# ON AN ASYMPTOTIC EXPANSION APPROACH TO NUMERICAL PROBLEMS IN FINANCE

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ABSTRACT. This paper reviews an asymptotic expansion approach to numerical problems in pricing financial assets and securities.

## 1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ denote a probability space with filtration, on which a $m$-dimensional standard Wiener process $w$ is defined where $P$ is the equivalent martingale measure (the risk neutral measure) in finance, and $T$ denotes some positive constant. Now, let $F(\omega)$ be a Wiener functional, and then $\mathbf{V}$, the value of a portfolio or a security can be expressed as $\mathbf{V} = \mathbf{E}[F(\omega)]$ under certain conditions. Evaluating this expectation is one of the main problems in finance. Moreover, if $F$ depends on the parameter $\theta$, evaluation of $\frac{\partial \mathbf{V}}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbf{E}[F(\omega;\theta)]$, the change of a security value caused by a minimal change of this parameter is also an important task in practice. As an example, consider a $d$-dimensional diffusion process $X^{(\epsilon)}$ which is obtained as a strong solution to the stochastic differential equation;

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)}, \epsilon)dt + V(X_t^{(\epsilon)}, \epsilon)dw_t,\ t \in [0,T];\ X_0^{(\epsilon)} = x_0,$$

where $\epsilon \in [0,1]$ is a known parameter. Here, the coefficients are assumed to be smooth and to satisfy some regularity conditions. In finance, the problems of evaluating the present value of derivatives or the portfolio value in investment theory are mostly reduced to the problems of computing $\mathbf{E}[f(X_T^{(\epsilon)})]$, the expectation of $f(X_T^{(\epsilon)})$, a function of $X_T^{(\epsilon)}$. In financial applications, it is important to deal with the case not only the function $f(x)$ is smooth but also the case it is not. For example, when various options are evaluated, it is expressed as $f = T \circ g$, where $T(x) = \max\{x, 0\}$ and $g$ denotes a smooth function of $\mathbf{R}^d \mapsto \mathbf{R}$. In general, it is difficult to represent this expectation explicitly except for special cases. Therefore, methods such as Monte Carlo simulation or numerical solutions of partial differential equations are applied and various speeding up techniques are developed, since fast computation is required as well as accuracy for practical purposes. As another approach, an approximation of the expectation by an asymptotic expansion of the stochastic differential equation around $\epsilon = 0$ can be considered. Further, because $\frac{\partial}{\partial x_0}\mathbf{E}[f(X_T^{(\epsilon)})]$ and $\frac{\partial}{\partial \epsilon}\mathbf{E}[f(X_T^{(\epsilon)})]$, the changes of the security value caused by the minimal changes of the initial value $x_0$ and the parameter $\epsilon$, are important indicators for practical purposes, obtaining the approximations with high accuracy

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are very useful. Moreover, some methods which combines Monte Carlo simulation and an asymptotic expansion with low orders are developed since the asymptotic expansion up to the first or second order can be easily evaluated. By this way, the efficiency of Monte Carlo simulation or the accuracy of approximation obtained by the asymptotic expansion can be improved.

The application of the asymptotic expansion to finance is closely related to the study in mathematical statistics, which is on asymptotic expansions of statistics for stochastic differential equations, though it seems not to be so relevant to finance. In particular, it is based on the asymptotic expansions of estimators developed by Yoshida [47], [48], [49], which applies Watanabe theory(Watanabe [46]) in Malliavin calculus to unknown parameters of small diffusions. For further readings on asymptotic expansions of small diffusions in mathematical statistics, see Dermoune-Kutoyants [4], Kutoyants [18], [19], Masuda-Yoshida [22], Sakamoto-Yoshida [28], [29], Sakamoto-Takada-Yoshida [27], Taniguchi-Kakizawa [41], Uchida-Yoshida [42], [43], and Yoshida [50], [51], [52].

To my knowledge, the asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets. [12] and [31] derive the approximation formulas for an average option by an asymptotic method based on log normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion. [48] applies a formula derived more generally by the asymptotic expansion of small diffusion processes. Thereafter, the asymptotic expansion is applied to a broad class of problems in finance. For basic theory in finance, see for instance Karatzas-Shreve [8] and Bjӧrk [1], and for the general relationship between finance and the asymptotic expansion, see [15]. In what follows, more concrete applications of the asymptotic expansion to numerical problems in finance are introduced.

## 2. Foundation of an Asymptotic Expansion

This section provides basic methodology to approximate the values of financial assets or securities after a summary of the framework of the asymptotic expansion approach based on [47] and [48] in Section 2.1. As for details on Watanabe theory([46]) in Malliavin calculus that is a core theory of this method, as well as the asymptotic expansion, see Takanobu [39], Takanobu-Watanabe [40] and Uemura [44] for instance besides the literatures on mathematical statistics mentioned above. Kusuoka-Strook [17] also derives an asymptotic expansion of a certain Wiener functional by Malliavin calculus. Further, see lecture notes or textbooks such as Watanabe [45], Ikeda-Watanabe [6], Nualart [25], Malliavin [21] and Shigekawa [30] for general references for Malliavin calculus.

### 2.1. **The Framework of an Asymptotic Expansion.**

First, I consider a $d$-dimensional diffusion process $X^{(\epsilon)}$, which is the strong solution to the following stochastic differential equation:

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)})dt + \epsilon V(X_t^{(\epsilon)})dw_t;\ X_0^{(\epsilon)} = x_0,\ t \in [0,T], \tag{2.1}$$

where $w$ denotes a $m$-dimensional standard Wiener process and $\epsilon \in [0,1]$ is a known parameter. Suppose that coefficients $V_0 : \mathbf{R}^d \mapsto \mathbf{R}^d,\ V : \mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^m$ are smooth and satisfies regularity conditions. Let $V_i$ denote the $i$-th column of the $V$ and a $\mathbf{R}^d \otimes \mathbf{R}^d$-valued stochastic process $Y$ denote the solution to the stochastic differential equation;

$$dY_t^{(\epsilon)} = \partial V_0(X_t^{(\epsilon)})Y_t^{(\epsilon)}dt + \epsilon\sum_{i=1}^{m}\partial V_i(X_t^{(\epsilon)})Y_t^{(\epsilon)}dw_{it};\ Y_0^{(\epsilon)} = I_d,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, and $\partial V_i(i = 0, 1, \cdots, m)$ denotes the $d \times d$ matrix whose $(j,k)$-element is $\partial_k V_{ji}$. $I_d$ denotes the $d \times d$ identity matrix. Moreover, define $Y_t$ as $Y_t = Y_t^{(0)}$.

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, for $\epsilon \downarrow 0$, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots \quad in\ \mathbf{D}_\infty, \tag{2.2}$$

where $g_{0T}, g_{1T}, g_{2T}, \cdots \in \mathbf{D}_\infty$. For any $k \in \mathbf{N}$, $q \in (1,\infty)$ and $s > 0$, this expansion means that

$$\frac{1}{\epsilon^k}\|g(X_T^{(\epsilon)}) - (g_{0T} + \epsilon g_{1T} + \cdots + \epsilon^{k-1}g_{k-1,T})\|_{q,s} = O(1)\ (as\ \epsilon \downarrow 0),$$

where $\|G\|_{q,s}$ represents the sum of the $L^q$ -norms of Malliavin derivatives of a Wiener functional $G$ up to the $s$-th order. Further, a Banach space $\mathbf{D}_{q,s} = \mathbf{D}_{q,s}(\mathbf{R})$ can be regarded as the totality of random variables bounded with respect to $(q,s)$-norm $\|\cdot\|_{q,s}$, and $\mathbf{D}_\infty = \cap_{s>0}\cap_{1<q<\infty\ q,s}$.

