A Remark on a Singular Perturbation Method for Option Pricing under a Stochastic Volatility Model

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Abstract

This paper studies the accuracy of a singular perturbation method for option pricing under a stochastic volatility model (Fouque, Papanicolaou and Sircar (2000c)). First, through numerical experiments we confirm that the first order approximation provides sufficiently accurate option prices in a fast mean-reversion case of the volatility process while it does not in a non-fast mean-reversion case. Then, we derive the second order approximation formula and examine the improvement of the approximation.

Keywords: option pricing, stochastic volatility, partial differential equation, singular perturbation, approximation accuracy

1 Introduction

In this paper, we examine the accuracy of a singular perturbation method for option pricing in a stochastic volatility environment. Singular perturbation is a powerful technique to obtain the approximate solution of a differential equation when the true analytical solution is difficult to find. Recently, this methodology has been applied to option pricing under a stochastic volatility model. (See, for example, Fouque, Papanicolaou and Sircar (2000c).)

In the framework of Fouque *et al* (2000c), the partial differential equation (PDE) derived from the stochastic differential equation (SDE), which describes the stochastic process of the underlying asset, is asymptotically expanded around the invariant distribution of volatility process. The theoretical justification of the approximation is argued in Fouque, Papanicolaou, Sircar and Solna (2003a). (Mathematical background of the relationships between diffusion process and PDE, and between invariant distribution and ergodic property are described in Varadhan (1981). Mathematical theory for asymptotic analysis of stochastic equation can be found in Papanicolaou (1978).) Therefore, when the mean-reverting of volatility is fast or the time to maturity of option is long, the option price approximation is valid.

Fouque *et al* (2000c) showed, by using high-frequency data, the empirical fact that S&P 500 has fast mean-reverting volatility. The mean-reverting speed parameter $1/\epsilon$ in equation (3) is considered to be about 200. They judged the speed level is so high that the first order stochastic volatility correction works. Then, the research group also applied the method to many options other than plain vanilla European options. Fouque, Papanicolaou and Sircar (2001) is its application to American options. Ilhan, Jonsson and Sircar (2004) calculated barrier, lookback, and passport option prices with the arguments of boundary problem. Fouque and Han (2003) and (2005) valuated Asian and compound options respectively. Cotton, Fouque, Papanicolaou and Sircar (2004) is an application to interest rate derivatives. As for other application, Yamamoto, Sato, Takahashi (2008) studied the probability distribution and pricing options for drawdown. However, there are also other empirical results for volatility process. For example, Boswijk (2002) estimated the parameter $1/\epsilon = 5.074$ using the daily data of Amsterdam stock exchange (AEX) index. Fouque, Papanicolaou, Sircar and Solna (2003c) found fast and slow varying factors, and proposed option pricing methods for this case by the combination of regular and singular perturbations approach.

This paper studies the approximation accuracy of the singular perturbation method through numerical experiments for the cases of fast and non-fast mean-reverting mean-reverting volatility. First, we calculate 1 month, 3 month, and 6 month at-the-money (ATM) and two different depths of out-of-the-money (OTM) European call option prices by Black-Scholes and the first order stochastic volatility correction by the singular perturbation method, and compare them with the estimates by Monte Carlo simulation. For the case of fast mean-reverting volatility, the first order correction improves the price accuracy from Black-Scholes prices except for one exception. The errors of the first order approximations are small for relatively long maturities and near ATM options. As for non-fast mean-reverting volatility cases, we cannot conclude that the first order correction works.

In response to the result, we present the second order correction term. Since it has been considered that the first order correction works sufficiently, the second order correction term has not been derived so far. We examine, by numerical experiments, whether it works or not. For the case of fast mean-reverting volatility, it improves the accuracy from the first order correction for 1 month or deeper OTM options, for which the errors of the first order approximations are relatively large. For non-fast mean reverting volatility cases, the second order correction term succeed in improving the accuracy for 6 month or ATM options.

