Pricing Average Options on Commodities [1]

Kenichiro Shiraya [2],

Akihiko Takahashi [3]

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Abstract

This paper proposes a new approximation formula for pricing average options on commodities under a stochastic volatility environment. In particular, it derives an option pricing formula under Heston and an extended λ-SABR stochastic volatility models (which includes an extended SABR model as a special case). Moreover, numerical examples support the accuracy of the proposed average option pricing formula.

1   Introduction

Average options are very popular derivatives in over-the-counter (OTC) commodity markets. [1] On the contrary, analytic approximation formulas for pricing average price commodity options are unknown except for the Black-Scholes model.

It is fairly common in market practice to price exotic options based on calibration to option prices in a more liquid market (e.g. see pp.48-49 in [1]). For instance, average price options on West Texas Intermediate (WTI) crude oil futures contracts are evaluated based on a model whose parameters are obtained by calibration to the prices of the more liquid American vanilla options on WTI futures traded on the NYMEX division of the CME Group.

However, it is almost impossible for one parameter set under the Black-Scholes option pricing model to reproduce market prices with various strikes for a given maturity. Hence, it is well-known in practice that the Black-Scholes option pricing model is not appropriate for valuation of average price options with calibration to the prices of more liquid plain-vanilla options (e.g. see p.1 in [2]), and that more complex models such as stochastic volatility models are necessary for the calibration.

This paper applies stochastic volatility models and proposes a new approximation formula for pricing average option under Heston model as well as an extension of the -SABR (SABR) model (Labordere [11] (Haganet. al.[7])). To the best of our knowledge, this paper is the first one that provides an analytic (approximation) formula under a stochastic volatility environment for the valuation of average options traded in the commodity markets. Moreover, it gives estimates of average option prices under stochastic volatility markets based on the parameters obtained by calibration to the market prices of American options on WTI futures under stochastic volatility models.

In addition, our pricing models take the specific feature of the WTI average price option into consideration: the underlying price of an average option is the average of the settlement prices of the first nearby WTI futures contract during the last one month prior to the maturity of the average option. Note also that the expiration of an average option is the last business day of a calendar month, while trading of a WTI futures contract usually ceases on the third business day prior to the twenty-fifth calendar day. Hence, the settlement prices of WTI futures contracts with two consecutive maturities are relevant for the the underlying price of the average option. (See (2) in Section 2.) Similarly, American options on WTI futures with two consecutive maturities are relevant for calibration to plain-vanilla option prices. The proposed models in this paper incorporate those features of the WTI average price option.

Although Yoshida [24] and Takahashi[13],[14] apply the asymptotic expansion method to pricing average options, the underlying price they consider is a continuous average of an asset price that does not represent the underlying price of a contract in the real world precisely. Hence the formula they derived can not be used directly for valuation of average options traded in actual markets. Also they do not investigate the numerical

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[1]Average options are traded mainly in OTC markets (e.g. see P.40 in [9]), while they are illiquid in the listed markets such as CME Group.

accuracies of the third order expansion under stochastic volatility models.

As for pricing average options with stochastic volatility models, there are some exiting works: for instance, Fouque and Han[4] have derived asymptotic solutions to arithmetic average options under a class of fast mean-reverting stochastic volatility models. Wong and Cheung[22] extended their works to geometric average options pricing and calibration. Fouque and Han[5] used the solution of Wong and Cheung[22] to construct an efficient simulation method for arithmetic average options. Although these works did not consider the types of options in this paper directly, their methods may be applicable to the product considered here.

This paper takes Heston model(Heston [8]) and the extended λ-SABR model that includes the extended SABR model as a special case: They are very popular models for calibration to plain-vanilla option prices in the various markets. Numerical experiments show the sufficient accuracies of our approximation formulas for pricing average options under those stochastic volatility models. Moreover, calibration to the market prices of American options on WTI futures are implemented and average option prices in the WTI market are estimated based on the calibrated parameters.

Note that it is not efficient in numerical computation to calibrate directly to the American options under a stochastic volatility model. Alternatively, an implied volatility for each strike and maturity is estimated from a market price of the American option under a binomial version of Black-Scholes model. Then, the corresponding European option price is computed with its implied volatility. Next, given a maturity, calibration is implemented against those European prices with different strikes simultaneously under Heston, SABR and λ-SABR models. [2] Especially, vanilla option prices under λ-SABR model are computed by the third order expansion formula of the asymptotic method, which makes calibration very efficient.

Finally, the prices of average options traded in the WTI market are evaluated with the calibrated parameters by the asymptotic expansion formula as well as Monte Carlo simulations. Further, those estimated prices are compared with the settlement prices of average options traded in CME Group. Note that average options in CME Group are illiquid since they are traded mainly in OTC market. However, there are no official prices in the OTC market, while the settlement prices in CME Group are announced every day for the contracts with positive open interests. Thus, the settlement prices of average options are used as the reference prices in our study.

The organization of the paper is as follows: The next section briefly explains the structures of average options traded in commodity markets and then describes the extended λ-SABR and Heston models. Section 3 derives a new asymptotic expansion formula for the average options. Section 4 presents series of numerical experiments. Section 5 concludes. Appendix summarizes formulas of the conditional expectations used for the asymptotic expansion up to the third order.

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[2]The similar procedure is taken in Trolle and Schwartz [19].

2   Average Options under Stochastic Volatility Models

This section briefly describes the structure of average options on commodities, where the underlying assets are the averages of spot prices or future prices. Then, it presents two stochastic volatility models, an extended λ-SABR model and Heston model ([8]) used in the subsequent analysis: the extended λ-SABR model adds a drift term to the original model introduced by [11]. Note that the model with $\lambda = 0$ is reduced to the extended SABR model. (The original SABR model is introduced by [7].)

2.1   Average Option on Commodities

The underlying asset price of an average option contract on commodities is the average of settlement prices from the first business day to the last business day of a calendar month. In addition, there are two standard types of contracts.

One type is mostly applied to average options on metals in OTC markets(e.g. Copper). It calculates an average of a commodity's spot-price every business day through the month. Let the reference dates $t_1, \cdots, t_n$ ($t_1 < \cdots < t_n = T$), and the spot price S. Then, the underlying asset's price of this type is expressed as follows:

$$X(T) \;\; = \;\; \frac{1}{n} \left(\sum_{i=1}^{n} S(t_i) \right). \tag{1}$$

Another standard type is mainly traded in OTC oil markets(e.g. WTI). It refers to a futures price every business day. As the expiration of trading a futures contract is about a week before the end of a calendar month, the futures contracts with two consecutive maturities become the underlying assets of an average option. More precisely, let the reference dates $t_1, \cdots, t_{n_1}$ for the first contract and $t'_1, \cdots, t'_{n_2}$ for the subsequent contract ($t_1 < \cdots < t_{n_1} < t'_1 < \cdots < t'_{n_2} = T$). Also denote the relevant two underlying futures prices as S_i, $i = 1, 2$. Then, the average price of this type is expressed as follows:

$$X(T) \;\; = \;\; \frac{1}{n_1 + n_2} \left(\sum_{i=1}^{n_1} S_1(t_i) + \sum_{i=1}^{n_2} S_2(t'_i) \right). \tag{2}$$

Thus, the payoff functions of average call and put options with strike price K and maturity T are given by $\max\{X(T) - K, 0\}$ and $\max\{K - X(T), 0\}$, respectively.

The subsequent sections will concentrate on the second type because it includes the first type as a special case in terms of the way of averaging. However, there is a difference in the underlying asset, a spot or futures. In order that the difference does not cause any problems, this paper will present the underlying assets' models and pricing formulas which can be applied to both types.

Moreover, it is common in practice that pricing an average option is based on the calibration to the prices of the liquid vanilla options whose underlying assets are the same spot or futures contracts as those of the average option. Hence, in order for calibration, this paper will apply stochastic volatility models to the underlying assets' dynamics.

2.2 Stochastic Volatility Models

We take an extended λ-SABR model and Heston model as the underlying asset model. As remarked before, the extended λ-SABR model becomes the extended SABR model when $\lambda = 0$.

First, the extended λ-SABR model is described. For an average option on a commodity, the dynamics of the related two underlying asset prices S_i ($i = 1, 2$) are described under the equivalent martingale measure as follows:

$$dS_1(t) = \alpha_1(t)S_1(t)dt + v_1\sigma_1(t)S_1(t)^{\beta_1}dZ_1(t), \tag{3}$$

$$dS_2(t) = \alpha_2(t)S_2(t)dt + v_2\sigma_2(t)S_2(t)^{\beta_2}dZ_2(t), \tag{4}$$

$$d\sigma_1(t) = \lambda_1(\theta_1 - \sigma_1(t))dt + \nu_1\sigma_1(t)dZ_3(t), \tag{5}$$

$$d\sigma_2(t) = \lambda_2(\theta_2 - \sigma_2(t))dt + \nu_2\sigma_2(t)dZ_3(t), \tag{6}$$

where α_i, $\beta_i \in [0, 1]$, $\lambda_i > 0$, $\theta_i > 0$ and $\nu_i > 0$ ($i = 1, 2$) are constants, and Z_1, Z_2, Z_3 are correlated Brownian motions with correlations $\rho_{12}, \rho_{23}, \rho_{31} \in [-1, 1]$.

Note that Brownian motion Z_3 driving a stochastic volatility component $\sigma_i(i = 1, 2)$ is common for two assets while the correlation (ρ_{13} or ρ_{23}) between the underlying asset price $S_i(i = 1, 2)$ and its volatility $\sigma_i(i = 1, 2)$ is allowed to be different for each asset. Moreover, positive constants v_1 and v_2 represent different levels of two assets' volatilities. When the underlying assets are futures contracts, the prices are martingales under the equivalent martingale measure and hence $\alpha_1(t) = \alpha_2(t) = 0$ in the corresponding stochastic differential equations described above.

Let W_1, W_2, W_3 independent Brownian motions. Then, the underlying asset prices' processes are rewritten as follows:

$$dS_1(t) = \alpha_1(t)S_1(t)dt + \mu_1\sigma_1(t)S_1(t)^{\beta_1}dW_1(t), \tag{7}$$

$$\begin{aligned} dS_2(t) = &\alpha_2(t)S_1(t)dt + \mu_{2,1}\sigma_2(t)S_2(t)^{\beta_2}dW_1(t) \\ &+\mu_{2,2}\sigma_2(t)S_2(t)^{\beta_2}dW_2(t), \end{aligned} \tag{8}$$

$$\begin{aligned} d\sigma_1(t) = &\lambda_1(\theta_1 - \sigma_1(t))dt + \nu_{1,1}\sigma_1(t)dW_1(t) \\ &+\nu_{1,2}\sigma_1(t)dW_2(t) + \nu_{1,3}\sigma_1(t)dW_3(t), \end{aligned} \tag{9}$$

$$\begin{aligned} d\sigma_2(t) = &\lambda_2(\theta_2 - \sigma_2(t))dt + \nu_{2,1}\sigma_1(t)dW_1(t) \\ &+\nu_{2,2}\sigma_2(t)dW_2(t) + \nu_{2,3}\sigma_2(t)dW_3(t), \end{aligned} \tag{10}$$

where

$$\mu_1 = v_1, \mu_{2,1} = v_2\rho_{12}, \mu_{2,2} = v_2 c_{22} := v_2\sqrt{1 - \rho_{12}^2},$$

$$\nu_{1,1} = \nu_1\rho_{31}, \nu_{1,2} = \nu_1 c_{23} := \nu_1\frac{(\rho_{23} - \rho_{31}\rho_{12})}{c_{22}}, \nu_{1,3} = \nu_1\sqrt{1 - \rho_{31}^2 - c_{23}^2},$$

$$\nu_{2,1} = \nu_2\rho_{31}, \nu_{2,2} = \nu_2 c_{23} := \nu_2\frac{(\rho_{23} - \rho_{31}\rho_{12})}{c_{22}}, \nu_{2,3} = \nu_2\sqrt{1 - \rho_{31}^2 - c_{23}^2}.$$

Remark . *SABR and λ-SABR models are introduced by Hagan et. al. [7] and Labordere [11], respectively. They are defined as follows:*

$$
\begin{aligned}
dF(t) &= \alpha(t)F(t)^\beta dZ_1(t), \\
d\alpha(t) &= \lambda(\bar\lambda - \alpha(t))dt + \nu\alpha(t)dZ_2(t) \quad (\lambda\text{-}SABR), \\
&\text{or} \\
d\alpha(t) &= \nu\alpha(t)dZ_2(t) \quad\quad\quad\quad\quad\quad (SABR),
\end{aligned}
$$

with $F(0) = F$, $\alpha(0) = \alpha$ and $dZ_1 dZ_2 = \rho dt$. The term of "SABR model" stands for "stochastic-$\alpha\beta\rho$ model". This paper extended these models by adding drifts.

