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An Asymptotic Expansion Scheme for the Optimal Portfolio for Investment *

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Abstract

We shall propose a new computational scheme for the evaluation of the optimal portfolio for investment. Our method is based on an extension of the asymptotic expansion approach which has been recently developed for the pricing problems of the contingent claims' analysis by Kunitomo-Takahashi(1992,1995,1998), Yoshida(1992), Takahashi(1995,1999) and others. In particular, we will explicitly derive the formula of the optimal portfolio associated with maximizing utility from terminal wealth for a power utility function in a financial market with Markovian coefficients.

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1 Introduction

We shall propose a new computational scheme for the evaluation of the optimal portfolios for investment. Our method is based on the asymptotic expansion approach, a unified method of efficient computation justified by Malliavin-Watanabe(1987) theory, which has been recently developed for the pricing problems of the contingent claims' analysis by Kunitomo-Takahashi(1992,1995,1998), Yoshida(1992), Takahashi(1995,1999), Kunitomo and Kim(1999), Sorensen and Yoshida(1998) and Kashiwakura and Yoshida(2001). They have developed the method through deriving formulas for practical examples such as average options, basket options, and options with stochastic volatility and with stochastic interest rates in a Markovian setting, as well as bond options(swaptions), average options on interest rates, and average options on foreign exchange rates with stochastic interest rates in the Heath-Jarrow-Morton(1992) framework. In this paper, we extend the method for portfolio problems. In particular, we will explicitly derive the formula of the optimal portfolio associated with maximizing utility from terminal wealth for a power utility function in a financial market with Markovian coefficients. In general, it is quite difficult to compute an optimal portfolio explicitly when the investment oppotunity is stochastic in a multiperiod setting. The stochastic control approach initiated by Merton(1969,1971) gives a solution in terms of the derivatives of the value function: While the solution can be evaluated numerically based on the Hamilton-Jacobi-Bellman equation, the implementation is not easy especially for the case of multiple assets. In the martingale approach initiated by Karazas et al.(1987) and Cox and Huang(1989), Ocone and Karatzas(1991) proposed the representation of optimal portfolios by utilizing the Clark formula. Although their representation formulas were derived in general setting, explicit evaluation was obtained only for logarithmic utility functions or a financial market with deterministic coefficients, which were already known without their formulas. Starting with the Clark formula, we will present an explicit expression for the optimal portfolio in a financial market with Markovian coefficients which is more concrete but practically sufficient setting. Moreover, our method can be easily extended to the optimal portfolios associated with maximizing utility from both consumption and terminal wealth, and to the hedging portfolios associated with contingent claims. The organization of this paper is as follows. In Section 2 we explain the problem of the optimal portfolio for investment, and in Section 3 we restate our problem in a Markovian setting. In Section 4 we derive the second order scheme explicitly for a case of power utility function through explaining the asymptotic expansion approach, and in the appendix we show the result of the third order scheme.

2 The Representation of Optimal Portfolio

We will breifly describe the financial market, and introduce the representation of the optimal portfolio for investment derived by Ocone and Karatazas(1

Let $(\Omega, \mathcal{F}, P)$ probability space, $w(t) = (w^{\alpha}(t), \cdots, w^{r}(t))^{*}$ for $0 \leq t \leq T$, R^{r}-valued Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}$ for $0 \leq t \leq T$ P-augmentation of the natural filtration, $\mathcal{F}_t^w = \sigma(w(s); 0 \leq s \leq t)$. Here, we use the notation of x^* as the transpose of x. $S_i(t)$, $i = 1, \cdots, r$ and $S_0(t)$ denote the prices at time $t \in [0, T]$ of the risky asset i and of the risless asset respectively. The prices are assumed to follow the stochastic processes: For $t \in [0, T]$,

$$\begin{aligned} dS_i &= S_i(t)[b_i(t)dt + \sum_{j=1}^{r} \sigma_{ij}(t)dw_j(t)];\ S_i(0) = s_i\ i = 1, \cdots, r \\ dS_0 &= r(t)S_0(t)dt;\ S_0(0) = 1 \end{aligned}$$

where we suppose that $r(t)$,$b_i(t)$ and $\sigma_{ij}(t)$, $i, j = 1, \cdots, r$ are bounded and progressively measurable with respect to $\{\mathcal{F}_t\}$. We also assume the nondegeneracy condition; for the $r \times r$ matrix $\sigma(t) \equiv \{\sigma_{ij}(t)\}_{1 \leq i,j \leq r}$ there exists a real number $\varepsilon > 0$ such that

$$\xi^* \sigma(t, \omega) \sigma(t, \omega)^* \xi \geq \epsilon |\xi|^2;\ \forall \xi \in \mathbf{R}^r,\ (t, \omega) \in [0, T] \times \Omega.$$

Then, the stochastic process of an investor's wealth denoted by $W(t)$ are expressed as

$$dW(t) = [r(t)W(t) - c(t)]dt + \pi(t)^*[(b(t) - r(t)\mathbf{1})dt + \sigma(t)dw(t)]$$

where $W(0) = W > 0$ is the initial capital, $\mathbf{1}$ denotes the vector in $\mathbf{R}^r$ with all elements equal to 1, $c(t)$ denotes the consumption rate, and $\pi(t) = \{\pi_i(t)\}_{i=1,\cdots,r}^*$ denotes the portfolio. $c(t)$ and $\pi(t)$ satisfy the integrability condition;

$$\int_0^T \{|\pi(t)|^2 + c(t)\}dt < \infty \quad a.s.$$

Next, let $\mathcal{A}(W)$ denote the set of stochastic processes (π, c) which generate $W(t) \geq 0$ for all $t \in [0, T]$ given $W(0) = W$. We call (π, c) is admissible for W if $(\pi, c) \in \mathcal{A}(W)$.

