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Pricing Convertible Bonds with Default Risk: A Duffie-Singleton Approach

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Abstract

We propose a new method to value convertible bonds(CBs). In particular, we explicitly take default risk into consideration based on Duffie-Singleton(1999), and provide a consistent and practical method for relative pricing of securities issued by a firm such as CBs, non-convertible corporate bonds and equities. Moreover, we show numerical examples using Japanese CBs' data, and compare our model with other practical models.

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1 Introduction

1.1 Convertible Bonds

Convertible bonds(CBs), corporate bonds that can be converted to equities are widely issued and traded in the financial markets of various countries. It is well known that many types of options are embedded in a CB: The most important and basic option is a call option on the equity and the option is usually American or Bermudan type. Call or/and put conditions may be added, and recently CBs whose conversion prices depend on the current stock prices become popular in the market.

It is important to determine the underlying state variables to value and hedge such a complicated option because the future value of a CB is subject to several risk sorces. Looking at the structure of the product, we easily notice that the state variables should be the stock price, the default-free interest rate and credibility of the issuer. More essentially, CBs, equities and corporate bonds are all derivatives of the underlying firm value. Hence, if we choose the firm value as a state variable focusing on the capital structure of the firm, we are able to evaluate consistently all the securities issued by the firm. On the other hand, this approach cannot be easily implemented since the firm value and its stochastic process, unobservable in the market, must be specified. It is usually difficult to evaluate CBs based on the firm value by utilizing the stock price and the prices of corporate bonds that are observable in the market because those prices themselves depend on the firm value nonlinearly. Thus, in this paper, we take the underlying stock price as a state variable. In addition to the stock price, we focus on default event as a state variable. CBs issued by "new economy" firms that have low credibility might increase since they can be used in financing for active growth, which is the case in the United States. Thus, pricing CB with default risk will become more important. We also remark that the variables explaining the process of a default-free rate as in term structure models can be easily added as state variables. Finally, we notice that although pricing CBs without passing through the firm value, we should value CBs, non-convertible corporate bonds and equities consistently within the model in terms of relative pricing.

1.2 Modeling Default Risk

Valuing financial products with default risk can be separated into two approaches such as *Structural Approach*(A1) and *Reduced-form Approach*(A2). A1 regards default as a endogenous event by focusing on the capital structure of the firm. Merton[1974] initiated this approach where he expressed the firm value as a diffusion process and characterized default by the event that the value of the firm went below the face value of a debt. In this formulation, corporate bonds are options on the firm value and options on the corporate bonds are compound options on the firm value. However, to value a corporate bond in practice, the debts that are senior to it must be simultaneously valued, which raises computational difficulty. The problem is more serious when valuing options because of its compound option feature. To overcome the problem, Longstaff-Schwartz[1995] modified the approach in that default was characterized by a stopping time of a firm value to a certain boundary which was common among all the debts of the firm. They also introduced a default-free interest rate as the second state variable emphasizing the relation between default and the interest rate. In practical implementation, however, there are still problems in the model. First, the firm value is not directly tradable and is unobservable in the market, which makes parameters' estimation quite difficult. Second, some argue that the event of default should be modeled by a jump process because diffusion processes are not able to explain the empirical observation that there are large credit spreads even just before maturity. (See Madan-Unal[1993] for instance.) Forcusing on the problems, Duffie-Singleton[1999] and Jarrow-Turnbell[1994] proposed models in A2. They did not explain the event of default endogenously, but characterized it exogenously by a jump process, and derived the arbitrage-free prices of securities subject to common default risk. Further, the parameters of models could be estimated or calibrated more directly from observable prices of trading securities such as corporate bonds.

In pricing CBs with default risk, most of existing literatures took *the structual approach*(A1); Ingersoll[1977] and Brenann-Schwartz[1977,1980] developed models within the class. Recently, Tsiveriotis-Fernandes[1998] proposed a model with exogenously given credit spreads. However, they just used spreads observed in markets without any theoretical consideration, and there may exist some inconsistency in the model because default risk embedded in equities was not taken into account. We will propose a new model in *reduced-form approach*(A2) to price CBs with default risk. In particular, we explicitly take default risk into consideration based on Duffie-Singleton[1999], and provide a consistent method for relative pricing of CBs, non-convetible corporate bonds and equities issued by a firm. Moreover, we give numerical examples using Japanese CBs' data, and confirm that our model is valid through comparison with other practical models. From practical view point, an advantage of Duffie-Singleton[1999] is that a jump term does not explicitly appear in the valuation formula and hence that computational methods developed for diffusion models can be utilized. The reason is that they derive the valuation formula for the pre-default values of defaultable assets; the pre-default value is defined by the value whose process is equivalent to the price process of an asset before default, and it can be modeled by a diffusion process under some reguRality conditions. As a result, the valuation formula is similar to the one for default-free assets except in that the discout rate is not a default-free short-rate, but *the default-adjusted short-rate*, which is determined by a default-free rate, a default hazard rate and the fractional loss of market value of the claim at default. In this approach, all the securities issued by a company have the same hazard rate while they could have different loss rates. We model the hazard rate as a decreasing function of the stock price since the stock price is easily observed and traded most frequently among the securities issued by a company, and it seems natural that the probability of default is negatively related with the level of the stock price. We also assume that the pre-default process of the stock price follows a diffusion process noting that the stock price itself is subject to default risk. Then we can obtain a consistent pricing method for any securities by specifying their payoffs and their fractional loss rates at default under an assumption for the stochastic process of the default-free rate.

The organization of the paper is as follows. Section 2 describes our model in the framework of Duffie and Singleton[1999]. Section 3 shows comparison of our model with other models by using data of the Japanese CB market, and the final section gives conclusion.

2 A CB model in Duffie-Singleton Approach

Under an appropriate mathematical setting, Duffie and Singleton[1999] shows that the pre-default value V_t of a defaultable security characterized by the final payoff X at time T and the cumulative dividend process $\{D_t : 0 \leq t \leq T\}$, is expressed as

$$V_t = \mathbf{E}_t^Q \left[e^{-\int_t^T R(u)du} X + \int_t^T e^{-\int_t^s R(u)du} dD_s \right]. \tag{1}$$

Here, $R(t)$ is the default-adjusted discount rate defined by

$$R(t) := r(t) + L(t)\lambda(t) \tag{2}$$

where $r(t)$ is the default-free short rate, $L(t)$ is the fractional loss rate of market value at default, $\lambda(t)$ is the default hazard rate, and $\mathbf{E}_t[\cdot]$ denotes the conditional expectation under a risk-neutral measure Q, given available information at time t.