The coefficients in the expansion, $g_{0T}, g_{1T}, g_{2T}\cdots$ can be obtained by Taylor's formula and represented as multiple Wiener-Ito integrals. In particular, let $D_t = \frac{\partial X_t^{(\epsilon)}}{\partial\epsilon}|_{\epsilon=0}$ and $E_t = \frac{\partial^2 X_t^{(\epsilon)}}{\partial\epsilon^2}|_{\epsilon=0}$, then $g_{0T}, g_{1T}$ and $g_{2T}$ can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}),\ g_{1T} = \sum_{i=1}^{d}\partial_i g(X_T^{(0)})D_{iT}, \\ g_{2T} &= \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})D_{iT}D_{jT} + \frac{1}{2}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})E_{iT}. \end{aligned}$$

Here, $D_{it}$ and $E_{it}$, $i = 1, \cdots, d$, which are the elements of $D_t$ and $E_t$ respectively are represented by

$$\begin{aligned} D_{it} &= \int_0^t Y_t^{(i,\cdot)}Y_u^{-1}V(X_u^{(0)})dw_u, \\ E_{it} &= \int_0^t Y_t^{(i,\cdot)}Y_u^{-1}[\sum_{j,k=1}^{d}\partial_j\partial_k V_0(X_u^{(0)})D_{ju}D_{ku}du + 2\sum_{j=1}^{d}\partial_j V(X_u^{(0)})D_{ju}dw_u], \end{aligned}$$

where $Y_t^{(i,\cdot)}$ denote the $i$-th row of $Y_t$.

Next, normalize $g(X_T^{(\epsilon)})$ as

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0,1]$. Then, for $h \in H$(where $H$ denotes the Cameron-Martin subspace of the $m$-dimensional Wiener space), the $H$-derivative of $G^{(\epsilon)}$ is expressed as

$$\mathbf{D}_h G^{(\epsilon)} = \frac{1}{\epsilon}\sum_{i=1}^{d}\partial_i g(X_T^{(\epsilon)})\mathbf{D}_h X_{iT}^{(\epsilon)} = \sum_{i=1}^{d}\partial_i g(X_T^{(\epsilon)})\int_0^T[Y_T^{(\epsilon)}(Y_t^{(\epsilon)})^{-1}V(X_t^{(\epsilon)})\dot{h}_t]_i\ dt.$$

With a notation $a_t^{(\epsilon)}$,

$$a_t^{(\epsilon)} = (\partial g(X_T^{(\epsilon)}))^{'} [Y_T^{(\epsilon)} (Y_t^{(\epsilon)})^{-1} V(X_t^{(\epsilon)})],$$

the Malliavin (co)variance of $G^{(\epsilon)}$ is given by

$$\sigma_{G^{(\epsilon)}} = \int_0^T a_t^{(\epsilon)} (a_t^{(\epsilon)})^{'} dt, \tag{2.3}$$

where $x^{'}$ denotes the transpose of $x$. Moreover, let

$$a_t = a_t^{(0)} = (\partial g(X_T^{(0)}))^{'} [Y_T Y_t^{-1} V(X_t^{(0)})]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T a_t a_t^{'} dt > 0. \tag{2.4}$$

Since $\Sigma_T$ is the variance of the random variable $g_{1T}$, which follows a normal distribution, Assumption 1 means the condition that the distribution of $g_{1T}$ does not degenerate. In application, it is easy to check this condition in most cases, hence it plays an important role for practical purposes. In what follows, I derive asymptotic expansions useful for finance under Assumption 1. For detailed arguments such as the proofs omitted here because of the space limitation, see [47], [48] and [14].

Under Assumption 1, $\sigma_{G^{(\epsilon)}}$ is uniformly non-degenerate for $\{|\eta_c^{(\epsilon)}| \leq 1\}$; that is, it can be shown that there exists a positive real number $c_0 > 0$ such that for any $c > c_0$ and $p > 1$,

$$\sup_{\epsilon \in (0,1]} \mathbf{E}[1_{\{|\eta_c^{(\epsilon)}| \leq 1\}} (|\sigma_{G^{(\epsilon)}}|)^{-p}] < \infty, \tag{2.5}$$

where $\eta_c^{(\epsilon)} = c \int_0^T |a_t^{(\epsilon)} - a_t|^2 dt$. Then, for a measurable function with polynomial growth $T : \mathbf{R} \mapsto \mathbf{R}$, a composite function $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)} = \psi(\eta_c^{(\epsilon)}) T(G^{(\epsilon)})$ is well-defined as an element of $\tilde{\mathbf{D}}_{-\infty} = \cup_{s<0} \cap_{1<p<\infty} \mathbf{D}_{p,s}$, where $\psi(x),\ x \in R$ denotes a smooth function $0 \leq \psi(x) \leq 1$, defined as $\psi(x) = 1$ for $|x| \leq 1/2$ and $\psi(x) = 0$ for $|x| \geq 1$. Here, a Banach space $\mathbf{D}_{p,s},\ s < 0$ is the dual space of $\mathbf{D}_{q,-s}(\mathbf{R}) (q = p/(p-1))$. Moreover, the coupling with the function 1 is well-defined, which is called as generalized expectation and is written as $\mathbf{E}[\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}]$. Further, $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}$ can be expanded in $\tilde{\mathbf{D}}_{-\infty}$. Since a function $T$ such as $T(x) = \max\{x, 0\}$ that is measurable but not smooth appears frequently in finance, the framework mentioned above is necessary for the asymptotic expansion.

Moreover, it also can be shown that for any $k \geq 1$,

$$\lim_{\epsilon \downarrow 0} \frac{P(|\eta_c^{(\epsilon)}| > \frac{1}{2})}{\epsilon^k} < \infty.$$

This means that the probability of the events truncated by $\psi(\eta_c^{(\epsilon)})$ is smaller than any polynomial orders of $\epsilon$. Then, in the expansion of $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}$, the coefficients expressed as generalized Wiener functionals belonging to $\tilde{\mathbf{D}}_{-\infty}$ can be written by applying Taylor's formula to $T(g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots)$. Therefore, the asymptotic expansion of the expectation $\mathbf{E}[T(G^{(\epsilon)})]$ can be obtained relatively easily.

Now let us consider a more specific case. For a smooth function $\phi^{(\epsilon)} : [0,1] \times \mathbf{R} \mapsto \mathbf{R}$, of which all derivatives with respect to $x$ have polynomial growth orders uniformly in $\epsilon$, the following holds uniformly in $B \in \mathbf{B}$ for a positive large number $c > 0$:

$$\psi(\eta_c^{(\epsilon)})\phi^{\epsilon}(G^{(\epsilon)})I_B(G^{(\epsilon)}) \sim \Phi_0 + \epsilon\Phi_1 + \cdots \quad in\ \tilde{\mathbf{D}}_{-\infty}\ (as\ \epsilon \downarrow 0), \tag{2.6}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{R})$ denotes a Borel set in $\mathbf{R}$. Here, the coefficients $\Phi_0, \Phi_1, \cdots$ are generalized Wiener functionals and they are obtained by applying formal Taylor's formula to $\phi^{\epsilon}(G^{(\epsilon)})I_B(G^{(\epsilon)})$; in particular, $\Phi_0$ and $\Phi_1$ are expressed as

$$\begin{aligned}
\Phi_0 &= \phi^{(0)}(g_{1T})I_B(g_{1T}) \\
\Phi_1 &= \{g_{2T}\partial\phi^{(0)}(g_{1T}) + \phi^{(1)}(g_{1T})\}I_B(g_{1T}) + \phi^{(0)}(g_{1T})g_{2T}\partial I_B(g_{1T}),
\end{aligned}$$

where $\phi^{(1)} = \frac{\partial\phi^{(\epsilon)}}{\partial\epsilon}|_{\epsilon=0}$.