The problem of *maximizing utility from terminal wealth* is formulated as follows: With $c \equiv 0$,

$$\sup_{(\pi,c) \in \mathcal{A}(W)} E[U(W(T))]$$

where $U : (0, \infty) \to \mathbf{R}$ denotes a utility function, and $E[\cdot]$ denotes the expectation operator under P. We assume U is a strictly increasing, strictly concave function of class C^2, with $U(0+) \equiv \lim_{c \downarrow 0} U(c) \in [-\infty, \infty)$, $U'(0+) \equiv \lim_{c \downarrow 0} U'(c) = \infty$ and $U'(\infty) \equiv \lim_{c \to \infty} U'(c) = 0$.

Let the market price of risk $\theta(t)$, $t \in [0, T]$ an R^r−valued progressively measurable bounded process defined by

$$\theta(t) = \sigma(t)^{-1}[b(t) - r(t)\mathbf{1}].$$

Then, the martingale measure denoted by P_0 is defined: $P_0(A) =$
for all $A \in \mathcal{F}_T$ where

$$Z(t) = \exp\left(-\int_0^t \theta(s)^* dw(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds\right);\ 0 \le t \le$$

We note that under P_0 $w_0(t) \equiv w(t) + \int_0^t \theta(u)du$, $0 \le t \le T$ is a
Brownian motion.

Regarding the problem of *maximizing utility from terminal wealth*
known that the optimal wealth level of terminal wealth given b
$I(\mathcal{Y}(W)H_0(T))$, and that the value function $V(W) := \sup_{(\pi,c)\in\mathcal{A}(W}$
can be computed as $V(W) = G(\mathcal{Y}(W))$, where $G(y) := E[U(I(yH_0$
$y < \infty$. (See for instance Theorem 7.6 in Karatzas and Shreve(199
Here, $I \in C^1((0,\infty);(0,\infty))$ denotes the inverse of $U'(\cdot)$, and $\mathcal{Y}($
the inverse of the continuous, decreasing function:

$$\mathcal{X}(y) = \mathbf{E}_0[\beta(T)I(yH_0(T))] = \mathbf{E}[H_0(T)I(yH_0(T))];\ 0 < y <$$

which we assume maps $(0,\infty)$ into $(0,\infty)$, where $\beta(t) \equiv 1/S_0(t)$
$\beta(t)Z(t)$ denotes the state price density at t, and $E_0[\cdot]$ denotes th
tion operator under P_0.

Ocone and Karazas(91) gives the following theorem by utilizing
formula regarding the problem of the optimal portfolio for invest
ciated with *maximizing utility from terminal wealth.*

Theorem 1 *Suppose that*

$$I(y) + |I'(y)| \le K(y^\alpha + y^{-\beta}),\ 0 < y < \infty$$

holds for some real, positive, constants α, β, K.
Then the optimal portfolio admits the repesenatation

$$\begin{aligned}\pi^*(t)\sigma(t) &= -\frac{1}{\beta(t)}\left\{\theta^*(t)\mathbf{E}_0[\beta(T)\mathcal{Y}(W)H_0(T)I'(\mathcal{Y}(W)H_0(T))|\right.\\ &+ \mathbf{E}_0\left[\beta(T)\phi'(\mathcal{Y}(W)H_0(T))\left(\int_t^T D_t r(u)du + \sum_{\alpha=1}^{r}\int_t^T\right.\right.\end{aligned}$$

where $\phi(y) \equiv yI(y)$,$0 < y < \infty$, and $D_t r(u)$ and $D_t\theta_\alpha(u)$, $\alpha =$
deonote the Malliavin derivatives of $r(u)$ and $\theta_\alpha(u)$.

Here we suppose that θ and r satisfy the following conditions:

- *$\mathbf{R}$-valued progressivlely measurable process r is bounded; fo*
$[0,T]$ $r(s,\cdot) \in D_{1,1}$ where $D_{1,1}$ denotes the Sobolev space
$(p,s) = (1,1)$, $(s,\omega) \to Dr(s,\omega) \in (L^2([0,T]))^r$ admits pro
measurable version, and

$$\|r\|_{1,1}^{a} = \mathbf{E}\left[\left(\int_0^T |r(s)|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^T \|Dr(s)\|^2 ds\right)^{\frac{1}{2}}\right]$$

where $\|\cdot\|$ denotes the $L^2([0,T])$ norm, and $\|Dr(s)\|^2 = \sum_{i=1}^{r}$

- *$\mathbf{R}^r$-valued progressively measurable process θ is bounded; for a.e. $s \in [0,T]$ $\theta(s,\cdot) \in (\mathbf{D}_{1,1})^r$, $(s,\omega) \to D\theta(s,\omega) \in (L^2([0,T]))^{r^2}$ admits a progressively measurable version, and*

$$\|\theta\|_{1,1}^a = \mathbf{E}\left[\left(\int_0^T |\theta(s)|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^T \|D\theta(s)\|^2 ds\right)^{\frac{1}{2}}\right] < \infty$$

  where $\|D\theta(s)\|^2 = \sum_{i,j=1}^r \|D^i\theta_j(s)\|^2$.

- *For some $p > 1$ we have*

$$\mathbf{E}\left[\left(\int_0^T \|Dr(s)\|^2 ds\right)^{\frac{p}{2}}\right] < \infty, \ \mathbf{E}\left[\left(\int_0^T \|D\theta(s)\|^2 ds\right)^{\frac{p}{2}}\right] < \infty.$$