Next, Heston model is described. In the similar manner as above, the dynamics of the underlying asset prices in Heston model under the equivalent martingale measure with Brownian motion Z_1, Z_2, Z_3 is given as follows:

$$
\begin{aligned}
dS_1(t) &= \alpha_1(t)S_1(t)dt + v_1\sqrt{V_1(t)}S_1(t)dZ_1(t), & (11) \\
dS_2(t) &= \alpha_2(t)S_2(t)dt + v_2\sqrt{V_2(t)}S_2(t)dZ_2(t), & (12) \\
dV_1(t) &= \lambda_1(\theta_1 - V_1(t))dt + \nu_1\sqrt{V_1(t)}dZ_3(t), & (13) \\
dV_2(t) &= \lambda_2(\theta_2 - V_2(t))dt + \nu_2\sqrt{V_2(t)}dZ_3(t). & (14)
\end{aligned}
$$

The expression with independent Brownian motions W_1, W_2, W_3 with the same transformation of parameters as the extended λ-SABR are obtained:

$$
\begin{aligned}
dS_1(t) &= \alpha_1(t)S_1(t)dt + \mu_1\sqrt{V_1(t)}S_1(t)dW_1(t), & (15) \\
dS_2(t) &= \alpha_2(t)S_1(t)dt + \mu_{2,1}\sqrt{V_2(t)}S_2(t)dW_1(t) \\
&\quad + \mu_{2,2}\sqrt{V_2(t)}S_2(t)dW_2(t), & (16) \\
dV_1(t) &= \lambda_1(\theta_1 - V_1(t))dt + \nu_{1,1}\sqrt{V_1(t)}dW_1(t) \\
&\quad + \nu_{1,2}\sqrt{V_1(t)}dW_2(t) + \nu_{1,3}\sqrt{V_1(t)}dW_3(t), & (17) \\
dV_2(t) &= \lambda_2(\theta_2 - V_2(t))dt + \nu_{2,1}\sqrt{V_2(t)}dW_1(t) \\
&\quad + \nu_{2,2}\sqrt{V_2(t)}dW_2(t) + \nu_{2,3}\sqrt{V_2(t)}dW_3(t). & (18)
\end{aligned}
$$

3 Approximation Formula of Average Options

This section derives an approximation formula for pricing average options on commodities explained in the previous section. Note that the formula based on an asymptotic expansion approach is general enough to be applied to pricing average options under all multi-dimensional diffusion processes. Specifically, this section presents concrete applications for the extended λ-SABR model and Heston model.

3.1 Approximation Formula for Extended λ-SABR Model

First, for given $\epsilon \in (0, 1)$, the extended λ-SABR model in an asymptotic expansion approach is described as follows:

$$
S_1^{(\epsilon)}(T) = S_1(0) + \int_0^T \alpha_1(t)S_1^{(\epsilon)}(t)dt
$$

$$+\epsilon \int_0^T \bar{\mu}_1 \sigma_1^{(\epsilon)}(t) S_1^{(\epsilon)}(t) dW_1(t), \tag{19}$$

$$S_2^{(\epsilon)}(T) = S_2(0) + \int_0^T \alpha_2(t) S_2^{(\epsilon)}(t) dt$$
$$+\epsilon \Bigg(\int_0^T \bar{\mu}_{2,1} \sigma_2^{(\epsilon)}(t) S_2^{(\epsilon)}(t) dW_1(t)$$
$$+ \int_0^T \bar{\mu}_{2,2} \sigma_2^{(\epsilon)}(t) S_2^{(\epsilon)}(t) dW_2(t) \Bigg), \tag{20}$$

$$\sigma_1^{(\epsilon)}(T) = \sigma_1(0) + \int_0^T \lambda_1(\theta_1 - \sigma_1^{(\epsilon)}(t)) dt$$
$$+\epsilon \Bigg(\int_0^T \bar{\nu}_{1,1} \sigma_1^{(\epsilon)}(t) dW_1(t) + \int_0^T \bar{\nu}_{1,2} \sigma_1^{(\epsilon)}(t) dW_2(t)$$
$$+ \int_0^T \bar{\nu}_{1,3} \sigma_1^{(\epsilon)}(t) dW_3(t) \Bigg), \tag{21}$$

$$\sigma_2^{(\epsilon)}(T) = \sigma_2(0) + \int_0^T \lambda_2(\theta_2 - \sigma_2^{(\epsilon)}(t)) dt$$
$$+\epsilon \Bigg(\int_0^T \bar{\nu}_{2,1} \sigma_2^{(\epsilon)}(t) dW_1(t) + \int_0^T \bar{\nu}_{2,2} \sigma_2^{(\epsilon)}(t) dW_2(t)$$
$$+ \int_0^T \bar{\nu}_{2,3} \sigma_2^{(\epsilon)}(t) dW_3(t) \Bigg), \tag{22}$$

$$X^{(\epsilon)}(T) = \frac{1}{n_1 + n_2} \left(\sum_{i=1}^{n_1} S_1^{(\epsilon)}(t_i) + \sum_{i=1}^{n_2} S_2^{(\epsilon)}(t_i') \right). \tag{23}$$

where $\epsilon \bar{\mu}_1 = \mu_1, \epsilon \bar{\mu}_{2,1} = \mu_{2,1}, \epsilon \bar{\mu}_{2,2} = \mu_{2,2}, \epsilon \bar{\nu}_1 = \nu_1, \epsilon \bar{\nu}_{2,1} = \nu_{2,1}, \epsilon \bar{\nu}_{2,2} = \nu_{2,2}$. Then, $S_i(T), \sigma_i(T), X(T)$ is expanded around 0 as

$$X^{(\epsilon)}(T) \approx X^{(0)}(T) + \epsilon X^{(1)}(T) + \cdots, \tag{24}$$
$$S_i^{(\epsilon)}(T) \approx S_i^{(0)}(T) + \epsilon S_i^{(1)}(T) + \cdots, \tag{25}$$
$$\sigma_i^{(\epsilon)}(T) \approx \sigma_i^{(0)}(T) + \epsilon \sigma_i^{(1)}(T) + \cdots, \tag{26}$$

where $X^{(n)}(T) := \frac{\partial^n X^{(\epsilon)}(T)}{\partial \epsilon^n}|_{\epsilon=0}, S_i^{(n)}(T) := \frac{\partial^n S_i^{(\epsilon)}(T)}{\partial \epsilon^n}|_{\epsilon=0}, \sigma_i^{(n)}(T) := \frac{\partial^n \sigma_i^{(\epsilon)}(T)}{\partial \epsilon^n}|_{\epsilon=0}$.

Remark . *The validity of asymptotic expansions appearing in what follows is justified by Watanabe Theory(Watanabe[20]) in Malliavin calculus. In particular, Theorem 2.3 of Watanabe[20] and its truncated version, Theorem 2.2 of Yoshida[23] provided a justification of asymptotic expansions for generalized Wiener functionals. Subsequent works such as Takahashi[13],[14], Kunitomo and Takahashi[10] and Takahashi-Yoshida[17],[18] have applied Watanabe theory to finance for pricing derivatives and deriving optimal portfolios as well as hedging portfolios. For more detailed explanation and references, see Takahashi[16].*

For example, when the underlying factors are generated by (multi-dimensional) general diffusion processes, Kunitomo-Takahashi[10] provided a justification of the asymptotic expansion. See Section 3(pp. 920-931) of the paper. Especially, Theorem 3.3. and Corollary 3.1.(p.925) stated it for the case of average options. Since this paper concentrates on the practical applications of an asymptotic expansion, the details are omitted.

The next proposition provides the first four coefficients in the asymptotic expansion of $X^{(\epsilon)}(T)$.

Proposition 3.1. *We define $\mathbb{N}_1$ and $\mathbb{N}_2$ as*

$$\mathbb{N}_1(t) = 1_{t \leq t_1} e^{\int_0^{t_1} \alpha_1(s)ds} + \cdots + 1_{t \leq t_{n_1}} e^{\int_0^{t_{n_1}} \alpha_1(s)ds},$$

$$\mathbb{N}_2(t) = 1_{t \leq t'_1} e^{\int_0^{t'_1} \alpha_2(s)ds} + \cdots + 1_{t \leq t'_{n_2}} e^{\int_0^{t'_{n_2}} \alpha_2(s)ds}.$$

Set $M = n_1 + n_2$ and $W(t) = (W_1(t), W_2(t), W_3(t))'$. Then,

$$X^{(0)}(T) = \frac{\mathbb{N}_1(0)S_1(0) + \mathbb{N}_2(0)S_2(0)}{M}, \tag{27}$$

$$X^{(1)}(T) = \int_0^T f_{11}(s)'dW(s), \tag{28}$$

$$X^{(2)}(T) = 2 \sum_{i=1}^{4} \int_0^T \int_0^s f_{2i}(u)'dW(u)g_{2i}(s)'dW(s), \tag{29}$$

$$X^{(3)}(T) = 6\left(\sum_{i=1}^{6} \int_0^T \int_0^s \int_0^u f_{3i}(v)'dW(v)g_{3i}(u)'dW(u)h_{3i}(s)'dW(s) \right.$$

$$+ \sum_{i=1}^{4} \int_0^T \left(\int_0^s g_{4i}(u)'dW(u) \right) \left(\int_0^s f_{4i}(u)'dW(u) \right)$$

$$\left. \times h_{4i}(s)'dW(s) \right). \tag{30}$$

Also set $\eta_i(t) = \theta_i + (\sigma_i(0) - \theta_i)e^{-\lambda_i t}$, $\xi_i(t) = S_i(0)e^{\int_0^t \alpha_i(s)ds}$ and then, $f_{11}(t)$, $f_{2i}(t)$, $g_{2i}(t)$ $(i = 1, \cdots, 4)$, $f_{3i}(t)$, $g_{3i}(t)$, $h_{3i}(t)$ $(i = 1, \cdots, 6)$, f_{4i}, g_{4i}, h_{4i} $(i = 1, \cdots, 4)$, are expressed as follows:

$$f_{11}(t) = \frac{1}{M} \begin{pmatrix} \mathbb{N}_1(t)\bar{\mu}_1 e^{-\int_0^t \alpha_1(s)ds} \xi_1^{\beta_1}(t)\eta_1(t) + \mathbb{N}_2(t)\bar{\mu}_{2,1} e^{-\int_0^t \alpha_2(s)ds} \xi_2^{\beta_2}(t)\eta_2(t) \\ \mathbb{N}_2(t)\bar{\mu}_{2,2} e^{-\int_0^t \alpha_2(s)ds} \xi_2^{\beta_2}(t)\eta_2(t) \\ 0 \end{pmatrix},$$

$$f_{21}(t) = f_{31}(t) = f_{41}(t) = g_{41}(t) = g_{43}(t) = \begin{pmatrix} \bar{\mu}_1 e^{-\int_0^t \alpha_1(s)ds} \xi_1^{\beta_1}(t)\eta_1(t) \\ 0 \\ 0 \end{pmatrix},$$

$$f_{22}(t) = f_{32}(t) = f_{42}(t) = g_{42}(t) = g_{44}(t) = \begin{pmatrix} \bar{\mu}_{2,1} e^{-\int_0^t \alpha_2(s)ds} \xi_2^{\beta_2}(t) \eta_2(t) \\ \bar{\mu}_{2,2} e^{-\int_0^t \alpha_2(s)ds} \xi_2^{\beta_2}(t) \eta_2(t) \\ 0 \end{pmatrix},$$