Therefore, the expectation $\mathbf{E}[\phi(G^{(\epsilon)})I_B(G^{(\epsilon)})]$ has its asymptotic expansion uniformly in $B$ as

$$\mathbf{E}[\phi^{(\epsilon)}(G^{(\epsilon)})I_B(G^{(\epsilon)})] \sim \mathbf{E}[\Phi_0] + \epsilon\mathbf{E}[\Phi_1] + \cdots \ (as\ \epsilon \downarrow 0), \tag{2.7}$$

where each term of the expansion can be expressed as the expectation of a multiple Wiener-Ito integral conditional on a normal random variable. For example, $\mathbf{E}[\Phi_0]$ and $\mathbf{E}[\Phi_1]$ are written as

$$\begin{aligned}
\mathbf{E}[\Phi_0] &= \int_B \phi^{(0)}(x)n[x;0,\Sigma_T]dx, \\
\mathbf{E}[\Phi_1] &= \int_B (\phi^{(1)}(x)n[x;0,\Sigma_T] - \phi^{(0)}(x)\partial\{\mathbf{E}[g_{2T}|g_{1T} = x]n[x;0,\Sigma_T]\})dx,
\end{aligned}$$

where $n[x;0,\Sigma_T]$ denotes the density function of the normal distribution with mean 0 and variance $\Sigma_T$. Moreover, $E[g_{2T}|g_{1T} = x]$ is a polynomial function of $x$ and hence the computations of the expectations become easier. It is known that precise approximations are obtained by applying asymptotic expansions up to such low orders in financial examples.

### 2.2. **Valuation of Option Values by an Asymptotic Expansion.**

Now approximations for values of various derivatives can be provided by the framework explained above. First, note that $X^{(\epsilon)}$ in the previous subsection represents the key variable such as the underlying asset's price, which is an important factor to determine values of derivatives. Next, I show the application of the asymptotic expansion to finance, specifically by using an example of a standard European vanilla option which is examined in [32]. Let $w$ be a one-dimensional Wiener process, $\alpha$ be a constant, $\epsilon \in [0,1]$, and $\sigma(x,t)$ be a smooth function of $x$ satisfying appropriate regularity conditions. Suppose that under the equivalent martingale measure, the underlying asset price of the option $S^{(\epsilon)}$ follows a stochastic process;

$$dS_t^{(\epsilon)} = \alpha S_t^{(\epsilon)}dt + \epsilon\sigma(S_t^{(\epsilon)},t)dw_t;\ S_0^{(\epsilon)} = s(>0). \tag{2.8}$$

Let a strike price $K$ be some positive constant, then the values of vanilla call and put options at the expiration date $T$ are respectively written as

$$V_c(T) = \max\left\{S_T^{(\epsilon)} - K, 0\right\} \text{ and } V_p(T) = \max\left\{K - S_T^{(\epsilon)}, 0\right\}. \tag{2.9}$$

Moreover, the values at the contract date($t = 0$) are respectively expressed as

$$V_c(0) = e^{-rT}\mathbf{E}[V_c(T)] \text{ and } V_p(0) = e^{-rT}\mathbf{E}[V_p(T)], \tag{2.10}$$

where $r$, a nonnegative constant denotes the (instantaneous) short-term interest rate. In what follows, I explain the case of call options, which can be also applied to the case of a put option. First, in the setting of Section 2.1, let $d = 2$, $m = 1$, and $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, and $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$. Next, with $x = (x_0, x_1)$ and $g(x) = x_1 - K$, $G^{(\epsilon)}$ is given by

$$G^{(\epsilon)} = \frac{X_{1T}^{(\epsilon)} - X_{1T}^{(0)}}{\epsilon},$$

where $\epsilon \in (0,1]$ and $X_{1T}^{(0)} = S_T^{(0)} = e^{\alpha T} s$. Then, the value at expiration can be written as

$$\begin{aligned} V_c(T) &= \max\left\{S_T^{(\epsilon)} - K, 0\right\} = \max\left\{\epsilon G^{(\epsilon)} + X_{1T}^{(0)} - K, 0\right\} \\ &= \max\left\{\epsilon(G^{(\epsilon)} + y^{(\epsilon)}), 0\right\}, \end{aligned}$$

where $y^{(\epsilon)} = \frac{X_{1T}^{(0)} - K}{\epsilon}$. Note also that $\Sigma_T$ is expressed as

$$\Sigma_T = \int_0^T e^{2\alpha(T-u)} \sigma^2(X_{1u}^{(0)}, X_{0u}^{(0)}) du = \int_0^T e^{2\alpha(T-u)} \sigma^2(S_u^{(0)}, u) du. \tag{2.11}$$

Although $\phi^{(\epsilon)}(x)$ and $B$ can be set as $\phi^{(\epsilon)}(x) = \epsilon x + (X_{1T}^{(0)} - K)$ and $B = \{X_{1T}^{(\epsilon)} \geq K\} = \{G^{(\epsilon)} \geq -y^{(\epsilon)}\}$ respectively, I consider this problem under the following assumption for more practical purposes:

(Assumption 2)

For any $y^{(\epsilon)}$ there exists a real number $y$ such that $y^{(\epsilon)} = y + o(\epsilon)$ .

Assumption 2 means that $K = K^{(\epsilon)} = S_T^{(0)} - \epsilon y + o(\epsilon^2)$. Thus, I consider the strike price to be close to the forward price $S_T^{(0)}$ with the same expiration date as the option's one, which is based on the fact that such trade is common in practice. This assumption can be relaxed, but in order to avoid complexity I admit it in the following. Now I set

$$\phi^{(\epsilon)}(x) = \epsilon(x + y^{(\epsilon)}),$$

and $B = \{G^{(\epsilon)} \geq -y^{(\epsilon)}\}$. Then, by the equation (2.7), an approximation formula up to the $\epsilon^2$-order for the call option's value at the contract date is derived as

$$\begin{aligned} V_c(0) &= e^{-rT}\{\epsilon \int_{-y}^{\infty} (x+y) n[x; 0, \Sigma_T] dx \\ &+ \epsilon^2 \int_{-y}^{\infty} (cx^2 + f) n[x; 0, \Sigma_T] dx\} + o(\epsilon^2), \end{aligned} \tag{2.12}$$

where the constants $c$ and $f$ are given by

$$\begin{aligned} c &= \frac{1}{\Sigma_T^2} \int_0^T e^{\alpha(T-u)} \sigma(S_u^{(0)}, u) \partial\sigma(S_u^{(0)}, u) \int_0^u e^{2\alpha(T-v)} \sigma^2(S_u^{(0)}, v) dv du , \\ f &= -c\Sigma_T. \end{aligned} \tag{2.13}$$

Moreover, a more concrete approximation formula can be obtained by integration by parts.