We note that the optimal portfolio is also expressed under P:

$$\begin{aligned}
\pi^*\sigma(t) &= -\mathbf{E}\left[\frac{H_0(T)}{H_0(t)}\mathcal{Y}(W)H_0(T)I'(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t\right]\theta^*(t) \\
&\quad -\mathbf{E}\Bigg[\frac{H_0(T)}{H_0(t)}\phi'(\mathcal{Y}(W)H_0(T))\times \\
&\quad \left(\int_t^T D_t r(u)du + \sum_{\alpha=1}^r\left\{\int_t^T \{D_t\theta_\alpha(u)\}dw^\alpha(u) + \int_t^T \{D_t\theta_\alpha(u)\}\theta_\alpha(u)du\right\}\right)|\mathcal{F}_t\Bigg] \\
&= W(t)\theta^*(t) - \mathbf{E}\left[\frac{H_0(T)}{H_0(t)}\phi'(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t\right]\theta^*(t) \\
&\quad -\mathbf{E}\Bigg[\frac{H_0(T)}{H_0(t)}\phi'(\mathcal{Y}(W)H_0(T))\times \\
&\quad \left(\int_t^T D_t r(u)du + \sum_{\alpha=1}^r\left\{\int_t^T \{D_t\theta_\alpha(u)\}dw^\alpha(u) + \int_t^T \{D_t\theta_\alpha(u)\}\theta_\alpha(u)du\right\}\right)|\mathcal{F}_t\Bigg]
\end{aligned}$$

where $W(t)$ denotes the optimal wealth at time t, and

$$W(t) = \mathbf{E}\left[\frac{H_0(T)}{H_0(t)}I(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t\right].$$

It is well known that the optimal portfolio $\pi(t)$ is easily derived for two simple cases: (See for instance chapter 3 in Karatzas and Shreve(1998).) For the case of a log utility function $U(x) = \log x$,

$$\pi^*(t) = \theta^*(t)\sigma(t)^{-1}W(t)$$

where $\theta(t) = \sigma(t)^{-1}[b(t) - r(t)1]$. For the case of a power utility function $U(x) = \frac{x^\delta}{\delta}$, $\delta < 1$, $\delta \neq 0$, if $r(\cdot)$ and $\theta(\cdot)$ are deterministic,

$$\pi^*(t) = \frac{1}{(1-\delta)}\theta^*(t)\sigma(t)^{-1}W(t).$$

However, if $r(\cdot)$ and $\theta(\cdot)$ are *not* deterministic, it is difficult to evaluate $\pi(t)$ explicitly for a power utility function.

3 The Optimal Portfolio Problem for Investment in a Markovian Setting

In the spirit of Ocone and Karazas (1991), we will a consider more concrete but sufficiently general setting for practical purpose in the sequel.

Let X_u^ϵ be a d-dimensional diffusion process defined by the stochastic differential equation:

$$\begin{cases} dX_u^\epsilon = V_0(X_u^\epsilon, \epsilon)du + V(X_u^\epsilon, \epsilon)dw_u, \quad X_t^\epsilon = x, \\ dS_u^\epsilon = I_S b(X_u^\epsilon)du + I_S\sigma(X_u^\epsilon)dw_u, \quad S_t^\epsilon = s \\ dS_{0u}^\epsilon = S_{0u} r(X_u^\epsilon)du, \quad S_{0t}^\epsilon = s_0 \end{cases}$$

for $u \in [t, T]$ where I_S denotes the $r \times r$ diagonal matrix with i-th diagonal element of S_i. Here we suppose $\epsilon \in (0,1]$, $V_0 \in C_b^\infty(\mathbf{R}^d \times (0,1]; \mathbf{R}^d)$ and $V = (V_\beta)_{\beta=1}^r \in C_b^\infty(\mathbf{R}^d \times (0,1]; \mathbf{R}^d \otimes \mathbf{R}^r)$ where $C_b^\infty(R^d \times (0,1]; E)$ denotes a class of smooth mappings $f : \mathbf{R}^d \times (0,1] \to E$ whose derivatives $\partial_x^{\mathbf{n}} \partial_\epsilon^m f(x, \epsilon)$ are all bounded for $\mathbf{n} \in \mathbf{Z}_+^d$ such that $|\mathbf{n}| \geq 1$ and $m \in \mathbf{Z}_+$. Note that time-dependent-coefficient diffusion processes are included in the above equation if we enlarge the process to a higer-dimensional one. We also assume that $b \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^r)$, $r \in C_b^\infty(\mathbf{R}^d; \mathbf{R}_+)$ and $\sigma \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^r \otimes \mathbf{R}^r)$ are bounded, and that $\sigma \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^r \otimes \mathbf{R}^r)$ is non-singular. Then, θ is defined as

$$\theta(X_u^\epsilon) = \sigma(X_u^\epsilon)^{-1}[b(X_u^\epsilon) - r(X_u^\epsilon)\mathbf{1}],$$

and $\theta \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^r)$ is bounded.

Let $Y_{t,u}^\epsilon$ be a unique solution of the $d \times d$-matrix valued stochastic differential equation:

$$\begin{cases} dY_{t,u}^\epsilon = \sum_{\alpha=0}^r \partial_x V_\alpha(X_u^\epsilon, \epsilon) Y_{t,u}^\epsilon dw_u^\alpha \\ Y_{t,t}^\epsilon = \underline{I} \end{cases}$$

It is then known that

$$D_t X_u^\epsilon \;=\; Y_{t,u}^\epsilon V(X_t^\epsilon, \epsilon) = Y_{t,u}^\epsilon V(x_t, \epsilon), \; u \geq t.$$

(See for instance pp.109 of Nualart(1995).)

Let $f \in C_b^\infty(\mathbf{R}^d; \mathbf{R})$ and utilizing the fact

$$D_t f(X_u^\epsilon) = \partial f(X_u^\epsilon)[D_t X_u^\epsilon] = \partial f(X_u^\epsilon) Y_{t,u}^\epsilon V(x_t, \epsilon), \; u \geq t$$

we give the representation of the optimal portfolio $\pi(t)$ in our Markovian setting:

$$\pi^*(t)\sigma(x) = W\theta^*(x) - \mathbf{E}\left[H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})\right]\theta^*(x)$$
$$-\mathbf{E}\Bigg[H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})\Bigg(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon$$
$$+ \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du\Bigg)\Bigg]$$

where W is the wealth at time t,

$$\begin{aligned} H_{0,t,T} &\equiv \frac{H_0(T)}{H_0(t)} \\ &= \exp\left(-\int_t^T \theta(X_u^\epsilon)^* dw(u) - \frac{1}{2}\int_t^T |\theta(X_u^\epsilon)|^2 du - \int_t^T r(X_u^\epsilon) du\right), \end{aligned}$$

and the relation between W and $\mathcal{Y}$ is given by the equation:

$$W = \mathbf{E}[H_{0,t,T} I(\mathcal{Y} H_{0,t,T})].$$

X_u^ϵ for $u \in [t,T]$ is generated by the SDE:

$$dX_u^\epsilon = V_0(X_u^\epsilon, \epsilon)du + V(X_u^\epsilon, \epsilon)dw_u,$$

with the initial value $X_t^\epsilon = x$.

