3.8}$$

The coefficients $A(s,t)$ and $C(s,t)$ in (3.7) and (3.8) are defined by

$$A(s,t) = \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v,t)dB_i(v), \tag{3.9}$$

$$C(s,t) = \int_0^s b^{(0)}(v,t)dv + \int_0^s \sum_{i=0}^n A(v,t)\partial\sigma_i^{(0)}(v,t)dB_i(v), \tag{3.10}$$

where

$$b^{(0)}(v,t) = b(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0}, \tag{3.11}$$

$$\partial\sigma_i^{(0)}(v,t) = \frac{\partial\sigma_i(f^{(\varepsilon)}(v,t),v,t)}{\partial f^{(\varepsilon)}(v,t)}|_{\varepsilon=0}, \tag{3.12}$$

and

$$b(f^{(\varepsilon)}(v,t),v,t) = \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t)\int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy\ . \tag{3.13}$$

In the representations above, the first terms of (3.7) and (3.8) are deterministic functions. The second term $A(s,t)$ in (3.7) follows the Gaussian distribution with zero mean and variance $\Sigma(s,t)$, which corresponds to the limit of the Malliavin-covariance for the normalized forward rate processes in the theory of Malliavin-Watanabe calculus when $\varepsilon \to 0$. The stochastic expansion method around the Gaussian distribution is standard in statistical asymptotic theory.

The next step in the asymptotic expansion approach is to obtain the stochastic expansions of the bond price process and the discount factor. For this purpose, we utilize the relationship between the zero-coupon bond price and the forward rates:

$$P^{(\varepsilon)}(t,T) = \exp\left[-\int_t^T f^{(\varepsilon)}(t,u)du\right] . \tag{3.14}$$

Using (3.7), we immediately have a stochastic expansion of the bond price process $\{P^{(\varepsilon)}(t,T)\}$ as

$$P^{(\varepsilon)}(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\varepsilon\int_t^T A(t,u)du - \varepsilon^2 \int_t^T C(t,u)du + o_p(\varepsilon^2)\right] , \tag{3.15}$$

where $P(0,T)$ and $P(0,t)$ are the observable initial discount bond prices. Because the coupon bond price $\{P^{(\varepsilon)}_{m,\{T_j\},\{c_j\}}\}$ defined by using (2.3) from $P^{(\varepsilon)}(t,T)$ with a parameter ε is a linear combination of zero-coupon bond prices, it has a stochastic expansion given by

(3.16)

$$\begin{aligned}
P^{(\varepsilon)}_{m,\{T_j\},\{c_j\}}(t) &= \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,t)} \exp\left[-\varepsilon\int_t^{T_j} A(t,u)du - \varepsilon^2\int_t^{T_j} C(t,u)du + o_p(\varepsilon^2)\right] \\
&= \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,t)} \Bigg[1 - \varepsilon\int_t^{T_j} A(t,u)du - \varepsilon^2\int_t^{T_j} C(t,u)du \\
&\qquad + \varepsilon^2 \frac{1}{2}\left(\int_t^{T_j} A(t,u)du\right)^2 + o_p(\varepsilon^2)\Bigg] .
\end{aligned}$$

By using a Fubini-type result [6] , we can write

$$\begin{aligned}
\int_t^T A(t,u)du &= \int_t^T \left[\int_0^t \sum_{i=1}^{n} \sigma_i^{(0)}(v,u)dB_i(v)\right] du \\
&= \int_0^t \boldsymbol{\sigma}^{(0)}_{tT}(v)d\boldsymbol{B}(v),
\end{aligned} \tag{3.17}$$

[6] We can use *Lemma* 4.1 of Ikeda and Watanabe (1989) under Assumption I as a generalized Fubini-type theorem.

where $\boldsymbol{B}(v) = (B_i(v))$ is an $n \times 1$ vector of standard (i.e. mutually independent) Brownian motions and $\boldsymbol{\sigma}_{tT}^{(0)}$ is a $1 \times n$ vector

$$\boldsymbol{\sigma}_{tT}^{(0)}(v) = \left[\int_t^T \sigma_i^{(0)}(v,u)du\right] . \tag{3.18}$$

Since (3.17) is a linear combinations of $\{B_i(v)\}$ with deterministic coefficients, it follows a Gaussian distribution. Also we have

(3.19)

$$\int_t^T C(t,u)du = k_1(t,T) + \int_t^T \left[\int_0^t \left[\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)d\boldsymbol{B}(v)\right] \partial\boldsymbol{\sigma}^{(0)}(s,u)d\boldsymbol{B}(s)\right] du,$$

where $\boldsymbol{\sigma}^{(0)}(v,u) = (\sigma_i^{(0)}(v,u))$ and $\partial\boldsymbol{\sigma}^{(0)}(s,u) = (\partial\sigma_i^{(0)}(s,u))$ are $1 \times n$ vectors of deterministic functions, and

$$k_1(t,T) = \int_0^t \left[\int_t^T b^{(0)}(v,u)du\right] dv. \tag{3.20}$$

Hence we notice that (3.19) is a quadratic functional of n standard Brownian motions. Similarly, by making use of (3.8), the stochastic expansion of the discount factor process is given by

(3.21)

$$\begin{aligned} e^{-\int_0^T r^{(\varepsilon)}(s)ds} &= P(0,T)\exp\left[-\varepsilon\int_0^T A(s,s)ds - \varepsilon^2\int_0^T C(s,s)ds + o_p(\varepsilon^2)\right] \\ &= P(0,T)\left[1 - \varepsilon\int_0^T A(s,s)ds - \varepsilon^2\int_0^T C(s,s)ds \right. \\ &\quad \left. +\varepsilon^2\frac{1}{2}\left(\int_0^T A(s,s)ds\right)^2\right] + o_p(\varepsilon^2) . \end{aligned}$$

The second term of the discount factor process in (3.21) can be expressed as

$$\begin{aligned} \int_0^T A(t,t)dt &= \int_0^T \int_v^T \boldsymbol{\sigma}_i^{(0)}(v,t)dtdB_i(v), \\ &= \int_0^T \boldsymbol{\sigma}_T^{(0)}(v)d\boldsymbol{B}(v), \end{aligned} \tag{3.22}$$

where $\boldsymbol{\sigma}_T^{(0)}(v)$ is a $1 \times n$ vector

$$\boldsymbol{\sigma}_T^{(0)}(v) = \left[\int_v^T \sigma_i^{(0)}(v,t)dt\right] . \tag{3.23}$$

Since (3.22) is also a linear combination of $\{B_i(v)\}$ with deterministic coefficients, the second term of (3.21) follows a Gaussian distribution. The third term of (3.21) can be expressed as

$$\int_0^T C(t,t)dt = k_2(T) + \int_0^T \left[\int_0^t \left[\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)dB(v)\right] \partial\boldsymbol{\sigma}^{(0)}(s,t)dB(s)\right] dt, \tag{3.24}$$

where

$$k_2(T) = \int_0^T \left[\int_v^T b^{(0)}(v,t)dt\right] dv. \tag{3.25}$$

The third step in our approach is to obtain the asymptotic expansion of the discounted payoff functional on the expiration date. We shall illustrate this procedure by using the two examples we have mentioned in Section 2. By using (3.16) and (3.21), the asymptotic expansion of the discounted coupon bond price minus the strike price is given by

$$\begin{aligned} g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s)ds}\left[P_{m,\{T_j\},\{c_j\}}(T) - K\right] \\ &= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + o_p(\varepsilon^2), \end{aligned} \tag{3.26}$$

where the coefficients g_i $(i = 0,1,2)$ are given by

$$g_0 = \sum_{j=1}^m c_j P(0,T_j) - KP(0,T), \tag{3.27}$$

$$g_1 = \int_0^T \boldsymbol{\sigma}^*_{g_1}(v)dB(v), \tag{3.28}$$

$$\boldsymbol{\sigma}^*_{g_1}(v) = -g_0\boldsymbol{\sigma}^{(0)}_T(v) - \sum_{i=1}^m c_j P(0,T_j)\boldsymbol{\sigma}^{(0)}_{T,T_j}(v) \tag{3.29}$$

and

(3.30)

$$\begin{aligned} g_2 &= \tfrac{1}{2}g_0\{\textstyle\int_0^T A(s,s)ds\}^2 + \{\int_0^T A(s,s)ds\}\displaystyle\sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} A(T,u)du\} \\ &+ \tfrac{1}{2}\sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} A(T,u)du\}^2 - g_0\int_0^T C(s,s)ds \\ &- \sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} C(T,u)du\}. \end{aligned}$$

What we really need is not to derive the stochastic expansion of the random variable $g^{(\varepsilon)}$, but to obtain the asymptotic expansion of its density function. We note that $g^{(\varepsilon)} \to g_0$ as $\varepsilon \to 0$ in the sense of probability. Thus we consider the normalized random variable :

$$X_T^{(\varepsilon)} = \frac{1}{\varepsilon}(g^{(\varepsilon)} - g_0). \tag{3.31}$$