**Theorem 2.1.** *Suppose that the underlying asset price follows the stochastic process (2.8) under the equivalent martingale measure. Then, under Assumptions 1 and 2, the asymptotic expansion up to the* $\epsilon^2$*-order of* $V_c(0)$*, the price of a vanilla call option at the contract date with the maturity date* $T$ *and the strike price* $K$*, is given by*

$$
\begin{aligned}
V_c(0) &= e^{-rT}\epsilon\left\{\Sigma_T n[y;0,\Sigma_T] + yN\left(\frac{y}{\sqrt{\Sigma_T}}\right)\right\} \\
&\quad + e^{-rT}\epsilon^2 c\left\{\Sigma_T N\left(\frac{y}{\sqrt{\Sigma_T}}\right) - y\Sigma_T n[y;0,\Sigma_T] + fN\left(\frac{y}{\sqrt{\Sigma_T}}\right)\right\} + o(\epsilon^2), \qquad (2.14)
\end{aligned}
$$

*where* $\Sigma_T$ *and* $c, f$ *are respectively defined as (2.11) and (2.13), and* $N(x)$ *denotes the distribution function of a standard normal distribution.*

Finally, it is remarkable that in practice, by computation of the option price under the assumption that $y^{(\epsilon)}$ is a constant and $y = y^{(\epsilon)}$, a precise approximation can be still obtained in general even with the strike price far from $S_T^{(0)}$, the forward price with the same maturity as the option's one.

The values of various option contracts can be evaluated by using the similar method. In particular, with redefinition of $\Sigma_T$, $c$ and $f$ in (2.14), almost the same approximation formula can be applied to various options. In what follows, some examples for call options are presented. For more details, see [32] and [36].

(1) A Basket Option
Suppose that $S_{it}^{(\epsilon)}$, $i = 1, \cdots, n$, the underlying asset prices of a basket option follow the stochastic processes;

$$dS_{it}^{(\epsilon)} = \alpha_i S_{it}^{(\epsilon)} dt + \epsilon \sum_{j=1}^{n} \sigma_{ij}(S_{it}^{(\epsilon)}, t) dw_{jt};\ S_{i0}^{(\epsilon)} = s_i (> 0),$$

where $\alpha_i, i = 1, \cdots, n$ are constants, and $w = (w_1, \cdots, w_n)$ is a $n$-dimensional Wiener process. Then, the value of the basket option at the maturity date $T$ is given by

$$V_c(T) = \max\left\{\sum_{i=1}^{n} \beta_i S_{iT}^{(\epsilon)} - K, 0\right\},$$

where the strike price $K$ is a positive constant and $\beta_i$, $i = 1, \cdots, n$ are constants. Further, let $r$ be a nonnegative constant. Then, the price at the contract date($t = 0$) is expressed as $V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$. In this case, it is specified that $d = n + 1$, $m = n$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, \cdots, X_{nt}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{it}^{(\epsilon)} = S_{it}^{(\epsilon)} (i = 1, \cdots, n)$, and $g(x) = \sum_{i=1}^{n} \beta_i x_i - K$, $x = (x_0, x_1, \cdots, x_n)$ in the setting of Section 2.1.

In the following examples, $\alpha$ and $r$ are set to be a constant and a nonnegative constant respectively as well as in this example, unless otherwise stated.

(2) An Average Option
The stochastic processes which describe the dynamics of the underlying variables of an average option are expressed as

$$
\begin{aligned}
dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + \epsilon\sigma(S_t^{(\epsilon)}, t) dw_t;\ S_0^{(\epsilon)} = s (> 0) \\
dZ_t^{(\epsilon)} &= S_t^{(\epsilon)} dt;\ Z_0^{(\epsilon)} = 0,
\end{aligned}
$$

where $w$ denotes a one-dimensional Wiener process. Then, the maturity value of an average option is expressed as

$$V_c(T) \quad = \quad \max\left\{\frac{1}{T}Z_T^{(\epsilon)} - K, 0\right\}.$$

Thus the price at the contract date is given by

$V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$, where $d = 3$, $m = 1$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, X_{2t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, and $g(x) = \frac{1}{T}x_2 - K$, $x = (x_0, x_1, x_2)$ in the setting of Section 2.1.

(3) A Vanilla Option with the Stochastic Volatility
Assume that the underlying variables of a vanilla option with the stochastic volatility follow the stochastic processes below;

$$\begin{aligned}
dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)}dt + \epsilon\sum_{j=1}^{2}\sigma_{1j}(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t)dw_{jt};\ S_0^{(\epsilon)} = s(>0)\\
dZ_t^{(\epsilon)} &= \mu(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t)dt + \epsilon\sum_{j=1}^{2}\sigma_{2j}(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t)dw_{jt};\ Z_0^{(\epsilon)} = z(>0),
\end{aligned}$$

where $w = (w_1, w_2)$ denotes a two-dimensional Wiener process.

The payoff at maturity is expressed as $V_c(T) = \max\{S_T^{(\epsilon)} - K, 0\}$, and the price at the contract date is evaluated by $V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$.

In this case, it is specified that $d = 3$, $m = 2$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, X_{2t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, and $g(x) = x_1 - K$, $x = (x_0, x_1, x_2)$ in the setting of Section 2.1.

(4) A Vanilla Option under the Stochastic Interest Rate
Assume that the stochastic processes determining the value of the option under the stochastic interest rate follow

$$\begin{aligned}
dS_t^{(\epsilon)} &= (r_t^{(\epsilon)} - \alpha)S_t^{(\epsilon)}dt + \epsilon\sum_{j=1}^{2}\sigma_{1j}(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t)dw_{jt};\ S_0^{(\epsilon)} = s(>0)\\
dZ_t^{(\epsilon)} &= -r_t^{(\epsilon)}Z_t^{(\epsilon)}dt;\ Z_0^{(\epsilon)} = 1\\
dr_t^{(\epsilon)} &= \mu(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t)dt + \epsilon\sum_{j=1}^{2}\sigma_{2j}(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t)dw_{jt};\ r_0^{(\epsilon)} = r(\geq 0),
\end{aligned}$$

where $w = (w_1, w_2)$ denotes a two-dimensional Wiener process.

The value at maturity is expressed as $V_c(T) = \max\{S_T^{(\epsilon)} - K, 0\}$.

Then the value at the contract date is evaluated by

$$V_c(0) \quad = \quad \mathbf{E}\left[e^{-\int_0^T r_u^{(\epsilon)}du}V_c(T)\right],$$

where $d = 4$, $m = 2$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, \cdots, X_{3t}^{(\epsilon)})'$ with $X_{0t} = t$, $X_{1T}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, $X_{3t}^{(\epsilon)} = r_t^{(\epsilon)}$, and $g(x) = x_2(x_1 - K)$, $x = (x_0, x_1, x_2, x_3)$ in the setting of Section 2.1.

### 2.3. **Approximations of the Values of Financial Assets and Securities under Diffusion Processes.**

The framework of the asymptotic expansion can be applied not only to the simple cases mentioned above, but also to evaluation of much broader range of assets' and securities' values. In particular, there are many cases where the asymptotic expansion can be applied to approximate that values when the underlying asset prices of financial assets or securities, cash flows and discount rates such as interest rates are expressed as a function of a random vector $X^{(\epsilon)}$ that follows a diffusion process. The method is almost the same as the one illustrated above and hence it is omitted. In this subsection, only the method to represent values of financial assets or securities will be reviewed.