Our objective is to evaluate $\pi(t)$ explicitly. In the present article, we will propose a practical and effcient scheme for computing the optimal portfolio by utilizing the asymptotic expansion approach.

4 An Asymptotic Expansion Scheme

4.1 Preparations

First, we will summarize the basic tools for the asymptotic expansion approach. We assume the **deterministic limit condition**:

$[A1]$ $V(\cdot, 0) \equiv 0$.

It follows from $[A1]$ that the limit process $(X_u^0)_{u \in [t,T]}$ is a unique (deterministic) solution of the ordinary differential equation:

$$X_u^0 = x + \int_t^u V_0(X_s^0, 0) ds.$$

We further assume $\sigma(X_u^0)$ is non-singular for all $u \in [t,T]$. Next, put $Y_{t,s} := Y_{t,s}^0$. Clearly, $Y_{t,s}$ is a unique (deterministic) solution of the ordinary differential equation:

$$\begin{aligned} dY_{t,s} &= \partial_x V_0(X_s^0, 0) Y_{t,s} ds \quad s \in [t,T] \\ Y_{t,t} &= \underline{I}. \end{aligned}$$

and $Y_{t,s} \in GL(d, \mathbf{R})$. Next, let $D(t;u) = \frac{\partial X_u^\epsilon}{\partial \epsilon}|_{\epsilon=0}$, $E(t;u) = \frac{\partial^2 X_u^\epsilon}{\partial \epsilon^2}|_{\epsilon=0}$ and $Y_{t,u}^{[1]} = \frac{\partial Y_{t,u}^\epsilon}{\partial \epsilon}|_{\epsilon=0}$. Then $D(t;u)$, $E(t,u)$ and $Y_{t,u}^{[1]}$ $(u \in [t,T])$ are determined by the following stochastic differential equations:

$$\begin{cases} dD(t;u) = \partial_x V_0(X_u^0, 0) D(t;u) du + \sum_{\alpha=0}^r \partial_\epsilon V_\alpha(X_u^0, 0) dw^\alpha \\ D(t;t) = 0, \end{cases}$$

$$\begin{cases} dE(t;u) &= \partial_x V_0(X_u^0,0)E(t;u)du + \partial_x^2 V_0(X_u^0,0)[D(t;u),D(t;u)]du \\ & \quad +2\sum_{\alpha=0}^r \partial_x\partial_\epsilon V_\alpha(X_u^0,0)D(t;u)dw^\alpha \\ & \quad +\sum_{\alpha=0}^r \partial_\epsilon^2 V_\alpha(X_u^0,0)dw^\alpha \\ E(t;t) &= 0 \end{cases}$$

and

$$\begin{cases} dY_{t,s}^{[1]} &= \partial_x V_0(X_s^0,0)Y_{t,s}^{[1]}ds + \partial_x^2 V_0(X_s^0,0)[D(t;s)]Y_{t,s}ds \\ & \quad +\sum_{\alpha=0}^r \partial_\epsilon\partial_x V_\alpha(X_s^0,0)Y_{t,s}dw_s^\alpha \\ Y_{t,t}^{[1]} &= 0. \end{cases}$$

Here we used the fact that $\partial_x V_\alpha(\cdot,0)=0$ for $\alpha=1,...,r$. Moreover, we use a conventions $dw^0=du$, $\partial_x^i=\partial^i/\partial(X_u^\epsilon)^i$, $\partial_\epsilon^i=\partial^i/\partial\epsilon^i$, and notations:

$$\partial_x^2 V_0(X_u^0,0)[D(t;u),D(t;u)] = \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} V_0(X_u^0,0)D^{(i)}(t;u)D^{(j)}(t;u),$$

and

$$\partial_x^2 V_0(X_s^0,0)[D(t;s)]Y_{t,s}ds = \sum_{i,j=1}^d \partial_{x^i}\partial_{x^j} V_0(X_s^0,0)D^{(j)}(t;s)(Y_{t,s})^{(i,\cdot)}ds.$$

where $D^{(i)}(t;s)$ denotes the i-th element of $D(t;s)$ and $(Y_{t,s})^{(i,\cdot)}$ denotes the i-th row of $Y_{t,s}$. We will use the following abbreviations:

$$X_u = X_u^0, \quad Y_u = Y_u^0, \quad V_{\alpha u} = V_{\alpha u}^{[0]} = V_\alpha(X_u,0), \ \alpha = 0,1,\cdots,r.$$

We then have representations of $D(t;u)$, $E(t;u)$ and $Y_{t,u}^{[1]}$ from the above set of stochastic differential equations:

$$\begin{aligned} D(t;u) &= Y_{t,u}\int_t^u Y_{t,s}^{-1}\sum_{\alpha=0}^r \partial_\epsilon V_{\alpha s}dw_s^\alpha \\ E(t;u) &= Y_{t,u}\int_t^u Y_{t,s}^{-1}\{\partial_x^2 V_{0s}[D(t;s),D(t;s)]ds \\ & \quad +2\sum_{\alpha=0}^r \partial_x\partial_\epsilon V_{\alpha s}D(t;s)dw^\alpha + \sum_{\alpha=0}^r \partial_\epsilon^2 V_{\alpha s}dw^\alpha\} \\ Y_{t,u}^{[1]} &= Y_{t,u}\int_t^u (Y_{t,s})^{-1}\left[\partial_x^2 V_{0s}[D(t;s)]Y_{t,s}ds + \sum_{\alpha=0}^r \partial_\epsilon\partial_x V_{\alpha s}Y_{t,s}dw_s^\alpha\right]. \end{aligned}$$

Next, we will illustrate our approach by using an example of a power utility function.