The leading term of (3.31) is g_1 in (3.28) which is a simple function of Brownian motions. Hence the random variable $(g^{(\varepsilon)} - g_0)/\varepsilon = g_1 + \varepsilon g_2 + o_p(\varepsilon)$ converges to g_1, which has a non-degenerate Gaussian distribution, as $\varepsilon \to 0$. Notice that Assumption II is sufficient for the non-degeneracy for the limiting distribution of (3.31). We expect that the exact distribution of (3.31) can be represented as the limiting Gaussian distribution plus some adjustment terms with smaller orders with respect to ε when ε is small. In order to find the explicit formula of its asymptotic expansion, we formally expand the characteristic function of $X_T^{(\varepsilon)}$ with respect to ε as

$$\begin{aligned} \varphi_X(t) &= \mathbf{E}\left[e^{itX_T^{(\varepsilon)}}\right] \\ &= \mathbf{E}\left[e^{itg_1}(1 + \varepsilon it\mathbf{E}[g_2|g_1])\right] + o(\varepsilon), \end{aligned} \tag{3.32}$$

where $\mathbf{E}[g_2|g_1]$ is the conditional expectation operator.
Here it is possible to show the existence of $p-$th order moments for (3.31) for any $p > 1$ under Assumption I by using *Lemma 6.4* and a similar argument used in *Lemma 6.3* in Section 6. Then by applying the conditional expectation formulae in *Lemma 6.1* in Section 6 with $k = 1$ and $g_1 = x$ to each term of g_2 in (3.30), we can evaluate $\mathbf{E}[g_2|g_1 = x]$ and find that it is a quadratic function of x by using *Lemma 6.1.* Applying the inversion formula for the characteristic function given in *Lemma 6.2* for the second term of the second equation in (3.32), and noting $g(x)$ is a quadratic function of x with $h(-it) = \varepsilon it$, we obtain the next result.

Theorem 3.2 *: Under Assumptions I and II, the density function of $X_T^{(\varepsilon)}$ for (3.26) as $\varepsilon \to 0$ can be expressed as*

$$f_X^{(\varepsilon)}(x) = \phi_\Sigma(x) + \varepsilon\left[\frac{c}{\Sigma}x^3 + (\frac{f}{\Sigma} - 2c)x\right]\phi_\Sigma(x) + O(\varepsilon^2), \tag{3.33}$$

where $\phi_\Sigma(x)$ stands for the Gaussian density function with zero mean and variance

$$\Sigma = \int_0^T \boldsymbol{\sigma}_{g_1}^*(t)\boldsymbol{\sigma}_{g_1}^{*'}(t)dt\ , \tag{3.34}$$

provided that $\Sigma > 0$. The coefficients c and f in (3.33) are determined by the integral equation

$$\mathbf{E}\left[g_2|g_1 = x\right] = cx^2 + f. \tag{3.35}$$

We note that the characteristic function method [7] we have used above is a formal one from a rigorous mathematical view because we need the regularity conditions to ensure the inversion procedure for the characteristic function of the random variable $X_T^{(\varepsilon)}$. It is not easy to show these conditions directly since $X_T^{(\varepsilon)}$ is a complicated functional of the solution of the stochastic differential equation given by (3.1). However, we can rigorously prove the validity of the procedure along the lines developed by Watanabe (1987) and Yoshida (1992a,b) using the Malliavin Calculus, which will be discussed at the end of Section 6.3. (See Kunitomo and Takahashi (1998) also on the related technical issues.)

The asymptotic variance Σ is the limit of the Malliavin-covariance when $\varepsilon \to 0$ for the call options of the coupon bond and the swaptions whose payoff function is given by (2.4). The explicit formulae of the coefficients in (3.35) are quite complicated in this problem. By using *Lemma* 6.1 from Section 6 with $k = 1$ and $x = g_1$, we can show that c and f for the call options of the coupon bond and swaptions in Example 1 are given by

(3.36)

$$\begin{aligned}
c &= \frac{1}{2}\frac{g_0}{\Sigma^2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 \\
&+ \frac{1}{\Sigma^2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]\sum_{j=1}^m c_jP(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right] \\
&+ \frac{1}{2}\frac{1}{\Sigma^2}\sum_{j=1}^m c_jP(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 \\
&- \frac{g_0}{\Sigma^2}\left[\int_0^T\left[\int_0^t \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]dt\right] \\
&- \frac{1}{\Sigma^2}\sum_{j=1}^m c_jP(0,T_j)\left[\int_T^{T_j}\left[\int_0^T \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]du\right]
\end{aligned}$$

[7] See Bhattacharya and Rao (1976) on the related mathematical problems and techniques of the standard characteristic function approach for the simple discrete i.i.d. case, for instance.

and

(3.37)

$$
\begin{aligned}
f &= -g_0 k_2(T) - \sum_{j=1}^{m} c_j P(0,T_j) k_1(T,T_j) \\
&- \frac{1}{2}\frac{g_0}{\Sigma}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv\right]^2 + \frac{g_0}{2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_T^{(0)}(v)' dv\right] \\
&+ \sum_{j=1}^{m} c_j P(0,T_j) \int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{T,T_j}^{(0)}(v)' dv \\
&- \frac{1}{\Sigma}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv\right] \sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv\right] \\
&- \frac{1}{2}\frac{1}{\Sigma}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv\right]^2 \\
&+ \frac{1}{2}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{T,T_j}^{(0)}(v)' dv\right] \\
&+ \frac{g_0}{\Sigma}\left[\int_0^T \left[\int_0^t \boldsymbol{\sigma}_{g_1}^{*}(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv) ds\right] dt\right] \\
&+ \frac{1}{\Sigma}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_T^{T_j}\left[\int_0^T \boldsymbol{\sigma}_{g_1}^{*}(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}_{g_1}^{*}(v)' dv) ds\right] du\right] .
\end{aligned}
$$

We can also treat Example 2 in Section 2 using the same method. After some tedious calculations for the call options of the average interest rates whose payoff function is given by (2.7), we can determine the stochastic expansion;

$$
\begin{aligned}
g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s)ds}\left(\tfrac{1}{T\tau}\right)\left[\int_0^T \tfrac{1}{P(t,t+\tau)}dt - k\right] \\
&= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + o_p(\varepsilon^2),
\end{aligned} \tag{3.38}
$$

where we use the notation $k = (1+K\tau)T$. In this expression, the coefficients g_i $(i = 0,1,2)$ are given by

$$
g_0 = \frac{P(0,T)}{T\tau}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right], \tag{3.39}
$$

$$
g_1 = \int_0^T \boldsymbol{\sigma}_{g_1}^{*}(v) dB(v), \tag{3.40}
$$

and

(3.41)

$$
\begin{aligned}
g_2 &= \frac{1}{2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)d\boldsymbol{B}(v)\right]^2 dt \\
&- \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)d\boldsymbol{B}(v)\right]\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)d\boldsymbol{B}(v)\right] dt \\
&+ \frac{1}{2}g_0\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)d\boldsymbol{B}(v)\right]^2 \\
&+ \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_t^{t+\tau} B(t,u)du\right] dt - g_0\left[\int_0^T B(t,t)dt\right],
\end{aligned}
$$

where we use the notations

(3.42)

$$
\boldsymbol{\sigma}^*_{g_1}(v) = \frac{P(0,T)}{T\tau}\int_v^T \frac{P(0,t)}{P(0,t+\tau)}\boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)dt - \frac{P(0,T)}{T\tau}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right]\boldsymbol{\sigma}^{(0)}_{T}(v)\ ,
$$

$$
\boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v) = \left[\int_t^{t+\tau}\boldsymbol{\sigma}^{(0)}_i(v,u)du\right]_i ,
$$

and

$$
\boldsymbol{\sigma}^{(0)}_{T}(v) = \left[\int_v^{T}\boldsymbol{\sigma}^{(0)}_i(v,u)du\right]_i .
$$

Then, the asymptotic variance Σ in (3.33) is given by the formula of (3.34) for Example 2, where we use (3.42) instead of (3.29). By a tedious but straightforward calculation in present case, we arrive at

(3.43)

$$
\begin{aligned}
c &= \frac{1}{2}\frac{1}{\Sigma^2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 dt \\
&- \frac{P(0,T)}{T\tau}\frac{1}{\Sigma^2}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}(\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv)dt\right]\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right] \\
&+ \frac{1}{2}\frac{1}{\Sigma^2}g_0\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 \\
&+ \frac{P(0,T)}{T\tau\Sigma^2}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_t^{t+\tau}\left[\int_0^t \boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]du\right]dt \\
&- \frac{g_0}{\Sigma^2}\int_0^T\left[\int_0^t \boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]dt,
\end{aligned}
$$

and

(3.44)

$$
\begin{aligned}
f &= -\frac{1}{2}\frac{P(0,T)}{T\tau}\frac{1}{\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 dt \\
&+ \frac{1}{2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)'dv\right] dt \\
&+ \frac{P(0,T)}{T\tau}\frac{1}{\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right] dt \left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right] \\
&- \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{(0)}_{T}(v)'dv\right] dt \\
&- \frac{1}{2}\frac{1}{\Sigma}g_0\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 \\
&+ \frac{1}{2}g_0\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^{(0)}_{T}(v)'dv\right] \\
&+ \frac{P(0,T)}{T\tau}\left[\int_0^T k_1(t,t+\tau)\frac{P(0,t)}{P(0,t+\tau)}dt\right] - g_0k_2(T) \\
&- \frac{P(0,T)}{T\tau\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\int_t^{t+\tau}\left[\int_0^t\left[\boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]du\right]dt \\
&+ \frac{g_0}{\Sigma}\int_0^T\left[\int_0^t \boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]dt.
\end{aligned}
$$