$$g_{33}(t) = \frac{e^{-\lambda_1 t}}{\eta_1(t)} f_{21}(t), g_{34}(t) = \frac{e^{-\lambda_2 t}}{\eta_2(t)} f_{22}(t), g_{35}(t) = \begin{pmatrix} \bar{\nu}_{1,1} \\ \bar{\nu}_{1,2} \\ \bar{\nu}_{1,3} \end{pmatrix}, g_{36}(t) = \begin{pmatrix} \bar{\nu}_{2,1} \\ \bar{\nu}_{2,2} \\ \bar{\nu}_{2,3} \end{pmatrix},$$

$$f_{23}(t) = f_{33}(t) = f_{35}(t) = f_{43}(t) = \eta_1(t) e^{\lambda_1 t} g_{35}(t),$$

$$f_{24}(t) = f_{34}(t) = f_{36}(t) = f_{44}(t) = \eta_2(t) e^{\lambda_2 t} g_{36}(t),$$

$$g_{31}(t) = \begin{pmatrix} \bar{\mu}_1 \xi_1^{\beta_1-1}(t) \beta_1 \eta_1(t) \\ 0 \\ 0 \end{pmatrix}, g_{32}(t) = \begin{pmatrix} \bar{\mu}_{2,1} \xi_2^{\beta_2-1}(t) \beta_2 \eta_2(t) \\ \bar{\mu}_{2,2} \xi_2^{\beta_2-1}(t) \beta_2 \eta_2(t) \\ 0 \end{pmatrix},$$

$$g_{21}(t) = h_{31}(t) = h_{33}(t) = \frac{\mathbb{N}_1(t)}{M} g_{31}(t), g_{22}(t) = h_{32}(t) = h_{34}(t) = \frac{\mathbb{N}_2(t)}{M} g_{32}(t),$$

$$g_{23}(t) = h_{35}(t) = \frac{\mathbb{N}_1(t)}{M} g_{33}(t), g_{24}(t) = h_{36}(t) = \frac{\mathbb{N}_2(t)}{M} g_{34}(t),$$

$$h_{41}(t) = \frac{1}{2M} \begin{pmatrix} \mathbb{N}_1 \bar{\mu}_1 \xi_1^{\beta_1-2}(t) \beta_1(\beta_1-1) \eta_1(t) e^{\int_0^t \alpha_1(s)ds} \\ 0 \\ 0 \end{pmatrix},$$

$$h_{42}(t) = \frac{1}{2M} \begin{pmatrix} \mathbb{N}_2 \bar{\mu}_{2,1} \xi_2^{\beta_2-2}(t) \beta_2(\beta_2-1) \eta_2(t) e^{\int_0^t \alpha_2(s)ds} \\ \mathbb{N}_2 \bar{\mu}_{2,2} \xi_2^{\beta_2-2}(t) \beta_2(\beta_2-1) \eta_2(t) e^{\int_0^t \alpha_2(s)ds} \\ 0 \end{pmatrix},$$

$$h_{43}(t) = \frac{\mathbb{N}_1 e^{-\lambda_1 t}}{M \eta_1(t)} g_{31}(t), h_{44}(t) = \frac{\mathbb{N}_2 e^{-\lambda_2 t}}{M \eta_2(t)} g_{32}(t).$$

Proof. We will derive $X^{(0)}(T), X^{(1)}(T)$, and $X^{(2)}(T)$ explicitly. and $X^{(3)}(T)$ can be derived in the similar manner.

First, we calculate $X^{(0)}(T)$.

$$X^{(0)}(T) = \frac{1}{n_1 + n_2} \left(\sum_{i=1}^{n_1} S_1^{(0)}(t_i) + \sum_{i=1}^{n_2} S_2^{(0)}(t_i') \right),$$

$$S_1^{(0)}(t) = \left(\int_0^t \alpha_1(s)(S_1^{(0)}(s) + \epsilon S_1^{(1)}(s) + \cdots)ds \right.$$

$$\left. + \epsilon \int_0^t \bar{\mu}_1(\sigma_1^{(0)}(s) + \epsilon \sigma_1^{(1)}(s) + \cdots)(S_1^{(0)}(s) + \epsilon S_1^{(1)}(s) + \cdots)^{\beta_1} dW_1(s) + O(\epsilon^2) \right)\Big|_{\epsilon=0}$$

$$= \int_0^t \alpha_1(s) S_1^{(0)}(s)ds.$$

$S_1^{(0)}(t)$ can be solved as $S_1^{(0)}(t) = e^{\int_0^t \alpha_1(s)ds} S_1(0)$, and $S_2^{(0)}$ is derived in the same way. Then substituting $S_i^{(0)}(t)$ for $X^{(0)}(T)$.

Next, we calculate $X^{(1)}(T)$.

$$
\begin{aligned}
X^{(1)}(T) &= \left.\frac{\partial X^{(\epsilon)}(T)}{\partial \epsilon}\right|_{\epsilon=0} \\
&= \frac{1}{n_1+n_2}\left(\sum_{i=1}^{n_1}\left.\frac{\partial S_1^{(\epsilon)}(t_i)}{\partial \epsilon}\right|_{\epsilon=0} + \sum_{i=1}^{n_2}\left.\frac{\partial S_2^{(\epsilon)}(t_i')}{\partial \epsilon}\right|_{\epsilon=0}\right) \\
&= \frac{1}{n_1+n_2}\left(\sum_{i=1}^{n_1}S_1^{(1)}(t_i) + \sum_{i=1}^{n_2}S_2^{(1)}(t_i')\right),
\end{aligned}
$$

$$
\begin{aligned}
S_1^{(1)}(t) &= \left.\frac{\partial S_1^{(\epsilon)}(t)}{\partial \epsilon}\right|_{\epsilon=0} \\
&= \left(\int_0^t \alpha_1(s)(S_1^{(1)}(s) + \epsilon S_1^{(2)}(s) + \cdots)ds\right. \\
&\quad + \int_0^t \bar{\mu}_1(\sigma_1^{(0)}(s) + \epsilon\sigma_1^{(1)}(s) + \cdots)(S_1^{(0)}(s) + \epsilon S_1^{(1)}(s) + \cdots)^{\beta_1}dW_1(s) \\
&\quad + \epsilon\int_0^t \bar{\mu}_1(\sigma_1^{(0)}(s) + \epsilon\sigma_1^{(1)}(s) + \cdots)\beta_1 S_1^{(1)}(s)(S_1^{(0)}(s) + \epsilon S_1^{(1)}(s) + \cdots)^{\beta_1-1}dW_1(s) \\
&\quad \left.+ \epsilon\int_0^t \bar{\mu}_1(\sigma_1^{(1)}(s) + \epsilon\sigma_1^{(2)}(s) + \cdots)(S_1^{(0)}(s) + \epsilon S_1^{(1)}(s) + \cdots)^{\beta_1}dW_1(s) + O(\epsilon^2)\right)\Bigg|_{\epsilon=0} \\
&= \int_0^t \alpha_1(s)S_1^{(1)}(s)ds + \int_0^t \bar{\mu}_1\sigma_1^{(0)}(s)S_1^{(0)}(s)^{\beta_1}dW_1(s),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1^{(0)}(t) &= \left(\sigma_1(0) + \int_0^t \lambda_1(\theta_1 - (\sigma_1^{(0)}(s) + \epsilon\sigma_1^{(1)}(s) + \cdots))ds\right. \\
&\quad \left.+ \epsilon\sum_{i=1}^{3}\int_0^t \nu_{1,i}(\sigma_1^{(0)}(s) + \epsilon\sigma_1^{(1)}(s) + \cdots)dW_i(t) + O(\epsilon^2)\right)\Bigg|_{\epsilon=0} \\
&= \sigma_1(0) + \int_0^t \lambda_1(\theta_1 - \sigma_1^{(0)}(s))ds.
\end{aligned}
$$

Those equations can be solved by method of variation of constants as:

$$
\begin{aligned}
\sigma_1^{(0)}(t) &= \theta_1 + (\sigma_1(0) - \theta_1)e^{-\lambda_1 t}, \\
S_1^{(1)}(t) &= \int_0^t \bar{\mu}_1\sigma_1^{(0)}(s)e^{\int_s^t \alpha(u)du}S_1^{(0)}(s)^{\beta_1}dW_1(s).
\end{aligned}
$$

$S_2^{(1)}$ is derived in the same way and the coefficient f_{11} is derived.

In the same way, to calculate $X^{(2)}(T)$, we get the following equations.

$$
\begin{aligned}
X^{(2)}(T) &= \left.\frac{\partial^2 X^{(\epsilon)}(T)}{\partial \epsilon^2}\right|_{\epsilon=0} \\
&= \frac{1}{n_1+n_2}\left(\sum_{i=1}^{n_1}S_1^{(2)}(t_i) + \sum_{i=1}^{n_2}S_2^{(2)}(t_i')\right),
\end{aligned}
$$

$$S_1^{(2)}(t) = \left. \frac{\partial^2 S_1^{(\epsilon)}(t)}{\partial \epsilon^2} \right|_{\epsilon=0}$$

$$= \int_0^t \alpha_1(s) S_1^{(2)}(s) ds + 2 \int_0^t \bar{\mu}_1 \beta_1 \sigma_1^{(0)}(s) S_1^{(0)}(s)^{\beta_1-1} S_1^{(1)}(s) dW_1(s)$$

$$+ 2 \int_0^t \bar{\mu}_1 \sigma_1^{(1)}(s) S_1^{(0)}(s)^{\beta_1} dW_1(s),$$

$$\sigma_1^{(1)}(t) = \left. \frac{\partial \sigma_1^{(\epsilon)}(t)}{\partial \epsilon} \right|_{\epsilon=0}$$

$$= - \int_0^t \lambda_1 \sigma_1^{(1)}(s) ds + \sum_{i=1}^3 \int_0^t \nu_{1,i} \sigma_1^{(0)}(s) dW_i(t).$$

Those equations can be solved again as follows:

$$\sigma_1^{(1)}(t) = \sum_{i=1}^3 \int_0^t \nu_{1,i} e^{-\lambda_1(t-s)} \sigma_1^{(0)}(s) dW_i(t),$$

$$S_1^{(2)}(t) = 2 \int_0^t \bar{\mu}_1 \sigma_1^{(0)}(s) e^{\int_s^t \alpha_1(u)du} \beta_1 S_1^{(0)}(s)^{\beta_1-1} \int_u^s \bar{\mu}_1 \sigma_1^{(0)}(u) e^{\int_u^s \alpha_1(w)dw} S_1^{(0)}(u)^{\beta_1} dW_1(u) dW_1(s)$$

$$+ 2 \int_0^t \bar{\mu}_1 e^{\int_s^t \alpha_1(u)du} S_1^{(0)}(s)^{\beta_1} \left(\sum_{i=1}^3 \int_0^s \nu_{1,i} e^{-\lambda_1(s-u)} \sigma_1^{(0)}(u) dW_i(u) \right) dW_1(s).$$

$S_2^{(2)}$ is derived in the similar manner, again. Then we can derive $f_{2i}(t), g_{2i}(t), i = 1, \cdots, 4$.

The remaining coefficients can be easily derived in the same way, and the details of derivation are omitted.

□

Let the riskless interest rate $r > 0$. Then, the asymptotic expansion of the call price of an average option up to the ϵ^3-order with maturity date T and strike price K is given as follows:

$$C(0) = e^{-rT} \left(\epsilon \left(y \int_{-y}^\infty n[x; 0, \Sigma] dx + \int_{-y}^\infty x \, n[x; 0, \Sigma] dx \right) \right.$$

$$+ \epsilon^2 \frac{1}{2} \int_{-y}^\infty \mathbf{E} \left[X^{(2)}(T) \big| X^{(1)}(T) = x \right] n[x; 0, \Sigma] dx$$

$$+ \epsilon^3 \left(\frac{1}{6} \int_{-y}^\infty \mathbf{E} \left[X^{(3)}(T) \big| X^{(1)}(T) = x \right] n[x; 0, \Sigma] dx \right.$$

$$\left. \left. + \frac{1}{8} \mathbf{E} \left[\left(X^{(2)}(T) \right)^2 \big| X^{(1)}(T) = y \right] n[y; 0, \Sigma] \right) \right) + o(\epsilon^3),$$

where $y = \frac{X^{(0)}(T) - K}{\epsilon}$, $\Sigma = \int_0^T f_{11}(s)' f_{11}(s) ds$ and $n[x; 0, \Sigma] = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(\frac{-x^2}{2\Sigma} \right)$.