Hereafter V_t denotes the value of a CB provided that default has not occured by time t; if t is the date of a coupon payment, V_t denotes the ex-coupon value. For modeling CBs in this framework, we take the pre-defalut value of the underlying stock, S_t as a state variable, and suppose that the default hazard rate is a nonnegative function of S_t and t, $\lambda(S,t)$.

$$\lambda(S,t) : R_+ \times R_+ \to R_+$$

We also assume that $\lambda(S,t)$ is a decreasing function of S because it seems natural that the probability of default becomes higher when the stock price becomes lower, and vice versa. More specifically, noting that the stock price itself is subject to default risk and should satisfy the equation (1) with zero recovery rate(that is, the fractional loss rate is 1), we suppose that S_t follows a diffusion process under the assumption of a deterministic default-free interest rate:

$$dS = \{\alpha(S,t) + \lambda(S,t)\} S dt + \sigma S dw_t \tag{3}$$

where $\alpha \equiv r(t) - d(S,t)$, $r(t)$ is a function of the time parameter t, σ is a positive constant, and $d(S,t)$ denotes the dividend rate which could be a function of the stock price S and the time parameter t. The process can be justified by the fact that the pre-default value of the stock with its cumulative dividend discounted by the default-adjusted rate is a martingale under the risk-neutral measure; that is,

$$e^{-\int_0^t R(u)du} S_t + \int_0^t e^{-\int_0^u R(s)ds} d(S,u)du \tag{4}$$

is a martingale under Q where $R(t) = r(t) + \lambda(t)$ because $L(t) \equiv 1$. The assumption that the risk-free rate is deterministic is just for simplicity; clearly, we can easily extend the model to the one in which the risk-free rate is a function of the time parameter and a vector of random variables $Y (\in \mathbf{R}^n)$, $r = r(Y, t)$, and Y could be described by a multi-dimensional Markov process. In that case, S together with Y also follows a multi-dimensional Markov process and any Markovian term structure model could be combined.

Next, we characterize the valuational equation (1) through payoffs of a CB. If the conversion is allowed only at maturity, X and D_t in the equation (1) for CB are given by

$$X = \max[aS_T, F]$$

and

$$D_t = \sum_i c_i 1_{\{t \geq T_i\}} \tag{5}$$

respectively where a is a positive constant representing the conversion ratio, F denotes the face value and c_i is the coupon payment at time T_i. We assume for simplicity that the fractional loss of the market value of a CB is a constant, $L(t) = L$ while we can also define $L(t)$ as a function of CB's value itself V_t, the stock price S_t and the time parameter t. In this basic case, a CB is regarded as a non-convertible corporate bond plus a call option on the underlying stock.

$$V_t = \mathbf{E}_t^Q \left[\sum_i e^{-\int_t^{T_i} R(u)du} c_i + e^{-\int_t^T R(u)du} F \right]$$

$$+ \quad a\mathbf{E}_t^Q \left[e^{-\int_t^T R(u)du} \max[S_T - k, 0] \right] \tag{6}$$

where $k \equiv \frac{F}{a}$; the first term and the second term represent the price of non-convertible corporate bond and the price of a call option on the stock respectively. The CB can be evaluated by standard numerical technique such as Monte Carlo simulations, and in particular, if the dividend rate $d(S,t)$ and the hazard rate $\lambda(S,t)$ are non-stochastic, the price is obtained explicitly by utilizing the Black-Scholes formula, a similar formula to the one obtained in the non-defaultable case.

$$\begin{aligned} V_t &= \left[\sum_i e^{-\int_t^{T_i} R(u)du} c_i + e^{-\int_t^T R(u)du} F \right] \qquad (7) \\ &+a \left[e^{-\int_t^T \{d(u)-(1-L)\lambda(u)\}du} S_t \Phi(d_1) - e^{-\int_t^T R(u)du} k \Phi(d_1 - \sigma\sqrt{T-t}) \right] \end{aligned}$$

where

$$d_1 = \frac{\log \frac{S_t}{k} + \int_t^T \{\alpha(s) + \lambda(s)\}ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

and $\Phi(x)$ denotes the standard normal distribution function evaluated at x.

If the conversion is allowed before maturity, the valuation problem is formulated as an optimal stopping problem: The value of a CB at time t provided that default has not occured by time t is expressed by

$$\begin{aligned} V_t = \sup_{\tau \in \mathcal{S}} \mathbf{E}_t^Q &\left[e^{-\int_t^\tau R(u)du} 1_{\{\tau < T\}} a S_\tau + e^{-\int_t^T R(u)du} 1_{\{\tau = T\}} \max[aS_T, F] \right. \\ &+ \left. \sum_{t < T_i \le \tau} e^{-\int_t^{T_i} R(u)du} c_i \right] \end{aligned} \tag{8}$$

where $\mathcal{S}$ denotes the set of feasible conversion strategies which are stopping times taking values in $[t, T]$. See chapter 2 of Karatzas-Shreve[1998] for rigorous mathematical argument. Practically, by standard argument as in American options, this can be solved with a recursive backward algorithm in a discretized setting: Starting with $V_T = \max[aS_{t_N}, F]$ at maturity $t_N \equiv T$, we implement the following alogorithm at each descritized time point t_j, $j = N-1, N-2, \cdots, 0$, where t_0 denotes the date of valuation;

$$V_{t_j} = \max\{\mathbf{E}_{t_j}^{Q}[e^{-R(t_j)\Delta t} V_{t_{j+1}}],\ aS_{t_j}\},\ \Delta t \equiv t_{j+1} - t_j. \tag{9}$$

Further, at each date of the coupon payment T_i, V_{T_i} is replaced by $V_{T_i} + c_j$.

We can also derive the associated partial differential equation(PDE) by noting that in the equation (1) the pre-default value with the cumulative dividend discounted by the default-adjusted rate is a martingale under Q:

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + \{\alpha(S,t) + \lambda(S,t)\} S V_S + V_t - \{r(t) + L\lambda(S,t)\} V + f(t) = 0, \tag{10}$$

where V_S, V_{SS}, and V_t denote the first or second order partial derivatives with respect to S or t. $f(t)$ represents the coupon payments, $f(t) = \sum_i c_i \delta(t - T_i)$ where $\delta(\cdot)$ denotes the delta function. In the similar way as in non-defaultable securities, the boundary condtions are given by $V(S,T) = \max[aS_T, F]$ at maturity, and $V(S,t) \geq aS_t$ for conversion when the conversion is allowed before maturity. Further, if there are call and/or put conditions, we add the boundary condtions such as $V(S,t) \leq \max[cp(t), aS_t]$ in the callable period and $V(S,t) \geq pp(t)$ at the redemption date, where $cp(t)$ and $pp(t)$ denote the call and put prices at time t respectively. See Brenann-Schwartz[1977,1980] and McConnell-Schwartz[1986] for the details of boundary conditions. In practice, we can utilize standard numerical schemes such as finite difference methods for solving the PDE as in Brenann-Schwartz[1977] of *Structural Approach.*

We emphasize that our PDE unlike the one derived in Tsiveriotis and Fernandes[1998], includes the hazard rate $\lambda(S,t)$ in the coefficient of V_S because we explicitly take into account that the underlying stock price itself is subject to default risk, which is theoretically more consistent.