The organization of the paper is as follows. The next section describes the economy, in which the option pricing will be discussed. In section 3, we review the general framework of the singular perturbation method for option pricing. Section 4 derives the Black-Scholes price term and the first order stochastic volatility correction term. Then, the approximation accuracy is examined through numerical experiments. Section 5 presents the second order correction formula, and then its approximation improvements are verified. Section 6 concludes.

2 Economy

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \le t \le T < \infty})$ be a complete probability space with a filtration satisfying the usual conditions. There are a risk-free asset with a constant risk-free rate r, and a risky asset. In $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, the risky asset price $\{X_t\}$ follows the stochastic differential equation (SDE)

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t^1, \qquad X_0 = x_0, \tag{1}$$

where $\{W_t^1\}$ is a standard Brownian motion, and μ is a constant. The volatility σ_t is the stochastic process expressed as follows.

$$\sigma_t = f(Y_t), \tag{2}$$

$$dY_t = \frac{1}{\epsilon}(\theta - Y_t)dt + \nu\sqrt{\frac{2}{\epsilon}}\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \qquad Y_0 = y_0, \tag{3}$$

where f is some positive function, and $\{W_t^2\}$ is a standard Brownian motion that is independent of $\{W_t^1\}$. We give explanations about the parameters shortly. ϵ and ν are positive constants. $1/\epsilon$ represents the speed of mean-reversion of $\{Y_t\}$. As shown in Fouque *et al* (2000c), $\{Y_t\}$ has the normal invariant distribution $N(\theta, \nu^2)$, which represents Gaussian distribution with mean θ and variance ν^2. Finally, ρ is a constant that expresses the instantaneous correlation between $\{X_t\}$ and $\{Y_t\}$.

Since we calculate the approximate price of options on the risky asset, a risk-neutral measure is required. Since the market is incomplete, there is more than one equivalent martingale measure $\mathbb{P}^{*(\gamma)}$; the non uniqueness is denoted by the dependence γ. We assume that γ_t is a bounded function of Y_t: $\gamma_t = \gamma(Y_t)$. By Maruyama-Girsanov's theorem, when we define

$$W_t^{1*(\gamma)} = W_t^1 + \int_0^t \frac{\mu - r}{f(Y_s)}ds \quad \text{and} \quad W_t^{2*(\gamma)} = W_t^2 + \int_0^t \gamma(Y_s)ds,$$

$(W_t^{1*(\gamma)}, W_t^{2*(\gamma)})$ are independent Brownian motions under $P^{*(\gamma)}$ defined by

$$\frac{d\mathbb{P}^{*(\gamma)}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{\mu - r}{f(Y_s)}dW_s^1 - \int_0^T \gamma(Y_s)dW_s^2 - \frac{1}{2}\left[\left(\frac{\mu - r}{f(Y_s)}\right)^2 + \gamma^2(Y_s)\right]ds\right).$$

X follows the SDE

$$\begin{cases} dX_t = rX_t dt + f(Y_t)X_t dW_t^{1*(\gamma)}, \\ dY_t = \left[\frac{1}{\epsilon}(\theta - Y_t) - \nu\sqrt{\frac{2}{\epsilon}}\Lambda(Y_t)\right]dt + \nu\sqrt{\frac{2}{\epsilon}}\left(\rho dW_t^{1*(\gamma)} + \sqrt{1-\rho^2}dW_t^{2*(\gamma)}\right), \end{cases}$$

where $\Lambda(Y_t) = \rho\frac{\mu - r}{f(Y_t)} + \sqrt{1-\rho^2}\gamma(Y_t)$, which represents the market price of volatility risk.