A more concrete approximation of the price is obtained by the application of conditional expectation formulas shown in the appendix.

Remark . *The convergence of the option price with the asymptotic expansion was guaranteed by Theorem 3.5 and Corollary 3.1 of Kunitomo and Takahashi[10].*

Finally, the following theorem summarizes an analytic approximation formula for pricing average options under the extended λ-SABR model.

Theorem 3.2. *The asymptotic expansion up to the ϵ^3-order of $C(0)$, the price of an average call option at the contract date with maturity date T and strike price K is given as follows:*

$$
\begin{aligned}
C(0) \;=\; & e^{-rT}\Bigg[\epsilon\Big\{yN\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma n[y;0,\Sigma]\Big\} \\
& +\epsilon^2\int_{-y}^{\infty}C_1\frac{H_2(x;\Sigma)}{\Sigma^2}n[x;0,\Sigma]dx \\
& +\epsilon^3\Bigg\{\int_{-y}^{\infty}C_2\frac{H_3(x;\Sigma)}{\Sigma^3}n[x;0,\Sigma]dx+C_3n[y;0,\Sigma] \\
& \quad +\left(C_4\frac{H_4(y;\Sigma)}{\Sigma^4}+C_5\frac{H_2(y;\Sigma)}{\Sigma^2}+C_6\right)n[y;0,\Sigma]\Bigg\}\Bigg]+o(\epsilon^3),
\end{aligned}
\tag{31}
$$

where $y=\frac{X^{(0)}(T)-K}{\epsilon}$, $\Sigma=\int_0^T f_{11}(s)'f_{11}(s)ds$, $n[x;0,\Sigma]=\frac{1}{\sqrt{2\pi\Sigma}}\exp\left(\frac{-x^2}{2\Sigma}\right)$ and $N(x)$ denotes the standard normal distribution function. Moreover, $H_k(x;\Sigma)$ denotes k-th order Hermite polynomial:

$$
H_k(x;\Sigma):=(-\Sigma)^k e^{x^2/2\Sigma}\frac{d^k}{dx^k}e^{-x^2/2\Sigma}.
$$

For example, $H_0(x;\Sigma)=1$, $H_1(x;\Sigma)=x$, $H_2(x;\Sigma)=x^2-\Sigma$, $H_3(x;\Sigma)=x^3-3\Sigma x$, $H_4(x;\Sigma)=x^4-6\Sigma x^2+3\Sigma^2$. Moreover,

$$
\Sigma \;=\; \int_0^T f_{11}(s)'f_{11}(s)ds,
$$

$$
C_1 \;=\; \sum_{i=1}^{4}\int_0^T f_{11}(s)'g_{2i}(s)\int_0^s f_{11}(u)'f_{2i}(u)duds,
$$

$$
C_2 \;=\; \sum_{i=1}^{6}\int_0^T f_{11}(s)'h_{3i}(s)\int_0^s f_{11}(u)'g_{3i}(u)\int_0^u f_{11}(v)'f_{3i}(v)dvduds
$$

$$
+\sum_{i=1}^{4}\int_0^T f_{11}(s)'h_{4i}(s)\int_0^s f_{11}(u)'g_{4i}(u)du\int_0^s f_{11}(u)'f_{4i}(u)duds,
$$

$$
C_3 \;=\; \sum_{i=1}^{4}\int_0^T f_{11}(s)'h_{4i}(s)\int_0^s g_{4i}(u)'f_{4i}(u)duds,
$$

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$$C_4 = \frac{1}{2}\sum_{i=1}^{10}\left(\int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{11}(s)'f_{5i}(u)duds\right)$$
$$\times\left(\int_0^T f_{11}(s)'k_{5i}(s)\int_0^s f_{11}(s)'h_{5i}(u)duds\right),$$

$$C_5 = \frac{1}{2}\sum_{i=1}^{10}\left(\int_0^T f_{11}(s)'k_{5i}(s)\int_0^s f_{11}(u)'g_{5i}(u)\int_0^u f_{5i}(v)'h_{5i}(v)dvduds\right.$$
$$+\int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{11}(u)'k_{5i}(u)\int_0^u f_{5i}(v)'h_{5i}(v)dvduds$$
$$+\int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{5i}(u)'k_{5i}(u)\int_0^u f_{11}(v)'h_{5i}(v)dvduds$$
$$+\int_0^T g_{5i}(s)'k_{5i}(s)\int_0^s f_{11}(u)'h_{5i}(u)du\int_0^s f_{11}(u)'f_{5i}(u)duds$$
$$\left.+\int_0^T f_{11}(s)'k_{5i}(s)\int_0^s g_{5i}(u)'h_{5i}(u)\int_0^u f_{11}(v)'f_{5i}(v)dvduds\right),$$

$$C_6 = \frac{1}{2}\sum_{i=1}^{10}\int_0^T g_{5i}(s)'k_{5i}(s)\int_0^s f_{11}(u)'h_{5i}(u)duds,$$

where $f_{11}(t)$, $f_{2i}(t)$ ($i = 1,\cdots,4$), $f_{3i}(t)$ ($i = 1,\cdots,6$), f_{4i} ($i = 1,\cdots,4$), $g_{2i}(t)$ ($i = 1,\cdots,4$), $g_{3i}(t)$ ($i = 1,\cdots,6$), $g_{4i}(t)$ ($i = 1,\cdots,4$), $h_{3i}(t)$ ($i = 1,\cdots,6$) and $h_{4i}(t)$ ($i = 1,\cdots,4$) are defined above and $f_{5i}(t)$, $g_{5i}(t)$, $h_{5i}(t)$, $k_{5i}(t)$ ($i = 1,\cdots,10$) are given as follows:

$$f_{51}(t) = h_{51}(t) = f_{55}(t) = f_{56}(t) = f_{57}(t) = f_{21}(t),$$
$$f_{52}(t) = h_{52}(t) = h_{55}(t) = f_{58}(t) = f_{59}(t) = f_{22}(t),$$
$$f_{53}(t) = h_{53}(t) = h_{56}(t) = h_{58}(t) = f_{510}(t) = f_{23}(t),$$
$$f_{54}(t) = h_{54}(t) = h_{57}(t) = h_{59}(t) = h_{510}(t) = f_{24}(t),$$
$$g_{51}(t) = k_{51}(t) = g_{55}(t) = g_{56}(t) = g_{57}(t) = g_{21}(t),$$
$$g_{52}(t) = k_{52}(t) = \frac{1}{2}k_{55}(t) = g_{58}(t) = g_{59}(t) = g_{22}(t),$$
$$g_{53}(t) = k_{53}(t) = \frac{1}{2}k_{56}(t) = \frac{1}{2}k_{58}(t) = g_{510}(t) = g_{23}(t),$$
$$g_{54}(t) = k_{54}(t) = \frac{1}{2}k_{57}(t) = \frac{1}{2}k_{59}(t) = \frac{1}{2}k_{510}(t) = g_{24}(t).$$

Remark . *The integrals on the right hand side of (31) are evaluated by the following relation:*

$$\int_{-y}^{\infty}\frac{1}{\Sigma^k}H_k(x;\Sigma)n[x;0,\Sigma]dx = \frac{1}{\Sigma^{k-1}}H_{k-1}(-y;\Sigma)n[y;0,\Sigma]\ (k \geq 1).$$

Remark . *A plain-vanilla option of European type is regarded as a special case of an average option. Hence, set $n_1 = 1$, $n_2 = 0$ and $t_1 = T$ in (23). Then, an approximation*

formula for pricing plain-vanilla options with maturity T under the extended model is obtained by application of this theorem.

3.2 Approximation Formula for Heston Model

An approximation formula for Heston model is obtained in a similar manner as above: Only thing to be done is that the coefficients in Theorem 3.2 are changed to the following.

Set $\zeta_i(t) = \sqrt{\theta_i + (V_i(0) - \theta_i) e^{-\lambda_i t}}$ and for given $\epsilon \in (0,1)$, set $\epsilon \bar{\mu}_1 = \mu_1, \epsilon \bar{\mu}_{2,1} = \mu_{2,1}, \epsilon \bar{\mu}_{2,2} = \mu_{2,2}, \epsilon \bar{\nu}_1 = \nu_1, \epsilon \bar{\nu}_{2,1} = \nu_{2,1}, \epsilon \bar{\nu}_{2,2} = \nu_{2,2}$ and

$$
\begin{aligned}
S_1^{(\epsilon)}(T) &= S_1(0) + \int_0^T \alpha_1(t) S_1^{(\epsilon)}(t) dt \\
&\quad + \epsilon \int_0^T \bar{\mu}_1 \sqrt{V_1^{(\epsilon)}(t)} S_1^{(\epsilon)}(t)^\beta dW_1(t), \tag{32}
\end{aligned}
$$

$$
\begin{aligned}
S_2^{(\epsilon)}(T) &= S_2(0) + \int_0^T \alpha_2(t) S_2^{(\epsilon)}(t) dt \\
&\quad + \epsilon \Bigg(\int_0^T \bar{\mu}_{2,1} \sqrt{V_2^{(\epsilon)}(t)} S_2^{(\epsilon)}(t)^\beta dW_1(t) \\
&\quad + \int_0^T \bar{\mu}_{2,2} \sqrt{V_2^{(\epsilon)}(t)} S_2^{(\epsilon)}(t)^\beta dW_2(t) \Bigg), \tag{33}
\end{aligned}
$$

$$
\begin{aligned}
V_1^{(\epsilon)}(T) &= V_1(0) + \int_0^T \lambda_1(\theta_1 - V_1^{(\epsilon)}(t)) dt \\
&\quad + \epsilon \Bigg(\int_0^T \bar{\nu}_{1,1} \sqrt{V_1^{(\epsilon)}(t)} dW_1(t) + \int_0^T \bar{\nu}_{1,2} \sqrt{V_1^{(\epsilon)}(t)} dW_2(t) \\
&\quad + \int_0^T \bar{\nu}_{1,3} \sqrt{V_1^{(\epsilon)}(t)} dW_3(t) \Bigg), \tag{34}
\end{aligned}
$$

$$
\begin{aligned}
V_2^{(\epsilon)}(T) &= V_2(0) + \int_0^T \lambda_2(\theta_2 - V_2^{(\epsilon)}(t)) dt \\
&\quad + \epsilon \Bigg(\int_0^T \bar{\nu}_{2,1} \sqrt{V_2^{(\epsilon)}(t)} dW_1(t) + \int_0^T \bar{\nu}_{2,2} \sqrt{V_2^{(\epsilon)}(t)} dW_2(t) \\
&\quad + \int_0^T \bar{\nu}_{2,3} \sqrt{V_2^{(\epsilon)}(t)} dW_3(t) \Bigg), \tag{35}
\end{aligned}
$$

$$
X^{(\epsilon)}(T) = \frac{1}{n_1 + n_2} \left(\sum_{i=1}^{n_1} S_1^{(\epsilon)}(t_i) + \sum_{i=1}^{n_2} S_2^{(\epsilon)}(t_i') \right). \tag{36}
$$