3 Numerical Examples

3.1 Implementation and Sensitivity Analyses

In this subsection, we briefly describe how to implement our model in the subsequent numerical analysis. We take the function of $\lambda(S_t,t)$ as

$$\lambda(S_t,t)=\lambda(S_t)=\theta+\frac{c}{S_t^b} \tag{11}$$

where $\theta\geq 0$, b, and c are some constants. To estimate the credit parts of a CB we utilize information implied in a non-convertible corporate bond(SB) issued by the same company. Notice that the price of a non-convertible bond is consistently evaluated in our framework. That is, in the equation (1), X and D_t are specified by the face value and coupon payments respectively. The default-adjsuted short term rate R consists of three parts; the default-free short rate r and the hazard rate λ are common while the fractional loss rates L are generally different in convertible bonds and non-convertible bonds. Practically, we take the following approach; assume that the fractional loss rate of a non-convertible corporate bond denoted by L_{SB} is a constant and is the same as that of a CB, that is $L = L_{\mathrm{SB}}$ since it is usually difficult to estimate λ and L, or L and L_{SB} separately. Then, given θ, c, and L, estimate b in the equation (11) by calibrating the price of a non-convertible corporate bond of which maturity is the closest to that of the CB. For computation, we numerically implement a discretized scheme of (9) illustrated in the previous section by utilizing a recombining binomial tree as in Nelson-Ramaswamy[1990] and Takahashi-Tokioka[1999]: Note first that the process $Z_t \equiv \log S_t$ is described as

$$dZ_t = -\frac{1}{2}\sigma^2 dt + \beta(e^{Z_t}, t)dt + \sigma dw_t,\ Z_0 = \log S_0 \tag{12}$$

where $\beta(e^{Z_t}, t) \equiv \alpha(e^{Z_t}, t) + \lambda(e^{Z_t})$. Then, we discretize the process Z_t by N time steps, and approximate it by a binomial lattice in the following;

$$Z(j', i+1) - Z(j, i) = -\frac{1}{2}\sigma^2 h + \sigma\sqrt{h}Y,\ Z(0,0) = \log S_0 \tag{13}$$

where $h \equiv \frac{T}{N}$, $Z(j,i)$ denotes the value of Z at time ih and state j, and states are characterized by a random variable Y such that $j' = j+1$ when $Y = 1$ and $j' = j$ when $Y = -1$ where

$$Y = \begin{cases} \ \ 1 & \text{with probability } q(j,i) \\ -1 & \text{with probability } 1 - q(j,i). \end{cases} \tag{14}$$

Here, $q(j,i)$ is defined so that the expectation of $\sigma\sqrt{h}Y$ is equal to $\beta(j,i)h$ where $\beta(j,i) \equiv \beta(e^{Z(j,i)}, ih)$. Notice also that the second moment, the expectation of $(\sigma\sqrt{h}Y)^2$ is automatically equal to $\sigma^2 h$. Hence, we define $q(j,i)$ with the adjustment for the case that the probability is not in $[0,1]$.

$$q(j,i) = \begin{cases} q'(j,i), & \text{if } q'(j,i) \in [0,1] \\ 0, & \text{if } q'(j,i) < 0 \\ 1, & \text{otherwise,} \end{cases} \tag{15}$$

where

$$q^{'}(j,i) = \frac{1}{2}\left(1 + \frac{\sqrt{h}}{\sigma}\beta(j,i)\right). \tag{16}$$

Using the binomial scheme and actual market data of corporate bonds, we provide sensitivity analyses of our model. In addition, for comparative purpose, we introduce another new model called *boundary model* which is similar to the model proposed by Longstaff-Shwartz[1995] except in that the underlying state variable is not a firm value, but a stock price. Before showing sensitivity analyses, we describe the structure of *boundary model.*

- **Boundary Model**

  We suppose that the stock price follows the stochastic process,

$$dS = \alpha(S,t)Sdt + \sigma S dw_t \tag{17}$$

  where $\alpha(S,t) \equiv r(t) - d(S,t)$, the risk-free rate $r(t)$ is a function of the time parameter t, the volatility σ is some positive constant, the dividend rate d is a function of S and t, and w is a standard Brownian motion under the risk-neutral measure Q. Next, define the default time τ as

$$\tau = \inf\{t : S_t \leq S_t^{'}\}. \tag{18}$$

  given the critical price of default $S^{'}(< S_0)$. In the model, the way of recognizing default event is whether the stock price reaches the critical price. We next show that the model also gives a consistent pricing of non-convertible bonds and non-convertible corporate bonds. In this setting, the fundamental partial differential equation(PDE) to price securities issued by a firm is expressed as

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + \alpha(S,t) S V_S + V_t - r(t)V + f(t) = 0 \tag{19}$$

where $f(t)$ represents the coupon payments, $f(t) = \sum_i c_i \delta(t - T_i)$. For pricing a CB, the boundary condtions are given by $V(S,T) = \max[aS_T, F]$ at maturity, $V(S', \tau) = (1 - L')(F + \sum_{\tau \leq T_i} c_i)$ at the time of default, and $V(S,t) \geq aS_t$ for conversion when the conversion is allowed before maturity, where L' denotes the fractional loss rate of the principal and coupons.

