An Asymptotic Expansion Approach to Computing Greeks *

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Abstract

We developed a new scheme for computing "Greeks"of derivatives by *an asymptotic expansion approach*. In particular, we derived analytical approximation formulae for Deltas and Vegas of plain vanilla and average call options under general Markovian processes of underlying asset prices. We also derived approximation formulae for Gammas of plain vanilla and average call options, and for Deltas of digital options under CEV(Constant Elasticity of Variance) processes of underlying assets' prices. Moreover, we introduced a new variance reduction method of Monte Carlo simulations based on the asymptotic expansion scheme. Finally, several numerical examples under CEV processes confirmed the validity of our method.

*This paper presents the authors' personal views, and these are not the official views of the Financial Services Agency, the Financial Research and Training Center, and of Tokio Marine & Nichido.

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1 Introduction

We propose a new approximation formulae for computing Greeks, such as the Delta, the Gamma and the Vega that indicate risk indices, the first or second order derivatives of the value of an asset with respect to its parameters. In particular, we derive analytic approximation formulae of the Delta and the Vega of plain-vanilla and average options where the underlying assets' prices follow general diffusion processes. We also show formulae of the Gamma of plain-vanilla and average options, and a formula of the Delta of digital options where the underlying assets' prices follow CEV processes. Our method is based on the asymptotic expansion approach developed by Takahashi[1995,1999], Kunitomo and Takahashi[1992,2001,2003a] and Takahashi and Yoshida[2004]. We also introduce a new variance reduction method as an extension of Takahashi and Yoshida[2005] to increase efficiency of Monte Carlo simulation in computation of Greeks. Moreover, we presented series of numerical examples where the underlying prices follow the constant elasticity of variance (CEV) processes and showed effectiveness of our method.

Monitoring and controlling the risks of derivative securities are as important as pricing derivative securities in the practical world. In Black-Scholes(BS) model, option prices and their Greeks are obtained analytically. However, in the more realistic models, it is usually very hard to evaluate both of option prices and their Greeks analytically. Then, numerical methods are applied.

We suppose that the underlying asset's price S_t follows a stochastic differential equation(SDE);

$$\begin{cases} dS_t = \mu\, S_t\, dt + \sigma\left(S_t\right)\, dW_t \\ S_0 = s_0\ (>0)\,, \end{cases} \tag{1}$$

where W_t is a one-dimensional Brownian motion, $\mu = r - q$. r and q denote a risk-free rate and a divident rate respectively which are assumed to be constants. Next, we consider a derivative security whose payoff function at maturity time T is given by ϕ such as

$$\begin{array}{cc} \phi\left(S_T\right) = \left(S_T - K\right)_+ & \text{plain vanilla call option} \\ \phi\left(\tilde{S}_T\right) = \left(\tilde{S}_T - K\right)_+ & \text{average call option,} \end{array} \tag{2}$$

where $\tilde{S}_T = \frac{1}{T}\int_0^T S_t\, dt$. The price of the derivative security can be represented by

$$\begin{aligned} u(s_0) &= E\left[e^{-rT}\,\phi\left(S_T\right)\right] \\ &= e^{-rT} E\left[\phi\left(S_T\right)\right], \end{aligned}$$

where $E\left[\cdot\right]$ denotes the expectation operator under the risk-neutral measure. To obtain the first order derivative of the price with respect to the underlying asset's price at the initial price s_0, a natural method is computing

$$u'\left(s_0\right) \sim \frac{u\left(s_0 + \Delta\right) - u\left(s_0 - \Delta\right)}{2\Delta},$$

where Δ is a sufficiently small positive number, and $u\left(s_0 - \Delta\right)$ and $u\left(s_0 + \Delta\right)$ are generated by Monte Carlo simulations. However, its convergence speed is sometimes very slow due to the irregularity of ϕ. Moreover, convergence to the true value may not be achieved because its convergent value depends on Δ.

