An Asymptotic Expansion Approach to Computing Greeks *

Ryosuke Matsuoka † and Akihiko Takahashi ‡

Abstract

We developed a new scheme for computing "Greeks" of derivatives by an asymptotic expansion approach. In particular, we derived analytical approximation formulae for Deltas and Vegas of plain vanilla and average call options under general Markovian processes of underlying asset prices. We also derived approximation formulae for Gammas of plain vanilla and average call options, and for Deltas of digital options under CEV (Constant Elasticity of Variance) processes of underlying assets’ prices. Moreover, we introduced a new variance reduction method of Monte Carlo simulations based on the asymptotic expansion scheme. Finally, several numerical examples under CEV processes confirmed the validity of our method.

---

*This paper presents the authors’ personal views, and these are not the official views of the Financial Services Agency, the Financial Research and Training Center, and of Tokio Marine & Nichido.
†Tokio Marine & Nichido Fire Insurance Co., Ltd.
‡Graduate School of Economics, The University of Tokyo, Special Research Fellow at the Financial Research and Training Center, Financial Services Agency
1 Introduction

We propose a new approximation formulae for computing Greeks, such as the Delta, the Gamma and the Vega that indicate risk indices, the first or second order derivatives of the value of an asset with respect to its parameters. In particular, we derive analytic approximation formulae of the Delta and the Vega of plain-vanilla and average options where the underlying assets’ prices follow general diffusion processes. We also show formulae of the Gamma of plain-vanilla and average options, and a formula of the Delta of digital options where the underlying assets’ prices follow CEV processes. Our method is based on the asymptotic expansion approach developed by Takahashi[1995,1999], Kunitomo and Takahashi[1992,2001,2003a] and Takahashi and Yoshida[2004]. We also introduce a new variance reduction method as an extension of Takahashi and Yoshida[2005] to increase efficiency of Monte Carlo simulation in computation of Greeks. Moreover, we presented series of numerical examples where the underlying prices follow the constant elasticity of variance (CEV) processes and showed effectiveness of our method.

Monitoring and controlling the risks of derivative securities are as important as pricing derivative securities in the practical world. In Black-Scholes(BS) model, option prices and their Greeks are obtained analytically. However, in the more realistic models, it is usually very hard to evaluate both of option prices and their Greeks analytically. Then, numerical methods are applied.

We suppose that the underlying asset’s price \( S_t \) follows a stochastic differential equation (SDE);

\[
\begin{align*}
    dS_t &= \mu S_t \, dt + \sigma (S_t) \, dW_t \\
    S_0 &= s_0 (>0),
\end{align*}
\]

where \( W_t \) is a one-dimensional Brownian motion, \( \mu = r - q \). \( r \) and \( q \) denote a risk-free rate and a dividend rate respectively which are assumed to be constants. Next, we consider a derivative security whose payoff function at maturity time \( T \) is given by \( \phi \) such as

\[
\begin{align*}
    \phi (S_T) &= (S_T - K)_+ \quad \text{plain vanilla call option} \\
    \phi (S_T) &= (\tilde{S}_T - K)_+ \quad \text{average call option},
\end{align*}
\]

where \( \tilde{S}_T = \frac{1}{T} \int_0^T S_t \, dt \). The price of the derivative security can be represented by

\[ u(s_0) = E [ e^{-rT} \phi (S_T) ] = e^{-rT} E [ \phi (S_T) ], \]

where \( E [ \cdot ] \) denotes the expectation operator under the risk-neutral measure. To obtain the first order derivative of the price with respect to the underlying asset’s price at the initial price \( s_0 \), a natural method is computing

\[ u'(s_0) \sim \frac{u(s_0 + \Delta) - u(s_0 - \Delta)}{2\Delta}, \]

where \( \Delta \) is a sufficiently small positive number, and \( u(s_0 - \Delta) \) and \( u(s_0 + \Delta) \) are generated by Monte Carlo simulations. However, its convergence speed is sometimes very slow due to the irregularity of \( \phi \). Moreover, convergence to the true value may not be achieved because its convergent value depends on \( \Delta \).

To overcome the problem, we may utilize a representation such as

\[ u'(s_0) = e^{-rT} E [ \phi' (S_T) Y_T ] , \]
where $Y_t$ satisfies the following SDE:

$$
\begin{align*}
\begin{cases}
    dY_t &= \mu Y_t \, dt + \sigma'(S_t) \, Y_t \, dW_t \\
    Y_0 &= 1
\end{cases}
\end{align*}
$$

(For the derivation, see Imamura, Uchida and Takahashi [2005] for instance.) Although this representation makes Monte Carlo simulations more efficient, using Monte Carlo simulation itself may be time-consuming, which cause many difficulties in the practical world such as trading business. Therefore, in the case that analytic formulae can not be obtained, ideally we need analytic approximation schemes which can generate values precise enough for practical purpose.

The asymptotic expansion approach have been applied successfully to a broad class of Itô processes appearing in finance. Takahashi[1999] presented a third-order pricing formula for plain options and second-order formulae for more complicated derivatives such as average options, basket options, and options with stochastic volatility in a general Markovian setting. Kunitomo and Takahashi[2001] provided pricing formulae for bond options(swap options) and average options based on an interest rate model in the class of Heath-Jarrow-Morton[1992] which is not necessarily Markovian. Takahashi[1995] also presented a second order scheme for average options on foreign exchange rates with stochastic interest rates in Heath-Jarrow-Morton framework.


The organization of this paper is as follows. In section 2 and 3, we derive approximation formulae for the Delta of plain-vanilla and average call options. In section 4, we introduce a variance reduction technique in computation of the Delta. Sections 5 and 6 treat approximations for computing the Vega of plain vanilla and average call options. In section 7, we propose a variance reduction technique in computation of the Vega. In section 8, when the underlying prices follow CEV processes, we show another derivation of approximation formulae for the Delta and the Vega, and derive approximation formulae for the Gamma of plain vanilla and average options as well as for the Delta of digital options. We present numerical examples in section 9. Finally, some concluding remarks are made in section 10.

### 2 The Delta of Plain Vanilla Call Option

First, we make the following assumption and definitions.

**Assumption 1.** We assume that the underlying asset price $S_t$ follows the SDE:

$$
\begin{align*}
\begin{cases}
    dS_t &= \mu S_t \, dt + \sigma (S_t) \, dW_t \\
    S_0 &= s_0 \ (> 0)
\end{cases}
\end{align*}
$$

where $W_t$ is 1-dimensional Brownian Motion, and $\mu \neq 0$. In addition we assume that $\sigma (\cdot)$ is a $C^2$ class function and its derivative $\sigma' (\cdot)$ is bounded.

We define the "differentiation of $S_t$ by the initial value $s_0$".
Definition 1. We define $Y_t$ as a stochastic process following the SDE:

$$
\begin{align*}
\{ & dY_t = \mu Y_t dt + \sigma Y_t dW_t \\
& Y_0 = 1
\end{align*}
$$

Definition 2. We define the defined indicator function:

$$
1_{\{x \geq a\}} = \begin{cases} 
1 & \text{if } x \geq a \\
0 & \text{if } x < a
\end{cases}
$$

Then, the payoff of Plain Vanilla Call Option is rewritten as $(S_T - K)_+ = (S_T - K) 1_{\{S_T \geq K\}}$.

Takahashi [1999] derived, the asymptotic expansion of $S_t$:

$$
S_t = A_{0t} + \epsilon A_{1t} + \frac{\epsilon^2}{2} A_{2t} + o(\epsilon^2),
$$

where

$$
\begin{align*}
dA_{0t} &= \mu A_{0t} dt, & A_{00} &= s_0 \\
dA_{1t} &= \mu A_{1t} dt + \sigma (A_{0t}) dW_t, & A_{10} &= 0 \\
dA_{2t} &= \mu A_{2t} dt + 2 \sigma' (A_{0t}) A_{1t} dW_t, & A_{20} &= 0.
\end{align*}
$$

Then, we can easily obtain $A_{0t}$, $A_{1t}$ and $A_{2t}$ as

$$
A_{0t} = s_0 e^{\mu t} \\
A_{1t} = \int_0^t e^{\mu (t-s)} \sigma (A_{0s}) dW_s \\
A_{2t} = 2 \int_0^t e^{\mu (t-s)} \sigma' (A_{0s}) A_{1s} dW_s.
$$

Now we put $X_t = (S_t - A_{0t}) / \epsilon$, $g_1 = A_{1T}$ and $g_2 = A_{2T} / 2$. Then,

$$
X_T = g_1 + \epsilon g_2 + o(\epsilon^2).
$$

The asymptotic expansion of the Plain Vanilla Call Option $e^{-rT} E [(S_T - K)_+]$ is given by

$$
\begin{align*}
e^{-rT} E [(S_T - K)_+] &= \epsilon e^{-rT} E [(y + X_T)_+] \\
&= \epsilon e^{-rT} E [(y + g_1 + \epsilon g_2 + \cdots) 1_{\{g_1+\epsilon g_2+\cdots \geq -y\}}],
\end{align*}
$$

where $y = (A_{0T} - K) / \epsilon$. Under an appropriate assumption such as Assumption 6.2 in Kunitomo and Takahashi[2003b], We obtain its approximated value as

$$
\begin{align*}
e^{-rT} E [(S_T - K)_+] &= \epsilon e^{-rT} E [(y + g_1) 1_{\{g_1 \geq -y\}}] + \epsilon^2 e^{-rT} E [g_2 1_{\{g_1 \geq -y\}}] \\
&+ \epsilon^2 e^{-rT} E [(y + g_1) g_2 (\partial_1)_{\{g_1 \geq -y\}}] + o(\epsilon^2).
\end{align*}
$$
where \((\partial l)_{l \geq -y}\) denotes the derivative of \(1_{l \geq -y}\). Next, by the property of Brownian Motion, we know the distribution of \(g_1\) follows a normal distribution \(N(0, \Sigma)\), where

\[
\Sigma = E \left[ \left( \int_0^T e^{\mu(T-t)} \sigma(A_{0t}) dW_t \right)^2 \right] = \int_0^T e^{2\mu(T-t)} \sigma(A_{0t})^2 dt. \tag{4}
\]

In addition, the conditional expectation \(E[g_2|g_1 = x]\) is given by

\[
E[g_2|g_1 = x] = cx^2 + f,
\]

where

\[
c = \frac{1}{\Sigma^2} \int_0^T \int_0^T e^{\mu(T-s)} \sigma(A_{0s}) \sigma'(A_{0s}) e^{2\mu(T-v)} \sigma(A_{0v})^2 dv ds
\]

and \(f = -c \Sigma\). Accordingly,

\[
e^{-rT} E \left[ (S_T - K)^+ \right] = e^{-rT} \int_{-y}^\infty (y + x) n \left[ x; 0, \Sigma \right] dx + \epsilon^2 e^{-rT} \int_{-y}^\infty (cx^2 + f) n \left[ x; 0, \Sigma \right] dx + o(\epsilon^2),
\]

where \(n \left[ x; 0, \Sigma \right]\) is the density function of \(N(0, \Sigma)\), i.e.

\[
n \left[ x; 0, \Sigma \right] = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}}.
\]

Now we define an approximated value of the call option as

\[
C(0, T) = e^{-rT} \int_{-y}^\infty (y + x) n \left[ x; 0, \Sigma \right] dx + \epsilon^2 e^{-rT} \int_{-y}^\infty (cx^2 + f) n \left[ x; 0, \Sigma \right] dx
\]

where \(N(x)\) is cumulative distribution function of \(N(0, 1)\). Next, consider its differentiation by \(s_0\):

\[
\frac{\partial}{\partial s_0} C(0, T)
\]

\[
= e^{-rT} d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \left( y \left( d - \frac{y}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) + \frac{\partial \Sigma}{\partial s_0} + \Sigma \left( -\frac{1}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \left[ y; 0, \Sigma \right]
\]

\[
+ \epsilon^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} y + f d + f y \left( -\frac{1}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \left[ y; 0, \Sigma \right]
\]

\[
= e^{-rT} d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n \left[ y; 0, \Sigma \right]
\]

\[
+ \epsilon^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} y + f d + cy \left( \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \left[ y; 0, \Sigma \right],
\]

76
where \( d = \frac{\partial y}{\partial s_0} = e^{\mu T}/\epsilon \). The derivatives of coefficients are calculated as follows:

\[
\frac{\partial \Sigma}{\partial s_0} = \frac{\partial}{\partial s_0} \int_0^T e^{2\mu(T-t)} \sigma(A_0t)^2 \, dt \\
= 2 \int_0^T e^{2\mu(T-t)} e^{\mu t} \sigma(A_0t) \sigma'(A_0t) \, dt \\
= 2 \int_0^T e^{2\mu T - \mu t} \sigma(A_0t) \sigma'(A_0t) \, dt
\]

\[
\frac{\partial c}{\partial s_0} = \frac{\partial}{\partial s_0} \left\{ \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu(T-s)} \sigma(A_0s) \sigma'(A_0s) e^{2\mu(T-v)} \sigma(A_0v)^2 \, dv \, ds \right\} \\
= -\frac{2}{\Sigma^3} \frac{\partial \Sigma}{\partial s_0} \int_0^T \int_0^s e^{\mu(T-s)} \sigma(A_0s) \sigma'(A_0s) e^{2\mu(T-v)} \sigma(A_0v)^2 \, dv \, ds \\
+ \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu T} \sigma'(A_0s)^2 e^{2\mu(T-v)} \sigma(A_0v)^2 \, dv \, ds \\
+ \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu T} \sigma(A_0s) \sigma''(A_0s) e^{2\mu(T-v)} \sigma(A_0v)^2 \, dv \, ds \\
+ \frac{2}{\Sigma^2} \int_0^T \int_0^s e^{\mu(T-s)} \sigma(A_0s) \sigma'(A_0s) e^{2\mu T - \mu v} \sigma(A_0v) \sigma'(A_0v) \, dv \, ds
\]

\[
\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0} (c\Sigma) = -\frac{\partial c}{\partial s_0} \Sigma - \frac{\partial \Sigma}{\partial s_0}
\]

Although it looks quite complicated, simpler expressions of coefficients can be derived if we give \( \sigma(\cdot) \) specific forms such as CEV, which are shown in section 8.