3 General Framework

This section describes the general framework of singular perturbation method for option pricing. Further details are argued in Fouque *et al* (2000c) . The economical setting is presented in section 2. We consider the pricing of a derivative product of the underlying asset X with the expiry date T. Let $P(t, x, y)$ represent the price of the product as a function of time, underlying asset price and volatility state. $P(t, x, y)$ is equal to the conditional expectation of the discounted payoff of the product under a risk-neutral measure. By Feynman-Kac's theorem, P satisfies the following PDE;

$$\mathscr{L}^\epsilon P = 0 \quad \text{in} \ \ (0, T) \times O \times \mathbb{R} \tag{4}$$

where O is an open interval in $(0,\infty)$ and

$$\mathscr{L}^\epsilon = \frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2,$$

$$\begin{cases} \mathscr{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (\theta - y)\frac{\partial}{\partial y}, \\ \mathscr{L}_1 = \sqrt{2}\nu\rho f(y) x \frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \\ \mathscr{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot). \end{cases}$$

For the case of the plain vanilla European call option with strike price K, O is $(0,\infty)$, and we get the price by solving the PDE (4) with the terminal condition $(x-K)_+$. When a knock-out provision is assigned, O is the interval of the underlying asset price in which the derivative contract is valid. The price of the product is calculated by solving PDE with the boundary condition and the terminal condition. We assume that $P(t,x,y)$ have an asymptotic expansion

$$P = P^0 + \sqrt{\epsilon}P^1 + \epsilon P^2 + \epsilon\sqrt{\epsilon}P^3 + \cdots. \tag{5}$$

Singular perturbation method inserts this formal expansion into (4). Then, it derives the PDE that each coefficient of $\sqrt{\epsilon}$ power satisfies, and solves the PDEs one after another.

4 First Order Stochastic Volatility Correction

This section calculates the approximate option prices up to first order stochastic volatility correction. First, the Black-Sholes price is calculated, and next the first order correction term is obtained. Then, the approximation accuracy is evaluated thorough numerical experiments.

4.1 Black-Scholes price term

First, we calculate P^0 that appeared in (5). Inserting the formal expansion (5) into (4) and comparing the coefficients of ϵ^{-1} gives $\mathscr{L}_0 P^0 = 0$. $\mathscr{L}_0$ is the generator of an ergodic Markov process and acts only on y. Therefore, P^0 must be a constant with respect to y, which implies that we can write

$$P^0 = P^0(t,x).$$

Similarly, comparing the terms of order $\epsilon^{-1/2}$, we conclude that P^1 also does not depend on y.

Comparing the constant (with respect to ϵ) terms gives

$$\mathscr{L}_0 P^2 + \mathscr{L}_2 P^0 = 0, \tag{6}$$

which is a Poisson equation for P^2 with respect to the operator $\mathscr{L}_0$ in the variable y. The necessary condition for (6) to admit a solution is

$$\langle \mathscr{L}_2 P^0 \rangle = \langle \mathscr{L}_2 \rangle P^0 = 0, \tag{7}$$

which is referred to as centering condition in Fouque *et al* (2000c) . $\langle\cdot\rangle$ represents the expectation with respect to the invariant measure of Y, $N(\theta,\nu^2)$. Since P^0 does not depend on y, P^0 gets outside the bracket in the first equality. $\langle\mathscr{L}_2\rangle$ is represented as

$$\langle\mathscr{L}_2\rangle = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot),$$

where $\bar{\sigma}^2 = \langle f^2\rangle$. Therefore, P^0 is equal to the price of the product under the Black-Scholes economy with volatility $\bar{\sigma}$, whose square is equal to the expected instantaneous variance of X under the invariant measure of Y.

4.2 Derivation of the first order term

Next, we proceed to the calculation for the first order stochastic volatility correction term. As centering condition (7) is satisfied, we can write

$$\mathscr{L}_2 P^0 = \mathscr{L}_2 P^0 - \langle\mathscr{L}_2\rangle P^0 = \frac{1}{2}(f(y)^2 - \bar{\sigma}^2)x^2\frac{\partial^2 P^0}{\partial x^2}.$$

Then, from (6),

$$\mathscr{L}_0 P^2 = -\frac{1}{2}(f(y)^2 - \bar{\sigma}^2)x^2\frac{\partial^2 P^0}{\partial x^2}.$$