To overcome the problem, we may utilize a representation such as

$$u'\left(s_0\right) = e^{-rT} E\left[\phi'\left(S_T\right)\mathbf{Y}_T\right],$$

where Y_t satisfies the following SDE:

$$\begin{cases} dY_t = \mu Y_t \, dt + \sigma' (S_t) \; Y_t \; dW_t \\ Y_0 = 1. \end{cases}$$

(For the derivation, see Imamura, Uchida and Takahashi [2005] for instance.) Although this representation makes Monte Carlo simulations more efficient, using Monte Carlo simulation itself may be time-consuming, which cause many difficulties in the practical world such as trading business. Therefore, in the case that analytic formulae can not be obtained, ideally we need analytic approximation schemes which can generate values precise enough for practical pupose.

The asymptotic expansion approach have been applied successfully to a broad class of Itô processes appearing in finance. Takahashi[1999] presented a third-order pricing formula for plain options and second-order formulae for more complicated derivatives such as average options, basket options, and options with stochastic volatility in a general Markovian setting. Kunitomo and Takahashi[2001] provided pricing formulae for bond options(swap options) and average options based on an interest rate model in the class of Heath-Jarrow-Morton[1992] which is not necessarily Markovian. Takahashi[1995] also presented a second order scheme for average options on foreign exchange rates with stochastic interest rates in Heath-Jarrow-Morton framework.

Moreover, Takahashi and Yoshida[2004] extended the method to dynamic portfolio problems. Recently, Takahashi and Saito[2003] successfully applied the method to American options, and Takahashi and Yoshida[2005] proposed a new variance reduction scheme of Monte Carlo simulation with the asymptotic expansion. For details of mathematical validity based on the Malliavin calculus and of further applications, see Kunitomo and Takahashi[2003a, 2003b, 2004], Takahashi and Uchida[2004] and Takahashi and Yoshda[2004,2005].

The organization of this paper is as follows. In section 2 and 3, we derive approximation formulae for the Delta of plain-vanilla and average call options. In section 4, we introduce a variance reduction technique in computation of the Delta. Sections 5 and 6 treat approximations for computing the Vega of plain vanilla and average call options. In section 7, we propose a variance reduction technique in computation of the Vega. In section 8, when the underlying prices follow CEV processes, we show another derivation of approximation formulae for the Delta and the Vega, and derive approximation formulae for the Gamma of plain vanilla and average options as well as for the Delta of digital options. We present numerical examples in section 9. Finally, some concluding remarks are made in section 10.

2 The Delta of Plain Vanilla Call Option

First, we make the following assumption and definitions.

Assumption 1. *We assume that the underlying asset price S_t follows the SDE:*

$$\begin{cases} dS_t = \mu S_t \, dt + \epsilon\sigma (S_t) \; dW_t \\ S_0 = s_0 \; (> 0) \end{cases}$$

where W_t is 1-dimensional Brownian Motion, and $\mu \neq 0$. In addition we assume that $\sigma(\cdot)$ is a C^2 class function and its derivative $\sigma'(\cdot)$ is bounded.

We define the "differentiation of S_t by the initial value s_0".

Definition 1. *We define Y_t as a stocastic process following the SDE:*

$$\begin{cases} dY_t = \mu\, Y_t\, dt + \epsilon\sigma'\left(S_t\right)\, Y_t\, dW_t \\ Y_0 = 1 \end{cases}$$

Definition 2. *We define the defined indicator function:*

$$1_{\{x\geq a\}} = \begin{cases} 1 & \textit{if } x \geq a \\ 0 & \textit{if } x < a \end{cases}$$

Then, the payoff of Plain Vanilla Call Option is rewritten as $(S_T - K)_+ = (S_T - K)\, 1_{\{S_T \geq K\}}$.