Then, we summarize the result obtained above.

**Theorem 1.** The asymptotic expansion of the Delta of a plain vanilla call option of which payoff function with maturity time \( T \) is expressed as \((S_T - K)_+\) is given by

\[
D(0, T) + o(\epsilon^2),
\]

where

\[
D(0, T) = e^{-rT} \left( dN \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n[y; 0, \Sigma] \right) \\
+ \epsilon^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} y + f d + c y \left( \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) \right) n[y; 0, \Sigma],
\]

and the coefficients are given by

\[
y = \frac{s_0 e^{\mu T} - K}{\epsilon},
\]

77
\[ \Sigma = E \left[ \left( \int_0^T e^\mu (T-t) \sigma (A_{0t}) \, dW_t \right)^2 \right] \]
\[ = \int_0^T e^{2\mu (T-t)} \sigma (A_{0t})^2 \, dt, \]
\[ c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu (T-t)} \sigma (A_{0s}) \sigma' (A_{0s}) e^{2\mu (T-v)} \sigma (A_{0v})^2 \, dv \, ds \]

and
\[ f = -c\Sigma. \]

3 The Delta of Average Call Option

Utilizing the asymptotic expansion of \( S_t \), Takahashi [1999] derived
\[ \tilde{S}_t \sim \tilde{A}_{0T} + \epsilon \tilde{A}_{1T} + \frac{\epsilon^2}{2} \tilde{A}_{2T} + o(\epsilon^2), \]
where \( \tilde{S}_t = \frac{1}{T} \int_0^T S_u \, du, \tilde{A}_{0T} = \frac{1}{T} \int_0^T A_{0u} \, du \) and so on. Now we put \( \tilde{X}_t = (\tilde{S}_t - \tilde{A}_{0T}) / \epsilon, g_1 = \tilde{A}_{1T} \) and \( g_2 = \tilde{A}_{2T} / 2 \). Then,
\[ \tilde{X}_T \sim g_1 + \epsilon g_2 + o(\epsilon^2). \]

The asymptotic expansion of the average call option \( e^{-rT} E \left[ (\tilde{S}_T - K)^+ \right] \) is given by
\[ e^{-rT} E \left[ (\tilde{S}_T - K)^+ \right] = e^{-rT} E \left[ (y + \tilde{X}_T)^+ \right] \]
\[ = e^{-rT} E \left[ (y + g_1 + \epsilon g_2 + \cdots) 1_{\{g_1 + \epsilon g_2 + \cdots \geq -y\}} \right], \]
where \( y = (\tilde{A}_{0T} - K) / \epsilon \). We obtain its approximated value
\[ e^{-rT} E \left[ (\tilde{S}_T - K)^+ \right] \]
\[ = e^{-rT} E \left[ (y + g_1) 1_{\{g_1 \geq -y\}} \right] + \epsilon^2 e^{-rT} E \left[ g_2 1_{\{g_1 \geq -y\}} \right] \]
\[ + \epsilon^2 e^{-rT} E \left[ (y + g_1) g_2 (\partial 1)_{\{g_1 \geq -y\}} \right] + o(\epsilon^2), \]
where
\[ \Sigma = E \left[ \left( \frac{1}{T} \int_0^T \int_0^t e^{\mu (t-s)} \sigma (A_{0s}) \, dW_s \, dt \right)^2 \right] \]
\[ = \frac{1}{T^2} E \left[ \left( \int_0^T \int_s^T e^{\mu (t-s)} \sigma (A_{0s}) \, dt \, dW_s \right)^2 \right] \]
\[ = \frac{1}{T^2} E \left[ \left( \int_0^T \frac{1}{\mu} (e^{\mu (T-s)} - 1) \sigma (A_{0s}) \, dW_s \right)^2 \right] \]
\[ = \frac{1}{\mu^2 T^2} \int_0^T (e^{\mu (T-s)} - 1)^2 \sigma (A_{0s})^2 \, ds. \]
In addition, the coefficients of the conditional expectation $E \left[ g_2 | g_1 = x \right] = cx^2 + f$ are given by

$$c = \frac{1}{\Sigma^2 T^2} \int_0^T \int_0^t e^{\mu (t-s)} \left[ \frac{e^{\mu (T-s)} - 1}{\mu} \right] \sigma (A_{0s}) \sigma' (A_{0s})$$

$$\times \int_s^T e^{\mu (s-v)} \left[ \frac{e^{\mu (T-v)} - 1}{\mu} \right] \sigma (A_{0v})^2 dv \, ds \, dt$$

and $f = -c \Sigma$. Accordingly,

$$e^{-rT} E \left[ (S_T - K)_{+} \right] = e^{-rT} \int_{-y}^{\infty} (y + x) \, n \, dx + e^2 e^{-rT} \int_{-y}^{\infty} (cx^2 + f) \, n \, dx + o (e^2).$$

Now we define the approximated value as

$$C (0, T) = e^{-rT} \int_{-y}^{\infty} (y + x) \, n \, dx + e^2 e^{-rT} \int_{-y}^{\infty} (cx^2 + f) \, n \, dx$$

$$= e^{-rT} \left( y N \left( \frac{y}{\sqrt{\Sigma}} \right) + \Sigma n [y; 0, \Sigma] \right) + e^2 e^{-rT} f \, n \, [y; 0, \Sigma],$$

(10)

and consider its differentiation by $s_0$:

$$\frac{\partial}{\partial s_0} C (0, T) = e^{-rT} \left( d \, N \left( \frac{y}{\sqrt{\Sigma}} \right) + \left( y \left( d - \frac{y}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) + \frac{\partial \Sigma}{\partial s_0} + \Sigma \left( -\frac{1}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \, [y; 0, \Sigma] \right)$$

$$+ e^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} + f \, d + f \, y \left( -\frac{1}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \, [y; 0, \Sigma]$$

$$= e^{-rT} \left( d \, N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \Sigma \frac{\partial \Sigma}{\partial s_0} n \, [y; 0, \Sigma] \right)$$

$$+ e^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} + f \, d + c \, y \left( \frac{1}{2} \Sigma \frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) \right) n \, [y; 0, \Sigma],$$

where $d = \frac{\partial y}{\partial s_0} = (e^{\mu T} - 1) / (\mu T \epsilon)$. The derivatives of coefficients are calculated as

$$\frac{\partial \Sigma}{\partial s_0} = \frac{1}{\mu^2 T^2} \int_0^T \left( e^{\mu (T-s)} - 1 \right)^2 \sigma (A_{0s})^2 ds$$

$$= \frac{2}{\mu^2 T^2} \int_0^T \left( e^{\mu (T-s)} - 1 \right)^2 \mu^2 \sigma (A_{0s}) \sigma' (A_{0s}) ds$$

$$= \frac{2}{\mu^2 T^2} \int_0^T \left( e^{2 \mu T - \mu s} - 2 e^{\mu T} + e^{\mu s} \right) \sigma (A_{0s}) \sigma' (A_{0s}) ds.$$
\[
\frac{\partial c}{\partial s_0} = \frac{\partial}{\partial s_0} \left( \frac{1}{\Sigma^2 T^3} \right) \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma(A_0^v) \sigma'(A_0^v) \times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma(A_0^v) \sigma'(A_0^v) dv \, ds \, dt \}
\]

\[
= -\frac{2}{\Sigma^2} \frac{\partial \Sigma}{\partial s_0} \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma(A_0^v) \sigma'(A_0^v) \times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma(A_0^v) \sigma'(A_0^v) dv \, ds \, dt + \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-s)} - 1}{\mu} \right] \sigma'(A_0^v)^2 \times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma'(A_0^v)^2 dv \, ds \, dt
\]

\[
+ \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-s)} - 1}{\mu} \right] \sigma'(A_0^v)^2 \times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma'(A_0^v)^2 dv \, ds \, dt + \frac{2}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-s)} - 1}{\mu} \right] \sigma'(A_0^v) \sigma''(A_0^v) \times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma'(A_0^v) \sigma''(A_0^v) dv \, ds \, dt
\]

and

\[
\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0} (\Sigma) = -\frac{\partial c}{\partial s_0} \Sigma - \frac{\partial \Sigma}{\partial s_0}.
\]

Then we sum up the result as a theorem.

**Theorem 2.** The asymptotic expansion of the Delta of an average call option, whose payoff function with maturity time \(T\) is expressed as \(\left( \tilde{S}_T - K \right)_+\) is given by

\[
D(0, T) + o(\epsilon^2),
\]

where

\[
D(0, T) = e^{-rT} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n[y; 0, \Sigma] \right) + \epsilon^2 e^{-rT} \left( \frac{\partial f}{\partial s_0} y + f d + c y \left( \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} + y d - \frac{y^2}{2\Sigma} \frac{\partial \Sigma}{\partial s_0} \right) \right) n[y; 0, \Sigma],
\]

and the coefficients are given by

\[
y = \frac{s_0 (e^{\mu T} - 1)}{\epsilon \mu T - K},
\]

80
\[
d = \frac{e^{\mu T} - 1}{e\mu T},
\]

\[
\Sigma = E \left[ \left( \int_0^T \int_0^t e^{\mu (t-s)} \sigma (A_{0s}) \, dW_s \, dt \right)^2 \right]
\]

\[
= \frac{1}{\mu^2 T^2} \int_0^T \left( e^{\mu (T-s)} - 1 \right)^2 \sigma (A_{0s})^2 \, ds,
\]

\[
c = \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu (t-s)} \left[ \frac{e^{\mu (T-s)} - 1}{\mu} \right] \sigma (A_{0s}) \sigma' (A_{0s}) \times \int_0^s e^{\mu (s-v)} \left[ \frac{e^{\mu (T-v)} - 1}{\mu} \right] \sigma (A_{0v})^2 \, dv \, ds \, dt,
\]

and

\[
f = -c\Sigma.
\]

### 4 Variance Reduction Technique for Delta of Call Option

Previous section showed the validity of the asymptotic expansion technique for the Delta, but we sometimes need more precise values, especially when \(\epsilon\) is quite large or when the true value is quite small. For this purpose, we propose a variance reduction method, which is an extension of Takahashi and Yoshida [2004].

Although the value of Delta is obtained by calculating \(E \left[ e^{-rT} Y_T 1_{\{S_T \geq K\}} \right]\), it may be impossible to calculate it analytically. In the case, Monte Carlo simulation based on Euler-Maruyama Scheme is a standard method, which is called "crude Monte Carlo" in this paper. However, crude Monte Carlo is usually quite time-consuming because of its low convergence speed, and hence, Takahashi and Yoshida [2004] proposed the method based on an asymptotic approach to accelerate the simulation.