Let $\phi(y)$ is a solution of the Poisson equation

$$\mathscr{L}_0\phi = (f(y)^2 - \bar{\sigma}^2),$$

P^2 is given by

$$P^2(t,x,y) = -\frac{1}{2}\phi(y)x^2\frac{\partial^2 P^0}{\partial x^2} + c(t,x), \tag{8}$$

where $c(t,x)$ is a function of (t,x) that does not depend on y. We impose the condition $\phi(\theta) = 0$, then

$$\phi(y) = \int_\theta^y \frac{1}{\nu^2\Phi(u)}\int_{-\infty}^u (f(z)^2 - \bar{\sigma}^2)\Phi(z)dzdu,$$

$$\phi'(y) = \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^y (f(z)^2 - \bar{\sigma}^2)\Phi(z)dz, \tag{9}$$

, where $\Phi(y)$ is the probability density function of $N(\theta,\nu^2)$.

Comparing the coefficients of $\epsilon^{1/2}$,

$$\mathscr{L}_0 P^3 + \mathscr{L}_1 P^2 + \mathscr{L}_2 P^1 = 0, \tag{10}$$

which is again a Poisson equation for P^3 with respect to $\mathscr{L}_0$. The centering condition is

$$\langle\mathscr{L}_1 P^2 + \mathscr{L}_2 P^1\rangle = 0. \tag{11}$$

Since P^1 does not depend on y,

$$\langle\mathscr{L}_2\rangle P^1 = -\langle\mathscr{L}_1 P^2\rangle.$$

Inserting (8),

$$\langle \mathscr{L}_2 \rangle P^1 = \langle \mathscr{L}_1 \phi \rangle \frac{x^2}{2} \frac{\partial^2 P^0}{\partial x^2}.$$

Since

$$\langle \mathscr{L}_1 \phi \rangle = \sqrt{2}\rho\nu \langle f\phi' \rangle x \frac{\partial}{\partial x} - \sqrt{2}\nu \langle \Lambda\phi' \rangle,$$

P^1 satisfies

$$\langle \mathscr{L}_2 \rangle P^1 = H_1(t,x), \tag{12}$$

where

$$H_1(t,x) = V_{12} x^2 \frac{\partial^2 P^0}{\partial x^2} + V_{13} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P^0}{\partial x^2} \right), \tag{13}$$

$$V_{12} = -\frac{\nu}{\sqrt{2}} \langle \Lambda\phi' \rangle, \ V_{13} = \frac{\rho\nu}{\sqrt{2}} \langle f\phi' \rangle.$$

We get P^1 by solving the PDE (12) with terminal condition and boundary condition with respect to the product.

For plain vanilla European call option with strike K, the terminal condition is $P^1(T,x) = 0$ and boundary condition is not assigned, because the terminal condition for P is $(x-K)_+$, which does not depend ϵ and $O = (0,\infty)$. Then, Fouque *et al* (2000c) showed that the solution P^1 is given by

$$P^1(t,x) = -(T-t)H_1(t,x),$$

because $\langle \mathscr{L}_2 \rangle \left(x^m \frac{\partial^m P^0}{\partial x^m} \right) = 0$ for any $m \in \mathbb{N}$. However, this solution does not satisfy the terminal condition $P^1(T,x) = 0$ at $x = K$. $H_1(t,x)$ contains second and third order derivatives of $P_0(t,x)$, which are

$$\frac{\partial^2 P_0(T,x)}{\partial x^2} = \delta(x-K),$$

and $\frac{\partial^3 P_0(T,x)}{\partial x^3}$ is its derivative with respect to x. Fouque *et al* (2003a) justified the approximation theoretically, although there exists the inconsistency at $x = K$.

4.3 The accuracy of the price approximation up to the first order stochastic volatility correction

This subsection examines the accuracy of the price approximation up to the first order correction through some numerical examples. The approximate plain vanilla European call option prices are compared with their estimates by Monte Carlo simulations.