The last step is to derive the asymptotic expansion of the conditional expectation of the discounted terminal payoff based on the asymptotic expansion of the exact density function we obtained in *Theorem* 3.2. By taking the mathematical expectaion of (2.9) for (3.26) and (3.38) with respect to the asymptotic expansions of the density functions given in *Theorem 3.2*, we have

(3.45)

$$
\begin{aligned}
& E[g_0+\varepsilon X_T^{(\varepsilon)}]^+ \\
&= \int_{g_0+\varepsilon x\geq 0}(g_0+\varepsilon x)f_X^{(\varepsilon)}(x)dx \\
&= \int_{g_0+\varepsilon x\geq 0}(g_0+\varepsilon x)\phi_\Sigma(x)\{1+\varepsilon[\tfrac{c}{\Sigma}x^3+(\tfrac{f}{\Sigma}-2c)x]+\varepsilon^2 h(x)+o(\varepsilon^2)\}dx \ ,
\end{aligned}
$$

where $h(x)$ is a function of the polynomial order.
Then by applying the integration-by-parts formula, we immediately obtain the following result.

Theorem 3.3 *: In addition to Assumptions I and II, we assume that* $\Sigma > 0$ *. Then the asymptotic expansions of* $V_0(T)$ *in (2.9) for (2.4) and (2.7) as* $\varepsilon \to 0$

are given by

$$V_0(T) = g_0 \int_{-y_\varepsilon}^{+\infty} \phi_\Sigma(x)dx + \varepsilon \int_{-y_\varepsilon}^{+\infty} x\phi_\Sigma(x)dx \tag{3.46}$$
$$+\varepsilon^2 \int_{-y_\varepsilon}^{+\infty} (cx^2 + f)\phi_\Sigma(x)dx + o(\varepsilon^2) ,$$

where $y_\varepsilon = (1/\varepsilon)g_0$. *The coefficients* c *and* f *are given as the same as in* $Theorem$ 3.2.

Note that each term of the right-hand side of (3.46) depends on y_ε with the parameter ε, but such expression can be rigorously justified. The mathematical validity of our method in this section will be discussed in Section 6.3 and also Kunitomo and Takahashi (1998) in the details. It is also possible to obtain an asymptotic expansion with respect to polynomials of ε under an additional assumption [8] . All terms on the right hand side of (3.46) are known functions of the distribution function and the density function of $N(0, \Sigma)$. For instance, take the relation

$$\int_{-y}^{\infty} x^2\phi_\Sigma(x)dx = \Sigma\Phi(\Sigma^{-1/2}y) - y\Sigma\phi_\Sigma(y), \tag{3.47}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. This and similar formulae are useful for the numerical implementation of our procedure.

4 Numerical Examples

In this section, we will present some numerical results to illustrate the method introduced in Section 3. For this purpose, we use the pricing problem of swaptions and average options of interest rates in the term structure model explained in Section 2. For the simplicity of exposition, we assume that the instantaneous forward rate processes $\{f^{(\varepsilon)}(s,t)\}$ have a one factor volatility function [9] with $n = 1$. For $0 \le \beta \le 1$, we take

$$\sigma_1(\xi(s,t), s, t) = \xi(s,t)^\beta h_1(\xi(s,t), M) + (M+1)^\beta h_2(\xi(s,t), M) , \tag{4.1}$$

[8] Originally Kunitomo and Takahashi (1995) have presented such expression. However, the resulting approximations are the same as (3.46).

[9] This case has been used by Amin and Bodurtha (1995) as a numerical example on the forward interest rates.

where $M = 10,000$. The smooth functions are taken as $h_1(\xi, M) = h(M+1-\xi)/[h(\xi - M) + h(M+1-\xi)]$, $h_2(\xi, M) = h(\xi - M)/[h(\xi - M) + h(M+1-\xi)]$, and

$$h(\xi) = \begin{cases} e^{-\frac{1}{\xi}} & if\ \xi > 0 \\ 0 & if\ \xi \le 0 \end{cases}$$

When $\beta = 1$, (4.1) corresponds to the geometric Brownian motion case for the forward rates approximately and we have used its truncated version with smoothness because there can be some explosive solutions as we discussed on Assumption I in Section 3. When $\beta = 0$, $\sigma_1(\xi(s,t), s, t) = 1$ in (4.1) and it corresponds to a continuous analogue of the discrete model by Ho and Lee (1986).

For the first example, Tables 1 and 2 show the numerical values of call options on average interest rates for the case when $\beta = 0$ and T is .25 year or 1 year. The time to maturity of the underlying interest rates is one year and the average is taken over interest rates whose maturity are one year ($\tau = 1$) . For simplicity, the present term structure at $t = 0$ is assumed to be a flat 5% per year and the volatility parameter is assumed to be 150 basis points per year ($\epsilon = 0.015$). We can use the approximations based on the asymptotic expansions in Section 3 and measuring their accuracy by the Monte Carlo results. We have given the results for the out-of-the money case ($K = 5.5\%$ or 6%), the at-the-money case ($K = 5\%$), and the in-the-money case ($K = 4.5\%$ or 4%).

For comparative purposes, the numerical values obtained by Monte Carlo simulations and by the finite difference method via the PDE approach have been calculated. The number of simulations in our Monte Calro calculations is 500,000 and we expect that the results obtained by this method are very accurate. The finite difference values in Tables 1 and 2 are based on a numerical solution of the PDE for the average options of interest rate processes under the assumption that they follow (4.1) when $\beta = 0$. This method has been recently developed by He and Takahashi (1996). Since the number of Monte Carlo simulations is large, we expect that this method provides the benchmark values for this study. From Tables 1-2 we can find that the differences between the option values obtained by the asymptotic expansion approach and the Monte Carlo approach are very small - less than 1 percent of the underlying price levels.

Table 3 shows the results for the call options of a swap contract (the swaption) in the case when $\beta = 0$ and $\epsilon = .01$(100 basis points). Tables 4-6 contains the numerical values for the case when $\beta = 1$ and $\epsilon = .2$ (20%). In Tables 3 and 4 the term of the underlying interest rate swap is 5 years and the time to expiration, T, is also 5 years; T is 3 years in Table 5 and 1 year in Table 6. We set $\tau =$ 1 year, $T_1 = T + 1, \cdots, T_5 = T + 5$, $m = 5$, and the present term structure at $t = 0$ is assumed to be a flat 5% per year. Because in this case we have $c_j = S\tau$ $(j = 1, \cdots, m-1)$, $c_m = 1 + S\tau$, $K = 1$, and

$$g_0 = S\tau \sum_{j=1}^{m} P(0,T_j) + P(0,T_m) - P(0,T) \ , \tag{4.2}$$

we have set

$$S = \frac{P(0,T) - P(0,T_5)}{\tau \sum_{j=1}^{5} P(0,T_j)} = 0.051271 \tag{4.3}$$

for the at-the-money case. We have computed the results for the out-of-the money case ($S = 5.1271\% \times .8, 5.1271\% \times .6$, and $5.1271\% \times .4$), the at-the-money case ($S = 5.1271\%$), and the in-the-money case ($S = 5.1271\% \times 1.2, 5.1271\% \times 1.4$, and $5.1271\% \times 1.6$). The approximations based on the asymptotic expansions in Section 3 are presented versus the Monte Carlo results for the cases when $\beta = 0$ and $\beta = 1$.

From Table 3 we also find that the differences in the swaption values between the two approaches using Gaussian forward rates are very small and negligible in most cases. From Tables 4-6 we can see that the differences in the option values become larger under the geometric Brownian forward rates case relative to Table 3; this is due to the non-Gaussianity of the underlying forward rates and the spot rate. By examining these results with respect to different terms to expiration, we note that the approximations become more accurate when T is shorter. For practical purposes, the *difference percentage rate* can be ignored if it is large when the option value is quite close to zero. On the whole, in the swaption example, the approximations are sufficiently accurate in absolute terms and the differences between the approximations and the corresponding Monte Carlo results are within 3 bp.

As these two examples show, the values of our approximations are reliable to at least two digits. Thus, we can tentatively conclude that the approximation formulae we have obtained in Section 3 are accurate and useful for practical purposes.