4.2 The Case of a Power Utility Function

We assume a utility function to be so called a power function, that is $U(x) = \frac{x^\delta}{\delta}$, $\delta < 1$, $\delta \neq 0$.

Then, $I(y)$ and $\phi(y)$ are given by $I(y) = y^{\frac{-1}{(1-\delta)}}$, $\phi(y) = y^{\frac{-\delta}{(1-\delta)}}$, and $\phi'(y) = \frac{-\delta}{(1-\delta)} I(y)$.

Hence,

$$
\begin{aligned}
\pi^*(t)\sigma(x) \;=\; & \frac{1}{(1-\delta)} W\theta(x)^* + \frac{\delta}{(1-\delta)}(\mathcal{Y})^{(\frac{-1}{1-\delta})}\mathbf{E}\Bigg[(H_{0,t,T})^{(\frac{-\delta}{1-\delta})} \\
& \left(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)dw^\alpha(u) \right. \\
& \left. + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du\right)\Bigg]
\end{aligned}
$$

where

$$W = (\mathcal{Y})^{(\frac{-1}{1-\delta})}\mathbf{E}\left[(H_{0,t,T})^{(\frac{-\delta}{1-\delta})}\right].$$

Here, we use the abbreviations $r(u) = r(X_u^\epsilon)$ and $\theta_\alpha(u) = \theta_\alpha(X_u^\epsilon)$.

We set

$$
\begin{aligned}
E \;\equiv\; & \frac{\delta}{(1-\delta)}(\mathcal{Y})^{(\frac{-1}{1-\delta})}\mathbf{E}\Bigg[(H_{0,t,T})^{(\frac{-\delta}{1-\delta})} \\
& \left(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)dw^\alpha(u) \right. \\
& \left. + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)du\right)\Bigg].
\end{aligned}
$$

We start with slightly general setting. Define

$$\zeta_{t,u}^\epsilon \;:=\; \exp\left(\int_t^u a_0(X_s^\epsilon)ds + \int_t^u a(X_s^\epsilon)dw_s\right),$$

where $a_0 \in C_\uparrow^\infty(\mathbf{R}^d;\mathbf{R})$ and $a \in C_\uparrow^\infty(\mathbf{R}^d;\mathbf{R}^r)$. Here, $C_\uparrow^\infty(\mathbf{R}^d;\mathbf{R})(C_\uparrow^\infty(\mathbf{R}^d;\mathbf{R}^r))$ denotes a class of smooth functions $f : \mathbf{R}^d \to \mathbf{R}$ ($f : \mathbf{R}^d \to \mathbf{R}^r$) whose derivatives are of polynomial growth orders.

We assume the following integrability condition for $\zeta_{t,T}^\epsilon$.

[$A2$] For any $p \in (1,\infty)$, $\sup_{\epsilon\in(0,1]}\left\|\zeta_{t,T}^\epsilon\right\|_p < \infty$.

Under Condition [$A2$], it is easily seen that $\zeta_{t,T}^\epsilon$ has a stochastic expansion:

$$\zeta_{t,T}^\epsilon \;\sim\; \zeta_{t,T}^0 + \epsilon\zeta_{t,T}^{[1]} + \frac{\epsilon^2}{2}\zeta_{t,T}^{[2]} + \cdots$$

in $\mathbf{D}^\infty$ as $\epsilon \downarrow 0$. The first three coefficients are given by

$$\zeta^0_{t,T} = \exp\left(\int_t^T a_0(X_s)ds + \int_t^T a(X_s)dw_s\right),$$

$$\zeta^{[1]}_{t,T} = \zeta^0_{t,T}\left(\int_t^T \partial_x a_0(X_s)D(t;s)ds + \int_t^T \partial_x a(X_s)D(t;s)dw_s\right)$$

and

$$\begin{aligned}\zeta^{[2]}_{t,T} = \zeta^0_{t,T}\Bigg\{&\left(\int_t^T \partial_x a_0(X_s)D(t;s)ds + \int_t^T \partial_x a(X_s)D(t;s)dw_s\right)^2 \\ &+\int_t^T \partial_x a_0(X_s)E(t;s)ds + \int_t^T \partial_x a(X_s)E(t;s)dw_s \\ &+\int_t^T \partial_x^2 a_0(X_s)[D(t;s),D(t;s)]ds + \int_t^T \partial_x^2 a(X_s)[D(t;s),D(t;s)]dw_s\Bigg\}.\end{aligned}$$

For $f \in C^\infty_\uparrow(\mathbf{R}^d;\mathbf{R})$, put

$$g^{\alpha,\epsilon} = \int_t^T \partial f(X^\epsilon_u)Y^\epsilon_{t,u}V(x_t,\epsilon)dw^\alpha_u, \quad \alpha = 0,1,\cdots,r$$

Since $V(x,0) \equiv 0$ from $[A1]$, we see that

$$g^{\alpha,0} = 0 \quad (\alpha = 0,1,...,r). \tag{1}$$

The first derivative $g^{\alpha,[1]} \equiv \frac{\partial g^{\alpha,\epsilon}}{\partial \epsilon}|_{\epsilon=0}$ of $g^{\alpha,\epsilon}$ is given by

$$g^{\alpha,[1]} = \int_t^T \partial_x f(X_u)[Y_{t,u}\partial_\epsilon V(x_t,0)]dw^\alpha_u. \tag{2}$$