For pricing a non-convertible bond, the boundary condtions are given by $V(S,T) = F$ at maturity and $V(S', \tau) = (1 - L')(F + \sum_{\tau \leq T_i} c_i)$ at the time of default. We first estimate S' through the calibration of the market price of a corporate bond of which maturity is the closest to that of the CB in target, and then use that S' to price the CB. We also note that if the risk-free rate $r(t)$ and the dividend rate $d(S,t)$ are positive constants, the initial price of non-convertible bond, $SB(0)$ is obtained by the following explicit formula:

$$SB(0) = (1 - L')\{FA(T) + \sum_i c_i A(T_i)\} + e^{-rT} FB(T) + \sum_i e^{-rT_i} c_i B(T_i), \tag{20}$$

where

$$A(u) \equiv \left(\frac{S_0}{S'}\right)^{\{\frac{1}{2} - \frac{(\alpha+z)}{\sigma^2}\}} \left[\Phi\left(\frac{-\log(S_0/S') - zu}{\sigma\sqrt{u}}\right) + \left(\frac{S_0}{S'}\right)^{\frac{-2z}{\sigma^2}} \Phi\left(\frac{-\log(S_0/S') + zu}{\sigma\sqrt{u}}\right)\right],$$

$$B(u) \equiv \Phi\left(\frac{\log(S_0/S') + (\alpha - \sigma^2/2)u}{\sigma\sqrt{u}}\right) - \left(\frac{S_0}{S'}\right)^{1-\frac{2\alpha}{\sigma^2}} \Phi\left(\frac{-\log(S_0/S') + (\alpha - \sigma^2/2)u}{\sigma\sqrt{u}}\right)$$

and $z \equiv \{(\alpha - \sigma^2/2)^2 + 2r\sigma^2\}^{\frac{1}{2}}$. Here, $\Phi(x)$ denotes the standard normal distribution function evaluated at x. For numerical computation of CBs and SBs, we utilize a lattice model with efficient technique for barrier options as in Ritchken[1995].

In the following analyses, $r(t)$ is determined through calibration of the current term structure implied in the Japanese LIBOR and swap market. For the volatility parameter σ in the stock price process (3), we use the historical volatility computed from the last half-year. We also note that all the CBs used in the analyses have no call nor put conditions. Basic parameters' values and names of CBs used in the analyses are summarized below.

- $\theta = 0.001$, $c = 0.6$ in the equation (11).
- $L = 1$ except in the recovery rate sensitivity analysis (figures (5-a)$\sim$(5-c)).
- Sega Enterprise No.4 at 8/11/1999 in the credit spread sensitivity analysis (figures 1$\sim$ 4).
- In the recovery rate sensitivity analysis (figures (5-a)$\sim$(5-c)),
  - Sega Enterprise No.4 at 8/11/1999 for *out-of-the-money* (OTM),
  - Nissan Motor No.5 at 11/13/2000 for *at-the-money* (ATM),
  - Softbank No.1 at 10/1/1999 for *in-the-money* (ITM).

- Nissan Motor No.5 at 11/13/2000 in the volatility sensitivity analysis (figure 6).

The following figure shows the hazard rate functions calibrated for the different credit spreads of a non-convertible corporate bond with fixed parameters except b.

Figure 1

W can observed that when the credit spread is low, the hazard rate declines rapidly as stock price rises, which is due to the increase in the calibrated parameter b. The next figure shows the CB prices using these implied hazard rate functions.

Figure 2

We notice that when the stock prices reach a certain low level, the rapid decline in CB prices is caused by increasingly rise in the hazard rate.

For *boundary model*, the next figure shows the default boundary $S^{'}$ against the different credit spreads.

Figure 3

Clearly, the default boundary increases as the credit spread is widen. Using the default boundaries for the different credit spreads, we compute the CB prices.

Figure 4

We can observe that when the stock price is low, CB prices reflect the difference of the credit spreads while there is little difference in CB prices against the different credit spreads when the stock price is high.

The following three figures show the recovery rate sensitivity, where the recovery rate is defined by $1 - L$. We also call our model *intensity model* in the figures.

Figure 5-a,5-b,5-c

For our model, we can observe the positive sensitivity of the price against the recovery rate in the *at-the-money*(ATM) and *in-the-money*(ITM) cases while the price is not sensitive to the change in the recovery rate in the *out-of-the-money*(OTM) case. Given the constant credit spread, the hazard rate should rise as the recovery rate increases, which picks up the drift of the stock price process under the risk-neutral measure; as a result, the CB price increases and the effect is the strongest in the ITM case. On the other hand, in the boundary model, the CB price is not senstive to the change in the recovery rate for all cases.

The final figure shows the volatility sensitivity. In the figure, the market price is 129.5, and HV denotes the historical volatility computed from the last half-year. As expected, the CB price rises as the volatility increases for both models. However, in the boundary model, we are not able to find the default boundary for low levels of the volatility.

Figure 6

3.2 Comparison of Models in the Japanese CB Market

We next implement comparison of our model and other practical models using actual market data in Japan. We briefly illustrate three models used in the analysis other than our model and *the boundary model* explained in the last subsection, and hereafter call our model **Model 1** and the boundary model **Model 2**:

- **(Model 3)Tsiveriotis-Fernandes [1998] Model**

  In Tsiveriotis-Fernandes[1998] the valuation problem of CBs is formulated as a system of the following two coupled partial differential equations:

$$\frac{\sigma^2 S^2}{2} u_{SS} + r_g S u_S + u_t - r(u - v) - (r + r_c)v = 0$$
$$\frac{\sigma^2 S^2}{2} v_{SS} + r_g S v_S + v_t - (r + r_c)v = 0 \quad (21)$$

  where u is the value of the CB, v is the value of the COCB, S is the price of the underlying stock, r_g is the growth rate of the stock, r is the risk-free rate, r_c is the observable credit spread implied by non-convertible bonds of the same issuer for similar maturities with the CB.

Here, COCB is defined as follows; the holder of a COCB is entitled to all cash flows, and no equity flows, that an optimally behaving holder of the corresponding CB would receive. See their artice for the details.

- **(Model 4)Goldman Sachs [1994] Model**

This model modifies the binomial pricing model for American options so that the discount rate is adjusted to reflect a given credit spread with calculation of the probability of the conversion. we illustrate the part of the adjustment. Let $P(S,t)$ and $\pi(S,t)$ the probability of conversion and the discount rate resepectively when the stock price is S at time t. First, we notice that $P(S,t)$ is determined at maturity T as follows:

$$P(S,T) = \begin{cases} 1 & \text{for } aS > F \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

We next show the essential part of the backward scheme: Let i and j indexes denoting the time and the state respectively. Then, $P(i,j)$ is computed by using $P(i+1,\cdot)$ as

$$P(i,j) = qP(i+1,j+1) + (1-q)P(i+1,j) \tag{23}$$

where q is the risk-neutral probability for the stock to rise in the next time step. Once $P(i,j)$ is obtained, $\pi(i,j)$ can be calculated by using $P(i,j)$:

$$\pi(i,j) = P(i,j)r + \{1 - P(i,j)\}(r + r_c) \tag{24}$$

where r is the risk-free rate and r_c is the observed credit spread implied by the non-convertible bonds of the same issuer for similar maturities with the CB. Finally, $P(i,j)$ is changed to 1 if the conversion is optimal;

$$P(i,j) = \begin{cases} 1 & \text{for } aS > F \\ P(i,j) & \text{otherwise.} \end{cases} \tag{25}$$

Given $P(S,T)$, if the one-step scheme is recursively used with the algorithm for American options, the price is obtained.