Then,

$$
f_{11}(t) = \frac{1}{M} \left(\begin{array}{c} \mathbb{N}_1(t) \bar{\mu}_1 S_1(0) \zeta_1(t) + \mathbb{N}_2(t) \bar{\mu}_{2,1} S_2(0) \zeta_2(t) \\ \mathbb{N}_2(t) \bar{\mu}_{2,2} S_2(0) \zeta_2(t) \\ 0 \end{array} \right),
$$

$$f_{21}(t) = f_{31}(t) = g_{31}(t) = f_{41}(t) = \begin{pmatrix} \bar{\mu}_1 \zeta_1(t) \\ 0 \\ 0 \end{pmatrix},$$

$$f_{22}(t) = f_{32}(t) = g_{32}(t) = f_{42}(t) = \begin{pmatrix} \bar{\mu}_{2,1} \zeta_2(t) \\ \bar{\mu}_{2,2} \zeta_2(t) \\ 0 \end{pmatrix},$$

$$g_{21}(t) = h_{31}(t) = 2h_{33}(t) = \frac{\mathbb{N}_1(t)}{M} S_1(0) f_{21}(t),$$

$$g_{22}(t) = h_{32}(t) = 2h_{34}(t) = \frac{\mathbb{N}_2(t)}{M} S_2(0) f_{22}(t),$$

$$g_{35}(t) = \frac{1}{\zeta_1(t)} \begin{pmatrix} \bar{\nu}_{1,1} \\ \bar{\nu}_{1,2} \\ \bar{\nu}_{1,3} \end{pmatrix}, g_{36}(t) = \frac{1}{\zeta_2(t)} \begin{pmatrix} \bar{\nu}_{2,1} \\ \bar{\nu}_{2,2} \\ \bar{\nu}_{2,3} \end{pmatrix},$$

$$f_{23}(t) = f_{33}(t) = f_{35}(t) = g_{41}(t) = f_{43}(t) = g_{43}(t) = \zeta_1^2(t) e^{\lambda_1 t} g_{35}(t),$$

$$f_{24}(t) = f_{34}(t) = f_{36}(t) = g_{42}(t) = f_{44}(t) = g_{44}(t) = \zeta_2^2(t) e^{\lambda_2 t} g_{36}(t),$$

$$g_{33}(t) = \frac{e^{-\lambda_1 t}}{\zeta_1(t)} \begin{pmatrix} \bar{\mu}_1 \\ 0 \\ 0 \end{pmatrix}, g_{34}(t) = \frac{e^{-\lambda_2 t}}{\zeta_2(t)} \begin{pmatrix} \bar{\mu}_{2,1} \\ \bar{\mu}_{2,2} \\ 0 \end{pmatrix},$$

$$g_{23}(t) = 2h_{35}(t) = h_{41}(t) = \frac{\mathbb{N}_1(t)}{2M} S_1(0) g_{33}(t),$$

$$g_{24}(t) = 2h_{36}(t) = h_{42}(t) = \frac{\mathbb{N}_2(t)}{2M} S_2(0) g_{34}(t),$$

$$h_{43}(t) = \frac{-e^{-\lambda_1 t}}{4\zeta_1^2(t)} g_{23}, h_{44}(t) = \frac{-e^{-\lambda_2 t}}{4\zeta_2^2(t)} g_{24}.$$

$f_{5i}(t), g_{5i}(t), h_{5i}(t), k_{5i}(t)$ $(i = 1, \cdots, 10)$ are given as follows:

$$f_{51}(t) = h_{51}(t) = f_{55}(t) = f_{56}(t) = f_{57}(t) = f_{21}(t),$$
$$f_{52}(t) = h_{52}(t) = h_{55}(t) = f_{58}(t) = f_{59}(t) = f_{22}(t),$$
$$f_{53}(t) = h_{53}(t) = h_{56}(t) = h_{58}(t) = f_{510}(t) = f_{23}(t),$$
$$f_{54}(t) = h_{54}(t) = h_{57}(t) = h_{59}(t) = h_{510}(t) = f_{24}(t),$$
$$g_{51}(t) = k_{51}(t) = g_{55}(t) = g_{56}(t) = g_{57}(t) = g_{21}(t),$$
$$g_{52}(t) = k_{52}(t) = \frac{1}{2}k_{55}(t) = g_{58}(t) = g_{59}(t) = g_{22}(t),$$
$$g_{53}(t) = k_{53}(t) = \frac{1}{2}k_{56}(t) = \frac{1}{2}k_{58}(t) = g_{510}(t) = g_{23}(t),$$
$$g_{54}(t) = k_{54}(t) = \frac{1}{2}k_{57}(t) = \frac{1}{2}k_{59}(t) = \frac{1}{2}k_{510}(t) = g_{24}(t).$$

4 Numerical Examples

This section examines effectiveness of our approximation formula through numerical examples.

The asymptotic expansion method is applied to the equations (19) - (23) and (32) - (36) with $\epsilon = 0.0001$ for the extended λ-SABR(SABR) model and Heston model, respectively. [3]

4.1 Accuracy of the Approximation Formula

This subsection compares the approximate prices with estimated prices by Monte Carlo simulations.

The setup of the numerical experiments is listed as follows:

- Benchmark Prices by Monte Carlo Simulations: [4]

  time steps: 2,500/year

  number of trials: 10 million

- The underlying asset prices: $S_1(0) = 96$, $S_2(0) = 106$

  the riskless interest rate: $0(r = 0)$,

  the drift of the underlying asset prices: $0(\alpha_1(t) = \alpha_2(t) = 0)$,

  $v_1 = 1.1$, $v_2 = 0.9$, $\sigma_1(0) = \sigma_2(0)$, $V_1(0) = V_2(0)$, $\beta_1 = \beta_2$, $\lambda_1 = \lambda_2$, $\theta_1 = \theta_2$, $\nu_1 = \nu_2$

- The term of the reference to the first contract(S_1) is from 0.9 year to 0.96 year, and the term of the reference to the second contract(S_2) is from 0.96 year to 1 year for an one year maturity option.

  The term of the reference to the first contract(S_1) is from 1.9 year to 1.96 year, and the term of the reference to the second contract(S_2) is from 1.96 year to 2 year for a two year maturity option.

  Then, the terms are equally divided into 15 for the first contract and 10 for the second contract. $(n_1 = 15, n_2 = 10, M = (n_1 + n_2) = 25)$

- Call option prices with strike prices: K=70,90,100,120,150

- λ-SABR model parameters (Table 1)

  Case i is a benchmark case: In particular, $\sigma(0)$ and θ are adjusted so that the initial volatility is equal to that of the log-normal case where the volatility is 30%. The other cases differ from the benchmark case in the following parameters:

  ii: β

  iii: correlations between the underlying assets' prices and their volatilities

  iv, v: level of volatility

---

[3] The equations (19) - (23) and (32) - (36) are transformed from (7) - (10) and (15) - (18), respectively.

[4] In λ-SABR model, if volatility goes to non-positive due to discretization, set $\sigma(t_{i+1}) = \sigma(t_i) + \lambda\theta\Delta t$ and $S(t_{i+1}) = S(t_i) + \alpha S(t_i)\Delta t$. If the underlying price goes to non-positive due to discretization, set $S(t_{i+1}) = 0$. The same procedure is taken for Heston model.

Table 1: λ-SABR

Parameter	$\sigma(0)$	β	λ	θ	ν	ρ_{12}	ρ_{23}	ρ_{31}	T
i	3.0	0.5	1.0	3.0	0.3	0.9	-0.1	-0.2	1.0
ii	0.3	1.0	1.0	0.3	0.3	0.9	-0.1	-0.2	1.0
iii	3.0	0.5	1.0	3.0	0.3	0.0	0.0	0.0	1.0
iv	1.0	0.5	1.0	1.0	0.3	0.9	-0.1	-0.2	1.0
v	5.0	0.5	1.0	5.0	0.3	0.9	-0.1	-0.2	1.0
vi	3.0	0.5	1.0	3.0	0.1	0.9	-0.1	-0.2	1.0
vii	3.0	0.5	1.0	3.0	0.7	0.9	-0.1	-0.2	1.0
viii	3.0	0.5	0.1	3.0	0.3	0.9	-0.1	-0.2	1.0
ix	3.0	0.5	2.0	3.0	0.3	0.9	-0.1	-0.2	1.0
x	3.0	0.5	1.0	3.0	0.3	0.9	-0.1	-0.2	2.0

vi, vii: level of volatility on volatility

viii, ix: speed of mean-reversion

x: maturity

- Heston model parameters (Table 2)

Table 2: Heston

Parameter	$V(0)$	λ	θ	ν	ρ_{12}	ρ_{23}	ρ_{31}	T
i	0.09	1.0	0.09	0.3	0.9	-0.1	-0.2	1.0
ii	0.09	1.0	0.09	0.3	0.0	0.0	0.0	1.0
iii	0.01	5.0	0.01	0.3	0.9	-0.1	-0.2	1.0
iv	0.25	1.0	0.25	0.3	0.9	-0.1	-0.2	1.0
v	0.09	1.0	0.09	0.1	0.9	-0.1	-0.2	1.0
vi	0.09	1.0	0.25	0.7	0.9	-0.1	-0.2	1.0
vii	0.09	0.5	0.09	0.3	0.9	-0.1	-0.2	1.0
viii	0.09	2.0	0.09	0.3	0.9	-0.1	-0.2	1.0
ix	0.09	1.0	0.09	0.3	0.9	-0.1	-0.2	2.0

Case i is a benchmark case: In particular, $V(0)$ and θ are adjusted so that the initial variance is equal to that of the log-normal case($\beta = 1$) where the volatility is 30%. The other cases differ from the benchmark case in the following parameters:

ii: correlations between the underlying assets' prices and their variances

iii, iv: level of variance

(Also, λ in Case iii is adjusted for satisfying Feller condition ($2\lambda\theta \geq \nu^2$, see [3]) that guarantees a positive variance process.)

v, vi: level of volatility on variance

vii, viii: speed of mean-reversion

ix: maturity

Finally, the results of numerical experiments are shown. Table 3 and Table 4 show the effectiveness of our formula: In particular, the third order expansion generally improves the lower order expansions and provides sufficient accuracies in practice for all cases.

Table 3: Result: λ-SABR Model

Case	Strike	MC	Asymptotic Expansion 1st	2nd	3rd	Difference 1st	2nd	3rd
i	70	31.84	32.22	31.83	31.86	0.37	-0.01	0.02
	90	16.97	17.18	16.97	16.98	0.21	0.00	0.01
	100	11.49	11.49	11.49	11.49	-0.00	-0.00	0.00
	110	7.39	7.18	7.38	7.39	-0.21	-0.01	-0.00
	130	2.63	2.22	2.60	2.62	-0.42	-0.03	-0.01
ii	70	31.37	32.20	31.32	31.37	0.83	-0.05	-0.00
	90	16.66	17.15	16.68	16.66	0.50	0.02	0.00
	100	11.44	11.47	11.47	11.45	0.02	0.02	0.00
	110	7.61	7.15	7.63	7.61	-0.46	0.02	0.00
	130	3.13	2.20	3.08	3.14	-0.93	-0.05	0.01
iii	70	30.59	30.82	30.54	30.60	0.23	-0.05	0.01
	90	14.34	14.53	14.32	14.35	0.19	-0.02	0.00
	100	8.65	8.63	8.63	8.65	-0.02	-0.02	0.00
	110	4.78	4.53	4.75	4.78	-0.24	-0.03	0.00
	130	1.15	0.82	1.09	1.15	-0.34	-0.06	-0.00
iv	70	30.00	30.00	30.00	30.00	-0.00	-0.00	0.00
	90	10.75	10.74	10.72	10.75	-0.01	-0.02	0.00
	100	3.84	3.83	3.83	3.85	-0.01	-0.01	0.00
	110	0.78	0.74	0.75	0.78	-0.04	-0.03	-0.00
	130	0.01	0.00	0.00	0.00	-0.00	-0.00	-0.00
v	70	36.57	37.77	36.75	36.62	1.20	0.18	0.05
	90	24.02	24.56	24.16	24.04	0.55	0.14	0.03
	100	19.02	19.15	19.15	19.04	0.13	0.13	0.02
	110	14.85	14.56	14.97	14.85	-0.28	0.12	0.01
	130	8.68	7.77	8.79	8.66	-0.91	0.11	-0.01
vi	70	31.71	32.22	31.75	31.71	0.51	0.04	0.00
	90	16.88	17.18	16.92	16.89	0.29	0.04	0.00
	100	11.46	11.49	11.49	11.46	0.03	0.03	0.00
	110	7.39	7.18	7.43	7.39	-0.22	0.04	-0.00
	130	2.65	2.22	2.68	2.65	-0.43	0.03	-0.00
vii	70	32.28	32.22	32.00	32.38	-0.07	-0.28	0.09
	90	17.27	17.18	17.06	17.32	-0.09	-0.21	0.05
	100	11.69	11.49	11.49	11.71	-0.20	-0.20	0.03
	110	7.54	7.18	7.29	7.55	-0.36	-0.25	0.01
	130	2.82	2.22	2.43	2.80	-0.61	-0.39	-0.02
viii	70	31.92	32.22	31.87	31.94	0.30	-0.05	0.03
	90	17.00	17.18	16.99	17.02	0.17	-0.02	0.01
	100	11.50	11.49	11.49	11.51	-0.01	-0.01	0.01
	110	7.39	7.18	7.36	7.39	-0.21	-0.03	0.00
	130	2.64	2.22	2.56	2.63	-0.43	-0.08	-0.01
ix	70	31.80	32.22	31.81	31.81	0.42	0.01	0.01
	90	16.94	17.18	16.95	16.95	0.23	0.01	0.01
	100	11.48	11.49	11.49	11.49	0.01	0.01	0.00
	110	7.39	7.18	7.40	7.39	-0.22	0.00	-0.00
	130	2.63	2.22	2.62	2.63	-0.42	-0.01	-0.01
x	70	34.85	35.69	34.90	34.89	0.84	0.05	0.04
	90	21.66	22.01	21.68	21.67	0.35	0.02	0.01
	100	16.52	16.53	16.53	16.53	0.01	0.01	0.00
	110	12.34	12.01	12.34	12.33	-0.32	0.01	-0.00
	130	6.48	5.69	6.47	6.46	-0.79	-0.00	-0.02