The convergence speed depends on the variance of sample paths. The smaller the variance, the higher its convergence speed. Let \(X\) be a random variable. We take another random variable \(Y\) whose expectation is 0. Then, the random variable \(X - Y\) has the same expectation as \(X\), and its variance is given by

\[
Var (X - Y) = E \left[ (X - E [X] - Y)^2 \right]
\]

\[
= E \left[ (X - E [X])^2 \right] - 2 E [(X - E [X]) Y] + E [Y^2]
\]

\[
= Var (X) + Var (Y) - 2 Corr (X, Y) \sqrt{Var (X) Var (Y)}
\]

\[
= Var (X) - \sqrt{Var (Y)} \left( 2 Corr (X, Y) \sqrt{Var (X)} - \sqrt{Var (Y)} \right),
\]

where \(Corr (X, Y)\) denotes the correlation between \(X\) and \(Y\). Let \(a > 0\), and we use \(aY\) instead of \(Y\). Then, \(Var (X - aY)\) is minimized when \(a = Corr (X, Y) \sqrt{\frac{Var (X)}{Var (Y)}}\), and we obtain

\[
Var (X - aY) = Var (X) - Corr (X, Y)^2 Var (X)
\]

\[
= \left( 1 - Corr (X, Y)^2 \right) Var (X).
\]

Thus, we have to find \(Y\) whose correlation with \(X\) is close to one.
We utilize the approximated Delta denoted by $D(0,T)$. In the derivation of the theorem 1 (or theorem 2), we obtain the approximated option price in an integral representation:

$$C(0,T) = e^{-rT} \int_{-y}^{\infty} (y + x) \ n [x; 0, \Sigma] \ dx + e^{2rT} \int_{-y}^{\infty} (cx^2 + f) \ n [x; 0, \Sigma] \ dx.$$

The candidate for a random variable $Y$ is obtained by differentiating it with respect to $s_0$.

Next, we prepare two lemmas for the ease of calculation.

**Lemma 1.** Let $\varphi(x)$ be a $C^\infty_0$ class function. Then

$$\int_{-\infty}^{\infty} (\partial_1)_{\{x \geq y\}} \varphi(x) \ dx = \varphi(-y). \quad (12)$$

**Proof.**

$$\int_{-\infty}^{\infty} (\partial_1)_{\{x \geq y\}} \varphi(x) \ dx = - \int_{-\infty}^{\infty} 1_{\{x \geq y\}} \varphi'(x) \ dx = \varphi(-y)$$

** Lemma 2.** The differentiation of $n \ [x; 0, \Sigma]$ with respect to $s_0$ is

$$\frac{\partial}{\partial s_0} n \ [x; 0, \Sigma] = \frac{1}{2} \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} n \ [x; 0, \Sigma] \quad (13)$$

**Proof.**

$$\frac{\partial}{\partial s_0} n \ [x; 0, \Sigma] = \frac{\partial}{\partial s_0} \left\{ \frac{1}{\sqrt{2\pi}} \Sigma^{-1/2} \exp \left( -\frac{1}{2} x^2 \Sigma^{-1} \right) \right\}$$

$$= -\frac{1}{2} \frac{1}{\sqrt{2\pi}} \Sigma^{-3/2} \frac{\partial \Sigma}{\partial s_0} \exp \left( -\frac{1}{2} x^2 \Sigma^{-1} \right) + \frac{1}{\sqrt{2\pi}} \Sigma^{-1/2} \exp \left( -\frac{1}{2} x^2 \Sigma^{-1} \right) \frac{x^2 \Sigma^{-2}}{\partial s_0}$$

$$= \frac{1}{2} \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \frac{1}{\sqrt{2\pi}} \Sigma^{-1/2} \exp \left( -\frac{1}{2} x^2 \Sigma^{-1} \right)$$

$$= \frac{1}{2} \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} n \ [x; 0, \Sigma]$$
Using lemmas 1 and 2, the approximation of Delta is obtained as

\[
D(0, T) = e^{-rT} \int_{-y}^{y} \left( d + \frac{1}{2} (y + x) \left( \frac{x^2}{\Sigma} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \right) n \left[ x; 0, \Sigma \right] dx + \epsilon^2 e^{-rT} \int_{-y}^{y} \left( cx^2 + f \right) n \left[ x; 0, \Sigma \right] dx
\]

On the other hand, \( D(0, T) \) is expressed as

\[
D(0, T) = e^{-rT} E \left[ d + \frac{1}{2} (y + g_1) \left( \frac{g_1^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \right] + \epsilon \left( d (cy^2 + f) n \left[ y; 0, \Sigma \right] + \left( \frac{\partial c}{\partial s_0} g_1^2 + \frac{\partial f}{\partial s_0} \right) + \frac{1}{2} (cg_1^2 + f) \left( \frac{g_1^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \right) 1_{\{y \geq -y\}}
\]

where

\[
\varphi(x) = e^{-rT} \left( d + \frac{1}{2} (y + x) \left( \frac{x^2}{\Sigma} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \right) + \epsilon \left( \left( \frac{\partial c}{\partial s_0} x^2 + \frac{\partial f}{\partial s_0} \right) + \frac{1}{2} (cx^2 + f) \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \frac{\partial \Sigma}{\partial s_0} \right) 1_{\{x \geq -y\}} + cd (cy^2 + f) n \left[ y; 0, \Sigma \right].
\]

Finally, we set \( Y = \varphi(g_1) - D(0, T) \). We call this new method "Hybrid Monte Carlo".

5 The Vega of Plain Vanilla Call Option

In the previous section, we obtained an approximated value of Plain Vanilla Call Option price as

\[
C(0, T) = e^{-rT} \int_{-y}^{y} (y + x) n \left[ x; 0, \Sigma \right] dx + \epsilon^2 e^{-rT} \int_{-y}^{y} (cx^2 + f) n \left[ x; 0, \Sigma \right] dx
\]

\[
= e^{-rT} \left( \frac{y N \left( \frac{y}{\sqrt{\Sigma}} \right)}{\sqrt{\Sigma}} + \Sigma n \left[ y; 0, \Sigma \right] \right) + \epsilon^2 e^{-rT} f y n \left[ y; 0, \Sigma \right],
\]

\[83\]
and consider its differentiation by $\epsilon$ to obtain an approximation of the Vega.

$$\frac{\partial}{\partial \epsilon} C (0, T) = \frac{\partial}{\partial \epsilon} \left( e^{-rT} \left( y N \left( \frac{y}{\sqrt{\Sigma}} \right) + \Sigma n [y; 0, \Sigma] \right) + e^{-rT} f y n [y; 0, \Sigma] \right)$$

$$= e^{-rT} \left( y N \left( \frac{y}{\sqrt{\Sigma}} \right) + \Sigma n [y; 0, \Sigma] - y N \left( \frac{y}{\sqrt{\Sigma}} \right) - y^2 n [y; 0, \Sigma] + y^2 n [y; 0, \Sigma] \right)$$

$$+ 2e^{-rT} f y n [y; 0, \Sigma] - \epsilon e^{-rT} f y n [y; 0, \Sigma] + \epsilon \frac{f y^3}{\Sigma} n [y; 0, \Sigma]$$

$$= e^{-rT} \left( \Sigma + \epsilon \left( f y + \frac{f y^3}{\Sigma} \right) \right) n [y; 0, \Sigma].$$

Then, we summarize the result as a theorem.

**Theorem 3.** The asymptotic expansion of the Vega of Plain Vanilla Call option of which payoff function with maturity time $T$ is expressed as $(S_T - K)_+$ is given by

$$V (0, T) + o (\epsilon),$$

where

$$V (0, T) = e^{-rT} \left( \Sigma + \epsilon \left( f y + \frac{f y^3}{\Sigma} \right) \right) n [y; 0, \Sigma], \quad (15)$$

and the coefficients are given by

$$y = \frac{s_0 e^{\mu T} - K}{\epsilon},$$

$$\Sigma = E \left[ \left( \int_0^T e^{\mu (T-t) \sigma (A_0 t) \sigma W_t} \right)^2 \right]$$

$$= \int_0^T e^{2\mu (T-t) \sigma (A_0 t)^2} dt,$$

$$c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu (T-s) \sigma (A_0 s) \sigma' (A_0 t) e^{2\mu (T-t) \sigma (A_0 t)^2} - dv ds,$$

and

$$f = -c \Sigma.$$

### 6 The Vega of Average Call Option

Repeating the similar argument used in the previous sections, we can obtain the approximated value of the Vega of Average Call Option. Hence, we only state the result as a theorem.
Theorem 4. The asymptotic expansion of the Vega of Average Call option of which payoff function with maturity time $T$ is expressed as $(S_T - K)_+$, is given by

$$V(0, T) + o(\epsilon),$$

where

$$V(0, T) = e^{-rT} \left( \Sigma + \epsilon \left( f y + \frac{f y^3}{\Sigma} \right) \right) n [y; 0, \Sigma], \tag{16}$$

and the coefficients are given by

$$y = \frac{s_0 \left( e^{\mu T} - 1 \right)}{\epsilon} / \mu T - K,$$

$$d = \frac{e^{\mu T} - 1}{e \mu T},$$

$$\Sigma = E \left[ \left( \frac{1}{T} \int_{0}^{T} \int_{0}^{t} e^{\mu(t-s)} \sigma(A_{0s}) dW_s dt \right)^2 \right],$$

$$= \frac{1}{\mu^2 T^2} \int_{0}^{T} \left( e^{\mu(t-S)} - 1 \right)^2 \sigma(A_{0s})^2 ds,$$

$$c = \frac{1}{\Sigma^2 T^3} \int_{0}^{T} \int_{0}^{t} e^{\mu(t-s)} \left[ \frac{e^{\mu(T-s)} - 1}{\mu} \right] \sigma(A_{0s}) \sigma'(A_{0s})$$

$$\times \int_{0}^{s} e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)} - 1}{\mu} \right] \sigma(A_{0c})^2 dv ds dt,$$

and

$$f = -c \Sigma.$$

7 Variance Reduction Technique for Vega of Call Option

Previous section showed the validity of the asymptotic expansion technique for the Vega. However, we sometimes need more precise values, especially when $\epsilon$ is quite large or when the true value is quite small. For this purpose, we propose a variance reduction method.

Although this method is quite similar to the case of Delta, we need some adjustments. We define the "differentiation with respect to the volatility coefficient $\epsilon$" of $S_t$ satisfying the following SDE:

$$\begin{cases} dS_t = \mu S_t dt + \epsilon \sigma(S_t) dW_t \\ S_0 = s_0 (>0) \end{cases}$$
Definition 3. We define $Z_t$ as a stochastic process which satisfies the SDE:

$$
\begin{align*}
&dZ_t = \mu Z_t \, dt + [\sigma (S_t) + \epsilon \sigma' (S_t) \, Z_t] \, dW_t \\
&Z_0 = 0
\end{align*}
$$

(17)

The value of Vega is obtained by calculating $E \left[ e^{-rT} Z_T 1_{\{s_T \geq K\}} \right]$, but it is sometimes impossible to calculate it analytically. Although we utilize Monte Carlo simulations by using Euler-Maruyama Scheme, it is sometimes time-consuming. In a similar way as Variance Reduction Technique for Delta, let $X$ be the sample path and we introduce a new random variable $Y$.