The functional forms and the parameters used in these examples are as follows. First, we assign

$$f(y) = \begin{cases} e^y & (y < 0) \\ 2 - e^{-y} & (y \geq 0) \end{cases} \tag{14}$$

and $\gamma(y)$ to be a constant γ. Second, we set the parameters. As stated in Fouque *et al* (2000c), the approximation method described in this article is justified under fast mean-reverting stochastic volatility circumstances. Concretely, it considers the speed of mean-reverting volatility is about $1/\epsilon = 200$. As previously mentioned, Boswijk (2002) estimated the parameter $1/\epsilon = 5.074$ using the daily data of Amsterdam stock exchange (AEX) index. Therefore, we examine the two cases: (i) fast mean-reverting volatility and (ii) non-fast mean-reverting volatility. In both cases, we set $r = 0.02$, $\mu = 0.1$, $\rho = -0.2$, and $\gamma = 0$. The volatility-related parameters for (i) fast mean-reverting volatility case are $1/\epsilon = 200$, $\bar{\sigma} = 0.07$, and $\nu = 0.26$, which are based on Fouque *et al* (2000c). Those for (ii) non-fast mean-reverting volatility case are $1/\epsilon = 5.07$, $\bar{\sigma} = 0.21$, and $\nu = 0.40$, which are based on Boswijk (2002).

In order to obtain the estimate value of the options for the two cases, we conduct Monte Carlo simulations with antithetic variables method. The number of the simulation is 1,000,000. For the case of (i) fast mean-reverting volatility, the volatility of Y is very high; $\nu\sqrt{\frac{2}{\epsilon}} = 5.2$. In order to converge the simulations of Y, we need to have the time step be very small. First, we confirm the convergence of the simulations of Y for the case of the market price of volatility risk to be zero and $f(y) = e^y$. In this case, we know the distribution of Y at the terminal date analytically. In order to match the distribution of simulations and analytic one, we need to take $\Delta t = 1/100,000$. Therefore, we use the time step for this cases. For the case of (ii) non-fast mean-reverting volatility, we choose $\Delta t = 1/20,000$ for the same reason.

We calculate the option values at time t for the case of $X_t = 100$ and $T - t = 1/12, 1/4, 1/2$. The strike prices are set to at-the-money (ATM), out-of-the-money (OTM) at $1\bar{\sigma}(T-t)$, and OTM at $2\bar{\sigma}(T-t)$. Hereinafter, they are referred to as $1\bar{\sigma}$ OTM and $2\bar{\sigma}$ OTM, respectively.

Table 1 shows the results of numerical experiments. The prices calculated by Black-Scholes formula, the first order approximation, and Monte Carlo simulations are reported. In addition, we exhibit difference rates, which are given by $(Analytic\ value - Monte\ Carlo)/(Monte\ Carlo)$.

At a glance, we can see that the difference rates for the case of (ii) are much higher than those for the case of (i) generally. In the case of (ii), the difference rates of Black-Scholes prices are very high, and we cannot find the approximation method to reduce them. Therefore, the first order approximation does not work for the case of non-fast mean-reverting volatility. As for the case of fast mean-reverting volatility, the approximation method reduces difference rates except for 1 month ATM option. Reading down the columns of the table, the difference rates of the first order approximation prices increase in depth of OTM for a given time to maturity. The only exception is 6 month 1 $\bar{\sigma}$ OTM option. Reading across the rows of the table, the first order approximation prices decrease in time to maturity. The only exception is 3 month 1 $\bar{\sigma}$ OTM option.

In summary, the first order stochastic volatility correction term improves the approximation for the case of fast mean-reverting volatility, while it does not work for non-fast mean-reverting volatility case. For the case of fast mean-reverting volatility, the error of the first order approximation is small for relatively long maturities and near ATM options. However,the first order stochastic volatility correction term.

5 Second Order Stochastic Volatility Correction

The previous subsection confirmed that the first order approximation does not work for the non-fast mean-reverting volatility case. This section derives the second order stochastic volatility correction correction. Then, the improvement of accuracy is examined through numerical experiments.