4.2 Calibration

This subsection applies SABR, λ-SABR and Heston models to calibration of the market prices of the WTI futures American option on July 1, 2009. American options on WTI futures are plain-vanilla options in NYMEX that is a part of CME Group.

In particular, our goal in the next subsection is to evaluate average options with two-month maturity(AUG09), four-month maturity(OCT09) and sixteen-month ma-

Table 4: Result: Heston Model

Case	Strike	MC	Asymptotic Expansion			Difference		
			1st	2nd	3rd	1st	2nd	3rd
i	70	31.39	32.20	31.40	31.35	0.81	0.01	-0.04
	90	16.45	17.15	16.72	16.43	0.70	0.27	-0.02
	100	11.14	11.47	11.47	11.14	0.32	0.32	-0.01
	110	7.28	7.15	7.58	7.29	-0.13	0.30	0.01
	130	2.91	2.20	3.00	2.95	-0.71	0.09	0.04
ii	70	30.41	30.79	30.27	30.41	0.38	-0.14	0.01
	90	13.99	14.48	14.06	13.97	0.49	0.07	-0.02
	100	8.44	8.57	8.57	8.43	0.13	0.13	-0.00
	110	4.81	4.48	4.90	4.81	-0.33	0.09	0.00
	130	1.45	0.79	1.32	1.47	-0.66	-0.13	0.02
iii	70	30.00	30.00	30.00	30.01	-0.00	-0.00	0.01
	90	10.70	10.73	10.72	10.71	0.04	0.02	0.01
	100	3.70	3.82	3.82	3.68	0.12	0.12	-0.01
	110	0.75	0.73	0.75	0.74	-0.02	-0.00	-0.01
	130	0.02	0.00	0.00	0.01	-0.02	-0.01	-0.01
iv	70	35.25	37.74	35.54	35.18	2.49	0.29	-0.07
	90	23.26	24.52	23.65	23.23	1.27	0.39	-0.02
	100	18.70	19.11	19.11	18.69	0.41	0.41	-0.00
	110	14.96	14.52	15.40	14.98	-0.44	0.43	0.01
	130	9.52	7.74	9.94	9.58	-1.78	0.42	0.06
v	70	31.27	32.20	31.26	31.25	0.93	-0.01	-0.02
	90	16.56	17.15	16.65	16.56	0.59	0.08	-0.00
	100	11.38	11.47	11.47	11.38	0.08	0.08	-0.00
	110	7.57	7.15	7.66	7.58	-0.42	0.09	0.00
	130	3.11	2.20	3.14	3.13	-0.91	0.03	0.02
vi	70	32.83	34.25	33.10	32.67	1.42	0.27	-0.16
	90	18.77	20.15	19.63	18.63	1.37	0.86	-0.14
	100	13.62	14.61	14.61	13.52	0.98	0.98	-0.10
	110	9.73	10.15	10.66	9.67	0.42	0.93	-0.07
	130	4.94	4.25	5.40	4.98	-0.68	0.47	0.04
vii	70	31.42	32.20	31.43	31.38	0.78	0.01	-0.05
	90	16.40	17.15	16.74	16.37	0.75	0.34	-0.03
	100	11.05	11.47	11.47	11.05	0.41	0.41	-0.01
	110	7.19	7.15	7.57	7.20	-0.03	0.38	0.01
	130	2.87	2.20	2.97	2.91	-0.67	0.10	0.04
viii	70	31.35	32.20	31.36	31.32	0.85	0.01	-0.03
	90	16.51	17.15	16.70	16.50	0.65	0.19	-0.01
	100	11.25	11.47	11.47	11.25	0.22	0.22	-0.00
	110	7.40	7.15	7.61	7.41	-0.25	0.21	0.01
	130	2.98	2.20	3.05	3.01	-0.78	0.07	0.03
ix	70	33.77	35.66	34.04	33.66	1.88	0.27	-0.12
	90	20.75	21.98	21.30	20.68	1.23	0.55	-0.06
	100	15.89	16.50	16.50	15.86	0.60	0.60	-0.04
	110	12.05	11.98	12.66	12.04	-0.08	0.61	-0.01
	130	6.83	5.66	7.27	6.89	-1.17	0.45	0.06

turity(OCT10) on July 1, 2009. Note that the underlying price of an average option is the average of the settlement prices of the first nearby WTI futures contract during the last one month prior to the maturity of the average option. Also, the expiration of an average option is the last business day of a calendar month, while trading of a WTI futures contract usually ceases on the third business day prior to the twenty-fifth calendar day. Hence, the WTI futures and their American options with two consecutive maturities are relevant for pricing an average options with calibration to plain-vanilla option prices.

In this analysis, the underlying asset prices are the average prices during August 2009 for the two-month maturity(AUG09), October 2009 for the four-month maturity(OCT09) and during October 2010 for the sixteen-month maturity(OCT10) of relevant futures; the relevant futures are SEP09 and OCT09 for the two-month maturity, NOV09 and DEC09 for the four-month maturity, and NOV10 and DEC10 for the

sixteen-month maturity. [5] Prices of SEP09, OCT09, NOV09, DEC09, NOV10 and DEC10 on July 1, 2009 are 70.27, 71.08, 71.78, 72.36, 76.06 and 76.4, respectively.

American options with two and three-month maturities are traded almost everyday while options with longer maturities are traded occasionally except options on DEC futures. For example, the volumes of trades of American futures options on July 1, 2009 are listed in Table 5.

Table 5: Volumes of Trades on July 1 2009(1000 barrel)

Strike	SEP09	OCT09	NOV09	DEC09	NOV10	DEC10
50 (Put)	4,220	0	0	370	0	4
55 (Put)	262	410	0	2,250	0	400
60 (Put)	2,302	2,800	5	6,605	0	0
65 (Put)	2,270	1,200	40	900	0	100
70 (Put)	501	156	1	3,575	0	0
75 (Call)	350	105	100	5	0	0
80 (Call)	1,237	116	245	901	0	0
85 (Call)	1,300	0	0	1,001	0	0
90 (Call)	70	1,200	0	415	0	250
95 (Call)	0	0	0	1	0	0
100 (Call)	100	0	0	2,686	0	1,414

For pricing OTC average options, if there are no trades of the relevant American options, practitioners refer to the implied volatilities of the settlement prices of the American options. The settlement prices are announced every day for the contracts with positive open interests. Those settlement prices rarely show unreasonable movements, which indicates that those prices can be used for the targets of the calibration.

For computational efficiency, the settlement prices of American options are transformed to those of the European options before calibration: More precisely, after an implied volatility of each American option price is estimated under a binomial version of the Black-Scholes model, the corresponding European option price is computed. Hereafter, this European option price is called the "transformed CME" option price. Then, calibration is implemented against the "transformed CME" option prices with different strikes simultaneously, where out-of-the-money (OTM) and at-the money (ATM) prices are used for the calibration; the strikes of the options range usd 50 to usd 100 with every five dollars. [6]

Moreover, the stochastic volatility models used for calibration to the "transformed CME" option prices are the following:

(i) SABR model ($\beta_1 = \beta_2 = 0.5$ in (7) and (8) with $\lambda_1 = \lambda_2 = 0$ in (9) and (10))

(ii) SABR model ($\beta_1 = \beta_2 = 1$ in (7) and (8) with $\lambda_1 = \lambda_2 = 0$ in (9) and (10))

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[5]Trading of SEP09(OCT09) ceases before the end of August(September) 2009. Also, trading of NOV09(DEC09) ceases before the end of October(November) 2009 and trading of NOV10(DEC10) ceases before the end of October(November) 2010.

[6]The similar procedure is taken in [19].

(iii) λ-SABR model ($\beta_1 = \beta_2 = 0.5$ in (7) and (8))

(iv) λ-SABR model ($\beta_1 = \beta_2 = 1$ in (7) and (8))

(v) Heston model

Note that the drifts of the underlying asset processes are set to be zero ($\alpha_1(t) = \alpha_2(t) = 0$) due to the futures contracts. Also, v_1 and v_2 are fixed to be one ($v_1 = v_2 = 1$) because the initial values of volatilities can be adjusted. Then, plain-vanilla European option prices are computed by the formula of Hagan et.al.[7] for SABR model, our third order asymptotic expansion formula for λ-SABR model and the formula of [8] for Heston model.

The calibrated parameters are obtained through minimization of the sum of the squared differences between model-based and transformed CME call option prices. That is,

$$\min_{\Theta} \sum_{K} \left(\text{Model Based Option Price}(T, K) - \text{Transformed CME Option Price}(T, K) \right)^2,$$

where T and K denote maturities (SEP09, OCT09, NOV09, DEC09, Nov10 or DEC10) and strike prices (usd 50 to usd 100 with every five dollars), respectively. Also, Θ stands for the set of parameters for calibration. Note that our price-based calibration above is similar to the implied volatility-based calibration weighted by Vega for adjustment of illiquidity in West[21], because the values of options with closer to ATM are more sensitive to the volatility change.

Parameters from calibration are reported in Table 6, 7 and 8. The calibrated and transformed call option prices are shown in Table 9, 10 and 11: "-" in columns of Price and Difference in the tables means that the settlement prices are not announced.

Table 6: Parameters(SEP09, OCT09)

		$\sigma(0)$	$V(0)$	λ	θ	ν	ρ
SEP09	(i)	3.607	-	-	-	1.427	-0.225
	(ii)	0.433	-	-	-	1.519	-0.369
	(iii)	3.606	-	0.008	1.788	1.434	-0.209
	(iv)	0.434	-	0.093	0.140	1.522	-0.342
	(v)	-	0.209	0.334	0.607	1.478	-0.377
OCT09	(i)	3.623	-	-	-	1.140	-0.188
	(ii)	0.433	-	-	-	1.252	-0.363
	(iii)	3.616	-	0.016	1.792	1.175	-0.170
	(iv)	0.435	-	0.087	0.140	1.227	-0.335
	(v)	-	0.207	0.226	0.804	1.256	-0.372

It is observed that while all models generally provide good calibration at first glance, the Feller condition($2\lambda\theta \geq \nu^2$) necessary for the positive variance process in Heston model (v) is not satisfied for all contracts. In other words, the smiles of implied volatilities in American options on WTI futures are so extreme that Heston model has difficulty in calibration under the Feller condition.