The second derivative $g^{\alpha,[2]} \equiv \frac{\partial^2 g^{\alpha,\epsilon}}{\partial \epsilon^2}|_{\epsilon=0}$ of $g^{\alpha,\epsilon}$ is given by

$$\begin{aligned}g^{\alpha,[2]} = {}& 2\int_t^T \sum_{i,j=1}^d \partial_i\partial_j f(X_u)D^{(j)}(t;u)Y^{(i,\cdot)}_{t,u}\partial_\epsilon V(x_t,0)dw^\alpha_u \\ &+2\int_t^T \sum_{i=1}^d \partial_i f(X_u)Y^{[1],(i,\cdot)}_{t,u}\partial_\epsilon V(x_t,0)dw^\alpha_u \\ &+\int_t^T \sum_{i=1}^d \partial_i f(X_u)Y^{(i,\cdot)}_{t,u}\partial^2_\epsilon V(x_t,0)dw^\alpha_u\end{aligned}$$

After all, from (1) and (2), and by tedius routine work, we obtain the stochastic expansion of $g^{\alpha,\epsilon}$:

$$g^{\alpha,\epsilon} \sim \epsilon g^{\alpha,[1]} + \frac{\epsilon^2}{2}g^{\alpha,[2]} + \cdots$$

in $\mathbf{D}^{\infty}(\mathbf{R}^d)$ as $\epsilon \downarrow 0$.

Utilizing above results, we will derive an asymptotic expansion of the inside of the expectation of E.

First, we directly apply the expression of the expansion for $\zeta^{\epsilon}_{t,u}$ if we set $\zeta^{\epsilon}_{t,u} = (H_{0,t,T})^{(\frac{-\delta}{1-\delta})}$ where $a_0(X^{\epsilon}_s)$ and $a(X^{\epsilon}_s)$ are specified by

$$\begin{aligned}
a_0(X^{\epsilon}_s) &= (\frac{\delta}{1-\delta})r(X^{\epsilon}_s) + \frac{\delta}{2(1-\delta)}|\theta(X^{\epsilon}_s)|^2 \\
a(X^{\epsilon}_s) &= (\frac{\delta}{1-\delta})\theta(X^{\epsilon}_s).
\end{aligned}$$

Here, we note that [A2] is satisfied in this case because of the boundedness assumptions of $r(\cdot)$ and $\theta(\cdot)$.

Next, we show the expansions of

$$\begin{aligned}
g^{\epsilon}_r &\equiv \int_t^T D_t r(u)du = \int_t^T \partial r(u) Y^{\epsilon}_{t,u} V(x,\epsilon)du \\
g^{\alpha,\epsilon}_{\theta} &\equiv \int_t^T D_t \theta_{\alpha}(u)dw^{\alpha}(u) = \int_t^T \partial\theta_{\alpha}(u) Y^{\epsilon}_{t,u} V(x,\epsilon)dw^{\alpha}(u) \\
g^{\alpha,\epsilon}_{\theta^2} &\equiv \int_t^T D_t \theta_{\alpha}(u)\theta_{\alpha}(u)du = \int_t^T \partial\theta_{\alpha}(u) Y^{\epsilon}_{t,u} V(x,\epsilon)\theta_{\alpha}(u)du.
\end{aligned}$$

Replacing $f(\cdot)$ by $r(\cdot)$, $\theta_{\alpha}(\cdot)$, $\frac{1}{2}\theta^2_{\alpha}(\cdot)$ and, utilizing above result, we obtain the expansions of g^{ϵ}_r, $g^{\alpha,\epsilon}_{\theta}$, and $g^{\alpha,\epsilon}_{\theta^2}$:

$$\begin{aligned}
g^{\epsilon}_r &= \epsilon g^{[1]}_r + \frac{\epsilon^2}{2} g^{[2]}_r + o(\epsilon^2) \\
g^{\alpha,\epsilon}_{\theta} &= \epsilon g^{\alpha,[1]}_{\theta} + \frac{\epsilon^2}{2} g^{\alpha,[2]}_{\theta} + o(\epsilon^2) \\
g^{\alpha,\epsilon}_{\theta^2} &= \epsilon g^{\alpha,[1]}_{\theta^2} + \frac{\epsilon^2}{2} g^{\alpha,[2]}_{\theta^2} + o(\epsilon^2)
\end{aligned}$$

where

$$\begin{aligned}
g^{[1]}_r &= \int_t^T \partial r^{[0]}(u) Y_{t,u} \partial_{\epsilon} V(x,0)du \\
g^{\alpha,[1]}_{\theta} &= \int_t^T \partial\theta^{[0]}_{\alpha}(u) Y_{t,u} \partial_{\epsilon} V(x,0)dw^{\alpha}(u) \\
g^{\alpha,[1]}_{\theta^2} &= \int_t^T \theta^{[0]}_{\alpha}(u)\partial\theta^{[0]}_{\alpha}(u) Y_{t,u} \partial_{\epsilon} V(x,0)du
\end{aligned}$$

$$\begin{aligned}
g^{[2]}_r &= 2\left(\int_t^T \partial^2 r^{[0]}(u)[D(t;u)]Y_{t,u}du + \int_t^T \partial r^{[0]}(u) Y^{[1]}_{t,u}du\right)\partial_{\epsilon} V(x,0) \\
&+ \left(\int_t^T \partial r^{[0]}(u) Y_{t,u}du\right)\partial^2_{\epsilon} V(x,0)
\end{aligned}$$

$$
\begin{aligned}
g_{\theta}^{\alpha,[2]} &= 2\left(\int_t^T \partial^2\theta_{\alpha}^{[0]}(u)[D(t;u)]Y_{t,u}dw^{\alpha}(u) + \int_t^T \partial\theta_{\alpha}^{[0]}(u)Y_{t,u}^{[1]}dw^{\alpha}(u)\right) \\
&+ \left(\int_t^T \partial\theta_{\alpha}^{[0]}(u)Y_{t,u}dw^{\alpha}(u)\right)\partial_{\epsilon}^2V(x,0) \\
g_{\theta^2}^{\alpha,[2]} &= 2\left(\int_t^T \theta_{\alpha}^{[0]}(u)\partial^2\theta_{\alpha}^{[0]}(u)[D(t;u)]Y_{t,u}du + \int_t^T \theta_{\alpha}^{[0]}(u)\partial\theta_{\alpha}^{[0]}(u)Y_{t,u}^{[1]}du \right. \\
&+ \left(\int_t^T \theta_{\alpha}^{[0]}(u)\partial\theta_{\alpha}^{[0]}(u)Y_{t,u}du\right)\partial_{\epsilon}^2V(x,0)
\end{aligned}
$$

[The second order scheme(the asymptotic expansion upto the order)]

We will obtain the asymptotic expansion of the optimal portfolio upto th ϵ-order. In the appendix, we will also show the third order scheme.