- **(Model 5)Chen-Nelken [1994] Model**

  Chen-Nelken[1994] is a two-factor model in which factors are a stock price and the yield of a corporate bond. The main assumption is that there is no correlation between the yield and the return on the stock. See their article for the detail of the model. We take Hull-White model as the yield process.

Finally, using Japanese CBs'data, we compare our model with other four models explained above. We list below the details of CBs used for the analysis. We note that all the CBs used in the analysis have no call nor put conditions.

- Sega Enterprise, ltd. CB No.4
- All Nippon Airways co., ltd. CB No.5
- The Nomura Securities co., ltd. CB No.6
- Nissan Motor co., ltd. CB No.5

For calibration we choose a corporate bond of which maturity is the closest to the maturity of each CB. Further, as the indicator for comparison, we use *absolute error ratio* which is defined by

$$\text{absolute error ratio} \equiv \frac{|\text{model price} - \text{market price}|}{\text{market price}}.$$

The result is listed in the following table.

Table 1

We can observe that our model is the best to explain the market data in that the sum of *absolute error ratio* is the smallest; the absolute error is the smallest for *Sega Enterprise* and *Nomura Securities* while that is the third to the smallest for *All Nippon Airways* and *Nissan Motor*. Hence, we confirm that our model is relatively valid in practice.

4 Conclusion

We have developed a new model for convertible bonds, where we explicitly took default risk into account based on Duffie-Singleton(1999). The proposed model provides a consistent and practical method for relative valuation of securities issued by a firm such as CBs, non-convertible corporate bonds and equities. In addition, we have shown numerical comparison of our model and other practical models. Finally, we remark that the model can be easily extended to the one in which a risk-free interest rate is described by every Markovian term structure model while it is currently assumed to be deterministic for simplicity.

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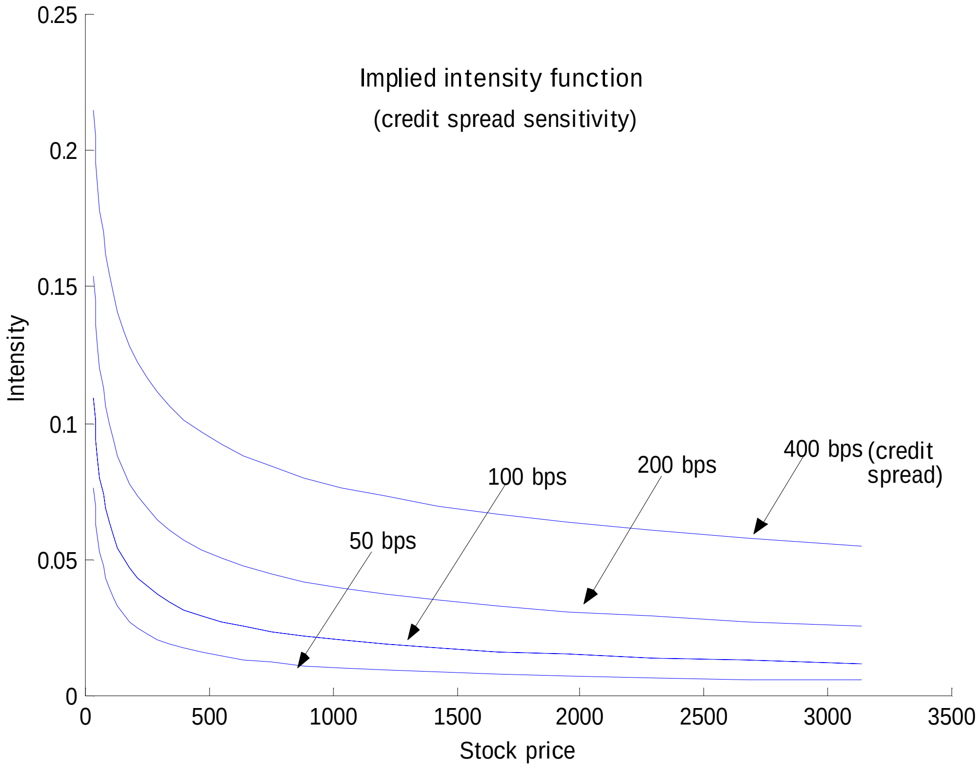
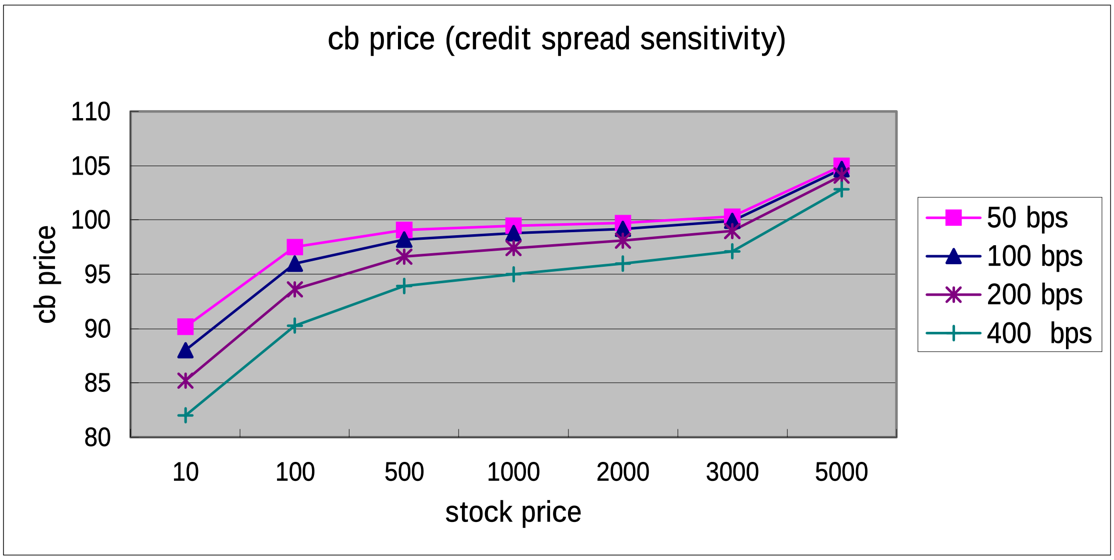
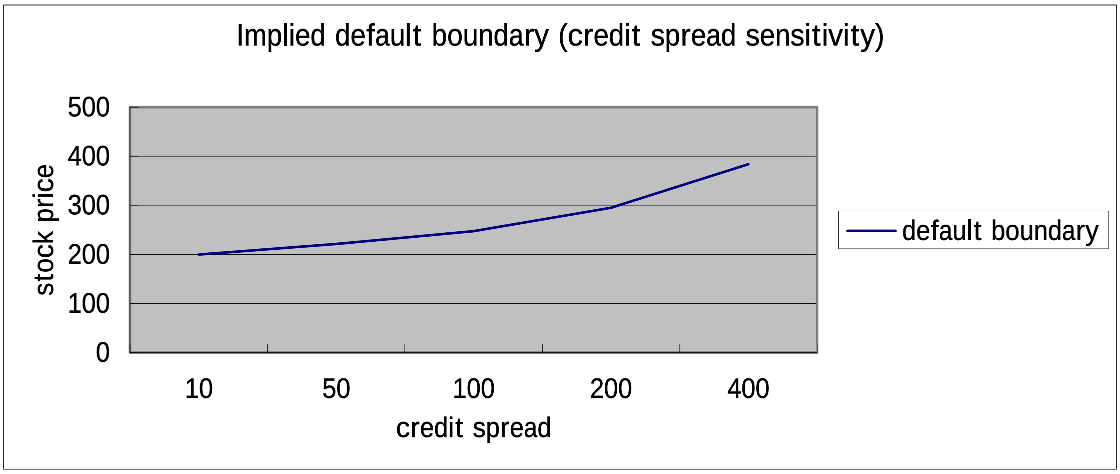