First, just as in the previous subsections, I consider a $d$-dimensional diffusion process $X^{(\epsilon)}$ defined as the strong solution to the stochastic differential equation (2.1). In general, the contract value $V$ of a financial asset which generates a cash flow at the maturity date $T$ is represented as

$$V = \mathbf{E}\left[e^{-\int_0^T R_2(X_u^{(\epsilon)})du} F(f(X_T^{(\epsilon)}))\right], \tag{2.15}$$

where $f$ denotes the underlying asset price and $F$ is the cash flow which characterizes the asset or security to be evaluated. Note that the dynamics of the underlying asset price $f$ is expressed by a diffusion process, whose drift term (the coefficient of the $dt$ term) is $R_1(X_t^{(\epsilon)})f - D(X_t^{(\epsilon)})$ under the equivalent martingale measure. Moreover, $R_1$ at time $t \in [0, T]$ is represented as

$$R_1(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_1} s_{1j}(X_t^{(\epsilon)}),$$

where $r$ is the risk-free interest rate and $s_{1j}$, $j = 1, \cdots, J_1$ are various spreads (the differences from the risk-free rate) such as credit spreads or liquidity spreads. Suppose that all of them are expressed as functions of the variable $X^{(\epsilon)}$. Further, $D(X_t^{(\epsilon)})$ denotes a payoff generated by the underlying asset such as a dividend or an interest rate and is also represented as a function of the variable $X^{(\epsilon)}$. Meanwhile, $R_2$, the discount rate of the objective asset or security to be evaluated at time $t$ is also expressed as

$$R_2(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_2} s_{2j}(X_t^{(\epsilon)}),$$

where $s_{2j}$, $j = 1, \cdots, J_2$ are various spreads related to the objective asset or security. Suppose also that all of them are expressed as some functions of the variable $X^{(\epsilon)}$.

As an example, let $F = 1$ in (2.15) for a zero-coupon bond with the face value 1 and the maturity date $T$. Next, let $V_i$ denote the price of the zero-coupon bond with the maturity $T_i$. Then, $V$, the value of a coupon bond with the maturity $T_N$ and coupon payments $c_i$ at $T_i (i = 1, \cdots, N,\ T_1 < \cdots < T_N)$ is represented by the equation $V = \sum_{i=1}^N c_i V_i$. Moreover, the present value of a call option on the coupon bond with the option maturity $T (< T_1)$ can be evaluated if I set $F(x) = (x - K)^+$ and $f(X_T^{(\epsilon)}) = \sum_{i=1}^N c_i f_i(X_T^{(\epsilon)})$ in the equation (2.15), where $f_i(X_T^{(\epsilon)})$, $i = 1, \cdots, N$ are given by

$$f_i(X_T^{(\epsilon)}) = \mathbf{E}\left[e^{-\int_T^{T_i} R_1(X_u^{(\epsilon)})du} | X_T^{(\epsilon)}\right].$$

## 3. Asymptotic Expansions in an Instantaneous Forward Rates Model and a Jump-Diffusion Model

Among main stochastic models in finance, there exist models where the stochastic processes of the underlying variables do not belong to the class of diffusion processes. This section illustrates two typical examples.

### 3.1. An Instantaneous Forward Rates Model.

Among stochastic models for evaluating the interest rate derivatives, there exists a model developed by Heath-Jarrow-Morton [5] which is formulated based on the forward rates with infinitesimal terms of the interest rates, that is the instantaneous forward rates $\{f(s,t) : 0 \leq s \leq t \leq T\}$. Here, $s$ is the time when the forward rate is determined and $t$ denotes the time when the forward rate starts to be applied.

The stochastic processes for the instantaneous forward rates are considered in the framework of the asymptotic expansion by introducing a parameter $\epsilon \in [0,1]$. For example, let $w$ be a $m$-dimensional standard Wiener process and let $f(0,t)$, $t \in [0,T]$ be a given Lipschitz continuous function of $t$. Then, under the equivalent martingale measure, the stochastic processes of $\{f^{(\epsilon)}(s,t) : 0 \leq s \leq t \leq T\}$ are written as

$$\begin{aligned} f^{(\epsilon)}(s,t) = f(0,t) \quad &+ \quad \epsilon^2 \int_0^s \sum_{i=1}^m \left[ \sigma_i(f^{(\epsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\epsilon)}(v,y),v,y) dy \right] dv \\ &+ \quad \epsilon \sum_{i=1}^m \int_0^s \sigma_i(f^{(\epsilon)}(v,t),v,t) dw_i(v) \ ; \epsilon \in [0,1], \end{aligned} \tag{3.1}$$

where the volatility functions $\{\sigma_i(f^{(\epsilon)}(s,t),s,t); i = 1,\cdots,m\}$ are smooth and satisfy the regularity conditions which guarantee that the equation (3.1) has its solution. It is to be noted that the drift term (the coefficient of the $dv$ term ) of $f^{(\epsilon)}(s,t)$ depends on $\{f^{(\epsilon)}(v,y); 0 \leq v < s, v \leq y < t\}$. Moreover, the stochastic process of the instantaneous short-term interest rate $r^{(\epsilon)}(t)$ is determined by the relationship, $r^{(\epsilon)}(t) = f^{(\epsilon)}(t,t)$.

Even for this model, the approximations of the values for interest rate derivatives can be still considered in a unified framework with derivation of asymptotic expansions of the instantaneous forward rates when $\epsilon \downarrow 0$ and with use of the relation between the instantaneous forward rates and a zero-coupon bond price;

$$P^{(\epsilon)}(t,T) = \exp\left\{ -\int_t^T f^{(\epsilon)}(t,u) du \right\}. \tag{3.2}$$

As an example, I consider a call option on a coupon bond, which is a standard interest rate derivative. The payoff at maturity of the option is given by

$$V_c(T) \quad = \quad \max\{\sum_{i=1}^n c_i P^{(\epsilon)}(t,T_i) - K, 0\},$$

where $0 \leq T \leq T_1 < \cdots < T_n$, $c_i$,$i = 1,\cdots,n$ are positive constants and $K(>0)$ is a strike price. Then, the contract value is evaluated by

$$V_c(0) \quad = \quad \mathbf{E}\left[ e^{-\int_0^T r_u^{(\epsilon)} du} V_c(T) \right].$$