According to Kunitomo and Takahashi [2003b], the asymptotic expansion of $S_t$ is obtained by

$$
S_t \sim A_{0t} + \epsilon A_{1t} + \frac{\epsilon^2}{2} A_{2t} + \cdots,
$$

where

$$
\begin{align*}
&A_{0t} = \mu A_{0t} \, dt, \quad A_{00} = s_0 \\
&A_{1t} = \mu A_{1t} \, dt + \sigma (A_{0t}) \, dW_t, \quad A_{10} = 0 \\
&A_{2t} = \mu A_{2t} \, dt + 2 \sigma' (A_{0t}) \, A_{1t} \, dW_t, \quad A_{20} = 0.
\end{align*}
$$

The asymptotic expansion of $Z_t$ is also given by

$$
Z_t \sim A_{1t} + \epsilon A_{2t} + \cdots.
$$

Now we put $X_t = (S_t - A_{0t}) / \epsilon$, $g_1 = A_{1T}$ and $g_2 = A_{2T} / 2$. Then, the asymptotic expansion of the Vega $E \left[ e^{-rT} Z_T 1_{\{s_T \geq K\}} \right]$ is given by

$$
E \left[ e^{-rT} Z_T 1_{\{s_T \geq K\}} \right] = e^{-rT} E \left[ \left( g_1 + 2 \epsilon g_2 + \cdots \right) 1_{\{g_1 + \epsilon g_2 + \cdots \geq y\}} \right],
$$

where $y = (A_{0T} - K) / \epsilon$. Thus, we obtain its approximated value as

$$
\begin{align*}
&E \left[ e^{-rT} Z_T 1_{\{s_T \geq K\}} \right] \\
&= e^{-rT} E \left[ g_1 1_{\{g_1 \geq y\}} \right] + \epsilon e^{-rT} E \left[ 2 g_2 1_{\{g_1 \geq y\}} \right] \\
&+ \epsilon e^{-rT} E \left[ g_1 g_2 (\partial 1)_{\{g_1 \geq y\}} \right] + o (\epsilon).
\end{align*}
$$

Finally, the approximated value of Vega $V (0, T)$ is expressed as

$$
V (0, T) = e^{-rT} E \left[ \left( g_1 + 2 \epsilon g_2 \right) 1_{\{g_1 \geq y\}} + \epsilon e^{-rT} E \left[ g_1 g_2 (\partial 1)_{\{g_1 \geq y\}} \right] \right]
$$

$$
= e^{-rT} E \left[ (g_1 + 2 \epsilon g_2) 1_{\{g_1 \geq y\}} \right] + \epsilon e^{-rT} E \left[ g_1 E \left[ g_2 | g_1 \right] (\partial 1)_{\{g_1 \geq y\}} \right]
$$

$$
= e^{-rT} E \left[ \left( g_1 + 2 \epsilon \left( c g_1^2 + f \right) \right) 1_{\{g_1 \geq y\}} \right] + \epsilon e^{-rT} E \left[ g_1 \left( c g_1^2 + f \right) (\partial 1)_{\{g_1 \geq y\}} \right]
$$

$$
= E \left[ \varphi (g_1) \right],
$$

where

$$
\varphi (x) = e^{-rT} \left( x + 2 \epsilon \left( c x^2 + f \right) \right) 1_{\{x \geq y\}} - e^{-rT} y \left( c y^2 + f \right) n [y; 0, \Sigma].
$$

(18)

Finally, we introduce a new random variable $Y = \varphi (g_1) - V (0, T)$, and implement the variance reduction technique described in section 4. We call this method "Hybrid Monte Carlo" as in section 4.
8 The Case of CEV processes

8.1 Another Derivation of Approximated Value of Delta

In this section we assume the price processes of the underlying asset to be CEV processes:
\[
\begin{aligned}
    dS_t &= \mu S_t dt + \sigma S_t^\gamma dW_t, \\ S_0 &= s_0 (> 0)
\end{aligned}
\] (19)

Then, we derive the approximation of the Delta in a more straightforward manner. We define \( Y_t \) as
\[
Y_t = \frac{\partial S_t}{\partial s_0}.
\]
Note that \( Y_t \) the SDE:
\[
\begin{aligned}
    dY_t &= \mu Y_t dt + \sigma Y_t^\gamma S_t^\gamma dW_t, \\ Y_0 &= 1
\end{aligned}
\] (20)

and the Delta of Plain Vanilla Call Option is obtained by
\[
e^{-rT} E \left[ Y_T 1_{\{S_T \geq K\}} \right].
\] (21)

See Imamura, Takahashi and Uchida [2005] for the derivation for instance. According to Kunitomo and Takahashi [2003b], the asymptotic expansion of \( S_t \) is given by
\[
S_t \sim A_0 t + \frac{\sigma^2}{2} A_1 t + \cdots
\]
where
\[
A_0 t = s_0 e^{\mu t}, \quad A_1 t = \frac{\sigma^2}{s_0} \int_0^t e^{\mu s} e^{-\mu (1-\gamma)s} dW_s,
\]
\[
A_2 t = 2 \sigma s_0^{\gamma-1} \frac{\sigma^2}{s_0} \int_0^t e^{\mu s} e^{\mu (2-\gamma)s} A_1 s dW_s.
\]

Similarly, the asymptotic expansion of \( Y_t \) is obtained by
\[
Y_t \sim B_0 t + \epsilon B_1 t + \frac{\sigma^2}{2} B_2 t + \cdots
\]
where \( B_0 t, B_1 t \) and \( B_2 t \) satisfy SDEs:
\[
\begin{aligned}
    dB_{0t} &= \mu B_{0t} dt, & B_{00} &= 1, \\
    dB_{1t} &= \mu B_{1t} dt + \gamma A_{0t}^{\gamma-1} B_{0t} dW_t, & B_{10} &= 0, \\
    dB_{2t} &= \mu B_{2t} dt + \left[ 2 \gamma (\gamma - 1) A_{0t}^{\gamma-2} A_{1t} B_{0t} + 2 \gamma A_{0t}^{\gamma-1} B_{1t} \right] dW_t, & B_{20} &= 0
\end{aligned}
\]

Then, we can easily obtain \( B_{0t}, B_{1t}, B_{2t} \) as
\[
\begin{aligned}
    B_{0t} &= \frac{1}{s_0} A_{0t}, \\
    B_{1t} &= \frac{\gamma}{s_0} A_{1t}, \\
    B_{2t} &= \frac{2\gamma - 1}{s_0} A_{2t}.
\end{aligned}
\]

Now we put \( X_t = (S_t - A_{0t}) / \epsilon, g_1 = A_{1t} \) and \( g_2 = A_{2t} \), then
\[
X_T \sim g_1 + g_2 + \cdots
\]

Therefore,
\[
e^{-rT} E \left[ Y_T 1_{\{S_T \geq K\}} \right] = \epsilon e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 + \frac{2\gamma - 1}{s_0} g_2 + \cdots \right) 1_{\{g_1 + g_2 + \cdots \geq -y\}} \right].
\]
where $y = (s_0 e^{\mu T} - K) / \epsilon$ and $d = \frac{\partial y}{\partial s_0} = e^{\mu T} / \epsilon$. We obtain its approximated value as

$$e^{-rT} E \left[ Y_T 1_{ \{ s_T \geq K \} } \right] = e^{e^{-rT} T} E \left[ \left( d + \frac{\gamma}{s_0} \right) 1_{ \{ g_1 \geq -y \} } \right] + e^{e^{-rT} T} E \left[ \left( \frac{2\gamma - 1}{s_0} g_2 1_{ \{ g_1 \geq -y \} } \right) \right] + o ( \epsilon^2 ) ,$$

where $(\partial 1)_{ \{ \geq -y \} }$ means the derivation of $1_{ \{ \geq -y \} }$ in the sense of distribution. By the property of Brownian Motion, we know the distribution of $g_1 \sim N (0, \Sigma)$, where

$$\Sigma = E \left[ \left( \int_0^T e^{\mu (T-t)} \sigma (A_{0t}) \, dW_t \right)^2 \right] = \int_0^T e^{2\mu (T-t)} \sigma (A_{0t})^2 \, dt = \begin{cases} \frac{\Sigma^2}{2 \mu (1-\gamma)} (e^{2\mu T} - e^{2\mu yT}) & \text{if } 0 < \gamma < 1, \\ \frac{\Sigma^2}{2} T e^{2\mu T} & \text{if } \gamma = 1, \end{cases}$$

In addition, the conditional expectation $E [g_2 | g_1 = x] = cx^2 + f$ where

$$e^{-rT} E \left[ Y_T 1_{ \{ s_T \geq K \} } \right] = e^{e^{-rT} T} E \left[ \left( d + \frac{\gamma}{s_0} \right) n [x; 0, \Sigma] \, dx \right] + e^{e^{-rT} T} \frac{2\gamma - 1}{s_0} \int_{-y}^\infty (cx^2 + f) \, n [x; 0, \Sigma] \, dx$$

Thus we obtain the approximation as

$$e^{e^{-rT} T} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \left( \frac{\gamma}{s_0} \Sigma + \epsilon \left( d - \frac{\gamma}{s_0} y \right) (cy^2 + f) + \epsilon \frac{2\gamma - 1}{s_0} fy \right) n [y; 0, \Sigma] \right) .$$

where $N (x)$ denotes a cumulative distribution function of standard normal distribution $N (0, 1)$.

Finally we see the agreement of the two approximated values derived in different manners. By the theorem 1, the approximated value of the Delta is given by

$$D (0, T) = e^{e^{-rT} T} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n [y; 0, \Sigma] \right)$$

Thus we obtain the approximation as

$$e^{e^{-rT} T} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n [y; 0, \Sigma] \right)$$

Finally we see the agreement of the two approximated values derived in different manners. By the theorem 1, the approximated value of the Delta is given by

$$D (0, T) = e^{e^{-rT} T} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n [y; 0, \Sigma] \right)$$

Finally we see the agreement of the two approximated values derived in different manners. By the theorem 1, the approximated value of the Delta is given by

$$D (0, T) = e^{e^{-rT} T} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{1}{2} \frac{\partial \Sigma}{\partial s_0} n [y; 0, \Sigma] \right)$$
In addition, the coefficients $\Sigma$, $c$, $f$ are monomials of $s_0$, and their differentiation with respect to $s_0$ are
\[
\frac{\partial \Sigma}{\partial s_0} = \frac{2\gamma \Sigma}{s_0}, \quad \frac{\partial c}{\partial s_0} = -\frac{c}{s_0}, \quad \frac{\partial f}{\partial s_0} = \frac{(2\gamma - 1) f}{s_0}.
\]
Substituting them, we obtain
\[
D(0, T)
= \varepsilon e^{-rT} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{\gamma \Sigma}{s_0} n [y; 0, \Sigma] \right)
\quad + \varepsilon^2 e^{-rT} \left( \frac{2\gamma - 1}{s_0} f y \right) + c y + f \left( \frac{\gamma \Sigma}{s_0} + y d - \frac{\gamma y^2}{s_0} \right) n [y; 0, \Sigma] \quad + \varepsilon^2 e^{-rT} \left( \frac{2\gamma - 1}{s_0} f y \right) + \left( c y^2 + f \right) \left( d - \frac{\gamma}{s_0} y \right) n [y; 0, \Sigma].
\]
Hence, the two approximated values agree.

Then we sum up the fact as a theorem.

**Theorem 5.** The asymptotic expansion of the Delta of Plain Vanilla Call option, whose payoff function with maturity time $T$ is expressed as $(S_T - K)_+$, is given by
\[
D(0, T) + o(\varepsilon^2),
\]
where
\[
D(0, T)
= \varepsilon e^{-rT} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{\gamma \Sigma}{s_0} n [y; 0, \Sigma] \right)
\quad + \varepsilon^2 e^{-rT} \left( \frac{2\gamma - 1}{s_0} f y \right) + \left( c y^2 + f \right) \left( d - \frac{\gamma}{s_0} y \right) n [y; 0, \Sigma],
\]
and the coefficients are given by
\[
y = \frac{s_0 e^{\mu T} - K}{\varepsilon},
\]
\[
\Sigma = E \left[ \left( \int_0^T e^{\mu (T-t)} \sigma(A_{0t}) \, dW_t \right)^2 \right]
\quad = \int_0^T e^{2\mu (T-t)} \sigma(A_{0t})^2 \, dt
\quad = \begin{cases}
\frac{s_0^2 \gamma}{2 \mu (1-\gamma)} \left( e^{2\mu T} - e^{2\mu^2 T} \right) & \text{if } 0 < \gamma < 1 \\
\frac{s_0^2 T^2 e^{2\mu T}}{2 \mu^2 (1-\gamma)^2} & \text{if } \gamma = 1,
\end{cases}
\]
\[
c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu (T-s)} \sigma(A_{0s}) \sigma'(A_{0v}) e^{2\mu (T-v)} \sigma(A_{0v})^2 \, dv \, ds
\quad = \begin{cases}
\frac{\gamma}{\Sigma^2} s_0 \int_0^T e^{3\mu T} \left( 1 - e^{2\mu (1-\gamma) T} \right)^2 \, dt & \text{if } 0 < \gamma < 1 \\
\frac{s_0^2 T^2 e^{3\mu T}}{2 \mu^2 (1-\gamma)^2} & \text{if } \gamma = 1,
\end{cases}
\]
\[89\]
and

\[ f = -c \Sigma. \]

Similarly, we gain the approximation formula for the Delta of the average option.