Implied intensity function
(credit spread sensitivity)
0.25
0.2
0.15
0.1
0.05
0
Intensity
0
500
1000
1500
2000
2500
3000
3500
Stock price
50 bps
100 bps
200 bps
400 bps (credit spread)

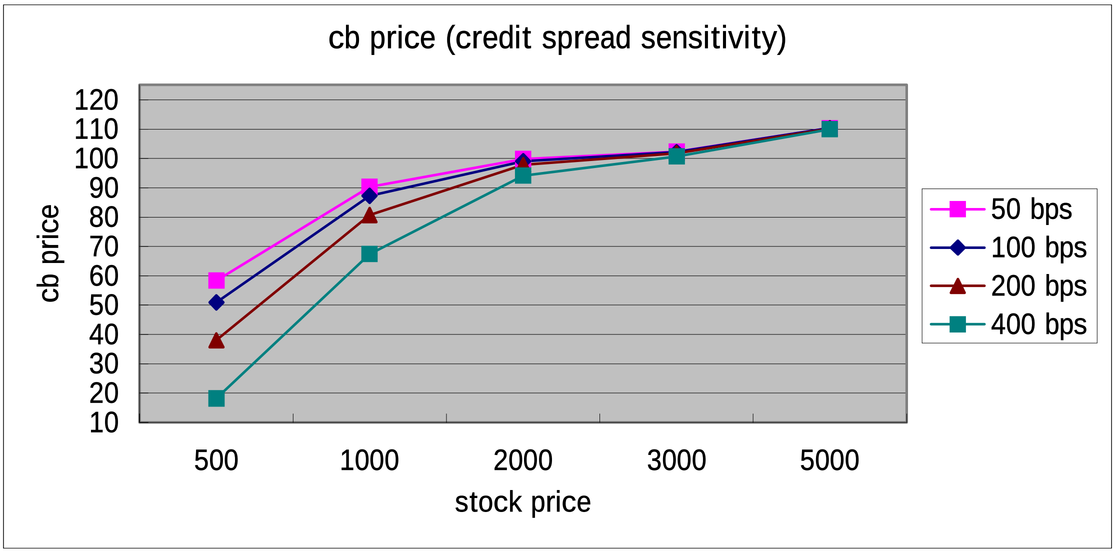
[Figure 1]


cb price (credit spread sensitivity)
110
105
100
95
90
85
80
cb price
10
100
500
1000
2000
3000
5000
stock price
50 bps
100 bps
200 bps
400 bps

[Figure 2]

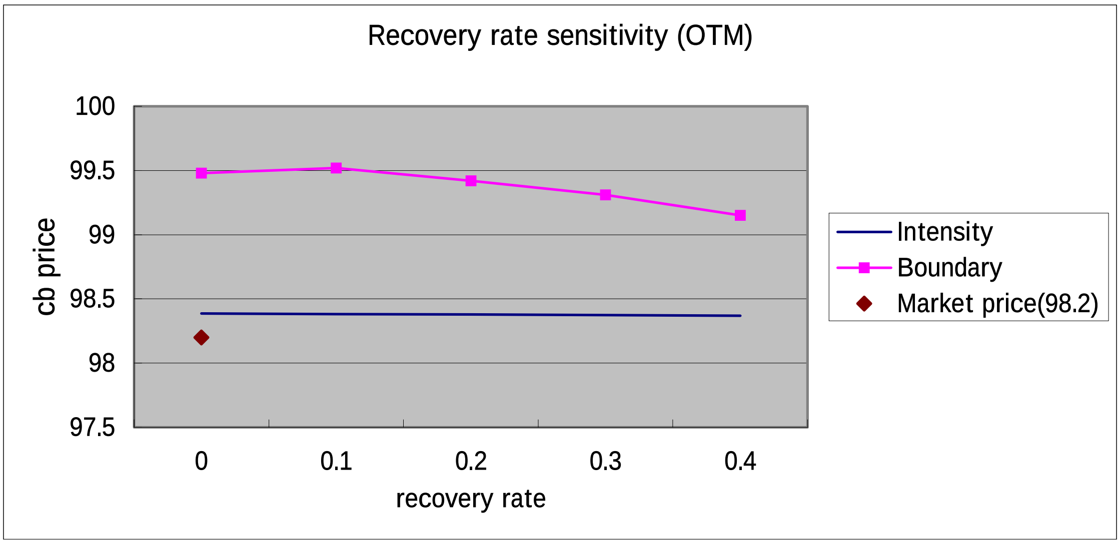

Implied default boundary (credit spread sensitivity)
stock price
500
400
300
200
100
0
10
50
100
200
400
credit spread
default boundary

[Figure 3]