Based on the previous expansions, we have

$$
\begin{aligned}
&\left(\int_t^T \partial r(X_u^{\epsilon})Y_{t,u}^{\epsilon}V(x,\epsilon)du + \sum_{\alpha=1}^{r}\int_t^T \partial\theta_{\alpha}(X_u^{\epsilon})Y_{t,u}^{\epsilon}V(x,\epsilon)dw^{\alpha}(u)\right. \\
&\left.+\sum_{\alpha=1}^{r}\int_t^T \theta_{\alpha}(X_u^{\epsilon})\partial\theta_{\alpha}(X_u^{\epsilon})Y_{t,u}^{\epsilon}V(x,\epsilon)du\right) \\
&= \epsilon(g_r^{[1]} + \sum_{\alpha=1}^{r} g_{\theta}^{\alpha,[1]} + \sum_{\alpha=1}^{r} g_{\theta^2}^{\alpha,[1]}) + o(\epsilon) \\
&= \epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du + \sum_{\alpha=1}^{r}\int_t^T \partial\theta_{\alpha}^{[0]}(u)Y_{t,u}dw^{\alpha}(u) + \sum_{\alpha=1}^{r}\int_t^T \theta_{\alpha}^{[0]}(u)\right. \\
&+o(\epsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
(H_{0,t,T})^{(\frac{-\delta}{1-\delta})} &= e^{(\frac{\delta}{1-\delta})\int_t^T r^{[0]}(u)du}e^{\frac{\delta}{2(1-\delta)^2}\int_t^T |\theta^{[0]}(u)|^2du} \times \\
&\quad e^{-\frac{1}{2}(\frac{\delta}{1-\delta})^2\int_t^T |\theta^{[0]}(u)|^2du+(\frac{\delta}{1-\delta})\int_t^T \theta^{[0]}(u)dw(u)} \times \\
&\quad \left(1+\epsilon(\frac{\delta}{1-\delta})\int_t^T \partial r^{[0]}(u)D(t;u)du + \epsilon(\frac{\delta}{1-\delta})\sum_{\alpha=1}^{r}\int_t^T \right. \\
&\quad \left.+\epsilon(\frac{\delta}{1-\delta})\sum_{\alpha=1}^{r}\int_t^T \theta_{\alpha}^{[0]}(u)\partial\theta_{\alpha}^{[0]}(u)D(t;u)du\right) + o(\epsilon).
\end{aligned}
$$

Then, E's expansion is obtained by

$$
\begin{aligned}
E &= \frac{\delta}{(1-\delta)}(\mathcal{Y})^{(\frac{-1}{1-\delta})}e^{(\frac{\delta}{1-\delta})\int_t^T r^{[0]}(u)du}e^{\frac{\delta}{2(1-\delta)^2}\int_t^T |\theta^{[0]}(u)|^2du} \times \\
&\quad \epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du + \frac{1}{(1-\delta)}\sum_{\alpha=1}^{r}\int_t^T \theta_{\alpha}^{[0]}(u)\partial\theta_{\alpha}^{[0]}(u)Y_{t,u}du\right)\partial_{\epsilon}V(
\end{aligned}
$$

If we utilize the relation;

$$(\mathcal{Y})^{(\frac{-1}{1-\delta})} = \frac{W}{\mathbf{E}\left[(H_{0,t,T})^{(\frac{-\delta}{1-\delta})}\right]},$$

and the expansion;

$$\mathbf{E}\left[(H_{0,t,T})^{(\frac{-\delta}{1-\delta})}\right] = e^{(\frac{\delta}{1-\delta})\int_t^T r^{[0]}(u)du} e^{\frac{\delta}{2(1-\delta)^2}\int_t^T |\theta^{[0]}(u)|^2 du} \times$$
$$\left(1 + \epsilon(\frac{\delta}{1-\delta})\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)du + \epsilon\frac{\delta}{(1-\delta)^2}\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)du\right) + o(\epsilon)$$

where

$$\hat{D}_1(t;u) \equiv Y_{t,u}\int_t^u Y_{t,s}^{-1}\{\partial_\epsilon V_0^{[0]}(s)ds + (\frac{\delta}{1-\delta})\partial_\epsilon V^{[0]}(s)\theta^{[0]}(s)ds\},$$

E's expression in terms of W is given by

$$\begin{aligned} E &= \frac{\delta}{(1-\delta)}W \times \\ & \epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du + \frac{1}{(1-\delta)}\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right)\partial_\epsilon V(x,0) + o(\epsilon). \end{aligned}$$

Then, we have the following theorem:

Theorem 2 *An asymptotic expansion of the optimal portfolio for investment for a power utility function is given by*

$$\begin{aligned} \pi^*(t) &= \frac{1}{(1-\delta)}W\,[\theta^*(x)+ \\ & \delta\epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du + \frac{1}{(1-\delta)}\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right)\partial_\epsilon V(x,0)\Big]\,\sigma^{-1}(x) + o(\epsilon). \end{aligned}$$

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5 Appendix

In this appendix, we will show the result of the third order scheme of the optimal portofolio for the case of a power utility function.