Takahashi [1999] derived , the asymptotic expantion of S_t:

$$S_t = A_{0t} + \epsilon A_{1t} + \frac{\epsilon^2}{2}\, A_{2t} + o(\epsilon^2), \tag{3}$$

where

$$\begin{aligned} dA_{0t} &= \mu\, A_{0t}\, dt, \qquad A_{00} = s_0 \\ dA_{1t} &= \mu\, A_{1t}\, dt + \sigma\left(A_{0t}\right)\, dW_t, \qquad A_{10} = 0 \\ dA_{2t} &= \mu\, A_{2t}\, dt + 2\,\sigma'\left(\, A_{0t}\right)\, A_{1t}\, dW_t, \qquad A_{20} = 0. \end{aligned}$$

Then, we can easily obtain $A_{0t},\ A_{1t}$ and A_{2t} as

$$\begin{aligned} A_{0t} &= s_0\, e^{\mu\, t} \\ A_{1t} &= \int_0^t e^{\mu(t-s)}\, \sigma\left(A_{0s}\right)\, dW_s \\ A_{2t} &= 2\int_0^t e^{\mu(t-s)}\, \sigma'\left(A_{0s}\right)\, A_{1s}\, dW_s. \end{aligned}$$

Now we put $X_t = \left(S_t - A_{0t}\right)/\epsilon$, $g_1 = A_{1T}$ and $g_2 = A_{2T}\,/\,2$. Then,

$$X_T = \ g_1 + \epsilon\, g_2 + o(\epsilon^2).$$

The asymptotic expantion of the Plain Vanilla Call Option $e^{-rT}E\left[(S_T - K)_+\right]$ is given by

$$\begin{aligned} e^{-rT}E\left[(S_T - K)_+\right] &= \epsilon\, e^{-rT}E\left[(y + X_T)_+\right] \\ &= \epsilon\, e^{-rT}\, E\left[(y + g_1 + \epsilon\, g_2 + \cdots)\ 1_{\{g_1 + \epsilon\, g_2 + \cdots \geq -y\}}\right], \end{aligned}$$

where $y = \left(A_{0T} - K\right)/\epsilon$. Under an appropriate assumption such as *Assumption 6.2* in Kunitomo and Takahashi[2003b], We obtain its approximated value as

$$\begin{aligned} & e^{-rT}E\left[(S_T - K)_+\right] \\ = \ & \epsilon\, e^{-rT}\, E\left[(y + g_1)\ 1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\, E\left[g_2\, 1_{\{g_1 \geq -y\}}\right] \\ & + \epsilon^2\, e^{-rT}\, E\left[(y + g_1)\ g_2\ (\partial 1)_{\{g_1 \geq -y\}}\right] + o\left(\epsilon^2\right) \end{aligned}$$

where $(\partial 1)_{\{\cdot\geq -y\}}$ denotes the derivative of $1_{\{\cdot\geq -y\}}$. Next, by the property of Brownian Motion, we know the distribution of g_1 follows a normal distribution $N(0,\Sigma)$, where

$$\begin{aligned} \Sigma &= E\left[\left(\int_0^T e^{\mu(T-t)}\,\sigma(A_{0t})\,dW_t\right)^2\right] \\ &= \int_0^T e^{2\mu(T-t)}\,\sigma(A_{0t})^2\,dt. \end{aligned} \tag{4}$$

In addition, the conditional expectation $E[g_2|g_1=x]$ is given by

$$E[g_2|g_1=x] = cx^2+f,$$

where

$$c = \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu(T-s)}\,\sigma(A_{0s})\,\sigma'(A_{0s})\,e^{2\mu(T-v)}\,\sigma(A_{0v})^2\,dv\,ds$$

and $f=-c\,\Sigma$. Accordingly,

$$\begin{aligned} & e^{-rT}E\left[(S_T-K)_+\right] \\ = {} & \epsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\,n[x;0,\Sigma]\,dx+\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n[x;0,\Sigma]\,dx+o\left(\epsilon^2\right), \end{aligned} \tag{5}$$

where $n[x;0,\Sigma]$ is the density function of $N(0,\Sigma)$, i.e.