5 Concluding Remarks

This paper proposes a new methodology for the valuation problem of financial contingent claims when the underlying forward rates follow a general class of continuous Itô processes, which are not necessarily in a class of continuous diffusion processes. Our method, called the *small disturbance asymptotic expansion* *approach*, can be applicable to a wide range of valuation problems including complicated contingent claims associated with the term structure of interest rates. We have illustrated our methdology by deriving some useful formulae for swaptions and average (or Asian) options for interest rates. We have also given some evidence that the resulting formulae are numerically accurate enough for practical applications. Since the asymptotic expansion approach can be rigorously justified by the Malliavin-Watanabe calculus in stochastic analysis, it is not an ad-hoc method to give numerical approximations. The asymptotic expansions explained in Section 3 can be made up to any order of precision $O(\varepsilon^k)(k = 1, 2, \cdots)$, in principle.

There are several advantage in our method over the PDE and the Monte Carlo methods, which have been extensively used in practical applications. First, our method is applicable in an unified manner to the pricing problem of various types of functionals of asset prices in the economy governed by the general class of continuous It$\hat{o}$ processes, which are not necessarily Markovian in the usual sense. This problem has been known to be difficult using the existing methods. Second, our method is computationally efficient in comparison with other methods since it is very fast to obtain numerical results with a computer. Third, the distributions of the underlying assets and their functionals at any date can be evaluated by our method. This aspect is quite useful in various kinds of simulations. For instance, the pricing formulae derived by our method can be used as control variates to improve the efficiency of Monte Carlo simulations and the PDE method. The PDE method, on the other hand, is difficult to implement, especially when the underlying assets or term structure of interest rates follow multi-factor processes. Monte Carlo simulations are often quite time consuming in this case. Takahashi (1995) has discussed some extensions of our method to the pricing problem of derivatives in more complicated multi-country and multi-factor situations when the forward rate processes are not necessarily Markovian in the usual sense.

Finally, we should mention that the asymptotic expansion approach in this paper can give a powerful and useful tool not only to the valuation problem of contingent claims associated with the term structure of interest rates, but also to other problems in financial economics. (See Kim and Kunitomo (1999), for instance.) Our method usually gives some explicit formulae which may shed some new light on the solution of the problem under consideration when the underlying asset prices follow a general class of continuous It$\hat{o}$ processes. Hence, we do not need to use simple stochastic processes among the class of diffusion or Markovian processes in the usual sense only because the resulting solutions are manageable. We suspect that there has been some work on interest rates, which have used simple but unreasonable stochastic processes mainly because the resulting analyses are mathematically convenient.

6 Mathematical Appendix

In this appendix, we gather some mathematical details which we have omitted in the previous sections. We also briefly discuss the validity of our method by the use of the Malliavin-Watanabe theory in stochastic analysis.

6.1 Two Useful Lemmas

We first give some formulae on the conditional expectation operations as Lemma 6.1, which is a slight generalization of Lemma 5.7 of Yoshida (1992a). The proof is a direct result of calculations by making use of the Gaussianity of continuous processes involved.

Lemma 6.1 : *Let $\boldsymbol{B}(t)$ be an $n \times 1$ vector of independent Brownian motions and $\boldsymbol{x}$ be a k dimensional vector. Let also $\boldsymbol{q}_1(t)$ be an $R^1 \mapsto R^{k\times n}$ non-stochastic function and*

$$\boldsymbol{\Sigma} = \int_0^T \boldsymbol{q}_1(t)\boldsymbol{q}_1^{'}(t)dt \tag{6.1}$$

is a positive definite matrix. (i) Suppose $\boldsymbol{q}_2(u)$ and $\boldsymbol{q}_3(u)$ are $R^1 \mapsto R^{m\times n}$ non-stochastic functions. Then for $0 \leq s \leq t \leq T$

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t \left[\int_0^s \boldsymbol{q}_2(u)d\boldsymbol{B}(u)\right]^{'} \boldsymbol{q}_3(s)d\boldsymbol{B}(s) | \int_0^T \boldsymbol{q}_1(u)d\boldsymbol{B}(u) = \boldsymbol{x}\right] \\
= &\ trace \int_0^t \boldsymbol{q}_1(s)\boldsymbol{q}_3(s)^{'}(\int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_1(u)^{'}du)ds\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}\boldsymbol{x}^{'} - \boldsymbol{\Sigma}\right]\boldsymbol{\Sigma}^{-1} .
\end{aligned} \tag{6.2}$$

(ii) Suppose $\boldsymbol{q}_2(u)$ and $\boldsymbol{q}_3(u)$ are $R^1 \mapsto R^n$ non-stochastic functions. Then for $0 \leq s \leq t \leq T$

(6.3)
$$\begin{aligned}
&\mathbf{E}\left[\left[\int_0^s \boldsymbol{q}_2(u)d\boldsymbol{B}(u)\right]\left[\int_0^t \boldsymbol{q}_3(v)d\boldsymbol{B}(v)\right] | \int_0^T \boldsymbol{q}_1(u)d\boldsymbol{B}(u) = \boldsymbol{x}\right] \\
= &\int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_3(u)^{'}du + \left[\int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_1(u)^{'}du\right]\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}\boldsymbol{x}^{'} - \boldsymbol{\Sigma}\right]\boldsymbol{\Sigma}^{-1}\left[\int_0^t \boldsymbol{q}_1(v)\boldsymbol{q}_3(v)^{'}dv\right] .
\end{aligned}$$

The second lemma is on the inversion formulae of the characteristic functions of some random variables. The proof is also a direct result of a calculation, which has been given in Fujikoshi et.al. (1982).

Lemma 6.2 (Fujikoshi et.al. (1982)) *: Suppose that $\boldsymbol{x}$ follows an n-dimensional Gaussian distribution with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. The density function of $\boldsymbol{x}$ is denoted by $\phi_{\Sigma}(\cdot)$. Then for any polynomial functions $g(\cdot)$ and $h(\cdot)$,*

$$\mathcal{F}^{-1}\left[h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right]\right]_{<\boldsymbol{\xi}>} = h\left[\frac{\partial}{\partial\boldsymbol{\xi}}\right]g(\boldsymbol{\xi})\phi_{\Sigma}(\boldsymbol{\xi}), \tag{6.4}$$

where $i=\sqrt{-1}$,

$$\mathcal{F}^{-1}\left[h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right]\right]_{<\boldsymbol{\xi}>} = (\frac{1}{2\pi})^n\int_{R^n} e^{-i\boldsymbol{t}'\boldsymbol{\xi}}h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right]d\boldsymbol{t}, \tag{6.5}$$

and the expectation opelation $\mathbf{E}[\cdot]$ is taken over $\boldsymbol{x}=(x_i)\in\boldsymbol{R}^n$. Also $\mathcal{F}^{-1}[\cdot]_{<\boldsymbol{\xi}>}$ denotes the $n-$dimensional Fourier inversion $\mathcal{F}^{-1}[\cdot]$ being evaluated at $\boldsymbol{\xi}$ and $\boldsymbol{t}'\boldsymbol{x}=\sum_{i=1}^n t_i x_i$ for $\boldsymbol{t}=(t_i)\in\boldsymbol{R}^n$.

6.2 A Sketch of Derivations of Asymptotic Expansions

We give a brief sketch on our derivations of the stochastic expansions of some random variables used in Section 3. The following derivations are purely formal, but the mathematical validity of our method will be discussed in the next subsection. From (3.1), the deterministic process of $\{f^{(\varepsilon)}(s,t)\}$ follows when $\varepsilon\to 0$ is given by

$$f^{(0)}(s,t)=\lim_{\varepsilon\to 0}f^{(\varepsilon)}(s,t)=f(0,t)\ . \tag{6.6}$$

Then we define the random variables $A(s,t)$ and $C(s,t)$ by

$$A(s,t)=\frac{\partial f^{(\varepsilon)}(s,t)}{\partial\varepsilon}|_{\varepsilon=0}, \tag{6.7}$$

and

$$C(s,t)=\frac{1}{2}\frac{\partial^2 f^{(\varepsilon)}(s,t)}{\partial^2\varepsilon}|_{\varepsilon=0}\ . \tag{6.8}$$

By a direct calculation of differentiation, we have

$$
\begin{aligned}
A(s,t) &= \int_0^s \left[2\varepsilon b(f^{(\varepsilon)}(v,t),v,t) + \varepsilon^2 \frac{\partial b(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon}\right]_{\varepsilon=0} dv \\
&+ \int_0^s \sum_{i=1}^n \left[\sigma_i(f^{(\varepsilon)}(v,t),v,t) + \varepsilon \frac{\partial \sigma_i(f^{\varepsilon}(v,t),v,t)}{\partial \varepsilon}\right]_{\varepsilon=0} dB_i(v) \\
&= \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v,t) dB_i(v) \ .
\end{aligned} \tag{6.9}
$$

Similarly, we have

(6.10)

$$
\begin{aligned}
C(s,t) &= \int_0^s \left[b(f^{(\varepsilon)}(v,t),v,t) + 2\varepsilon \frac{\partial b(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon} + \frac{\varepsilon^2}{2} \frac{\partial^2 b(f^{(\varepsilon)}(v,t),v,t)}{\partial \partial^2 \varepsilon}\right]_{\varepsilon=0} dv \\
&+ \int_0^s \sum_{i=1}^n \left[\frac{\partial \sigma_i(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon} + \frac{1}{2}\varepsilon \frac{\partial^2 \sigma_i(f^{(\varepsilon)}(v,t),v,t)}{\partial^2 \varepsilon}\right]_{\varepsilon=0} dB_i(v) \\
&= \int_0^s b^{(0)}(v,t)dv + \int_0^s \sum_{i=1}^n \partial \sigma_i^{(0)}(v,t) A(v,t) dB_i(v) \ .
\end{aligned}
$$

Hence we have obtained the stochastic differential equations which $\{A(s,t)\}$ and $\{C(s,t)\}$ must satisfy.