5.1 Derivation of the second order term

We calculate the second order stochastic volatility correction term P^2. Since it is given by (8), we will calculate $c(t,x)$. From (11),

$$\begin{aligned}
\mathscr{L}_1 P^2 + \mathscr{L}_2 P^1 &= \mathscr{L}_1 P^2 + \mathscr{L}_2 P^1 - \langle \mathscr{L}_1 P^2 + \mathscr{L}_2 P^1 \rangle \\
&= \sqrt{2}\rho\nu x \left(f(y) \frac{\partial^2 P_2}{\partial x \partial y} - \left\langle f \frac{\partial^2 P_2}{\partial x \partial y} \right\rangle \right) \\
&\quad -\sqrt{2}\nu \left(\Lambda(y) \frac{\partial P_2}{\partial y} - \left\langle \Lambda \frac{\partial P_2}{\partial y} \right\rangle \right) \\
&\quad + \frac{1}{2}(f(y)^2 - \langle f^2 \rangle) x^2 \frac{\partial^2 P_1}{\partial x^2} \\
&= (f(y)\phi'(y) - \langle f\phi' \rangle) q_1^x(t,x) \\
&\quad + (\Lambda(y)\phi'(y) - \langle \Lambda\phi' \rangle) q_2^x(t,x) \\
&\quad + (f(y)^2 - \langle f^2 \rangle) q_3^x(t,x)
\end{aligned}$$

, where

$$\begin{cases}
q_1^x(t,x) = -\frac{\rho\nu}{\sqrt{2}} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right), \\
q_2^x(t,x) = \frac{\nu}{\sqrt{2}} x^2 \frac{\partial^2 P_0}{\partial x^2}, \\
q_3^x(t,x) = \frac{1}{2} x^2 \frac{\partial^2 P_1}{\partial x^2}.
\end{cases}$$

Inserting to (10),

$$\begin{aligned}
\mathscr{L}_0 P^3 &= -(f(y)\phi'(y) - \langle f\phi' \rangle) q_1^x(t,x) \\
&\quad -(\Lambda(y)\phi'(y) - \langle \Lambda\phi' \rangle) q_2^x(t,x) \\
&\quad -(f(y)^2 - \langle f^2 \rangle) q_3^x(t,x).
\end{aligned}$$

Therefore,

$$\frac{\partial P^3}{\partial y} = -q_1^x(t,x) q_1^y(y) - q_2^x(t,x) q_2^y(y) - q_3^x(t,x) q_3^y(y),$$

where

$$
\begin{aligned}
q_1^y(y) &= \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^{y}(f(z)\phi'(z)-\langle f\phi'\rangle)\Phi(z)dz,\\
q_2^y(y) &= \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^{y}(\Lambda(z)\phi'(z)-\langle \Lambda\phi'\rangle)\Phi(z)dz,\\
q_3^y(y) &= \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^{y}(f(z)^2-\langle f^2\rangle)\Phi(z)dz.
\end{aligned}
$$

Comparing the coefficients of ϵ,

$$
\mathscr{L}_0P^4+\mathscr{L}_1P^3+\mathscr{L}_2P^2=0, \tag{15}
$$

The centering condition is

$$
\langle\mathscr{L}_1P^3+\mathscr{L}_2P^2\rangle=0. \tag{16}
$$

$$
\begin{aligned}
\langle\mathscr{L}_1P^3\rangle &= \sqrt{2}\rho\nu x\Big\langle f\frac{\partial^2P^3}{\partial x\partial y}\Big\rangle-\sqrt{2}\nu\Big\langle\Lambda\frac{\partial P^3}{\partial y}\Big\rangle\\
&= -\sqrt{2}\rho\nu x\sum_{i=1}^{3}\langle fq_i^y\rangle q_{ix}^x(t,x)+\sqrt{2}\nu\sum_{i=1}^{3}\langle\Lambda q_i^y\rangle q_i^x(t,x)
\end{aligned}
$$

$$
\begin{aligned}
\langle\mathscr{L}_2P^2\rangle &= \Big\langle\mathscr{L}_2\Big(-\frac{1}{2}\phi(y)x^2\frac{\partial^2P^0}{\partial x^2}+c(t,x)\Big)\Big\rangle\\
&= -\frac{1}{2}\langle\tilde{\mathscr{L}_2}\rangle\left(x^2\frac{\partial^2P^0}{\partial x^2}\right)+\langle\mathscr{L}_2\rangle c(t,x),
\end{aligned}
$$

where

$$
\langle\tilde{\mathscr{L}_2}\rangle=\langle\phi\rangle\frac{\partial}{\partial t}+\frac{\langle f^2\phi\rangle}{2}x^2\frac{\partial^2}{\partial x^2}+\langle\phi\rangle r(x\frac{\partial}{\partial x}-\cdot).
$$