**Theorem 6.** The asymptotic expansion of the Delta of Average Call option, whose payoff function with maturity time \( T \) is expressed as \( (\bar{S}_T - K)_+ \), is given by

\[ D(0, T) + o(e^2), \]

where

\[
D(0, T) = e^{rT} \left( d N \left( \frac{y}{\sqrt{\Sigma}} \right) + \frac{\gamma \Sigma}{s_0} n [y; 0, \Sigma] \right) + e^{rT} \left( \left( \left( 2\gamma - 1 \right) \frac{1}{s_0} f y + (c y^2 + f) \left( d - \frac{\gamma}{s_0} y \right) \right) n [y; 0, \Sigma] \right),
\]

and the coefficients are given by

\[
y = \frac{s_0 \left( e^{\mu T} - 1 \right) / \mu T - K}{e},
\]

\[
d = \frac{e^{\mu T} - 1}{e \mu T},
\]

\[
\Sigma = \int_0^T \left( \frac{1}{T} e^{\mu (T-t)} - 1 \right) \sigma (A_{0t})^2 dt
\]

\[
= \left\{ \begin{array}{ll}
\frac{e^{2\mu T}}{\mu^2 T^2} \left( \frac{e^{2\mu T}}{2\mu (1-\gamma)} - \frac{e^{2\mu T}}{\mu (1-2\gamma)} - \frac{1}{2\mu T} + \frac{e^{\mu T}}{2\mu T (1-\gamma)} (1-2\gamma) \right) & \text{if } 0 < \gamma < 1, \ \gamma \neq 1/2 \\
\frac{s_0}{\mu T} \left( \frac{e^{2\mu T}}{\mu^2 T^2} - 2T e^{\mu T} - \frac{1}{T} \right) + \frac{2e^{2\mu T}}{\mu T} e^{\mu T} - \frac{1}{2T} & \text{if } \gamma = 1/2 \\
\left( \frac{s_0}{\mu T} \right)^2 \left( e^{2\mu T} \left( T - \frac{3}{2T} \right) + \frac{2e^{\mu T}}{\mu T} - 1 \right) & \text{if } \gamma = 1,
\end{array} \right.
\]
\[
c = \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[ \frac{e^{\mu(T-s)-1}}{\mu} \right] \sigma(A_{0s}) \sigma'(A_{0s}) ds \]
\[
\times \int_0^s e^{\mu(s-v)} \left[ \frac{e^{\mu(T-v)-1}}{\mu} \right] \sigma(A_{0v})^2 \, dv \, ds \, dt
\]
\[
= \begin{cases}
\gamma \frac{\gamma s^{\gamma-1}}{\Sigma^2 \mu^3 T^3} \left( (2 - 2\gamma) - e^{\mu T} (\gamma - 1) \right) \\
\gamma \frac{s^2}{\Sigma^2 \mu^3 T^3} \left( e^{3\mu T} + e^{2\mu T} \left( \frac{1}{\mu^3} - \frac{2\mu^2}{\mu^3} \right) + e^{\mu T} \left( -\frac{1}{\mu^2} - \frac{T^2}{\mu^3} + \frac{T}{\mu^2} \right) \right) & \text{if } 0 < \gamma < 1, \gamma \neq \frac{1}{2}
\end{cases}
\]
\[
\frac{s^2}{\Sigma^2 \mu^3 T^3} \left( e^{3\mu T} + e^{\mu T} \left( \frac{2\mu^2}{\mu^3} - \frac{T^2}{\mu^3} - \frac{T}{\mu^2} \right) \right) & \text{if } \gamma = \frac{1}{2}
\]
\[
\frac{s^2}{\Sigma^2 \mu^3 T^3} \left( e^{3\mu T} \left( 2\mu T - 5 + \frac{17}{9} \right) + e^{2\mu T} \left( 2\mu T - 5 + \frac{5}{2} \right) \right) & \text{if } \gamma = 1,
\]

and
\[
f = -c\Sigma.
\]

### 8.2 The Gamma of Call Option

The approximated value of Delta of Plain Vanilla (Average) Call Option is given by
\[
D(0, T) + o\left( e^2 \right),
\]
where
\[
D(0, T) = e^{-rT} \left( dN \left( \frac{y}{\sqrt{\Sigma}} \right) + \gamma\Sigma s_0 n[y; 0, \Sigma] \right)
\]
\[
+ e^2 e^{-rT} \left( \frac{2\gamma - 1}{s_0} f y + (c y^2 + f) \left( d - \frac{\gamma}{s_0} y \right) \right) n[y; 0, \Sigma].
\]

Then, we can differentiate it with respect to \( s_0 \) again. We get an approximated value of the Gamma.

**Theorem 7.** The asymptotic expansion of the Gamma of Plain Vanilla Call option, whose payoff function with maturity time \( T \) is expressed as \( (S_T - K)_+ \), is given by
\[
G(0, T) + o\left( e^2 \right),
\]
where
\[ G(0, T) = e^{-rT} \left( d^2 - \frac{2\gamma}{s_0} dy - \frac{\gamma}{s_0} \Sigma + \frac{\gamma^2}{s_0^2} y \right) n[y; 0, \Sigma] \]
(24)
\[ -e^2 e^{-rT} \left( \left( d - \frac{\gamma}{s_0} y \right) (cy^2 + f) + \frac{2\gamma - 1}{s_0} f y \right) \left( \frac{\gamma}{s_0} + \left( y d - \frac{\gamma}{s_0} y^2 \right) \frac{1}{\Sigma} \right) n[y; 0, \Sigma] \]
\[ + e^2 e^{-rT} \left( \left( \frac{\gamma}{s_0} - \frac{\gamma}{s_0} y \right) (cy^2 + f) + \left( d - \frac{\gamma}{s_0} y \right) \left( - \frac{c}{s_0} y^2 + 2cy d + \frac{2\gamma - 1}{s_0} f \right) \right) \]
\[ + \left( \frac{2\gamma - 1}{s_0^2} \right) f y + \frac{2\gamma - 1}{s_0} f d) n[y; 0, \Sigma], \]
and the coefficients are given by
\[ y = \frac{s_0 e^{\mu T} - K}{\epsilon}, \]
\[ \Sigma = E \left[ \left( \int_0^T e^{\mu (T-t)} \sigma(A_{0t}) \, dW_t \right)^2 \right] \]
\[ = \int_0^T e^{2\mu (T-t)} \sigma(A_{0t})^2 \, dt \]
\[ = \begin{cases} \frac{s_0^2 e^{2\gamma} (e^{2\mu T} - e^{2\mu \gamma T})}{2\mu(1-\gamma)} & \text{if } 0 < \gamma < 1 \\ \frac{s_0^2 T e^{2\mu T}}{2} & \text{if } \gamma = 1 \end{cases} \]
\[ c = \frac{1}{\Sigma} \int_0^T \int_0^s e^{\mu (T-v)} \sigma(A_{0s}) \sigma'(A_{0s}) e^{2\mu (T-v)} \sigma(A_{0v})^2 \, dv \, ds \]
\[ = \begin{cases} \frac{\gamma e^{4\gamma-1}}{2\Sigma^2} \frac{s_0^3 T^2}{2} e^{3\mu T} (1-\epsilon e^{2\mu (1-\gamma) T})^2 & \text{if } 0 < \gamma < 1 \\ \frac{s_0^3 T^2}{2} e^{3\mu T} & \text{if } \gamma = 1 \end{cases} \]
and
\[ f = -c \Sigma. \]

### 8.3 The Delta of Digital Option

We consider the Digital Option whose payoff function at maturity time \( T \) is given by
\[ 1_{\{K_1 \leq S_T \leq K_2\}}. \]

Before considering the Delta of Digital Option, we note that its price is represented by \( e^{-rT} E \left[ 1_{\{K_1 \leq S_T \leq K_2\}} \right] \).

As above, \( X_T \sim g_1 + \epsilon g_2 + \cdots \), where \( X_t = (S_t - A_{0t}) / \epsilon \), and
\[ e^{-rT} E \left[ 1_{\{K_1 \leq S_T \leq K_2\}} \right] = e^{-rT} E \left[ 1_{\{-y_1 \leq X_T \leq -y_2\}} \right] \]
\[ = e^{-rT} E \left[ 1_{\{-y_1 \leq g_1 + \epsilon g_2 + \cdots \leq -y_2\}} \right], \]

92
where \( y_1 = (A_{0T} - K_1) / \epsilon \) and \( y_2 = (A_{0T} - K_2) / \epsilon \). We obtain its approximated value as

\[
e^{-rT} E \left[ 1_{\{K_1 \leq S_T \leq K_2\}} \right] \\
= e^{-rT} E \left[ 1_{\{-y_1 \leq y \leq -y_2\}} \right] + \epsilon e^{-rT} E \left[ g_2 (\partial 1)_{\{-y_1 \leq y \leq -y_2\}} \right] + o(\epsilon)
\]

Its differentiation corresponds to the Delta of the Digital option.

**Theorem 8.** The asymptotic expansion of the Delta of Digital option, whose payoff function of with maturity time \( T \) is expressed as \( 1_{\{K_1 \leq S_T \leq K_2\}} \), is given by

\[
D(0, T) + o(\epsilon^2),
\]

where

\[
D(0, T) = \left( d - \frac{\gamma}{s_0} y_1 \right)
\]

\[
+ \epsilon \left( -\frac{c}{s_0} y_1^2 + 2c y_1 d + \frac{2\gamma - 1}{s_0} f + (cy_1^2 + f) \left( -\frac{\gamma}{s_0} - \frac{y_1}{s_0} + \frac{y_2^2}{s_0} \right) \right) n [y_1; 0, \Sigma]
\]

\[
- \left( d - \frac{\gamma}{s_0} y_2 \right) \\
+ \epsilon \left( -\frac{c}{s_0} y_2^2 + 2c y_2 d + \frac{2\gamma - 1}{s_0} f + (cy_2^2 + f) \left( -\frac{\gamma}{s_0} - \frac{y_2}{s_0} + \frac{y_2^2}{s_0} \right) \right) n [y_2; 0, \Sigma],
\]

and the coefficients are given by

\[
y_1 = \frac{s_0 e^{\mu T} - K_1}{\epsilon}, \quad y_2 = \frac{s_0 e^{\mu T} - K_2}{\epsilon},
\]

\[
\Sigma = E \left[ \left( \int_0^T e^{\mu(T-t)} \sigma(A_{0t}) \, dW_t \right)^2 \right]
\]

\[
= \int_0^T e^{2\mu(T-t)} \sigma(A_{0t})^2 \, dt
\]

\[
= \begin{cases} 
\frac{e^{2\gamma}}{2\mu(1-\gamma)} \left( e^{2\mu T} - e^{2\mu T} \right) & \text{if } 0 < \gamma < 1 \\
\frac{s_0^2}{s_0^2 T} e^{2\mu T} & \text{if } \gamma = 1,
\end{cases}
\]

\[
c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu(T-s)} \sigma(A_{0s}) \, dW_t \sigma(A_{0s}) \, e^{2\mu(T-t)} \sigma(A_{0t})^2 \, dv \, ds
\]

\[
= \begin{cases} 
\frac{\gamma s_0 T^{\gamma-1}}{\Sigma^2} e^{2\mu T} \frac{(1-\epsilon e^{2\mu(T-s)} \sigma(A_{0s}) e^{2\mu(T-t)} \sigma(A_{0t})^2)}{8 \mu^2 (1-\gamma)^2} & \text{if } 0 < \gamma < 1 \\
\frac{s_0^3 T^2}{2 \Sigma^2} e^{3\mu T} & \text{if } \gamma = 1,
\end{cases}
\]

and

\[
f = -c \Sigma.
\]
9 Numerical Examples

The BS (Black-Scholes) model is represented as
\[
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
S_0 &= s_0 (> 0),
\end{aligned}
\]  

and the CEV (Constant Elasticity of Variance) model is represented as
\[
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t^\gamma dW_t, \\
S_0 &= s_0 (> 0),
\end{aligned}
\]  

0 < \gamma < 1

Hence, the volatility function of the models are expressed as \( \sigma(S_t) = S_t^\gamma (0 < \gamma \leq 1) \).