Next, we substitute $\{f^{(\varepsilon)}(t,u)\}$ in (3.14) and use the fact that $\{P^{(\varepsilon)}(s,t)\}$ are non-stochastic functions at $s = 0$, which lead to (3.15) and (3.16). The stochastic expansion of the discount factor can be obtained by using $\{r^{(\varepsilon)}(t)\}$ instead of $\{f^{(\varepsilon)}(s,t)\}$. In (3.17), (3.19), (3.22), and (3.24), we can utilize the Fubini-type theorem on the exchanges of integration operations. By expanding the exponential functions, we have

$$
\begin{aligned}
P^{(\varepsilon)}(t,T) = \frac{P(0,T)}{P(0,t)}[1 &- \varepsilon \int_t^T A(t,u)du - \varepsilon^2 \int_t^T C(t,u)du \\
&+ \varepsilon^2 \frac{1}{2}\{\int_t^T A(t,u)du\}^2\Big] + o_p(\varepsilon^2) \ ,
\end{aligned} \tag{6.11}
$$

and

$$
\begin{aligned}
e^{-\int_0^T r^{(\varepsilon)}(s)ds} = P(0,T)[1 &- \varepsilon \int_0^T A(t,t)dt - \varepsilon^2 \int_0^T C(t,t)dt \\
&+ \varepsilon^2 \frac{1}{2}\{\int_0^T A(t,t)dt\}^2\Big] + o_p(\varepsilon^2) \ ,
\end{aligned} \tag{6.12}
$$

respectively.

Finally, we multiply the stochastic expansions of the discount factor and the terminal payoff function. Then by rearranging each term in the resulting stochastic expansions, we can obtain the form of (3.26) and (3.38) in Example 1 and Example 2.

6.3 Validity of the Asymptotic Expansion Approach

The mathematical validity of the asymptotic expansion approach in this paper can be given along the line based on the remarkable work by Watanabe (1987) on the Malliavin calculus in stochastic analysis. Yoshida (1992a,b) have utilized the results and method originally developed by Watanabe (1987) and given some useful results on the validity of the asymptotic expansions of some functionals on continuous time homogenous diffusion processes. The validity of our method can be obtained by similar arguments to those used by Yoshida (1992a,b) and Chapter V of Ikeda and Watanabe (1989), but with substantial modifications. This is mainly because the continuous stochastic processes defined by (3.1) for the spot interest rate and forward rates are not necessarily Markovian in the usual sense.

Since the rigorous proofs of our claims in this section can be quite lengthy, but for the most part, are quite straightforward extensions of the existing results in stochastic analysis, we shall only give a rough sketch below. Our arguments on the validity of the asymptotic expansion approach for interest rate based contingent claims consist of four steps. The main aim in the following steps will be to check the truncated version of the non-degeneracy condition for the Malliavin-covariance in our situation.

[**Step 1**] : First, we shall prepare some notations. For this purpose, we shall freely use the notations of Ikeda and Watanabe (1989) as a standard textbook. We shall only discuss the validity of the asymptotic expansion approach based on a one-dimensional Wiener space without loss of generality. We only need more complicated notations in the general case. (See Ikeda and Watanabe (1989) for the details.) Let (W, P) be the $1-$dimensional Wiener space and let $\boldsymbol{H}$ be the Cameron-Martin subspace of W endowed with the norm

$$|h|_H^2 = \int_0^T |\dot{h}(t)|^2 dt \tag{6.13}$$

for $h \in \boldsymbol{H}$. The norm of $\boldsymbol{R}-$valued Wiener functional g for any $s \in \boldsymbol{R}$, and $p \in (1, \infty)$ is defined by

$$(6.14) \qquad \|g\|_{p,s} = \|(I-\mathcal{L})^{s/2}g\|_p \ ,$$

where $\mathcal{L}$ is the Ornstein-Uhlenbeck operator and $\|\cdot\|_p$ is the L_p-norm in the standard stochastic analysis. An $\boldsymbol{R}-$valued function $g : W \mapsto \boldsymbol{R}$ is called an $\boldsymbol{R}-$valued polynomial functional if $g = p([h_1](B),\cdots,[h_n](B))$, where $n \in \boldsymbol{Z}^+, h_i \in \boldsymbol{H}$, $p(x_1,\cdots,x_n)$ is a polynomial, and

$$[h](B) = \int_0^T \dot{h}(t)dB(t)$$

for $h \in \boldsymbol{H}$ are defined in the sense of stochastic integrals.

Let $\boldsymbol{P}(\boldsymbol{R})$ denote the totality of $\boldsymbol{R}-$valued polynomials on the Wiener space (W,P). Then $\boldsymbol{P}(\boldsymbol{R})$ is dense in $L_p(\boldsymbol{R})$. The Banach space $\boldsymbol{D}_p^s(\boldsymbol{R})$ is the completion of $\boldsymbol{P}(\boldsymbol{R})$ with respect to $\|\cdot\|_{p,s}$. The dual space of $\boldsymbol{D}_p^s(\boldsymbol{R})$ is $\boldsymbol{D}_q^{-s}(\boldsymbol{R})$, where $s \in \boldsymbol{R}, p>1$, and $1/p+1/q=1$. The space $\boldsymbol{D}^{\infty}(\boldsymbol{R}) = \cap_{s>0}\cap_{1<p<+\infty}\boldsymbol{D}_p^s(\boldsymbol{R})$ is the set of Wiener functionals and $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{R}) = \cup_{s>0}\cap_{1<p<+\infty}\boldsymbol{D}_p^{-s}(\boldsymbol{R})$ is a space of generalized Wiener functionals. For $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$, the derivative of F in the direction of h is defined by

$$(6.15) \qquad D_hF(B) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F(B+\varepsilon h) - F(B)\} \ .$$

Then for $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$ there exists $DF \in \boldsymbol{P}(\boldsymbol{H}\otimes\boldsymbol{R})$ such that $D_hF(B) =< DF(B), h >_H$, where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$ and DF is called the $H-$derivative of F. It is known that the norm $\|\cdot\|_{p,s}$ is equivalent to the norm $\sum_{k=0}^{s}\|D^k\cdot\|_p$. For $F \in \boldsymbol{D}^{\infty}(\boldsymbol{R})$, we can define the Malliavin-covariance by

$$(6.16) \qquad \sigma(F) =< DF(B), DF(B) >_H \ ,$$

where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$. It is known that the operator D can be well-defined in $\boldsymbol{D}^{\infty}(\boldsymbol{R})$. (See Chapter V of Ikeda and Watanabe (1989) for the details.)

[Step 2] : We set $n=1$ and $\sigma(f^{(\varepsilon)}(v,t),v,t) = \sigma_1(f^{(\varepsilon)}(v,t),v,t)$ in (3.1) without the loss of generality. We further set $\varepsilon = 1$ in Step 2. The starting point of our discussion is the result by Morton (1989) on the existence and uniqueness of the solution of the stochastic integral equation (3.1) for forward rate processes.

Theorem 6.1 : *Assume the conditions in Assumption I, except those on the smoothness and derivatives. Then there exists a jointly continuous process* $\{f^{(\varepsilon)}(s,t),$ $0 \le s \le t \le T\}$ *satisfying (3.1) with* $\varepsilon = 1$. *There is at most one solution of (3.1) with* $\varepsilon = 1$.

We shall consider the $H-$derivatives of the forward rate processes $\{f^{(1)}(s,t)\}$. For any $h \in H$, we successively define a sequence of random variables $\{\xi^{(n)}(s,t)\}$ using the integral equation:

(6.17)

$$\begin{aligned}\xi^{(n+1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\xi^{(n)}(v,t)\right]dv \\ &+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi^{(n)}(v,y)dy\right]dv \\ &+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)\xi^{(n)}(v,t)dB(v) \\ &+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}(v)dv\ ,\end{aligned}$$

where the initial condition is given by $\xi^{(0)}(s,t) = 0$. Then we have the next result by using the standard method in stochastic analysis.