Therefore,

$$
\langle\mathscr{L}_2\rangle c(t,x)=H_2(t,x), \tag{17}
$$

where

$$
H_2(t,x)=\frac{1}{2}\langle\tilde{\mathscr{L}_2}\rangle\left(x^2\frac{\partial^2P^0}{\partial x^2}\right)+\sqrt{2}\rho\nu x\sum_{i=1}^{3}\langle fq_i^y\rangle q_{ix}^x(t,x)-\sqrt{2}\nu\sum_{i=1}^{3}\langle\Lambda q_i^y\rangle q_i^x(t,x). \tag{18}
$$

We get $c(t,x)$ by solving the PDE (17) with terminal condition and boundary condition with respect to the product.

For plain vanilla European call option with strike K, the terminal condition is

$$
c(T,x)=\begin{cases}0 & x\neq K,\\ \frac{1}{2}\phi(y)x^2\delta(x-K) & x=K,\end{cases}
$$

by (8) and $P^2(T, x, y) = 0$. By neglecting the terminal condition at $x = K$ like the calculation for $P^1(t, x)$, we get

$$c(t, x) = -(T - t)H_2(t, x).$$

Therefore, we obtain

$$P^2(t, x, y) = -\frac{1}{2}\phi(y)x^2\frac{\partial^2 P^0}{\partial x^2} - (T - t)H_2(t, x).$$

5.2 The accuracy of the price approximation up to the second order stochastic volatility correction

This subsection examines the improvement of the approximation due to the second order term. The numerical experiments are conducted under the same condition with subsection 4.3. The result is shown Table 2. Firstly, we mention about (i) fast mean-reverting volatility case. The second order term improve the 1 month or 2 $\bar{\sigma}$ OTM prices, for which the errors of first order approximations are relatively large. For longer maturities and nearer ATM options, the differences between the first and second order approximations are slight. Next, as for (ii) non-fast mean-reverting volatility case, the second order approximation improves the accuracy for ATM or 6 month options.

6 Conclusion

This paper studied the accuracy of a singular perturbation method for option pricing under a stochastic volatility model (Fouque *et al* (2000c)) through numerical experiments. The first order approximation provided sufficiently accurate option prices in a fast mean-reversion case of the volatility process, while it did not work in a non-fast mean-reversion case. For the case of fast mean-reverting volatility, the errors of the first order approximations were small for relatively long maturities and near ATM options among 1 month, 3 month, and 6 month ATM and two different depths of OTM European call options. Then, we derived the second order approximation formula and examined the improvement of the approximation. For the case of fast mean-reverting volatility, it improved the accuracy from the first order correction for 1 month or deeper OTM options, for which the errors of first order approximations were relatively large. For non-fast mean reverting volatility cases, the second order correction term succeeded in improving the accuracy for 6 month or ATM options.

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Table 1: First order approximation accuracy

(i) Fast mean-reverting volatility case

	1 month	Diff. rate	3 month	Diff. rate	6 month	Diff. rate
ATM						
Monte Carlo	0.8923		1.6654		2.5167	
1st order app.	0.8975	0.58 %	1.6666	0.07 %	2.5165	-0.01 %
Black-Scholes	0.8915	-0.09 %	1.6563	-0.55 %	2.5021	-0.58 %
$1\bar{\sigma}$ OTM						
Monte Carlo	0.1954		0.3948		0.6356	
1st order app.	0.1961	0.35 %	0.3945	-0.07 %	0.6348	-0.13 %
Black-Scholes	0.2020	3.37 %	0.3976	0.71 %	0.6347	-0.15 %
$2\bar{\sigma}$ OTM						
Monte Carlo	0.0215		0.0502		0.0907	
1st order app.	0.0200	-6.99 %	0.0490	-2.46 %	0.0899	-0.92 %
Black-Scholes	0.0241	12.04 %	0.0528	5.23 %	0.0932	2.80 %