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Table 7: Parameters(NOV09, DEC09)

		$\sigma(0)$	$V(0)$	λ	θ	ν	ρ
NOV09	(i)	3.606	-	-	-	1.000	-0.190
	(ii)	0.430	-	-	-	1.109	-0.383
	(iii)	3.614	-	0.035	1.838	0.993	-0.172
	(iv)	0.430	-	0.016	0.163	1.086	-0.357
	(v)	-	0.211	0.084	0.887	1.058	-0.392
DEC90	(i)	3.519	-	-	-	0.993	-0.172
	(ii)	0.419	-	-	-	1.098	-0.365
	(iii)	3.523	-	0.007	1.590	0.977	-0.152
	(iv)	0.419	-	0.008	0.223	1.063	-0.340
	(v)	-	0.217	0.024	0.515	1.019	-0.378

Table 8: Parameters(NOV10, DEC10)

		$\sigma(0)$	$V(0)$	λ	θ	ν	ρ
NOV10	(i)	2.771	-	-	-	0.511	-0.126
	(ii)	0.323	-	-	-	0.630	-0.384
	(iii)	2.773	-	0.001	2.765	0.494	-0.112
	(iv)	0.322	-	0.005	0.326	0.610	-0.363
	(v)	-	0.016	0.367	0.618	0.834	-0.404
DEC10	(i)	2.742	-	-	-	0.581	-0.112
	(ii)	0.321	-	-	-	0.653	-0.371
	(iii)	2.750	-	0.000	2.753	0.555	-0.098
	(iv)	0.321	-	0.002	0.331	0.626	-0.350
	(v)	-	0.011	0.365	0.617	0.865	-0.390

Table 9: Calibration to Futures Call Option Prices(SEP09, OCT09)(usd)

Strike		Transformed CME	Price (i)	(ii)	(iii)	(iv)	(v)	Difference (i)	(ii)	(iii)	(iv)	(v)
SEP09	50	20.51	20.53	20.52	20.53	20.53	20.49	0.02	0.01	0.02	0.02	-0.02
	55	15.83	15.84	15.83	15.84	15.84	15.82	0.00	0.00	0.01	0.01	-0.02
	60	11.47	11.45	11.45	11.45	11.45	11.46	-0.02	-0.01	-0.01	-0.01	-0.01
	65	7.59	7.60	7.60	7.59	7.59	7.62	0.01	0.02	0.01	0.01	0.04
	70	4.56	4.54	4.54	4.54	4.54	4.55	-0.02	-0.01	-0.02	-0.02	-0.01
	75	2.44	2.43	2.43	2.44	2.44	2.42	-0.01	-0.01	0.00	0.00	-0.02
	80	1.16	1.19	1.19	1.20	1.20	1.18	0.03	0.03	0.04	0.04	0.02
	85	0.56	0.56	0.55	0.56	0.55	0.56	0.00	-0.01	0.00	-0.01	0.00
	90	0.27	0.26	0.26	0.25	0.25	0.26	-0.02	-0.01	-0.02	-0.02	-0.01
	95	0.13	0.12	0.12	0.10	0.10	0.13	-0.01	-0.01	-0.03	-0.03	0.00
	100	0.06	0.06	0.06	0.04	0.04	0.06	0.00	0.00	-0.02	-0.02	0.00
OCT09	50	21.63	21.66	21.65	21.68	21.67	21.62	0.03	0.02	0.06	0.04	-0.01
	55	17.16	17.15	17.15	17.17	17.15	17.15	-0.01	-0.01	0.00	-0.01	-0.02
	60	13.01	12.98	12.99	12.98	12.97	13.01	-0.03	-0.02	-0.02	-0.03	0.01
	65	9.32	9.31	9.32	9.30	9.30	9.34	-0.01	0.00	-0.02	-0.02	0.02
	70	6.30	6.29	6.29	6.28	6.28	6.29	-0.01	-0.01	-0.02	-0.01	-0.01
	75	3.99	4.00	3.99	4.00	4.01	3.97	0.01	0.00	0.01	0.02	-0.02
	80	2.38	2.42	2.41	2.43	2.43	2.39	0.04	0.03	0.05	0.05	0.01
	85	1.41	1.42	1.41	1.43	1.43	1.41	0.01	0.00	0.02	0.02	0.00
	90	0.84	0.82	0.82	0.82	0.82	0.83	-0.02	-0.02	-0.02	-0.02	-0.01
	95	0.50	0.47	0.48	0.46	0.46	0.49	-0.03	-0.02	-0.04	-0.04	0.00
	100	0.29	0.27	0.29	0.25	0.25	0.30	-0.02	0.00	-0.04	-0.04	0.01

When the Feller condition is not satisfied, the volatility process of Heston model is not well-defined. Thus, Heston model is not adopted for pricing average options in the

Table 10: Calibration to Futures Call Option Prices(NOV09, DEC09)(usd)

	Strike	Transformed CME	Price (i)	(ii)	(iii)	(iv)	(v)	Difference (i)	(ii)	(iii)	(iv)	(v)
NOV09	50	22.59	22.65	22.63	22.66	22.66	22.59	0.06	0.04	0.07	0.07	0.01
	55	18.26	18.26	18.26	18.26	18.26	18.25	0.01	0.00	0.01	0.01	-0.01
	60	14.24	14.21	14.22	14.20	14.20	14.24	-0.03	-0.02	-0.04	-0.04	0.00
	65	10.65	10.62	10.63	10.61	10.61	10.66	-0.03	-0.02	-0.04	-0.04	0.01
	70	7.61	7.60	7.60	7.60	7.60	7.62	-0.02	-0.01	-0.02	-0.02	0.00
	75	5.20	5.21	5.21	5.22	5.22	5.19	0.01	0.01	0.02	0.02	-0.01
	80	3.41	3.44	3.43	3.45	3.45	3.41	0.04	0.03	0.05	0.05	0.00
	85	2.19	2.22	2.21	2.23	2.22	2.20	0.03	0.02	0.04	0.04	0.01
	90	1.42	1.40	1.40	1.41	1.41	1.41	-0.01	-0.02	-0.01	-0.01	-0.01
	95	0.91	0.89	0.89	0.88	0.88	0.91	-0.03	-0.02	-0.04	-0.03	-0.01
	100	0.58	0.56	0.58	0.54	0.54	0.59	-0.02	0.00	-0.04	-0.04	0.01
DEC09	50	23.49	23.58	23.56	23.59	23.59	23.51	0.09	0.07	0.10	0.10	0.02
	55	19.28	19.29	19.29	19.29	19.29	19.28	0.01	0.01	0.01	0.01	0.00
	60	15.38	15.32	15.33	15.31	15.31	15.36	-0.06	-0.05	-0.07	-0.07	-0.02
	65	11.83	11.78	11.79	11.77	11.77	11.83	-0.05	-0.04	-0.07	-0.06	0.00
	70	8.78	8.75	8.76	8.75	8.75	8.78	-0.02	-0.01	-0.02	-0.02	0.00
	75	6.28	6.30	6.30	6.31	6.31	6.28	0.02	0.02	0.03	0.03	0.00
	80	4.38	4.42	4.41	4.44	4.44	4.38	0.05	0.04	0.06	0.06	0.01
	85	3.00	3.05	3.04	3.06	3.06	3.02	0.05	0.04	0.07	0.07	0.03
	90	2.11	2.09	2.09	2.10	2.10	2.09	-0.02	-0.02	-0.01	-0.01	-0.02
	95	1.47	1.43	1.44	1.43	1.43	1.46	-0.04	-0.03	-0.05	-0.04	-0.01
	100	1.02	0.99	1.01	0.96	0.97	1.03	-0.03	-0.01	-0.06	-0.05	0.01

Table 11: Calibration to Futures Call Option Prices(NOV10, DEC10)(usd)

	Strike	Transformed CME	Price (i)	(ii)	(iii)	(iv)	(v)	Difference (i)	(ii)	(iii)	(iv)	(v)
NOV10	50	27.98	28.00	28.02	28.01	28.06	28.05	0.02	0.04	0.03	0.08	0.07
	55	-	-	-	-	-	-	-	-	-	-	-
	60	20.47	20.39	20.42	20.37	20.39	20.48	-0.08	-0.05	-0.10	-0.08	0.01
	65	-	-	-	-	-	-	-	-	-	-	-
	70	14.08	14.05	14.06	14.04	14.03	14.09	-0.03	-0.02	-0.04	-0.05	0.01
	75	11.45	11.44	11.43	11.44	11.42	11.44	-0.01	-0.02	-0.01	-0.03	-0.01
	80	9.24	9.21	9.19	9.21	9.19	9.17	-0.03	-0.05	-0.03	-0.04	-0.07
	85	7.35	7.35	7.32	7.35	7.34	7.29	0.00	-0.03	0.00	-0.01	-0.06
	90	5.74	5.82	5.80	5.83	5.83	5.78	0.09	0.06	0.09	0.09	0.04
	95	-	-	-	-	-	-	-	-	-	-	-
	100	3.56	3.62	3.64	3.61	3.64	3.65	0.06	0.08	0.05	0.08	0.08
DEC10	50	28.46	28.59	28.55	28.60	28.60	28.53	0.13	0.09	0.14	0.14	0.07
	55	24.64	24.65	24.64	24.64	24.64	24.65	0.01	0.01	0.00	0.00	0.01
	60	21.07	20.98	20.99	20.96	20.96	21.01	-0.09	-0.08	-0.11	-0.11	-0.06
	65	17.73	17.62	17.64	17.60	17.61	17.67	-0.12	-0.09	-0.13	-0.13	-0.07
	70	14.61	14.61	14.63	14.60	14.61	14.65	-0.01	0.01	-0.01	-0.01	0.04
	75	11.91	11.97	11.98	11.98	11.98	12.00	0.06	0.07	0.07	0.07	0.09
	80	9.68	9.72	9.71	9.74	9.74	9.72	0.04	0.03	0.05	0.05	0.04
	85	7.81	7.84	7.82	7.86	7.85	7.82	0.03	0.01	0.05	0.04	0.01
	90	6.29	6.30	6.27	6.31	6.31	6.27	0.01	-0.03	0.02	0.01	-0.02
	95	5.04	5.06	5.03	5.06	5.06	5.04	0.02	-0.02	0.02	0.02	0.00
	100	4.15	4.07	4.05	4.06	4.06	4.07	-0.07	-0.10	-0.09	-0.09	-0.08

next subsection, where the estimated average prices based the calibration are compared with CME Group's settlement prices.

Also, note that $\lambda \approx 0$ in the calibrated λ-SABR for all of maturities and hence the model is not distinguished with SABR model. We also remark that the estimates of parameters(λ, θ) in λ-SABR and Heston models are sometimes are not stable because the estimates often converge to local optima.

4.3 Average Option Prices with Calibrated Parameters

This subsection shows the approximate prices of average options with calibrated parameters and compares the prices with the ones obtained by Monte Carlo simulations as well as with the settlement prices in CME Group.

Note that there are few trades of average options in CME Group since average options are traded mainly in OTC market; for instance, there were no trades for AUG09 , OCT09 and OCT10 on July 1, 2009 except only 125 barrel trades for 70 Put of AUG09. However, there are no official prices in the OTC market. On the other hand, the settlement prices in CME Group are announced every day for the contracts with positive open interests. Thus, we use the settlement prices of average options as the reference prices.

In Monte Caro simulation, the term is divided into 195 time-steps for AUG09, 305 time-steps for OCT09 and 600 time-steps for OCT10, and the number of trials are set to be $10,000,000$. [7]

The interest rates used for discounting are 0.41% for AUG09, 0.77% for OCT09 and 1.62% for OCT10. The correlation between two underlying futures prices, SEP09 and OCT09 is calculated based on the daily rates of returns from April 30, 2009 to July 1, 2009; the correlation is estimated as 0.99920. The correlation between NOV09 and DEC09 is calculated based on the daily rates of returns from February 27, 2009 to July 1, 2009; it is estimated as 0.99941. The correlation between NOV10 and DEC10 is calculated based on the daily rates of returns from February 29, 2008 to July 1, 2009; it is estimated as 0.99990. The average option prices computed under those conditions are reported in Table 12, 13 and 14.

Compared with prices computed by Monte Carlo simulations, the approximations with $\beta = 1$((ii),(iv)) are less accurate than those with $\beta = 0.5$((i),(iii)). Because the asymptotic expansion method is based on an expansion around a normal distribution and the distribution of the underlying asset price is closer to a normal when β is closer to zero, the smaller is β, the approximation becomes more accurate in general.