Lemma 6.3 : *For any* $p > 1$ *and* $0 \le s \le t \le T$,

$$E[|\xi^{(n)}(s,t)|^p] < \infty\ , \tag{6.18}$$

and as $n \to \infty$

$$\sup_{0\le s\le t\le T} E[\sup_{0\le u\le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] \to 0\ . \tag{6.19}$$

Proof of Lemma 6.3: **[i]** We use the induction argument for n. We have (6.18) when $n = 1$ because $\sigma(\cdot)$ is bounded and $\dot{h}(v)$ is a square-integrable function in (6.17). Suppose (6.18) hold for $n = m$. Then there exist positive constants $M_i (i = 1, \cdots, 4)$ such that

(6.20)

$$\begin{aligned}|\xi^{(m+1)}(s,t)|^p &\le M_1 \int_0^s |\xi^{(m)}(v,t)|^p dv + M_2[\sup_{0\le u\le s} |\int_0^u \xi^{(m)}(v,t)dB(v)|]^p \\ &\quad +M_3 \int_0^s \int_v^t |\xi^{(m)}(v,y)|^p dy dv + M_4[\int_0^s |\dot{h}(v)|^2 dv]^{p/2}\ .\end{aligned}$$

By a martingale inequality (Theorem III-3.1 of Ikeda and Watanabe (1989)), the expectation of the second term on the right hand side of (6.20) is less than

$$(6.21) \qquad M_2' E[(\int_0^s |\xi^{(m)}(v,t)|^2 dv)^{p/2}] \le M_2'' \int_0^s E[|\xi^{(m)}(v,t)|^p]dv \ ,$$

where M_2' and M_2'' are positive constants. Because $\dot{h}(v)$ is square-integrable, we have (6.18) when $n = m+1$.

[ii] From (6.17), there exist positive constants $M_i (i = 5, 6, 7)$ such that for $0 \le s \le t$,

(6.22)

$$\begin{aligned} |\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2 &\le M_5[\int_0^s |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dv]^2 \\ &+ M_6[\int_0^s \int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|dydv]^2 \\ &+ M_7[\int_0^s \partial\sigma(f^{(1)}(v,t),v,t)|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dB(v)]^2 \\ &\equiv \sum_{i=1}^{3} I_i^{(n)}(s,t) \ , \end{aligned}$$

where we have defined $I_i^{(n)}(s,t)$ by the last equality. Using the Cauchy-Schwartz inequality,

$$(6.23) \qquad E[\sup_{0 \le u \le s} I_1^{(n)}(u,t)] \le M_5 s \int_0^s E[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv \ .$$

By repeating the above argument to the second term of (6.22), we have

$$(6.24) \qquad \begin{aligned} I_2^{(n)}(u,t) &\le M_6 u \int_0^u [\int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|dy]^2 dv \\ &\le M_6 ut \int_0^u \int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2 dydv \ . \end{aligned}$$

Then

$$(6.25) \qquad E[\sup_{0 \le u \le s} I_2^{(n)}(u,t)] \le M_6 st \int_0^s \int_v^t E[|\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2]dydv \ .$$

For the third term of (6.22), we have

$$(6.26) \qquad E[\sup_{0 \le u \le s} I_3^{(n)}(u,t)] \le M_7' \int_0^s E[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv$$

because of the boundedness of $\partial\sigma(\cdot)$, where M_7' is a positive constant. By using (6.24), (6.25), and (6.26), we have

(6.27)

$$E[\sup_{0\le u\le s}|\xi^{(n+1)}(u,t)-\xi^{(n)}(u,t)|^2] \;\le\; M_8\left(\int_0^s E[\sup_{0\le v\le u}|\xi^{(n)}(v,t)-\xi^{(n-1)}(v,t)|^2]du \right.$$
$$\left. +\;\int_0^s\int_u^t E[\sup_{0\le v\le u}|\xi^{(n)}(v,y)-\xi^{(n-1)}(v,y)|^2]dydu\right),$$

where M_8 is a positive constant. By defining a sequence of $\{u^{(n)}(s,t)\}$ as

$$u^{(n+1)}(s,t)=E[\sup_{0\le u\le s}|\xi^{(n+1)}(u,t)-\xi^{(n)}(u,t)|^2]\ ,$$

we have the relation

$$u^{(n+1)}(s,t)\le M_8\int_0^s[\int_u^t u^{(n)}(u,y)dy+u^{(n)}(u,t)]du\ . \tag{6.28}$$

If we have an inequality

$$u^{(n+1)}(s,t)\le \frac{1}{(n+1)!}[M_8(t+1)s]^{n+1}\ , \tag{6.29}$$

we can show (6.19) as $n\to+\infty$. We use the induction argument for $n\ge 1$. When $n=1$, there exists a positive constant M_9 such that

$$\begin{aligned} u^{(1)}(s,t) &= E[\sup_{0\le u\le s}|\xi^{(1)}(u,t)-\xi^{(0)}(u,t)|^2] \\ &= E[\sup_{0\le u\le s}|\int_0^u \sigma(f^{(1)}(v,t),v,t)\dot{h}(v)dv|^2] \\ &\le M_8(1+t)s \end{aligned} \tag{6.30}$$

because $\xi^{(0)}(s,t)=0$, $\sigma(\cdot)$ is bounded, and $\dot{h}(v)$ is square$-$integrable. Suppose (6.29) holds for $m=n+1$. Then

$$\begin{aligned} u^{(m+1)}(s,t) &\le M_8\int_0^s[\int_u^t u^{(m)}(u,y)dy+u^{(m)}(u,t)]du \\ &\le M_8\int_0^s[\int_u^t M_8^m(t+1)^m\frac{s^m}{m!}dy+M_8^m(t+1)^m\frac{s^m}{m!}]du \\ &\le M_8^{m+1}(t+1)^{m+1}\frac{s^{m+1}}{(m+1)!}\ . \end{aligned} \tag{6.31}$$

Q.E.D.

By using (6.29) and the Chebyshev's inequality, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\{\sup_{0\le u\le s\le t} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)| > \frac{1}{2^n}\} \\
\le\ &\sum_{n=1}^{\infty} \frac{1}{n!}[4M_8(T+1)T]^n < +\infty
\end{aligned} \tag{6.32}
$$

uniformly for any $0 \le s \le t \le T$. Then by the Borel-Cantelli lemma, the sequence of random variables $\{\xi^{(n)}(u,t)\}$ converges uniformly on $0 \le u \le s \le t$ $(\le T)$. Hence we can establish the existence of the $H-$derivative of $f^{(1)}(s,t)$, which is given by the solution of the stochastic integral equation:

(6.33)

$$
\begin{aligned}
D_h f^{(1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy D_h f^{(1)}(v,t)\right] dv \\
&+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y) D_h f^{(1)}(v,y)dy\right] dv \\
&+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t) D_h f^{(1)}(v,t) dB(v) \\
&+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}(v)dv \ .
\end{aligned}
$$

We note that for the spot rate process the H-derivative can be well-defined by

$$
D_h r^{(1)}(t) = \lim_{s\to t} D_h f^{(1)}(s,t) \ . \tag{6.34}
$$

Now we define the random variables $\{\xi_{s,t}(u)\}$ for $0 \le u \le s \le t$ by

$$
\begin{aligned}
\xi_{s,t}(u) &= \int_u^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\,\xi_{v,t}(u)\right] dv \\
&+ \int_u^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi_{v,y}(u)dy\right] dv \\
&+ \int_u^s \partial\sigma(f^{(1)}(v,t),v,t)\xi_{v,t}(u)dB(v) \\
&+ \sigma(f^{(1)}(u,t),u,t) \ .
\end{aligned} \tag{6.35}
$$

Then we can show that

$$
\int_0^s \xi_{s,t}(u)\dot{h}(u)du = D_h f^{(1)}(s,t) \ . \tag{6.36}
$$

The rigorous proof for the existence of random variables $\{D_h f^{(1)}(s,t)\}$ and $\{\xi_{s,t}(u)\}$ can be given by extending the recursive method used in the proof of Proposition

10.1 in Ikeda and Watanabe (1989). Here we need to approximate the solution of the stochastic integral equations (6.33) and (6.35) in two arguments s and t by a sequence of solutions with step coefficients. Since it is straightforward to do this but quite lengthy, we omit the details.

Next, we examine the existence of higher order moments of $\{\xi_{s,t}(u)\}$ satisfying (6.35). To do this, we prepare the following inequality.