When $\epsilon \downarrow 0$, the forward rate $f^{(\epsilon)}(s,t)$ is expanded as

$$f^{(\epsilon)}(s,t) \sim f(0,t) + \epsilon f_1(s,t) + \epsilon^2 f_2(s,t) + \cdots \ in\ \mathbf{D}^\infty, \tag{3.3}$$

where $f_1(t,u), f_2(t,u) \cdots \in \mathbf{D}^\infty$. As a result, $P^{(\epsilon)}(t,T)$ and $\exp\left\{-\int_0^T r^{(\epsilon)}(t)dt\right\}$ are expanded as

$$\begin{aligned}
P^{(\epsilon)}(t,T) &\sim \frac{P(0,T)}{P(0,t)}\left[1 - \epsilon\int_t^T f_1(t,u)du - \epsilon^2\int_t^T f_2(t,u)du\right. \\
&+ \left.\epsilon^2\frac{1}{2}\left\{\int_t^T f_1(t,u)du\right\}^2\right] + \cdots \ in\ \mathbf{D}^\infty, \\
e^{-\int_0^T r^{(\epsilon)}(s)ds} &\sim P(0,T)\left[1 - \epsilon\int_0^T f_1(t,t)dt - \epsilon^2\int_0^T f_2(t,t)dt\right. \\
&+ \left.\epsilon^2\frac{1}{2}\left\{\int_0^T f_1(t,t)dt\right\}^2\right] + \cdots \ in\ \mathbf{D}^\infty,
\end{aligned}$$

where $f_i(s,t)$, $i = 1, 2$ are given by

$$\begin{aligned}
f_1(s,t) &= \tfrac{\partial f^{(\epsilon)}(s,t)}{\partial \epsilon}|_{\epsilon=0} = \int_0^s \sum_{i=1}^m \sigma_i^{(0)}(v,t)dw_i(v), \\
f_2(s,t) &= \tfrac{1}{2}\tfrac{\partial^2 f^{(\epsilon)}(s,t)}{\partial^2 \epsilon}|_{\epsilon=0} \\
&= \int_0^s b^{(0)}(v,t)dv + \int_0^s \sum_{i=1}^m \partial\sigma_i^{(0)}(v,t)f_1(v,t)dw_i(v).
\end{aligned}$$

Here, $\sigma_i^{(0)}(v,t) = \sigma_i(f^{(0)}(v,t),v,t)$, and $b^{(0)}(v,t)$ and $\partial\sigma_i^{(0)}(v,t)$ are defined as

$$\begin{aligned}
b^{(0)}(v,t) &= \sum_{i=1}^n \sigma_i(f^{(0)}(v,t),v,t)\int_v^t \sigma_i(f^{(0)}(v,y),v,y)dy, \\
\partial\sigma_i^{(0)}(v,t) &= \frac{\partial\sigma_i(x,v,t)}{\partial x}|_{x=f(0,t)}.
\end{aligned}$$

Therefore, in the framework of the previous section with definition of $X_{0t}^{(\epsilon)}$ and $X_{it}^{(\epsilon)}, i = 1, \cdots, n$ as

$$\begin{aligned}
X_{0t}^{(\epsilon)} &= \exp\left\{-\int_0^t r^{(\epsilon)}(u)du\right\} \\
X_{it}^{(\epsilon)} &= P^{(\epsilon)}(t,T_i) = \exp\left\{-\int_t^{T_i} f^{(\epsilon)}(t,u)du\right\},\ i = 1, \cdots, n,
\end{aligned}$$

the payoff at maturity of the call option on a coupon bond is written as

$$V_c(T) = \max\left\{\sum_{i=1}^n c_i X_{it}^{(\epsilon)} - K, 0\right\}.$$

Moreover, let $x = (x_0, x_1, \cdots, x_n)$ and define $g(x)$ as

$$g(x) = x_0\left(\sum_{i=1}^n c_i x_i - K\right).$$

Then, a similar method can be applied as in the case of diffusion processes. Consequently, with redefinition of $\Sigma_T$, $c$ and $f$, the approximation of this option can be obtained based on almost the same asymptotic expansion as for the approximation (2.14) given in Theorem 2.1. For more details, see [13], [14] and [34].

For evaluation of other various interest rate derivatives, approximations based on the asymptotic expansion can also be derived. Moreover, the approximate formula for the value of a derivative dependent on instantaneous forward rates and other variables following diffusion processes is given by [31].

### 3.2. Evaluation of the Values of Securities under Jump-Diffusion Processes.

So far, stochastic models where uncertainty is generated by Wiener processes have been used. However, I can also apply the asymptotic expansion to stochastic processes including jumps in their sample paths. Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ is a filtered probability space with an equivalent martingale measure $P$ and is equipped with a $m$-dimensional Wiener process $w$ and a stationary Poisson random measure $\mu$ on $[0,T]\times\mathbf{E}$, which are mutually independent. Let also $(\mathbf{E}, \mathcal{E})$ be a measurable space. Moreover, suppose that the intensity measure of the Poisson random measure $\mu$ is

$$\hat{\lambda}(dt,dx) = dt \times \nu(dx),$$

where $\nu$ is a positive $\sigma$-finite measure on $(\mathbf{E}, \mathcal{E})$ and the compensated Poisson measure is represented as

$$\tilde{\mu}(dt,dx) = \mu(dt,dx) - \hat{\lambda}(dt,dx).$$

In this setting, examples for evaluation of the values of a bond and an option are provided.

**Evaluation of a Bond Price**

Let $X^{(\epsilon)}$ denote the $R^d$-valued stochastic process defined as the solution to the following stochastic differential equation:

$$(3.4)\quad dX^{(\epsilon)}(t) = A(X^{(\epsilon)}(t),\epsilon)dt + \epsilon\sigma(X^{(\epsilon)}(t))dw(t) + \epsilon\int_E C(X^{(\epsilon)}(t-),x)\tilde{\mu}(dt,dx),$$

where $\epsilon\in[0,1]$ and assume that the coefficients $A:\mathbf{R}^d\times[0,1]\mapsto\mathbf{R}^d$ , $\sigma:\mathbf{R}^d\mapsto\mathbf{R}^d\otimes\mathbf{R}^m$ , and $C:\mathbf{R}^d\times\mathbf{R}^d\mapsto\mathbf{R}^d$ are smooth and satisfy regularity conditions. Define also $\mathbf{R}^d$-valued random vectors $D$ and $E$ as

$$D(t) = \frac{\partial X^{(\epsilon)}(t)}{\partial\epsilon}|_{\epsilon=0} \text{ and } E(t) = \frac{\partial^2 X^{(\epsilon)}(t)}{\partial\epsilon^2}|_{\epsilon=0}.$$

In this case, $D(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dD(t) \quad &= \quad \partial_\epsilon A(X^{(0)}(t),0)dt + \partial A(X^{(0)}(t),0)D(t)dt + \sigma(X^{(0)}(t))dw(t) \\ &\quad + \textstyle\int_E C(X^{(0)}(t-),x)\tilde{\mu}(dt,dx);\ D(0)=0. \end{aligned}$$

Therefore, the random vector $D(t)$ can be expressed as

$$D(t) = Y_t\int_0^t Y_s^{-1}[\partial_\epsilon A(X^{(0)}(s),0)ds + \sigma(X_s^{(0)})dw(s) + \int_E C(X^{(0)}(s-),x)\tilde{\mu}(ds,dx)],$$

where $Y$ is a $\mathbf{R}^d \otimes \mathbf{R}^d$-valued function and is the solution to the following ordinary differential equation:

$$dY_t = \partial A(X^{(0)}(t), 0) Y_t dt;\ Y_0 = I_d.$$

Similarly, $E$ satisfies the stochastic differential equation;

$$\begin{aligned} dE(t) &= \partial A(X_t^{(0)}, 0)E(t)dt + 2\partial_\epsilon \partial A(X_t^{(0)}, 0)D(t)dt + \partial_\epsilon^2 A(X_t^{(0)}, 0)dt \\ &\quad + \partial\partial A(X_t^{(0)}, 0)D(t)D(t)dt + 2\partial\sigma(X_t^{(0)})D(t)dw(t) \\ &\quad + 2\int_E \partial C(X_{t-}^{(0)}, x)D_{t-}\tilde{\mu}(dt, dx);\ E(0) = 0, \end{aligned}$$

where $\partial\partial A(X_t^{(0)}, 0)D(t)D(t)$ is a $d$-dimensional vector and is given by

$$\begin{aligned} &\partial\partial A(X_t^{(0)}, 0)D(t)D(t) \\ = &\left[D'\{\partial_i\partial_j A_1(X_t^{(0)}, 0)\}_{(i,j)}D, \cdots, D'\{\partial_i\partial_j A_d(X_t^{(0)}, 0)\}_{(i,j)}D\right]' \quad (i, j = 1, 2, \cdots, d). \end{aligned}$$