(ii) Non-fast mean-reverting volatility case

	1 month	Diff. rate	3 month	Diff. rate	6 month	Diff. rate
ATM						
Monte Carlo	2.1927		4.0426		6.0127	
1st order app.	2.5009	14.06 %	4.4317	9.63 %	6.4015	6.47 %
Black-Scholes	2.4520	11.83 %	4.3474	7.54 %	6.2828	4.49 %
$1\bar{\sigma}$ OTM						
Monte Carlo	0.3828		0.8305		1.3930	
1st order app.	0.3151	-17.68 %	0.8411	1.28 %	1.4421	3.53 %
Black-Scholes	0.5662	47.91 %	1.0753	29.48 %	1.6565	18.92 %
$2\bar{\sigma}$ OTM						
Monte Carlo	0.0366		0.1249		0.2537	
1st order app.	-0.0705	-292.58 %	0.0087	-93.00 %	0.1315	-48.18 %
Black-Scholes	0.0766	109.25 %	0.1732	38.71 %	0.3102	22.26 %

Table 2: Second order approximation accuracy

(i) Fast mean-reverting volatility case

	1 month	Diff. rate	3 month	Diff. rate	6 month	Diff. rate
ATM						
Monte Carlo	0.8923		1.6654		2.5167	
2nd order app.	0.8963	0.45 %	1.6659	0.03 %	2.5159	-0.03 %
1st order app.	0.8975	0.58 %	1.6666	0.07 %	2.5165	-0.01 %
Black-Scholes	0.8915	-0.09 %	1.6563	-0.55 %	2.5021	-0.58 %
$1\bar{\sigma}$ OTM						
Monte Carlo	0.1954		0.3948		0.6356	
2nd order app.	0.1957	0.17 %	0.3943	-0.12 %	0.6346	-0.15 %
1st order app.	0.1961	0.35 %	0.3945	-0.07 %	0.6348	-0.13 %
Black-Scholes	0.2020	3.37 %	0.3976	0.71 %	0.6347	-0.15 %
$2\bar{\sigma}$ OTM						
Monte Carlo	0.0215		0.0502		0.0907	
2nd order app.	0.0205	-4.61 %	0.0492	-1.93 %	0.0900	-0.73 %
1st order app.	0.0200	-6.99 %	0.0490	-2.46 %	0.0899	-0.92 %
Black-Scholes	0.0241	12.04 %	0.0528	5.23 %	0.0932	2.80 %

(ii) Non-fast mean-reverting volatility case

	1 month	Diff. rate	3 month	Diff. rate	6 month	Diff. rate
ATM						
Monte Carlo	2.1927		4.0426		6.0127	
2nd order app.	1.9825	-9.59 %	4.1309	2.18 %	6.1871	2.90 %
1st order app.	2.5009	14.06 %	4.4317	9.63 %	6.4015	6.47 %
Black-Scholes	2.4520	11.83 %	4.3474	7.54 %	6.2828	4.49 %
$1\bar{\sigma}$ OTM						
Monte Carlo	0.3828		0.8305		1.3930	
2nd order app.	0.1874	-51.04 %	0.7559	-8.98 %	1.3739	-1.37 %
1st order app.	0.3151	-17.68 %	0.8411	1.28 %	1.4421	3.53 %
Black-Scholes	0.5662	47.91 %	1.0753	29.48 %	1.6565	18.92 %
$2\bar{\sigma}$ OTM						
Monte Carlo	0.0366		0.1249		0.2537	
2nd order app.	0.1875	412.30 %	0.1547	23.83 %	0.2304	-9.17 %
1st order app.	-0.0705	-292.58 %	0.0087	-93.00 %	0.1315	-48.18 %
Black-Scholes	0.0766	109.25 %	0.1732	38.71 %	0.3102	22.26 %