Lemma 6.4 *: Suppose for* $k_0 \geq 0, k_1 > 0, A_N > 0$ *and* $0 < u \leq s \leq t \leq T$, *a function* $w_N(u,s,t)$ *satisfies (i)* $0 \leq w_N(u,s,t) \leq A_N$ *and (ii)*

$$w_N(u,s,t) \leq k_0 + k_1 \left[\int_u^s w_N(u,v,t)dv + \int_u^s \int_v^t w_N(u,v,y)dydv \right] . \tag{6.37}$$

Then

$$w_N(u,s,t) \leq k_0 e^{k_1(1+t)s} . \tag{6.38}$$

Proof of Lemma 6.4: By substituting (i) into the right hand side of (6.37), we have

$$\begin{aligned} w_N(u,s,t) &\leq k_0 + A_N k_1 \left[\int_u^s ds + \int_u^s \int_v^t dydv \right] \\ &\leq k_0 + A_N k_1 (1+t)s . \end{aligned} \tag{6.39}$$

By repeating the substitution of (6.39) into the right hand side of (6.37), we have

$$w_N(u,s,t) \leq k_0 \sum_{k=0}^{n} \frac{1}{k!}[k_1(1+t)s]^k + \frac{1}{(n+1)!} A_N [k_1(1+t)s]^{n+1} . \tag{6.40}$$

Then we have (6.38) by taking $n \to +\infty$. *Q.E.D.*

In order to use Lemma 6.4, we consider the truncated random variable

$$\zeta_{s,t}^N(u) = [\xi_{s,t}(u)]\, I_N(s,t) , \tag{6.41}$$

where $I_N(s,t) = 1$ if

$$\sup_{0 \leq u \leq v \leq s, v \leq y \leq t, s \leq t} |\xi_{v,y}(u)| \leq N$$

and $I_N(s,t) = 0$ otherwise. By using the boundedness conditions in Assumption I and the fact that $\dot{h}(s)$ is square-integrable, we can show that there exist positive constants $M_i (i = 9, \cdots, 12)$ such that for any $p > 1$

$$
\begin{aligned}
|\zeta_{s,t}^N(u)|^p &\leq M_9 \int_u^s |\zeta_{v,t}^N(u)|^p dv + M_{10}| \int_u^s \zeta_{v,t}^N(u) dB(v)|^p \\
&+ M_{11} \int_u^s \int_v^t |\zeta_{v,y}^N(u)|^p dy dv + M_{12} |\sigma(f^{(1)}(u,t),u,t)|^p . \\
&\equiv \sum_{i=1}^4 J_i^N(u,s,t) ,
\end{aligned} \tag{6.42}
$$

where we have defined $J_i^N(u,s,t) (i = 1, \cdots, 4)$ by the last equality. By using a martingale inequality (Theorem III-3.1 of Ikeda and Watanabe (1989)), we have

$$
\begin{aligned}
E[J_2^N(u,s,t)] &\leq M_{10}' E[\int_u^s |\zeta_{v,t}^N(u)|^2 dv]^{p/2} \\
&\leq M_{10}'' E[\int_u^s |\zeta_{v,t}^N(u)|^p dv] ,
\end{aligned} \tag{6.43}
$$

where M_{10}' and M_{10}'' are positive constants. Also $J_4^N(u,s,t)$ is bounded because $\sigma(\cdot)$ is bounded. If we set $w_N(u,s,t) = E[|\zeta_{s,t}^N(u)|^p]$, then we can directly apply Lemma 6.4. By taking the limit of the expectation function $w_N(u,s,t)$ as $N \to \infty$, we have the following result.

Lemma 6.5 : *For any $p > 1$ and $0 \leq u \leq s \leq t \leq T$,*

$$
E[|\xi_{s,t}(u)|^p] < +\infty . \tag{6.44}
$$

Through this lemma and the equivalence of two norms stated in Step 1, we can establish that

$$
f^{(1)}(s,t) \in \cap_{1<p<+\infty} \boldsymbol{D}_p^1(\boldsymbol{R}) . \tag{6.45}
$$

Then by repeating the above procedure [10] , we can derive the higher order $H-$derivatives of $f^{(1)}(s,t)$. Hence we can obtain the following result.

Theorem 6.2 : *Suppose Assumption I in Section 3 hold for the forward rate processes. Then for $0 < s \leq t \leq T$,*

$$
f^{(1)}(s,t) \in \boldsymbol{D}^{\infty}(\boldsymbol{R}) . \tag{6.46}
$$

[10] To be more rigorous mathematically, we have to use Lemma 2.1 of Kusuoka and Strook (1982) and Lemma 7 of Yoshida (1997) for obtaining higher order derivatives. See Kunitomo and Takahashi (1998) for the details.

Obviously, by a similar method as (6.33) and (6.34) we can establish the existence of the $H-$derivative of $\{f^{(\varepsilon)}(s,t)\}$ and $\{r^{(\varepsilon)}(t)\}$ with $0<\varepsilon\leq 1$. Then we have the corresponding results on $\{f^{(\varepsilon)}(s,t)\}$. By using the simple but lengthy arguments as *Theorem V.10.4* of Ikeda and Watanabe (1989), we have the mathematical validity on our formal derivation of *Theorem 3.1* in Section 3.

[Step 3] : Let a stochastic process $\{Y^{(\varepsilon)}(s,t), 0\leq s\leq t\leq T\}$ be the solution of the stochastic integral equation:

(6.47)
$$\begin{aligned} Y^{(\epsilon)}(s,t)=1 \quad &+\quad \varepsilon^2\int_0^s\left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t)\int_v^t\sigma(f^{(\varepsilon)}(v,y),v,y)dy\right]Y^{(\varepsilon)}(v,t)dv \\ &+\quad \varepsilon\int_0^s\partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y^{(\varepsilon)}(v,t)dB(v)\ . \end{aligned}$$

Since the coefficients of $Y^{(\varepsilon)}(s,t)$ on the right hand side of (6.47) are bounded by Assumption I, we can obtain the following.

Lemma 6.6 *: For any* $1<p<+\infty, 0<\varepsilon\leq 1,$ *and* $0<s\leq t\leq T,$

$$E[|Y^{(\varepsilon)}(s,t)|^p]+E[|Y^{(\varepsilon)-1}(s,t)|^p]<+\infty\ . \tag{6.48}$$

Proof of Lemma 6.6: We define a sequence of random variables $\{Y_n^{(\varepsilon)}(s,t)\}$ as

(6.49)
$$\begin{aligned} Y_{n+1}^{(\epsilon)}(s,t)=1 \quad &+\quad \varepsilon^2\int_0^s\left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t)\int_v^t\sigma(f^{(\varepsilon)}(v,y),v,y)dy\right]Y_n^{(\varepsilon)}(v,t)dv \\ &+\quad \varepsilon\int_0^s\partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y_n^{(\varepsilon)}(v,t)dB(v)\ , \end{aligned}$$

where the initial condition is given by $Y_0^{(\varepsilon)}(s,t)=1$. Then by the same argument used in the proof of Lemma 6.3, we have

$$E[|Y_n^{(\varepsilon)}(s,t)|^p]<\infty\ , \tag{6.50}$$

and as $n\to\infty$

$$\sup_{0\leq s\leq t\leq T}E[\sup_{0\leq u\leq s}|Y_{n+1}^{(\varepsilon)}(u,t)-Y_n^{(\varepsilon)}(u,t)|^2]\to 0\ . \tag{6.51}$$

Hence we can establish the existence of the random variables $\{Y^{(\varepsilon)}(s,t)\}$ satisfying (6.47). Then by the same argument as in (6.41)-(6.44), we have

$$E[|Y^{(\varepsilon)}(s,t)|^p]<\infty \tag{6.52}$$

for any $p > 1$. Let $Z^{(\varepsilon)}(s,t) = Y^{(\varepsilon)-1}(s,t)$. Then we can show that

$$d[Z^{(\varepsilon)}(s,t)Y^{(\varepsilon)}(s,t)] = 0 \tag{6.53}$$

and

(6.54)
$$\begin{aligned} Z^{(\epsilon)}(s,t) = 1 \quad &- \quad \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy \right] Z^{(\varepsilon)}(v,t)dv \\ &- \quad \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t) Z^{(\varepsilon)}(v,t)dB(v) \end{aligned}$$

by using Itô's Lemma and $Z^{(\varepsilon)}(0,t) = 1$. Hence by the similar argument as to $Y^{(\varepsilon)}(s,t)$, we can establish that

$$E[|Z^{(\varepsilon)}(s,t)|^p] < \infty \tag{6.55}$$

for any $p > 1$. *Q.E.D.*

Now consider the asymptotic behavior of a functional

$$F^{(\varepsilon)}(s,t) = \frac{1}{\varepsilon}[f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)] \tag{6.56}$$

as $\varepsilon \to 0$. By using the stochastic process $\{Y^{(\varepsilon)}(s,t)\}$, the $H-$derivative of $F^{(\varepsilon)}(s,t)$ can be represented as

$$D_h F^{(\varepsilon)}(s,t) = \int_0^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)C^{(\varepsilon)}(v,t)dv \ , \tag{6.57}$$

where

$$\begin{aligned} C^{(\epsilon)}(v,t) \quad &= \quad \sigma(f^{(\varepsilon)}(v,t),v,t)\dot{h}(v) + \varepsilon\sigma(f^{(\varepsilon)}(v,t),v,t) \\ &\quad \times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy \ . \end{aligned} \tag{6.58}$$

Then by re-arranging each term in the integrands of (6.57), we have the representation

$$D_h F^{(\varepsilon)}(s,t) = \int_0^s \nu_{s,t}^{(\varepsilon)}(u)\dot{h}(u)du \ , \tag{6.59}$$

where
(6.60)
$$\begin{aligned} \nu_{s,t}^{(\varepsilon)}(u) \quad &= \quad Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(u,t)\sigma(f^{(\varepsilon)}(u,t),u,t) \\ &+ \quad \varepsilon \int_u^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) \left(\int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)\xi_{v,y}^{(\varepsilon)}(u)dy \right) dv \ , \end{aligned}$$

and the random variable $\{\xi_{v,y}^{(\varepsilon)}(u)\}$ is defined as (6.35) with ε. Then the Malliavin-covariance for $F^{(\varepsilon)}(s,t)$ is given by

$$\begin{aligned}\sigma(F^{(\varepsilon)}(s,t)) &= |DF^{(\varepsilon)}(s,t)|^2_{H\otimes R} \\ &= \int_0^s [\nu_{s,t}^{(\varepsilon)}(u)]^2 du \ .\end{aligned} \tag{6.61}$$

Let

(6.62)

$$\begin{aligned}\eta_c^{(\varepsilon)}(s,t) &= c\int_0^s |\varepsilon \Big(\int_u^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) \\ &\quad \times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)\xi_{v,y}^{(\varepsilon)}(u)dydv\Big) |^2 du \\ &\quad + c\int_0^s |Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(u,t)\sigma(f^{(\varepsilon)}(u,t),u,t) - \sigma(f^{(0)}(u,t),u,t)|^2 du \ ,\end{aligned}$$

for a positive constant c. Then the condition in Assumption II in Section 3 is equivalent to the non-degeneracy condition:

$$\Sigma(s,t) = \int_0^s \sigma\left(f^{(0)}(v,t),v,t\right)^2 dv > 0 \tag{6.63}$$

because $Y^{(0)}(v,t) = 1$ for $0 < v \leq s \leq t$. The next lemma shows that the truncation by $\eta_c^{(\varepsilon)}(s,t)$ is negligible in probability.