Here, $\left\{\partial_i\partial_j A_k(X_t^{(0)}, 0)\right\}_{(i,j)}$ denotes a $d \times d$ matrix defined for each $k = 1, \cdots, d$. Further, $\partial\sigma(X_t^{(0)})D_t$ denotes a $d \times m$ matrix and is represented by

$$\partial\sigma(X_t^{(0)})D(t) = \left\{\sum_{k=1}^{d} \partial_k \sigma_{ij}(X_t^{(0)})D_k(t)\right\}_{i,j} \quad (i = 1, \cdots, d, j = 1, \cdots, m).$$

Moreover, a notation $\partial C(X_{t-}^{(0)}, x)D_{t-}$ denotes a $d$-dimensional vector whose $i$-th element is $\sum_{k=1}^{d} \partial_k C_i(X_{t-}^{(0)}, x)D_k(t-)$. Therefore, $E(t)$ is given by

$$\begin{aligned} E(t) = Y_t \int_0^t Y_s^{-1} &\Big[2\partial_\epsilon \partial A(X_s^{(0)}, 0)D(s)ds + \partial_\epsilon^2 A(X_s^{(0)}, 0)ds \\ + &\quad \partial\partial A(X_s^{(0)}, 0)D(s)D(s)ds + 2\partial\sigma(X_s^{(0)})D(s)dw(s) \\ + &\quad 2\int_E \partial C(X^{(0)}(s-), x)D_{s-}\tilde{\mu}(ds, dx)\Big]. \end{aligned}$$

On the other hand, it is known that the price of a zero-coupon bond is generally expressed as

$$P(0, T) = \mathbf{E}\left[e^{-\int_0^T g(X_s^{(\epsilon)})ds}\right].$$

Here, $g$ denotes the sum of the risk-free rate and the spreads determined by risk factors such as the credit risk and the liquidity risk, and is supposed to be represented by a function of $X^{(\epsilon)}(t)$. Then, an asymptotic expansion up to the $\epsilon^2$-order of $P(0, T)$, the zero-coupon bond price is given by

$$\begin{aligned} P(0, T) &= e^{-\int_0^T g_s^{(0)} ds}\left(1 - \epsilon\int_0^T \mathbf{E}\left[(\partial g_s^{(0)})' D(s)\right] ds\right. \\ &+ \quad \epsilon^2 \left\{-\frac{1}{2}\int_0^T \sum_{i,j=1}^{n} \partial_i\partial_j g_s^{(0)} \mathbf{E}[D_i(s)D_j(s)]ds - \frac{1}{2}\int_0^T \partial g_s^{(0)} \mathbf{E}[E(s)]ds\right. \\ &\quad \left.\left. + \frac{1}{2}\mathbf{E}[(\int_0^T \partial g_s^{(0)} D(s)ds)^2]\right\}\right) + o(\epsilon^2). \end{aligned} \tag{3.5}$$

With definition of $\partial_\epsilon A(\tau)$ and $C(t)$ as $\partial_\epsilon A(\tau) = \partial_\epsilon A(X^{(0)}(\tau),0)$ and $C(t) = C(X^{(0)}(t-),x)$, $\mathbf{E}[D_i(s)D_j(u)]$ is represented as the following:

$$
\begin{aligned}
(3.6)\ \ &\mathbf{E}[D_i(s)D_j(u)] \\
= \ &\left[Y^{(i,\cdot)}(s)\int_0^s Y^{-1}(\tau)\partial_\epsilon A(\tau)d\tau\right]\left[Y^{(j,\cdot)}(u)\int_0^u Y^{-1}(\tau)\partial_\epsilon A(\tau)d\tau\right] \\
&+Y^{(i,\cdot)}(s)\left(\int_0^{s\wedge u} Y^{-1}(\tau)\sigma(\tau)\sigma(\tau)'\left(Y^{-1}(\tau)\right)' d\tau\right)\left(Y^{(j,\cdot)}(u)\right)' \\
&+Y^{(i,\cdot)}(s)\left(\int_0^{s\wedge u} Y^{-1}(\tau)\left\{\int_E C(\tau)C(\tau)'\nu(dx)\right\}\left(Y^{-1}(\tau)\right)' d\tau\right)\left(Y^{(j,\cdot)}(u)\right)'.
\end{aligned}
$$

A more specific expression of (3.5) can be obtained based on the equations such as (3.6).

### Evaluation of an Option Value

Here, I consider a more specific example for evaluating the value of an option. Let $m = d = 1$, $\epsilon \in [0,1]$ and $\alpha$ be a constant. Further, under an equivalent martingale measure, suppose that the underlying asset price of an option follows a stochastic process;

$$
(3.7)\quad dS_t^{(\epsilon)} = \alpha S_t^{(\epsilon)}dt + \epsilon\sigma(S_t^{(\epsilon)},t)dw_t + \int_{\mathbf{R}} S_{t-}^{(\epsilon)}(e^{\epsilon x}-1)\tilde{\mu}(dtdx);\ S_0^\epsilon = s(>0),
$$

where $\sigma(x,t)$ is a function satisfying appropriate regularity conditions. Here, $\mu([0,t]\times A)$ is given by

$$
\mu([0,t]\times A) = \sum_{j=1}^{N_t} I_A(\xi_j^{(\epsilon)}),
$$

where $A \in \mathbf{B}(\mathbf{R})$, and $N_t$ denotes a Poisson process with a constant intensity $\lambda(>0)$. Define random variables $(\xi_j^{(\epsilon)})_{j\geq 1}$ determining jump sizes as

$$
\xi_j^{(\epsilon)} = e^{\epsilon\eta_j} - 1,\ j \geq 1,
$$

where $(\eta_j)_{j\geq 1}$ denotes a sequence of random variables which follow independent identical distributions ($i.i.d.$) and its probability law is $\nu$. Suppose also that $v = \mathbf{E}[\eta_j] < \infty$ and $\mathbf{E}[\xi_j^{(\epsilon)}] < \infty$. Note that $\xi_j^{(\epsilon)}$ is defined so that the underlying asset price does not become negative by its jumps. In this case, the compensated Poisson measure $\tilde{\mu}(dt,dx)$ is given by

$$
\tilde{\mu}(dt,dx) = \mu(dt,dx) - \lambda dt \times \nu(dx).
$$

Note that this model is an extension of Merton[24].