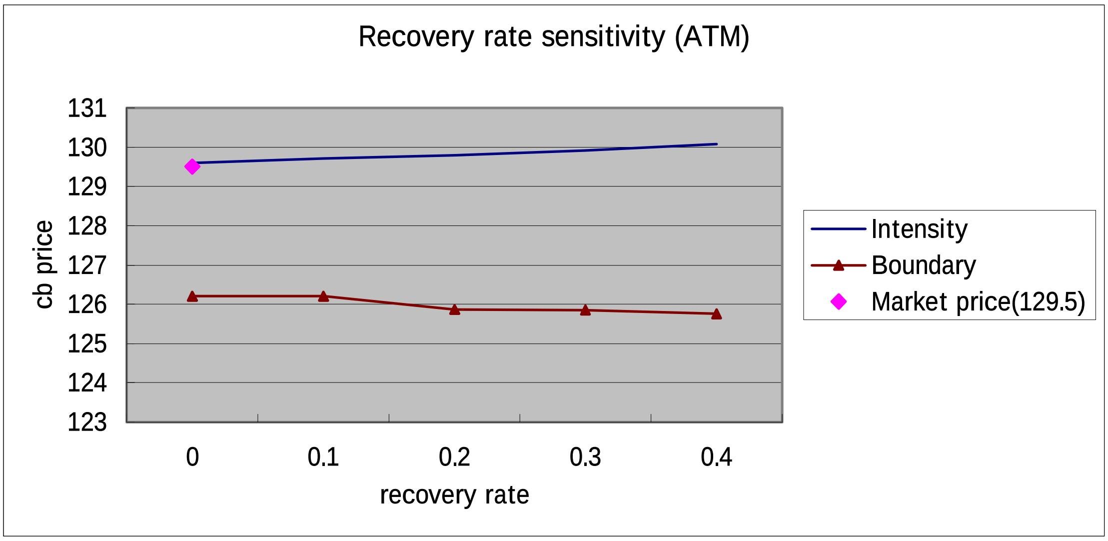


cb price (credit spread sensitivity)
cb price
120
110
100
90
80
70
60
50
40
30
20
10
500
1000
2000
3000
5000
stock price
50 bps
100 bps
200 bps
400 bps

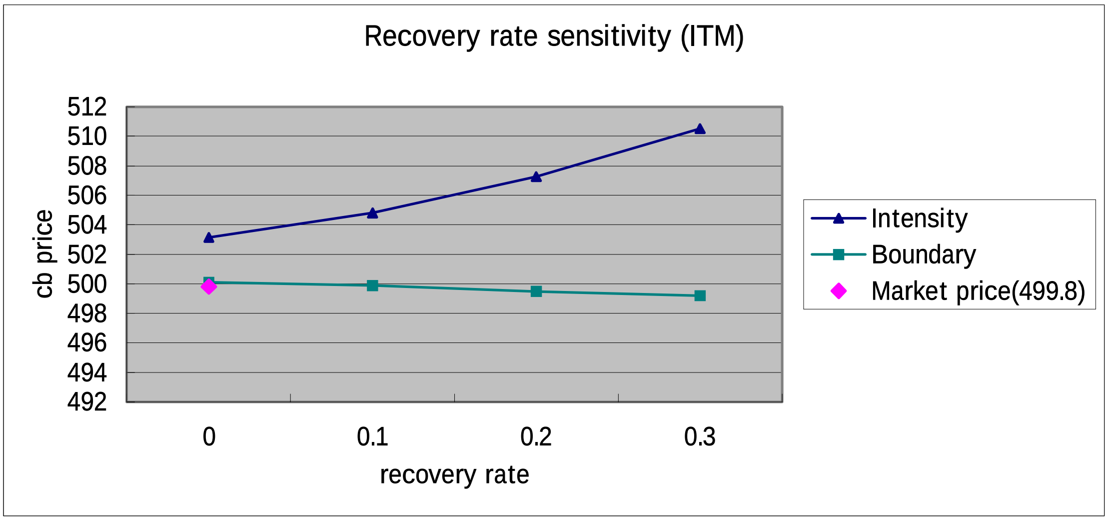
[Figure 4]


Recovery rate sensitivity (OTM)
100
99.5
99
98.5
98
97.5
cb price
0
0.1
0.2
0.3
0.4
recovery rate
Intensity
Boundary
Market price(98.2)

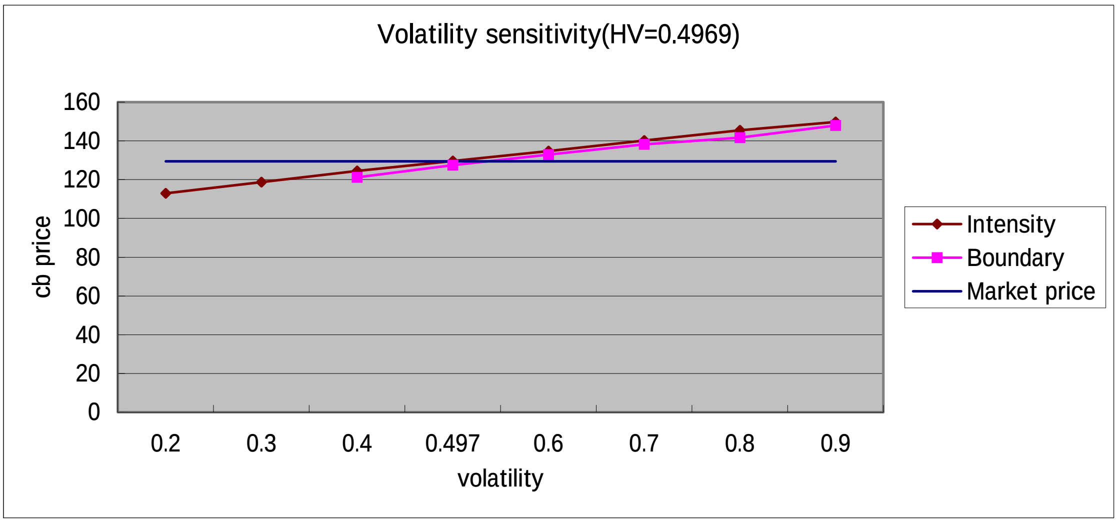
[Figure 5-a]


Recovery rate sensitivity (ATM)
131
130
129
128
127
126
125
124
123
cb price
0
0.1
0.2
0.3
0.4
recovery rate
Intensity
Boundary
Market price(129.5)

[Figure 5-b]


Recovery rate sensitivity (ITM)
cb price
512
510
508
506
504
502
500
498
496
494
492
0
0.1
0.2
0.3
recovery rate
Intensity
Boundary
Market price(499.8)