Lemma 6.7 *: For $0 < s \leq t \leq T$ and any $n \geq 1$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-n} P\{|\eta_c^{(\varepsilon)}(s,t)| > \frac{1}{2}\} = 0 \ . \tag{6.64}$$

Proof of Lemma 6.7: We re-write (6.62) as $\eta_c^{(\varepsilon)}(s,t) = \eta_1^{(\varepsilon)} + \eta_2^{(\varepsilon)}$. By using Assumption I, Lemma 6.5, Lemma 6.6, and the Markov inequality, it is straightforward to show that for any $p > 1$ and $c_1 > 0$ there exists a positive constant c_2 such that

$$P\{|\eta_1^{(\varepsilon)}| > c_1\} \leq c_2\varepsilon^{2p} \ . \tag{6.65}$$

By the continuity of the volatility function $\sigma(\cdot)$, there exist positive constants M_{13} and M_{14} such that

$$|\eta_2^{(\varepsilon)}| \leq M_{13}|f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| + M_{14}|Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t) - 1| \ . \tag{6.66}$$

Then by Lemma 10.5 of Ikeda and Watanabe (1989), for a positive c_3 and sufficiently small $\varepsilon > 0$, there exist positive constants c_4 and c_5 such that

$$(6.67) \qquad P\{\sup_{0\leq s\leq t\leq T} |f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| > c_3\} \leq c_4 exp(-c_5\varepsilon^{-2}) \ .$$

We rewrite the second term on the right hand side of (6.66) for $\eta_2^{(\varepsilon)}$,

$$(6.68) \qquad \eta_{22}^{(\varepsilon)} = M_{14} Y^{(\varepsilon)}(v,t)^{-1} |Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t)| \ ,$$

where

(6.69)

$$\begin{aligned} Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t) &= \varepsilon^2 \int_v^s \left[\partial\sigma(f^{(\varepsilon)}(u,t),u,t) \int_u^t \sigma(f^{(\varepsilon)}(u,y),u,y)dy\right] Y^{(\varepsilon)}(u,t)du \\ &+ \ \varepsilon \int_v^s \partial\sigma(f^{(\varepsilon)}(u,t),u,t) Y^{(\varepsilon)}(u,t)dB(u) \ . \end{aligned}$$

Then by Lemma 6.6, for any $p > 1$ and $c_6 > 0$ there exists a positive constant c_7 such that

$$(6.70) \qquad P\{|\eta_{22}^{(\varepsilon)}| > c_6\} \leq c_7 \varepsilon^{2p} \ .$$

By using (6.65), (6.67), and (6.70), we have (6.64). *Q.E.D.*

Thus by modifying the method developed by Yoshida (1992a) for the present case, we have the key result on the validity of the asymptotic expansion approach in this paper.

Theorem 6.3 *: Under Assumptions I and II in Section 3, the Malliavin-covariance $\sigma(F^{(\varepsilon)})$ of $F^{(\varepsilon)}$ is uniformly non-degenerate in the sense that there exists $c_0 > 0$ such that for any $c > c_0$ and any $p > 1$,*

$$(6.71) \qquad \sup_{\varepsilon} E[I(|\eta_c^{(\varepsilon)}| \leq 1)\sigma(F^{(\varepsilon)})^{-p}] < +\infty \ ,$$

where $I(\cdot)$ is the indicator function.

Hence we have obtained a truncated version of the non-degeneracy condition on the Malliavin-covariance for the spot interest rate and forward rates processes, which are the solutions of the stochastic integral equation (3.1).

It is also straightforward to obtain the truncated version of non-degeneracy condition on the Malliavin-covariance under the assumption $\Sigma > 0$ for the discount coupon bond price process and the average interest rate process as we have stated in Theorem 3.2 of Section 3. This is because under Assumption I in Section 3, we can show that for any $p > 1$ and $0 \leq s \leq t \leq T$,

$$E[|P^{(\varepsilon)}(s,t)|^p] < +\infty \, . \tag{6.72}$$

Then Using the Novikov condition (see Theorem III-5.3 of Ikeda and Watanabe (1989)), we can find a continuous martingale for the bond process and apply the same arguments as we did for the forward rate processes.

The rest of our arguments for the asymptotic expansion approach is based on Theorem 4.1 of Yoshida (1992a), which is an extension of Theorem 2.3 of Watanabe (1987) because it gives the validity of the asymptotic expansion of the distribution function of functionals with truncation under the non-degenacy condition on the Malliavin-covariance given by (6.71). Let $\psi : \boldsymbol{R} \to \boldsymbol{R}$ be a smooth function such that $0 \leq \psi(x) \leq 1, \psi(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\psi(x) = 0$ for $|x| \geq 1$. Then the composite functional $\psi(\eta^{(\varepsilon)}) I_A(F^{(\varepsilon)})$ is well-defined for any $A \in \boldsymbol{B}$ in the sense that it is in $\tilde{\boldsymbol{D}}^{-\infty}$, where $\boldsymbol{B}$ is the Borel $\sigma-$field in $\boldsymbol{R}$ and $I_A(\cdot)$ is the indicator function. By using Theorem 6.3, lemmas in this section, and Theorem 4.1 of Yoshida (1992a) it has a proper asymptotic expansion as $\varepsilon \to 0$ uniformly in $\tilde{\boldsymbol{D}}^{-\infty}$. Then we have a proper asymptotic expansion for the density function of our interest by taking the expectation operations.

At the end of this step, we should mention that the integrals in (3.46) depend on the parameter ε . Since Yoshida (1992b) has given *Theorem 2.2* and *Lemma 2.1* for the validity of asymptotic expansions when the bounded integral operators depend on a parameter, his arguments cover the situiation of our *Theorem 3.3.* Thus our formal derivations of *Theorems 3.2* and *Theorem 3.3* in Section 3 can be rigorously justified by applying the results of Yoshida (1992a,b).

[Step 4] : The inversion technique we have used is different from the one used by Yoshida (1992a,b). He has used the Schwartz's type distribution theory for the generalized Wiener functionals while our method is based on the simple inversion technique for the characteristic functions of random variables, which has been standard in the statistical asymptotic theory. Hence what we need to show is that the resulting formulae by our method are equivalent to his final formulae. In the notations of Yoshida (1992a), we take $\varphi(x) = 1$ in his Lemma 5.6. Then he has used

$$p_1^{'}(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}\left[\varphi(f_0) f_1^i \partial_i I_A(f_0)\right] , \tag{6.73}$$

and

$$p_1^{''}(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}\left[\{f_1^i \partial_i \varphi(f_0)\} I_A(f_0)\right], \tag{6.74}$$

where $I_A(f_0)$ is the indicator function and f_0 corresponds to the random variable of the order $O_p(1)$, which is similar to g_1 in our notation. The differentiation of the indicator function has some proper mathematical meaning in the sense of differentiation on a generalized Wiener functionals. (See Watanabe (1987) and Yoshida (1992a,b) for details.) By the use of the pull-back operation of the generalized Wiener functionals, Yoshida (1992a) has obtained the explicit expansion form of the density function for a particular functional in his problem as

$$p_1(x) = p_1^{'}(x) + p_1^{''}(x). \tag{6.75}$$

In our framework it is straightforward to show that

$$p_1^{'}(x) = (-1) \frac{d}{dx} \left[\mathbf{E}(g_2 | g_1 = x) \phi_\Sigma(x)\right] \tag{6.76}$$

and $p_1^{''}(x) = 0$ since $\partial_i \varphi(\cdot) = 0$ when $k = 1$ by using our notations in this paper. Then we notice that (6.76) is exactly what the inversion formula (Lemma 6.2) gives as the second order term in the asymptotic expansion of the density function of the normalized random variable $X_T^{(\varepsilon)}$ in (3.31).

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