9.1 Approximation for the Delta of Plain Vanilla Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as
\[
\Sigma = E \left[ \left( \int_0^T e^{\mu (T-t)} \sigma(A_0) \sigma(A_0_t) \sigma(A_0_v) \sigma(A_0_v_t) \sigma(A_0_v_v) \sigma(A_0_v_v_t) \sigma(A_0_v_v_v) \sigma(A_0_v_v_v_t) \sigma(A_0_v_v_v_v) \sigma(A_0_v_v_v_v_t) \sigma(A_0_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v) \sigma(A_0_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_v_t)ight)^2 \right]
\]
\[
= \int_0^T e^{2\mu (T-t)} \sigma(A_0)^2 dt
\]
\[
= \left\{ \begin{array}{ll}
\frac{s_0^{2\gamma}}{2\mu(1-\gamma)} \left( e^{2\mu T} - e^{2\mu \gamma T} \right) & \text{if } 0 < \gamma < 1 \\
\frac{s_0^2 T e^{2\mu T}}{8 \mu^2 (1-\gamma)^2} & \text{if } \gamma = 1
\end{array} \right.
\]

and
\[
c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu (T-s)} \sigma(A_0_s) \sigma'(A_0_s) e^{2\mu (T-v)} \sigma(A_0_v)^2 dv ds
\]
\[
= \left\{ \begin{array}{ll}
\frac{\gamma s_0^{2\gamma-1}}{\Sigma^2} e^{3\mu T} \frac{(1-e^{2\mu (1-\gamma) T})^2}{8 \mu^2 (1-\gamma)^2} & \text{if } 0 < \gamma < 1 \\
\frac{s_0^3 T^2}{\Sigma^2} e^{3\mu T} & \text{if } \gamma = 1
\end{array} \right.
\]

The coefficients \( \Sigma, c, \) and \( f \) are the function of \( s_0, T, \gamma \) and \( \mu. \) In particular, paying attention to \( \gamma, \) we represent \( \Sigma = \Sigma(\gamma), c = c(\gamma). \) We can easily check the continuity at \( \gamma = 1, \) i.e. \( \Sigma(\gamma) \to \Sigma(1) \) and \( c(\gamma) \to c(1), \) as \( \gamma \to 1. \)

We apply the theorem 5 and obtain the approximated value of the Delta.

We show some numerical examples (figure 1). The initial value \( s_0 \) is fixed to \( s_0 = 100. \) For the true values, we use the analytical values for BS model (\( \gamma = 1 \)), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model (\( \gamma < 1 \)). The error ratio is calculated as \( (\text{approximate value} - \text{true value}) / \text{true value}. \)

We can observe that the accuracy of this approximation is valid.
9.2 Approximation for the Delta of Average Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

\[ \Sigma = \int_0^T \left( \frac{e^{\mu(T-t)} - 1}{\mu} \right)^2 \sigma (A_{10}) dt \]

\[ = \begin{cases} 
\frac{e^{2\mu T}}{2\mu(1-\gamma)} - \frac{2e^{\mu T}}{\mu(1-2\gamma)} - \frac{1}{2\mu^2} + \frac{e^{\mu T}}{2\mu(1-\gamma)(1-2\gamma)} & \text{if } 0 < \gamma < 1, \gamma \neq \frac{1}{2} \\
\frac{s_0}{\mu^2 T} \left( \frac{e^{2\mu T}}{\mu} - 2T e^{\mu T} - \frac{1}{\mu} \right) & \text{if } \gamma = \frac{1}{2} \\
\left( \frac{s_0}{\mu T} \right)^2 \left( e^{2\mu T} \left( T - \frac{3}{2} \right) + 2e^{\mu T} - \frac{1}{2\mu} \right) & \text{if } \gamma = 1,
\end{cases} \]

and

\[ c = \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t \frac{e^{\mu(T-s)-1}}{\mu} \sigma (A_{0s}) \sigma' (A_{0s}) \]

\[ \times \int_0^s e^{\mu (t-s)} \left[ \frac{e^{\mu(T-s)-1}}{\mu} \right] \sigma (A_{0v})^2 dv ds dt \]

\[ = \begin{cases} 
\frac{\gamma^{s-1}}{\Sigma^2 \mu^2 T^3} \left( \frac{(-1 + e^{\mu T})(2 - 2\gamma + e^{\mu T}(-1 + 2\gamma))}{\mu} \right) & \text{if } 0 < \gamma < 1, \gamma \neq \frac{1}{2} \\
\frac{s_0}{2\mu^2 T} + \frac{e^{2\mu T}}{2\mu^3} \left( \frac{1}{\mu^3} - \frac{T}{\mu^2} + \frac{T^2}{\mu} \right) + \frac{e^{\mu T}}{2\mu^3} \left( 1 + \frac{T}{\mu} - \frac{1}{\mu^3} \right) & \text{if } \gamma = \frac{1}{2} \\
\frac{s_0}{2\mu^2 T} \left( e^{2\mu T} (\mu^2 T - 2 - 3\mu T + \frac{17}{3}) + e^{\mu T} (2\mu T - 5) + \frac{5}{2} e^{\mu T} - \frac{1}{3} \right) & \text{if } \gamma = 1.
\end{cases} \]

As with the Plain Vanilla case, the coefficients \( \Sigma, c, \) and \( f \) are the function of \( s_0, T, \gamma, \) and \( \mu \). We can easily check the continuity at \( \gamma = 1 \), i.e. \( \Sigma (\gamma) \rightarrow \Sigma (1) \) and \( c (\gamma) \rightarrow c (1) \), as \( \gamma \rightarrow 1 \). We apply the theorem 6 and obtain the approximated value of the Delta.

We show some numerical examples (figure 2). The initial value \( s_0 \) is fixed to \( s_0 = 100 \). For the true values, we use the analytical values for BS model (\( \gamma = 1 \)), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model (\( \gamma < 1 \)). The error ratio is calculated as (approximate value − true value) /true value.

We can observe that the accuracy of this approximation is valid in general.
9.3 Computation of Delta by the Variance Reduction Technique

In this section, we show some numerical examples of variance reduction method. We use the BS and CEV models as the dynamics of stock prices. The result of Plain Vanilla Delta is exhibited in figure 3. In theorem 5, the Delta of Plain Vanilla Call Option is obtained by

\[ e^{-rT} E \left[ Y_T 1_{\{s_T \geq K\}} \right] , \]

and its asymptotic expansion is expressed as

\[ e^{-rT} E \left[ Y_T 1_{\{s_T \geq K\}} \right] = e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 + \epsilon \frac{2 \gamma - 1}{s_0} g_2 + \cdots \right) 1_{\{g_1 + \epsilon g_2 + \cdots \geq -y\}} \right] , \]

where \( y = \left( s_0 e^{\mu T} - K \right) / \epsilon \) and \( \epsilon = \frac{\partial y}{\partial s_0} = e^{\mu T} / \epsilon \). Then, we obtained its approximated value

\[
D (0, T) = e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) 1_{\{g_1 \geq -y\}} \right] + e^{2} e^{-rT} E \left[ \frac{2 \gamma - 1}{s_0} g_2 1_{\{g_1 \geq -y\}} \right] \\
+ e^{2} e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) g_2 (\partial 1)_{\{g_1 \geq -y\}} \right] \\
= e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) 1_{\{g_1 \geq -y\}} \right] + e^{2} e^{-rT} E \left[ \frac{2 \gamma - 1}{s_0} E [g_2 | g_1] 1_{\{g_1 \geq -y\}} \right] \\
+ e^{2} e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) E [g_2 | g_1] (\partial 1)_{\{g_1 \geq -y\}} \right] \\
= e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) 1_{\{g_1 \geq -y\}} \right] + e^{2} e^{-rT} E \left[ \frac{2 \gamma - 1}{s_0} \left( c g_1^2 + f \right) 1_{\{g_1 \geq -y\}} \right] \\
+ e^{2} e^{-rT} E \left[ \left( d + \frac{\gamma}{s_0} g_1 \right) (c g_1^2 + f) (\partial 1)_{\{g_1 \geq -y\}} \right] \\
= E \left[ \varphi (g_1) \right],
\]

where

\[
\varphi (x) = e^{-rT} \left( d + \frac{\gamma}{s_0} x + \frac{2 \gamma - 1}{s_0} e \left( c x^2 + f \right) \right) 1_{\{x \geq -y\}} + e^{2} e^{-rT} \left( d - \frac{\gamma}{s_0} y \right) (c y^2 + f) n \left[ y; 0, \Sigma \right].
\]

(27)

Next, we define \( Y = \varphi (g_1) - D (0, T) \), and call this \( Y \) “Attendant” random variable of \( X \) generated by “Crude” Monte Carlo.

In the simulation, the initial value \( s_0 \) is fixed to \( s_0 = 100 \). For the true values, we use the analytical values for BS model (\( \gamma = 1 \)), and use the mean of 2,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model (\( \gamma < 1 \)). The errors 1,2 and 3 are defined by \( (X - \text{true value}) / \text{(true value)} \), \( Y / \text{(approximated value)} \) and \( (X - Y - \text{true value}) / \text{(true value)} \), respectively. We generated 1000 passes and took their average, and repeated it by 100 cases. We calculated the correlation between \( X \) and \( Y \), and the average, standard deviation, maximum and minimum of 100 cases for each of six random variables \( (X, Y, (X - Y) \) and errors 1,2,3).

The example of Average Delta is exhibited in figure 4. Items on the table are the same as above. We can observe that the standard deviation as well as maximum and minimum values of the errors of \( HybridMC(error\ 3) \) are reduced for each case relative to those of \( CrudeMC(error\ 1) \).
9.4 Approximation for the Vega of Plain Vanilla Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

\[
\Sigma = E \left[ \left( \int_0^T e^{\mu(T-t)} \sigma(A_{0t}) \, dW_t \right)^2 \right]
\]

\[
= \int_0^T e^{2\mu(T-t)} \sigma(A_{0t})^2 \, dt
\]

\[
= \begin{cases} 
\frac{s_0^{2\gamma}}{2\mu(1-\gamma)} \left( e^{2\mu T} - e^{2\mu \gamma T} \right) & \text{if } 0 < \gamma < 1 \\
\frac{s_0^2 T e^{2\mu T}}{s_0^2} & \text{if } \gamma = 1,
\end{cases}
\]

and

\[
c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu(T-s)} \sigma(A_{0s}) \sigma'(A_{0s}) e^{2\mu(T-v)} \sigma(A_{0v})^2 \, dv \, ds
\]

\[
= \begin{cases} 
\frac{\gamma s_0^{4\gamma-1}}{2^2} e^{3\mu T} \left( 1-e^{2\mu(1-\gamma)T} \right)^2 & \text{if } 0 < \gamma < 1 \\
\frac{s_0^3 T^2 e^{3\mu T}}{8 \mu^2 (1-\gamma)^2} & \text{if } \gamma = 1.
\end{cases}
\]

We apply the theorem 3 and get the approximated value of the Delta.

We show some numerical examples (figure 5). The initial value \(s_0\) is fixed to \(s_0 = 100\). For the true values, we use the analytical values for BS model (\(\gamma = 1\)), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model (\(\gamma < 1\)). The error ratio is calculated as \(\frac{\text{approximated value} - \text{true value}}{\text{true value}}\).

We can see that the accuracy of this approximation is generally valid.
9.5 Approximation for the Vega of Average Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

\[
\Sigma = \int_0^T \left( \frac{1}{T} e^{\mu (T-t)} - \frac{1}{\mu} \sigma (A_{0t}) \right)^2 dt
\]

\[
= \begin{cases} \frac{s^2 \gamma}{\mu^2 T^2} \left\{ \frac{e^{2\mu T}}{2\mu (1-\gamma)} - \frac{e^{2\mu T}}{\mu (1-2\gamma)} - \frac{1}{2\mu^2} + \frac{e^{\mu T}}{2\mu^2 (1-\gamma)(1-2\gamma)} \right\} & \text{if } 0 < \gamma < 1, \gamma \neq \frac{1}{2} \\ \frac{s_0}{\mu^2 T^2} \left\{ \frac{e^{2\mu T}}{\mu} - 2e^{\mu T} - \frac{1}{\mu} \right\} & \text{if } \gamma = \frac{1}{2} \\ \left( \frac{s_0}{\mu T} \right)^2 \left\{ e^{2\mu T} \left( T - \frac{3}{2\mu} \right) + 2e^{\mu T} - \frac{1}{2\mu} \right\} & \text{if } \gamma = 1, \end{cases}
\]

and

\[
c = \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu (t-s)} \left[ \frac{e^{\mu (T-s)-1}}{\mu} \right] \sigma (A_{0s}) \sigma' (A_{0s}) \\
\times \int_0^s e^{\mu (s-v)} \left[ \frac{e^{\mu (T-v)-1}}{\mu} \right] \sigma (A_{0v})^2 dv ds dt
\]

\[
= \begin{cases} \frac{\gamma s^{\gamma-1}}{\Sigma^2 \mu^2 T^3} \left\{ \left( \frac{-1 + e^{\mu T}}{\mu^3} \right) \left( 2 - 2\gamma + e^{\mu T} (-1 + 2\gamma) \right)^2 + \left( e^{\mu (2\gamma-1)T} \gamma (-7 + 10\gamma) + 4 \left( -1 + \gamma \right)^2 (3 - 13 + 12\gamma^2) \right) \right\} & \text{if } 0 < \gamma < 1, \gamma \neq \frac{1}{2} \\ \frac{s_0}{2\Sigma^2 \mu^2 T^3} \left\{ \frac{e^{3\mu T}}{2\mu^3} + e^{2\mu T} \left( \frac{1}{\mu^3} - \frac{2T}{\mu^2} \right) + e^{\mu T} \left( -\frac{1}{2\mu^2} - \frac{T}{\mu} + \frac{T^2}{\mu^2} \right) - \frac{1}{\mu^3} \right\} & \text{if } \gamma = \frac{1}{2} \\ \frac{s_0}{2\Sigma^2 \mu^2 T^3} \left\{ e^{3\mu T} \left( \mu^2 T^2 - 3\mu T + \frac{17}{6} \right) + e^{2\mu T} \left( 2\mu T - 5 \right) + \frac{5}{2} e^{\mu T} - \frac{1}{3} \right\} & \text{if } \gamma = 1. \end{cases}
\]

We apply the theorem 4 and obtained the approximated value of the Delta.