Here, if I assume $r$ to be a nonnegative constant, $V$, the contract price of a vanilla call option with the maturity $T$ and the strike price $K(>0)$ is represented as $V = e^{-rT}\mathbf{E}[(S_T^{(\epsilon)} - K)_+]$; $(x)_+$ denotes $\max\{x,0\}$. Here, in order to apply an asymptotic expansion, I define $X^{(\epsilon)}$ as

$$
X_t^{(\epsilon)} = \frac{S_t^{(\epsilon)} - S_t^{(0)}}{\epsilon},
$$

where $S_t^{(0)}$ is given by $S_t^{(0)} = S_0 e^{\alpha t}$ and $X^{(\epsilon)}$ follows the stochastic differential equation;

$$\begin{aligned} dX_t^{(\epsilon)} &= \alpha X_t^{(\epsilon)} dt + \sigma(\epsilon X_t^{(\epsilon)} + S_t^{(0)}) dw_t \\ &\quad + \int_{\mathbf{R}} (\epsilon X_{t-}^{(\epsilon)} + S_{t-}^0) \left(\frac{e^{\epsilon x}-1}{\epsilon}\right) \tilde{\mu}(dt,dx);\ X_0^{(\epsilon)} = 0. \end{aligned}$$

With this $X^{(\epsilon)}$, the option value $V$ is expressed as

$$V = e^{-rT} \epsilon \mathbf{E}[(X_T^{(\epsilon)} + k^{(\epsilon)})_+],$$

where $k^{(\epsilon)} = \frac{S_T^{(0)} - K}{\epsilon}$. As in the Assumption 2 in Section 2.2, assume that $k^{(\epsilon)}$ is represented as $k^{(\epsilon)} = k + O(\epsilon)$ by a real number $k$. Then, the asymptotic expansion up to the $\epsilon$ order of the call option value $V$ is expressed as

$$V = e^{-rT} \epsilon \mathbf{E}[(X_T^{(0)} + k)_+] + o(\epsilon).$$

$X^{(0)}$ in this equation follows the stochastic differential equation:

$$dX_t^{(0)} = \alpha X_t^{(0)} dt + \sigma(S_t^{(0)}, t) dw_t + \int_{\mathbf{R}} S_{t-}^{(0)} x \tilde{\mu}(dt,dx);\ X_0^{(0)} = 0.$$

Moreover, by further calculation, $\mathbf{E}[(X_T^{(0)} + k)_+]$ is expressed as

$$\begin{aligned} \mathbf{E}[(X_T^{(0)} + k)_+] &= \sum_{j=0}^{\infty} \mathbf{E}\left[\Sigma_T\, n\left(k_2 + S_0 e^{\alpha T} \sum_{i=1}^{j} \eta_i\right)\right. \\ &\quad + \left.(k_2 + S_0 e^{\alpha T} \sum_{i=1}^{j} \eta_i) N\left(\frac{k_2}{\sqrt{\Sigma_T}} + \frac{S_0 e^{\alpha T}}{\sqrt{\Sigma_T}} \sum_{i=1}^{j} \eta_i\right)\right] e^{-\lambda T} \frac{(\lambda T)^j}{j!}, \end{aligned}$$

where $\sum_{i=1}^{0} \eta_i = 0$. $k_2$, $n(x)$ and $\Sigma_T$ are given by

$$\begin{aligned} k_2 &= k - v(\lambda T) S_0 e^{\alpha T},\ n(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(\frac{-x^2}{2\Sigma_T}\right), \\ \Sigma_T &= \int_0^T e^{2\alpha(T-s)} \sigma^2(S_s^0, s) ds. \end{aligned} \tag{3.8}$$

Suppose also that $\Sigma_T > 0$. Moreover, with an assumption that the random variables $(\eta_i)_{i \geq 1}$ which determine the distributions of the jump sizes follow a normal one, a more concrete expression can be obtained.

**Theorem 3.1.** *Suppose that the underlying asset price follows the stochastic process in (3.7) and $(\eta_i)_{i\geq 1}$ follow a normal distribution, $\eta_i \sim N(v, \sigma^2)$. Then, under the assumption that $\Sigma_T > 0$, an asymptotic expansion up to the $\epsilon$-order of $V$, the contract price of a vanilla call option with the maturity date $T$ and the strike price $K$, is given by*

$$\begin{aligned} V &= e^{-rT} \epsilon \left\{ \sum_{j=0}^{\infty} \left[ \sqrt{\frac{\Sigma_T}{2\pi(c_{4j}^2 + 1)}} \exp\left(\frac{-c_{3j}^2}{2(c_{4j}^2+1)}\right) + c_{1j} N\left(\frac{c_{3j}}{\sqrt{c_{4j}^2 + 1}}\right) \right.\right. \\ &\quad + \left.\left. \frac{c_{2j} c_{4j}}{\sqrt{2\pi(c_{4j}^2+1)}} \exp\left(\frac{-c_{3j}^2}{2(c_{4j}^2+1)}\right)\right] e^{-\lambda T} \frac{(\lambda T)^j}{j!}) \right\} + o(\epsilon), \end{aligned} \tag{3.9}$$

*where $k_2$ and $\Sigma_T$ are defined as (3.8), and the constant coefficients $c_{1j}$, $c_{2j}$, $c_{3j}$ and $c_{4j}$ are defined as*

$$\begin{aligned} c_{1j} &\equiv k_2 + S_0 e^{\alpha T}(vj) \ , c_{2j} \equiv S_0 e^{\alpha T}(\sigma\sqrt{j}) \ , \\ c_{3j} &\equiv \frac{c_{1j}}{\sqrt{\Sigma_T}} \ , c_{4j} \equiv \frac{c_{2j}}{\sqrt{\Sigma_T}}. \end{aligned}$$

For the details such as numerical examples, see [15] and [16].

## 4. Conclusion

Finally, I briefly review other applications of the asymptotic expansion technique to numerical problems in finance, which can not be introduced in the paper because of the space limitation.

[37] applies an asymptotic expansion to a dynamic investment problem with utility maximization for the asset at the end of the investment period, and derives an approximation formula for evaluating the optimal portfolio. Although the optimal portfolio has been numerically evaluated as a function of derivatives of the solution to some Bellman equation except for special cases, it is a hard task to compute it when the number of assets is large. [37] provides its approximation based on the representation which Ocone-Karatzas [26] derives by using the Clark-Ocone formula. Moreover, [11] applies this method to a dynamic bond portfolio problem.

In evaluation of the expectation of a Wiener functional by Monte Carlo simulation, [38] proposes a new estimator using a random variable that has its expectation explicitly obtained by an asymptotic expansion and has a high correlation with the objective Wiener functional. The convergence of the simulation based on this estimator becomes faster and the approximation error due to the asymptotic expansion up to a low order is decreased. As for the extension of this method, see [34], [16], and [36].

[35] extends the decomposition formula for an American option value by Carr-Jarrow-Myneni [3] and proposes an approximation of the value applying the fact that the density function of the underlying asset can be approximated by the asymptotic expansion.

[23] provides approximations for the risk indicators of options by asymptotic expansions of the derivatives of a stochastic differential equation with respect to parameters as well as by extending the variance reduction method of Monte Carlo simulation in [38].

[43] studies the bias correction when unknown parameters in the representation of an option value are substituted by their estimators. Kawai [10] applies the asymptotic expansion method to the market models of interest rates developed by Brace-Gatarek-Musiela [2], Jamshidian [7] and others.

This paper illustrates the cases where the diffusion terms of the stochastic processes for the asset prices are 0 when $\epsilon = 0$. However, expansions for the cases where the diffusion processes are not 0 when $\epsilon = 0$, but become the processes that are relatively easy to be computed are also considered. For more details, see Kashiwakura-Yoshida [9], Lütkebohmert [20] and [37] for instance.

As introduced above, various applications of the asymptotic expansion method are developed for numerical problems in finance.

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