[Figure 5-c]



Volatility sensitivity(HV=0.4969)
160
140
120
100
80
60
40
20
0
cb price
0.2
0.3
0.4
0.497
0.6
0.7
0.8
0.9
volatility
Intensity
Boundary
Market price

[Figure 6]

[Table 1: Comparison of Models]

Model 1: Intensity model
Model 2: Boundary model
Model 3: Goldman Sachs (1994)
Model 4: Tsiveriotis and Fernandes (1998)
Model 5: Cheung and Nelken (1994)

	date	stock price	HV	bond price	libor cb	libor bond	bond yield	Market price	Model Number 1	2	3	4	5
(SEGA)	1999/7/5	1761	0.395	99.26	0.29	0.55	2.329	97.9	98.4	99.3	97.7	94	98.4
	1999/7/8	1782	0.395	99.07	0.292	0.582	2.415	97.9	98.3	99.2	97.4	94.2	98.1
	1999/7/13	1710	0.401	99.15	0.278	0.535	2.382	97.7	98.3	99.3	97.7	94	98.2
	1999/7/14	1685	0.402	99.1	0.265	0.532	2.405	98	98.3	99.6	97.7	93.9	98.2
	1999/7/26	1676	0.394	99.22	0.247	0.511	2.355	98.3	98.4	99.4	97.8	94.6	98.4
	1999/7/30	1610	0.405	98.89	0.274	0.571	2.51	98.4	98.2	99.6	97.7	94	98.4
	1999/8/2	1875	0.462	98.85	0.276	0.591	2.53	98.5	98.3	99.7	97.7	92.8	98.4
	1999/8/4	2085	0.518	98.76	0.294	0.624	2.573	98.6	98.4	99.8	97.7	91.8	98.5
	1999/8/9	2065	0.525	98.82	0.3	0.612	2.548	98.5	98.4	99.5	97.8	91.6	98.5
	1999/8/10	1979	0.526	98.71	0.29	0.63	2.601	98.4	98.4	99.5	97.7	91.2	98.5
	1999/8/11	1940	0.526	98.8	0.3	0.62	2.559	98.2	98.4	99.5	97.7	91.1	98.5
	1999/8/18	1935	0.53	98.82	0.298	0.622	2.554	98.4	98.4	99.8	97.8	91.3	98.5
	1999/8/31	2000	0.529	99.12	0.282	0.597	2.419	98.5	98.6	99.7	98	92	98.7
	1999/9/13	2290	0.546	99.25	0.25	0.53	2.363	96.5	98.7	99.9	98.3	93	98.9
	1999/10/13	1998	0.574	99.56	0.22	0.438	2.221	96.4	98.5	99.3	98.2	91.6	98.8
	1999/11/12	1900	0.564	99.71	0.22	0.41	2.151	96.4	99	99.9	98.4	92.4	99.1
	1999/12/27	3230	0.754	99.61	0.22	0.4	2.217	97.8	100.7	101.5	100.2	95.9	100.9
					sum of error ratio			⇒	0.131	0.308	0.153	0.867	0.134
(ANA)	1999/7/15	373	0.354	98.88	1.495	1.667	2.222	88.3	92.3	88.3	92.3	82.2	95.4
	1999/8/30	397	0.354	95.87	1.69	1.902	2.709	86.2	91.3	88.1	90.6	81.8	95.7
	1999/9/10	381	0.318	96.96	1.54	1.746	2.534	86.1	90.1	89.6	90.4	81.7	95.6
	1999/10/22	341	0.285	96.54	1.431	1.763	2.611	88.1	89.8	86.7	89.3	81.8	93.3
	1999/11/11	327	0.267	97.58	1.615	1.785	2.443	87.6	90.6	86.2	89.9	82.2	93.7
	1999/12/3	323	0.269	98.03	1.345	1.506	2.371	89.4	90.6	84.7	90.3	82.7	93.6
					sum of error ratio			⇒	0.218	0.147	0.196	0.379	0.477
(NOMURA)	1999/7/1	1425	0.201	94.56	0.986	1.876	2.874	100	99.9	97.7	95.1	94.4	99.7
	1999/7/19	1774	0.208	94.83	0.941	1.858	2.842	100.1	100.4	96.9	95.6	95.7	101.9
	1999/8/11	1490	0.215	94.91	1.035	2.082	2.835	98.9	99.8	97.9	95.6	94.8	100.6
	1999/8/23	1803	0.223	94.58	1.023	2.125	2.881	100.6	100.8	98.5	95.8	95.8	102.2
	1999/9/13	1676	0.218	95.94	0.914	1.891	2.704	99.4	99.7	97.3	96.4	96	102
	1999/10/15	1740	0.213	96.36	0.779	1.907	2.653	99.8	99.5	97.7	95.3	95.5	101.5
	1999/11/11	1730	0.211	97.34	0.728	1.976	2.526	99.9	99.6	96.5	95.9	96.1	101.9
	2000/1/24	2400	0.235	99.49	0.745	1.702	2.247	102.1	100.3	100.7	98.8	99.6	105.5
	2000/2/4	2995	0.243	98.71	0.79	1.802	2.351	108	104.4	102.7	101.6	102.6	110.1
					sum of error ratio			⇒	0.075	0.225	0.382	0.379	0.17
(NISSAN)	1999/7/5	582	0.453	99.19	0.879	0.871	2.405	108.8	119.8	115.5	119.6	122.1	125.7
	1999/9/30	645	0.387	100.1	0.701	0.691	2.147	116.9	122.4	116	120.6	124.4	126.5
	1999/10/4	690	0.393	100.2	0.825	0.818	2.117	127.7	125.6	120	124.4	128.8	130.5
	1999/11/4	624	0.397	100.6	0.807	0.8	1.992	116.2	120.4	116.8	119.7	123.4	125.1
	1999/11/16	520	0.462	100.6	0.8	0.793	2.003	104.6	115.6	109.7	114.9	116.4	119
	1999/12/27	405	0.525	100.4	0.776	0.768	2.042	98.5	110.5	104.2	109.8	107.4	112.8
	2000/11/2	760	0.507	101.3	0.715	0.711	1.618	128	134.2	130.6	133.7	136.2	136.5
	2000/11/13	720	0.497	101.3	0.709	0.705	1.598	129.5	129.6	126.2	129.1	132.1	132.1
					sum of error ratio			⇒	0.477	0.287	0.448	0.544	0.705
				total sum of error ratio				⇒	0.901	0.968	1.179	2.17	1.486