We show some numerical examples (figure 6). The initial value \( s_0 \) is fixed to \( s_0 = 100 \). For the true values, we use the analytical values for BS model (\( \gamma = 1 \)), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model (\( \gamma < 1 \)). The error ratio is calculated as (approximated value – true value) / (true value).

As above, we can see that the accuracy of this approximation is valid.
9.6 Computation of Vega by the Variance Reduction Technique

In this section, we show some numerical examples of variance reduction method. We use the BS and CEV models as the dynamics of stock prices. The example of Plain Vanilla Vega is exhibited in figure 7. The random variables $X$ and $Y$ correspond to Crude MC and Attendant, respectively. The initial value $s_0$ is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model ($\gamma = 1$), and use the mean of 2,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model ($\gamma < 1$). The error 1, 2, and 3 are defined by $(X - \text{true value})/(\text{true value})$, $Y/(\text{approximated value})$, and the error 3 is $(X - Y - \text{true value})/(\text{true value})$, respectively. We generated 1000 passes and took their average, and repeated it by 100 cases. We calculated the correlation between $X$ and $Y$, and average, standard deviation, maximum and minimum of 100 cases for each of the six random variables ($X$, $Y$, $X - Y$ and errors 1,2,3).

The example of Average Vega is exhibited in figure 8. Items on the table are the same as above. We can observe that the standard deviation as well as maximum and minimum values of the errors of HybridMC(error 3) are reduced for each case relative to those of CrudeMC(error 1).

10 Concluding Remarks

We developed an asymptotic expansion method for computing Greeks. Because it takes almost no time to compute the approximated values, it would be useful for monitoring Greeks changing rapidly in volatile markets. We also introduced so called “Hybrid Monte Carlo,” a variance reduction technique based on combination of Monte Carlo simulation and the asymptotic expansion. We showed that our variance reduction technique can accelerate Monte Carlo simulations. Although we mainly treated the Delta and the Vega, our method can also be applied to computing the other Greeks such as Gamma; it is obtained by differentiating the approximated Delta by the initial price of the underlying asset.

Clearly, the same method can be applied to term structure models and credit models such as Heath-Jarrow-Morton models, Market models and Duffie-Singleton models, which are topics in the next research.
<table>
<thead>
<tr>
<th>$r$</th>
<th>$T$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$K$</th>
<th>True Value</th>
<th>Approximation</th>
<th>Error Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>90</td>
<td>0.67614702</td>
<td>0.97608648</td>
<td>0.00%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>100</td>
<td>0.643342612</td>
<td>0.843653658</td>
<td>0.04%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>110</td>
<td>0.523547144</td>
<td>0.52326391</td>
<td>-0.05%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>120</td>
<td>0.190115249</td>
<td>0.190530823</td>
<td>0.21%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.843478784</td>
<td>0.84774386</td>
<td>-0.5%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.518581395</td>
<td>0.51748776</td>
<td>0.18%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.335376623</td>
<td>0.33685007</td>
<td>0.16%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>90</td>
<td>0.753731616</td>
<td>0.754586053</td>
<td>0.11%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>100</td>
<td>0.641329506</td>
<td>0.64280448</td>
<td>0.13%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>110</td>
<td>0.520993753</td>
<td>0.521408775</td>
<td>0.15%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>120</td>
<td>0.335726623</td>
<td>0.33685007</td>
<td>0.16%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>90</td>
<td>0.703381045</td>
<td>0.704125223</td>
<td>0.11%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>100</td>
<td>0.615017945</td>
<td>0.614621588</td>
<td>-0.06%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>110</td>
<td>0.520993753</td>
<td>0.521408775</td>
<td>0.15%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>120</td>
<td>0.335726623</td>
<td>0.33685007</td>
<td>0.16%</td>
</tr>
<tr>
<td>10%</td>
<td>0.1</td>
<td>20%</td>
<td>0.25</td>
<td>90</td>
<td>0.960879231</td>
<td>0.960924595</td>
<td>0.00%</td>
</tr>
<tr>
<td>10%</td>
<td>0.1</td>
<td>20%</td>
<td>0.25</td>
<td>100</td>
<td>0.565763231</td>
<td>0.56594217</td>
<td>0.03%</td>
</tr>
<tr>
<td>10%</td>
<td>0.1</td>
<td>20%</td>
<td>0.25</td>
<td>110</td>
<td>0.520993753</td>
<td>0.521408775</td>
<td>0.15%</td>
</tr>
<tr>
<td>10%</td>
<td>0.1</td>
<td>20%</td>
<td>0.25</td>
<td>120</td>
<td>0.335726623</td>
<td>0.33685007</td>
<td>0.16%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.853480625</td>
<td>0.853018697</td>
<td>-0.05%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.70867637</td>
<td>0.709151848</td>
<td>0.07%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.2</td>
<td>110</td>
<td>0.529413073</td>
<td>0.529076083</td>
<td>-0.06%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.351056665</td>
<td>0.350513712</td>
<td>-0.08%</td>
</tr>
<tr>
<td>10%</td>
<td>0.7</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.860423398</td>
<td>0.856186282</td>
<td>-0.09%</td>
</tr>
<tr>
<td>10%</td>
<td>0.7</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.715155518</td>
<td>0.71637899</td>
<td>0.01%</td>
</tr>
<tr>
<td>10%</td>
<td>0.7</td>
<td>20%</td>
<td>0.2</td>
<td>110</td>
<td>0.539513051</td>
<td>0.540332465</td>
<td>0.14%</td>
</tr>
<tr>
<td>10%</td>
<td>0.7</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.36442835</td>
<td>0.363713423</td>
<td>-0.20%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>20%</td>
<td>0.9</td>
<td>90</td>
<td>0.967898772</td>
<td>0.96735169</td>
<td>0.02%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>20%</td>
<td>0.9</td>
<td>100</td>
<td>0.572082155</td>
<td>0.57170026</td>
<td>0.01%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>20%</td>
<td>0.9</td>
<td>110</td>
<td>0.367441651</td>
<td>0.367154152</td>
<td>-0.21%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>20%</td>
<td>0.9</td>
<td>120</td>
<td>0.219722863</td>
<td>0.21685703</td>
<td>-1.5%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>90</td>
<td>0.661346865</td>
<td>0.66486152</td>
<td>-0.52%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>100</td>
<td>0.37745374</td>
<td>0.37761389</td>
<td>-0.2%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>110</td>
<td>0.21578712</td>
<td>0.21972286</td>
<td>-1.8%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>120</td>
<td>0.13781134</td>
<td>0.13758877</td>
<td>-0.2%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
<td>90</td>
<td>0.955318151</td>
<td>0.95581569</td>
<td>-0.05%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
<td>100</td>
<td>0.367844168</td>
<td>0.36715415</td>
<td>-0.21%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
<td>110</td>
<td>0.219722863</td>
<td>0.21685703</td>
<td>-1.5%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
<td>120</td>
<td>0.37745374</td>
<td>0.37761389</td>
<td>-0.2%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>90</td>
<td>0.725746935</td>
<td>0.72581461</td>
<td>0.01%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>100</td>
<td>0.549124363</td>
<td>0.54923168</td>
<td>0.03%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>110</td>
<td>0.37745374</td>
<td>0.37761389</td>
<td>-0.2%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>120</td>
<td>0.21578712</td>
<td>0.21972286</td>
<td>-1.8%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>120</td>
<td>0.13781134</td>
<td>0.13758877</td>
<td>-0.2%</td>
</tr>
<tr>
<td>1%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>120</td>
<td>0.45049192</td>
<td>0.44067279</td>
<td>-0.2%</td>
</tr>
</tbody>
</table>

Figure 1: Delta of Plain Vanilla Option
<table>
<thead>
<tr>
<th>r</th>
<th>T</th>
<th>σ</th>
<th>γ</th>
<th>K</th>
<th>True Value</th>
<th>Approximation</th>
<th>error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>90</td>
<td>0.946703688</td>
<td>0.94675927</td>
<td>0.01%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>100</td>
<td>0.768201338</td>
<td>0.768261843</td>
<td>0.01%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>110</td>
<td>0.3028172</td>
<td>0.30359663</td>
<td>0.28%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>120</td>
<td>0.10319023</td>
<td>0.1036615</td>
<td>0.45%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.858575119</td>
<td>0.85808016</td>
<td>-0.06%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.637483615</td>
<td>0.638108348</td>
<td>0.01%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>110</td>
<td>0.201951045</td>
<td>0.203651546</td>
<td>0.84%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.007273247</td>
<td>0.007444314</td>
<td>2.35%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>90</td>
<td>0.768435819</td>
<td>0.768741575</td>
<td>0.04%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>100</td>
<td>0.588937469</td>
<td>0.58929763</td>
<td>0.04%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>110</td>
<td>0.20435819</td>
<td>0.205940424</td>
<td>0.84%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>120</td>
<td>0.007875811</td>
<td>0.008174575</td>
<td>0.34%</td>
</tr>
<tr>
<td>5%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>90</td>
<td>0.9120356</td>
<td>0.911511296</td>
<td>-0.05%</td>
</tr>
<tr>
<td>5%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>100</td>
<td>0.56037022</td>
<td>0.560295853</td>
<td>-0.01%</td>
</tr>
<tr>
<td>5%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>110</td>
<td>0.153452354</td>
<td>0.153316963</td>
<td>-0.09%</td>
</tr>
<tr>
<td>5%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>120</td>
<td>0.015937687</td>
<td>0.016132116</td>
<td>1.22%</td>
</tr>
</tbody>
</table>

Figure 2: Delta of Average Option
Figure 3: Variance Reduction for Plain Vanilla Delta
<table>
<thead>
<tr>
<th>Crude/MC</th>
<th>Attendee</th>
<th>Hybrid/QMC</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>variance</td>
<td>0.9272646</td>
<td>0.9241266</td>
<td>0.202569</td>
<td>0.26973</td>
<td>0.79269</td>
</tr>
<tr>
<td>average</td>
<td>0.6840545</td>
<td>0.6957232</td>
<td>0.672345</td>
<td>0.645893</td>
<td>0.659093</td>
</tr>
<tr>
<td>stddev</td>
<td>0.202569</td>
<td>0.26973</td>
<td>0.79269</td>
<td>0.6840545</td>
<td>0.6957232</td>
</tr>
<tr>
<td>max</td>
<td>0.6840545</td>
<td>0.6957232</td>
<td>0.79269</td>
<td>0.6840545</td>
<td>0.6957232</td>
</tr>
<tr>
<td>min</td>
<td>0.202569</td>
<td>0.26973</td>
<td>0.79269</td>
<td>0.6840545</td>
<td>0.6957232</td>
</tr>
<tr>
<td>total</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
</tbody>
</table>