It is also observed in "MC - CME(Avg)" that the settlement prices of AUG09, OCT09 and OCT10 in CME Group are generally close to the average options prices with calibrated parameters, which implies that the settlement prices of average options may be given consistently with the prices of WTI futures American options.

As previously noted, $\lambda \approx 0$ is estimated in calibration under λ-SABR model and the estimates of parameters(λ, θ) are sometimes are unstable. Hence, at least for above examined cases, SABR model with $\beta = 0.5$ seems the better model for calibration to the market prices of the American option on WTI futures as well as for approximation of average options prices.

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[7]In the (λ-)SABR model, when the volatility process becomes non-positive due to discretization, we set $\sigma(t_{i+1}) = \sigma(t_i) + \lambda\theta\Delta t$, $S(t_{i+1}) = S(t_i) + \alpha S(t_i)\Delta t$ Also when the underlying asset price becomes non-positive, we $S(t_{i+1}) = 0$.

Table 12: AUG09 Average Option Price (Average Swap Price = 70.54)

			60 (Put)	65 (Put)	70 (Call)	75 (Call)	80 (Call)
Price	CME(Avg)		0.97	2.01	4.44	2.30	1.05
	MC	(i)	0.93	1.99	4.41	2.29	1.06
		(ii)	0.93	2.00	4.42	2.30	1.07
		(iii)	0.92	1.98	4.41	2.29	1.07
		(iv)	0.91	1.99	4.42	2.31	1.09
	AE	(i)	0.93	1.99	4.41	2.28	1.05
		(ii)	0.95	2.00	4.42	2.28	1.05
		(iii)	0.93	1.98	4.40	2.28	1.06
		(iv)	0.93	1.99	4.42	2.30	1.07
Difference	MC - CME(Avg)	(i)	-0.04	-0.02	-0.03	-0.01	0.01
		(ii)	-0.04	-0.01	-0.02	0.00	0.02
		(iii)	-0.05	-0.03	-0.03	-0.01	0.02
		(iv)	-0.06	-0.02	-0.02	0.01	0.04
	AE-MC	(i)	0.01	0.00	-0.01	-0.01	-0.01
		(ii)	0.02	0.01	0.00	-0.01	-0.02
		(iii)	0.01	0.00	-0.01	-0.01	-0.01
		(iv)	0.02	0.01	0.00	-0.01	-0.02
	AE - CME(Avg)	(i)	-0.04	-0.02	-0.03	-0.02	0.00
		(ii)	-0.02	-0.01	-0.02	-0.02	0.00
		(iii)	-0.04	-0.03	-0.04	-0.02	0.01
		(iv)	-0.04	-0.02	-0.02	0.00	0.02

Table 13: OCT09 Average Option Price (Average Swap Price = 71.99)

			65 (Put)	70 (Put)	75 (Call)	80 (Call)	85 (Call)
Price	CME(Avg)		2.29	3.64	7.54	5.10	3.30
	MC	(i)	2.25	3.61	7.53	5.12	3.35
		(ii)	2.24	3.61	7.53	5.12	3.35
		(iii)	2.22	3.58	7.52	5.13	3.36
		(iv)	2.21	3.58	7.52	5.13	3.38
	AE	(i)	2.27	3.62	7.54	5.12	3.35
		(ii)	2.30	3.64	7.55	5.13	3.34
		(iii)	2.25	3.60	7.53	5.13	3.36
		(iv)	2.26	3.61	7.54	5.14	3.37
Difference	MC - CME(Avg)	(i)	-0.04	-0.03	-0.01	0.02	0.05
		(ii)	-0.05	-0.03	-0.01	0.02	0.05
		(iii)	-0.07	-0.06	-0.02	0.03	0.06
		(iv)	-0.08	-0.06	-0.02	0.03	0.08
	AE-MC	(i)	0.02	0.01	0.01	0.00	0.00
		(ii)	0.05	0.03	0.02	0.01	-0.01
		(iii)	0.03	0.02	0.01	0.00	0.00
		(iv)	0.05	0.03	0.02	0.01	-0.01
	AE - CME(Avg)	(i)	-0.02	-0.02	0.00	0.02	0.05
		(ii)	0.01	0.00	0.01	0.03	0.04
		(iii)	-0.04	-0.04	-0.01	0.03	0.06
		(iv)	-0.03	-0.03	0.00	0.04	0.07

5   Conclusion

This paper derived a new approximation formula for pricing average options in commodity markets under Heston model and an extended λ-SABR model which includes

Table 14: OCT10 Average Option Price (Average Swap Price = 76.17)

			65 (Put)	70 (Put)	75 (Call)	80 (Call)	85 (Call)
Price	CME(Avg)		6.15	8.01	11.41	9.18	7.29
	MC	(i)	6.08	7.98	11.40	9.16	7.30
		(ii)	6.03	7.92	11.34	9.11	7.26
		(iii)	6.06	7.97	11.40	9.18	7.32
		(iv)	5.99	7.90	11.34	9.13	7.29
	AE	(i)	6.15	8.03	11.45	9.21	7.34
		(ii)	6.19	8.05	11.45	9.20	7.32
		(iii)	6.12	8.01	11.44	9.21	7.35
		(iv)	6.13	8.02	11.44	9.21	7.35
Difference	MC - CME(Avg)	(i)	-0.07	-0.03	-0.01	-0.02	0.01
		(ii)	-0.12	-0.09	-0.07	-0.07	-0.03
		(iii)	-0.09	-0.04	-0.01	0.00	0.03
		(iv)	-0.16	-0.11	-0.07	-0.05	0.00
	AE-MC	(i)	0.07	0.06	0.05	0.04	0.03
		(ii)	0.16	0.13	0.11	0.08	0.06
		(iii)	0.06	0.05	0.05	0.04	0.03
		(iv)	0.14	0.11	0.10	0.08	0.05
	AE - CME(Avg)	(i)	0.00	0.02	0.04	0.03	0.05
		(ii)	0.04	0.04	0.04	0.02	0.03
		(iii)	-0.03	0.00	0.03	0.03	0.06
		(iv)	-0.02	0.01	0.03	0.03	0.06

an extended SABR model as a special case.

Then, numerical experiments examined the effect of our formula and confirmed the sufficient accuracy of the third order expansion. Moreover, the paper presented an example for predicting average option prices with parameters obtained by calibration to the market prices of WTI futures American options; in particular, the analysis showed accurate approximations by our formula under SABR model with $\beta = 0.5$.

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A Formulas for the Conditional Expectations of the Wiener-Itô Integrals

This appendix summarizes conditional expectation formulas for explicit computation of the asymptotic expansions up to the third order.

In the following, W is a d-dimensional Brownian motion and $q_i = (\hat{q}_{i1}, \cdots, \hat{q}_{id})'$ where $\hat{q}_i \in L^2[0, T]$, $i = 1, 2, ..., 5$ and x' denotes the transpose of x. $H_n(x; \Sigma)$ denotes the Hermite polynomial of degree n and $\Sigma = \int_0^T |q_{1t}|^2 dt$. For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda (2009).

1.
$$\mathbf{E}\left[\int_0^T q_{2t}' dW_t | \int_0^T q_{1v}' dW_v = x\right] = \left(\int_0^T q_{2t}' q_{1t} dt\right) \frac{H_1(x; \Sigma)}{\Sigma}$$

2.
$$\mathbf{E}\left[\int_0^T \int_0^t q_{2u}' dW_u q_{3t}' dW_t | \int_0^T q_{1v}' dW_v = x\right] =$$
$$\left(\int_0^T \int_0^t q_{2u}' q_{1u} du q_{3t}' q_{1t} dt\right) \frac{H_2(x; \Sigma)}{\Sigma^2}$$

3.
$$\mathbf{E}\left[\left(\int_0^T q_{2u}' dW_u\right)\left(\int_0^T q_{3s}' dW_s\right) | \int_0^T q_{1v}' dW_v = x\right] =$$
$$\left(\int_0^T q_{2u}' q_{1u} du\right)\left(\int_0^T q_{3s}' q_{1s} ds\right) \frac{H_2(x; \Sigma)}{\Sigma^2}$$
$$+ \int_0^T q_{2t}' q_{3t} dt$$

4.
$$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2u}' dW_u q_{3s}' dW_s q_{4t}' dW_t | \int_0^T q_{1v}' dW_v = x\right] =$$
$$\left(\int_0^T q_{4t}' q_{1t} \int_0^t q_{3s}' q_{1s} \int_0^s q_{2u}' q_{1u} du ds dt\right) \frac{H_3(x; \Sigma)}{\Sigma^3}$$

5.

$$\mathbf{E}\left[\int_0^T \left(\int_0^t q'_{2u}dW_u\right)\left(\int_0^t q'_{3s}dW_s\right)q'_{4t}dW_t \Big| \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left\{\int_0^T \left(\int_0^t q'_{2u}q_{1u}du\right)\left(\int_0^t q'_{3s}q_{1s}ds\right)q'_{4t}q_{1t}dt\right\}\frac{H_3(x;\Sigma)}{\Sigma^3}$$

$$+\left(\int_0^T \int_0^t q'_{2u}q_{3u}du q'_{4t}q_{1t}dt\right)\frac{H_1(x;\Sigma)}{\Sigma}$$

6.

$$\mathbf{E}\left[\left(\int_0^T \int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T q'_{4u}dW_u\right)\Big| \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{1s}dsdt\right)\left(\int_0^T q'_{4u}q_{1u}du\right)\frac{H_3(x;\Sigma)}{\Sigma^3}$$

$$+\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{4s}dsdt + \int_0^T q'_{3t}q_{4t}\int_0^t q'_{2s}q_{1s}dsdt\right)\frac{H_1(x;\Sigma)}{\Sigma}$$

7.

$$\mathbf{E}\left[\left(\int_0^T \int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T \int_0^r q'_{4u}dW_u q'_{5r}dW_r\right)\Big| \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{1s}dsdt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\int_0^T q'_{3t}q_{1t}\int_0^t q'_{5r}q_{1r}\int_0^r q'_{2u}q_{4u}dudrdt + \int_0^T q'_{5t}q_{1t}\int_0^t q'_{3r}q_{1r}\int_0^r q'_{2u}q_{4u}dudrdt\right.$$

$$+\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2r}q_{5r}\int_0^r q'_{4u}q_{1u}dudrdt + \int_0^T q'_{3t}q_{5t}\left(\int_0^t q'_{2s}q_{1s}ds\right)\left(\int_0^t q'_{4u}q_{1u}du\right)dt$$

$$\left.+\int_0^T q'_{5r}q_{1r}\int_0^r q'_{3u}q_{4u}\int_0^u q'_{2s}q_{1s}dsdudr\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T \int_0^t q'_{2u}q_{4u}du q'_{3t}q_{5t}dt$$

8.

$$\mathbf{E}\left[\left(\int_0^T q'_{2t}dW_t\right)\left(\int_0^T q'_{3s}dW_s\right)\left(\int_0^T \int_0^r q'_{4u}dW_u q'_{5r}dW_r\right)\Big| \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{3s}q_{1s}ds\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\left(\int_0^T q'_{3s}q_{1s}ds\right)\left(\int_0^T q'_{2r}q_{5r}\int_0^r q'_{4u}q_{1u}dudr\right) + \left(\int_0^T q'_{2t}q_{3t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{4u}q_{1u}dudr\right)\right.$$

$$+\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{3s}q_{4s}dsdr\right) + \left(\int_0^T q'_{3t}q_{1t}dt\right)\left(\int_0^T q'_{5r}q_{1r}\int_0^r q'_{2u}q_{4u}dudr\right)$$

$$\left.+\left(\int_0^T q'_{2t}q_{1t}dt\right)\left(\int_0^T q'_{3r}q_{5r}\int_0^r q'_{4u}q_{1u}dudr\right)\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T q'_{2t}q_{5t}\int_0^t q'_{3s}q_{4s}dsdt + \int_0^T q'_{3r}q_{5r}\int_0^r q'_{2u}q_{4u}dudr$$

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