[The third order scheme(the asymptotic expansion upto the ϵ^2-order)]

$$\pi^*(t) = \frac{W}{(1-\delta)}[\theta^*(x) + \delta C\{\epsilon A + \epsilon^2 B - \epsilon^2 AD\}]\sigma^{-1}(x) + o(\epsilon^2)$$

where

$$A \equiv \left(\int_t^T \partial r^{[0]}(u) Y_{t,u} du + \frac{1}{(1-\delta)} \sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) Y_{t,u} du\right) \partial_\epsilon V(x,0),$$

$$C \equiv \exp\left((\frac{\delta}{1-\delta}) \int_t^T r^{[0]}(u) du + \frac{\delta}{2(1-\delta)^2} \int_t^T |\theta^{[0]}(u)|^2 du\right),$$

$$D \equiv (\frac{\delta}{1-\delta}) \int_t^T \partial r^{[0]}(u) \hat{D}_1(t;u) du + \frac{\delta}{(1-\delta)^2} \sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) \hat{D}_1(t;u) du,$$

and B is the sum of the following terms:

1.

$$(\frac{\delta}{1-\delta}) \left\{\int_t^T \partial r^{[0]}(u) \hat{D}_1(t;u) du\right\} \left\{\int_t^T \partial r^{[0]}(u) Y_{t,u} du\right\} \partial_\epsilon V(x,0)$$

2.

$$(\frac{\delta}{1-\delta})^2 \left\{\int_t^T \partial r^{[0]}(u) \hat{D}_1(t;u) du\right\} \left\{\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) Y_{t,u} du\right\} \partial_\epsilon V(x,0)$$
$$+(\frac{\delta}{1-\delta}) \left\{\sum_{\alpha=1}^{r} \int_t^T \partial\theta_\alpha^{[0]}(u) Y_{t,u} \left(\int_u^T \partial r^{[0]}(s) Y_{t,s} ds\right) Y_{t,u}^{-1} \partial_\epsilon V_u^{[0],(\cdot,\alpha)} du\right\} \partial_\epsilon V(x,0)$$

3.

$$(\frac{\delta}{1-\delta}) \left\{\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) Y_{t,u} du\right\} \partial_\epsilon V(x,0) \left\{\int_t^T \partial r^{[0]}(u) \hat{D}_1(t;u) du\right\}$$

4.

$$(\frac{\delta}{1-\delta}) \left\{\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) \hat{D}_1(t;u) du\right\} \left\{\int_t^T \partial r^{[0]}(u) Y_{t,u} du\right\}$$

5.

$$(\frac{\delta}{1-\delta})^2 \left\{\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) \hat{D}_1(t;u) du\right\} \left\{\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial\theta_\alpha^{[0]}(u) Y_{t,u} du\right\}$$
$$(\frac{\delta}{1-\delta}) \sum_{\alpha=1}^{r} \left\{\int_t^T \left(\sum_{\alpha'=1}^{r} \int_u^T \theta_{\alpha'}^{[0]}(s) \partial\theta_{\alpha'}^{[0]}(s) ds\right) Y_{t,u}^{-1} \partial_\epsilon V_u^{[0],(\cdot,\alpha)} \partial\theta_\alpha^{[0]}(u) Y_{t,u} du\right\} \partial_\epsilon V$$

$$(\frac{\delta}{1-\delta})\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right\}\partial_\epsilon V(x,0)\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)du\right\}$$

7.

$$(\frac{\delta}{1-\delta})^2\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)du\right\}\left\{\int_t^T \partial r^{[0]}(u)Y_{t,u}du\right\}\partial_\epsilon V(x,0)$$

8.

$$(\frac{\delta}{1-\delta})^2\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)du\right\}\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right\}\partial_\epsilon V(x,0)$$

9.

$$\begin{aligned}
&(\frac{\delta}{1-\delta})^3\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)du\right\}\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right\}\partial_\epsilon V(x,0)\\
&+(\frac{\delta}{1-\delta})\left\{\sum_{\alpha=1}^{r}\int_t^T \partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right\}\partial_\epsilon V(x,0)\\
&+(\frac{\delta}{1-\delta})^2\sum_{\alpha=1}^{r}\int_t^T\left(\sum_{\alpha'=1}^{r}\int_u^T \theta_{\alpha'}^{[0]}(s)\partial\theta_{\alpha'}^{[0]}(s)ds\right)Y_{t,u}^{-1}\partial_\epsilon V_u^{[0],(\cdot,\alpha)}du
\end{aligned}$$

10.

$$\begin{aligned}
&(\frac{1}{1-\delta})\left(\int_t^T \theta_\alpha^{[0]}(u)\partial^2\theta_\alpha^{[0]}(u)[\hat{D}_1(t;u)]Y_{t,u}du+\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)[\hat{Y}_{t,u}^{[1]}]du\right)\partial_\epsilon V(x,0)\\
&+(\frac{1}{1-\delta})\left(\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}du\right)\partial_\epsilon^2 V(x,0)
\end{aligned}$$

where

$$\hat{Y}_{t,u}^{[1]}\equiv\int_t^u Y_{t,u}Y_{t,s}^{-1}\{\partial_x^2V_{0s}[\hat{D}_1(t;s)]Y_{t,s}ds+\partial_\epsilon\partial_xV_{0s}Y_{t,s}ds+(\frac{\delta}{1-\delta})\sum_{\alpha=1}^{r}\theta_\alpha(s)\partial_\epsilon\partial_xV_{\alpha s}Y_{t,s}ds\}$$

11.

$$\begin{aligned}
&\left(\int_t^T \partial^2 r^{[0]}(u)[\hat{D}_1(t;u)]Y_{t,u}du+\int_t^T \partial r^{[0]}(u)\hat{Y}_{t,u}^{[1]}du\right)\partial_\epsilon V(x,0)\\
&+\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du\right)\partial_\epsilon^2 V(x,0)
\end{aligned}$$