Figure 4: Variance Reduction for Average Delta
<table>
<thead>
<tr>
<th>$r$</th>
<th>$T$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$K$</th>
<th>True Value</th>
<th>Approximation</th>
<th>Approximation error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>90</td>
<td>0.135620063</td>
<td>0.136145644</td>
<td>0.39%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>100</td>
<td>0.579057693</td>
<td>0.576043297</td>
<td>0.00%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>110</td>
<td>0.660242603</td>
<td>0.661726804</td>
<td>0.15%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>120</td>
<td>0.656887047</td>
<td>0.65763586</td>
<td>-0.02%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.580709278</td>
<td>0.582683079</td>
<td>0.34%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.83906647</td>
<td>0.842159142</td>
<td>0.37%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>110</td>
<td>0.662827562</td>
<td>0.662582342</td>
<td>-0.03%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.61984393</td>
<td>0.620976444</td>
<td>0.13%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>90</td>
<td>0.57697078</td>
<td>0.576246004</td>
<td>0.00%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>100</td>
<td>0.897567977</td>
<td>0.902646034</td>
<td>0.06%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>110</td>
<td>0.99334662</td>
<td>0.992740559</td>
<td>0.06%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>120</td>
<td>0.92734053</td>
<td>0.93017674</td>
<td>0.03%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>90</td>
<td>0.62872776</td>
<td>0.63715386</td>
<td>1.08%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>100</td>
<td>0.91784508</td>
<td>0.9282958</td>
<td>0.76%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>110</td>
<td>0.95635304</td>
<td>0.956286345</td>
<td>0.00%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>120</td>
<td>0.942198425</td>
<td>0.946094322</td>
<td>0.43%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>90</td>
<td>1.505240303</td>
<td>1.517243388</td>
<td>0.80%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>100</td>
<td>2.57930435</td>
<td>2.585420486</td>
<td>0.24%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>110</td>
<td>2.67514174</td>
<td>2.67697763</td>
<td>-0.07%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>120</td>
<td>1.71350119</td>
<td>1.71531578</td>
<td>0.12%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>90</td>
<td>2.232530619</td>
<td>2.25920744</td>
<td>0.85%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>100</td>
<td>3.332715923</td>
<td>3.35331757</td>
<td>0.62%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>110</td>
<td>3.67246366</td>
<td>3.68606179</td>
<td>0.35%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>120</td>
<td>3.59795660</td>
<td>3.62395206</td>
<td>0.71%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>90</td>
<td>2.93549686</td>
<td>2.95253178</td>
<td>0.56%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>100</td>
<td>3.55163245</td>
<td>3.56422902</td>
<td>0.09%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>110</td>
<td>3.86193548</td>
<td>3.88684106</td>
<td>0.64%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>120</td>
<td>3.71812029</td>
<td>3.81484873</td>
<td>0.97%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>130</td>
<td>8.62223284</td>
<td>8.71843779</td>
<td>1.12%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>140</td>
<td>13.26718447</td>
<td>13.35392573</td>
<td>0.65%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>150</td>
<td>15.64510472</td>
<td>15.6984559</td>
<td>0.34%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.5</td>
<td>160</td>
<td>14.82507253</td>
<td>14.88915555</td>
<td>0.50%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>90</td>
<td>28.16294977</td>
<td>28.71366052</td>
<td>1.96%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>100</td>
<td>35.45621563</td>
<td>35.9506663</td>
<td>1.28%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>110</td>
<td>39.359731</td>
<td>39.766357</td>
<td>1.32%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>1</td>
<td>120</td>
<td>39.58760491</td>
<td>40.07237771</td>
<td>1.23%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>1</td>
<td>90</td>
<td>21.14470728</td>
<td>21.44978946</td>
<td>1.44%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>1</td>
<td>100</td>
<td>33.32246029</td>
<td>33.54565167</td>
<td>0.67%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>1</td>
<td>110</td>
<td>39.59139563</td>
<td>39.7895782</td>
<td>0.50%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>1</td>
<td>120</td>
<td>38.0036401</td>
<td>38.23561123</td>
<td>0.61%</td>
</tr>
</tbody>
</table>

Figure 5: Vega of Plain Vanilla Option
<table>
<thead>
<tr>
<th>r</th>
<th>T</th>
<th>σ</th>
<th>γ</th>
<th>K</th>
<th>True Value</th>
<th>Approximation</th>
<th>Approximation error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>90</td>
<td>0.017620536</td>
<td>0.020081338</td>
<td>13.97%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>100</td>
<td>0.37292946</td>
<td>0.374267264</td>
<td>0.36%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>110</td>
<td>0.398718399</td>
<td>0.399866113</td>
<td>0.25%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>10%</td>
<td>0.2</td>
<td>120</td>
<td>0.028642431</td>
<td>0.029156347</td>
<td>1.76%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>90</td>
<td>0.233997165</td>
<td>0.236452456</td>
<td>1.05%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>100</td>
<td>0.42048552</td>
<td>0.43495716</td>
<td>3.66%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>110</td>
<td>0.506707856</td>
<td>0.517406352</td>
<td>2.16%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.2</td>
<td>120</td>
<td>0.265196439</td>
<td>0.267081362</td>
<td>0.71%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>90</td>
<td>0.370148458</td>
<td>0.372996781</td>
<td>0.77%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>100</td>
<td>0.51844131</td>
<td>0.52174468</td>
<td>0.53%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>110</td>
<td>0.52681847</td>
<td>0.530165585</td>
<td>0.66%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.2</td>
<td>120</td>
<td>0.39731001</td>
<td>0.40121822</td>
<td>0.88%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>90</td>
<td>0.43374062</td>
<td>0.437488462</td>
<td>0.92%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>100</td>
<td>0.52820917</td>
<td>0.530697986</td>
<td>0.47%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>110</td>
<td>0.536368397</td>
<td>0.538568445</td>
<td>0.41%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>40%</td>
<td>0.2</td>
<td>120</td>
<td>0.45876673</td>
<td>0.46214756</td>
<td>0.77%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>90</td>
<td>0.451809942</td>
<td>0.455691138</td>
<td>0.81%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>100</td>
<td>1.497145322</td>
<td>1.504188555</td>
<td>0.47%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>110</td>
<td>1.104185264</td>
<td>1.10946028</td>
<td>0.47%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>120</td>
<td>0.228404048</td>
<td>0.22978291</td>
<td>0.61%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>130</td>
<td>0.895201544</td>
<td>0.89967307</td>
<td>0.46%</td>
</tr>
<tr>
<td>10%</td>
<td>0.5</td>
<td>20%</td>
<td>0.5</td>
<td>140</td>
<td>1.966035268</td>
<td>1.969793649</td>
<td>0.20%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>110</td>
<td>2.045315868</td>
<td>2.055561023</td>
<td>0.50%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>120</td>
<td>1.133204923</td>
<td>1.142823644</td>
<td>0.79%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>130</td>
<td>1.429725891</td>
<td>1.43891795</td>
<td>0.62%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>140</td>
<td>2.056665164</td>
<td>2.07160649</td>
<td>0.74%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>150</td>
<td>2.126552438</td>
<td>2.14675012</td>
<td>0.90%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.5</td>
<td>160</td>
<td>1.656340874</td>
<td>1.662985408</td>
<td>0.39%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.6</td>
<td>100</td>
<td>3.34324547</td>
<td>3.36206635</td>
<td>0.55%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.6</td>
<td>110</td>
<td>7.802925684</td>
<td>7.81702161</td>
<td>0.61%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.6</td>
<td>120</td>
<td>8.234805853</td>
<td>8.337629253</td>
<td>1.29%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>20%</td>
<td>0.6</td>
<td>130</td>
<td>13.30358393</td>
<td>13.5367179</td>
<td>1.72%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.6</td>
<td>100</td>
<td>20.4947224</td>
<td>20.6641664</td>
<td>0.82%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.6</td>
<td>110</td>
<td>21.83972441</td>
<td>22.0762786</td>
<td>1.12%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.6</td>
<td>120</td>
<td>18.01543655</td>
<td>18.2916237</td>
<td>1.53%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.6</td>
<td>130</td>
<td>8.002372068</td>
<td>8.11376058</td>
<td>1.38%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.8</td>
<td>100</td>
<td>15.47497484</td>
<td>15.61230035</td>
<td>0.71%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.8</td>
<td>110</td>
<td>21.09200973</td>
<td>21.19619263</td>
<td>0.49%</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>30%</td>
<td>0.8</td>
<td>120</td>
<td>12.76683758</td>
<td>12.94213589</td>
<td>1.44%</td>
</tr>
</tbody>
</table>

Figure 6: Vega of Average Option
<table>
<thead>
<tr>
<th>Variance Reduction Method for Plain Vanilla Vega</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude MCM</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>correlation</td>
</tr>
<tr>
<td>average</td>
</tr>
<tr>
<td>stddev</td>
</tr>
<tr>
<td>max</td>
</tr>
<tr>
<td>min</td>
</tr>
<tr>
<td>total error</td>
</tr>
<tr>
<td>1000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crude MCM</th>
<th>Attendant MCM</th>
<th>Hybrid MCM</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation</td>
<td>0.999557467</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>average</td>
<td>3.1342408</td>
<td>0.0090959</td>
<td>3.3195792</td>
<td>0.1%</td>
<td>0.2%</td>
</tr>
<tr>
<td>stddev</td>
<td>0.2006168</td>
<td>0.25988</td>
<td>0.2517990</td>
<td>6.6%</td>
<td>0.6%</td>
</tr>
<tr>
<td>max</td>
<td>3.99646</td>
<td>0.614126</td>
<td>3.30658</td>
<td>10.7%</td>
<td>10.1%</td>
</tr>
<tr>
<td>min</td>
<td>2.47307</td>
<td>-0.57944</td>
<td>2.38591</td>
<td>-18.9%</td>
<td>-18.3%</td>
</tr>
<tr>
<td>total error</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>0</td>
<td>K</td>
</tr>
<tr>
<td>1000</td>
<td>100</td>
<td>0.1</td>
<td>1</td>
<td>0.2</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crude MCM</th>
<th>Attendant MCM</th>
<th>Hybrid MCM</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation</td>
<td>0.990472682</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>average</td>
<td>3.1517166</td>
<td>0.22529</td>
<td>3.3452947</td>
<td>-1.0%</td>
<td>-0.9%</td>
</tr>
<tr>
<td>stddev</td>
<td>2.96631003</td>
<td>2.890031</td>
<td>2.6555948</td>
<td>4.6%</td>
<td>0.1%</td>
</tr>
<tr>
<td>max</td>
<td>46.03922</td>
<td>7.91683</td>
<td>263573</td>
<td>25.1%</td>
<td>22.0%</td>
</tr>
<tr>
<td>min</td>
<td>28.12041</td>
<td>-29.8989</td>
<td>24.4306</td>
<td>-20.8%</td>
<td>-20.1%</td>
</tr>
<tr>
<td>total error</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>0</td>
<td>K</td>
</tr>
<tr>
<td>1000</td>
<td>100</td>
<td>0.1</td>
<td>1</td>
<td>0.2</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Crude MCM</th>
<th>Attendant MCM</th>
<th>Hybrid MCM</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation</td>
<td>0.990797188</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>average</td>
<td>1.3239511</td>
<td>0.0202615</td>
<td>1.2102273</td>
<td>0.2%</td>
<td>0.2%</td>
</tr>
<tr>
<td>stddev</td>
<td>0.200899330</td>
<td>0.0808245</td>
<td>0.0991528</td>
<td>6.0%</td>
<td>0.6%</td>
</tr>
<tr>
<td>max</td>
<td>1.438665</td>
<td>0.182951</td>
<td>1.34744</td>
<td>11.5%</td>
<td>11.0%</td>
</tr>
<tr>
<td>min</td>
<td>1.13236</td>
<td>-0.18802</td>
<td>1.21742</td>
<td>-14.7%</td>
<td>-14.8%</td>
</tr>
<tr>
<td>total error</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>0</td>
<td>K</td>
</tr>
<tr>
<td>1000</td>
<td>100</td>
<td>0.1</td>
<td>1</td>
<td>0.2</td>
<td>100</td>
</tr>
</tbody>
</table>

Figure 7: Variance Reduction for Plain Vanilla Vega

106
<table>
<thead>
<tr>
<th>Variance Reduction Method for Average Vega</th>
<th>Crude MC</th>
<th>Lyapunov</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation: 0.990039157</td>
<td>0.789309461</td>
<td>0.032531</td>
<td>0.020136</td>
<td>0.3%</td>
<td>0.1%</td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance Reduction Method for Average Vega</th>
<th>Crude MC</th>
<th>Lyapunov</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation: 0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance Reduction Method for Average Vega</th>
<th>Crude MC</th>
<th>Lyapunov</th>
<th>error 1</th>
<th>error 2</th>
<th>error 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation: 0.997132146</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
<td>0.990039157</td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Figure 8: Variance Reduction for Average Vega
Reference