$$n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{x^2}{2\Sigma}}.$$

Now we define an approximated value of the call option as

$$\begin{aligned} C(0,T) &= \epsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\,n[x;0,\Sigma]\,dx+\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n[x;0,\Sigma]\,dx \\ &= \epsilon\,e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n[y;0,\Sigma]\right)+\epsilon^2\,e^{-rT}\,f\,y\,n[y;0,\Sigma], \end{aligned} \tag{6}$$

where $N(x)$ is cumulative distribution function of $N(0,1)$. Next, consider its differentiation by s_0:

$$\begin{aligned} & \frac{\partial}{\partial s_0}C(0,T) \\ = {} & \epsilon\,e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\left(y\left(d-\frac{y}{2\,\Sigma}\frac{\partial\Sigma}{\partial s_0}\right)+\frac{\partial\Sigma}{\partial s_0}+\Sigma\left(-\frac{1}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}-\frac{yd}{\Sigma}+\frac{y^2}{2\Sigma^2}\frac{\partial\Sigma}{\partial s_0}\right)\right)n[y;0,\Sigma]\right) \\ & +\epsilon^2\,e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y+f\,d+f\,y\left(-\frac{1}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}-\frac{yd}{\Sigma}+\frac{y^2}{2\Sigma^2}\frac{\partial\Sigma}{\partial s_0}\right)\right)n[y;0,\Sigma] \\ = {} & \epsilon\,e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{1}{2}\frac{\partial\Sigma}{\partial s_0}n[y;0,\Sigma]\right) \\ & +\epsilon^2\,e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y+f\,d+c\,y\left(\frac{1}{2}\frac{\partial\Sigma}{\partial s_0}+yd-\frac{y^2}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}\right)\right)n[y;0,\Sigma], \end{aligned}$$

where $d = \frac{\partial y}{\partial s_0} = e^{\mu T}/\epsilon$. The derivatives of coefficients are calculated as follows:

$$\begin{aligned}\frac{\partial \Sigma}{\partial s_0} &= \frac{\partial}{\partial s_0}\int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\,dt \\ &= 2\int_0^T e^{2\mu\,(T-t)}\,e^{\mu\,t}\,\sigma\left(A_{0t}\right)\,\sigma'\left(A_{0t}\right)\,dt \\ &= 2\int_0^T e^{2\mu\,T-\mu\,t}\,\sigma\left(A_{0t}\right)\,\sigma'\left(A_{0t}\right)\,dt\end{aligned}$$

$$\begin{aligned}\frac{\partial c}{\partial s_0} &= \frac{\partial}{\partial s_0}\left\{\frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds\right\} \\ &= -\frac{2}{\Sigma^3}\frac{\partial \Sigma}{\partial s_0}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds \\ &\quad+\frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,T}\,\sigma'\left(A_{0s}\right)^2\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds \\ &\quad+\frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,T}\,\sigma\left(A_{0s}\right)\,\sigma''\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds \\ &\quad+\frac{2}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,T-\mu\,v}\,\sigma\left(A_{0v}\right)\,\sigma'\left(A_{0v}\right)\,dv\,ds\end{aligned}$$

$$\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0}\left(c\Sigma\right) = -\frac{\partial c}{\partial s_0}\Sigma - c\frac{\partial \Sigma}{\partial s_0}$$

Although it looks quite complicated, simpler expressions of coefficients can be derived if we give $\sigma\left(\cdot\right)$ specific forms such as CEV, which are shown in section 8.

Then, we summarize the result obtained above.

Theorem 1. *The asymptotic expansion of the Delta of a plain vanilla call option of which payoff function with maturity time T is expressed as $\left(S_T - K\right)_+$ is given by*

$$D\left(0,T\right) + o\left(\epsilon^2\right),$$

where

$$\begin{aligned}&D\left(0,T\right) \\ =\;& \epsilon\,e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{1}{2}\frac{\partial \Sigma}{\partial s_0}n\left[y;0,\Sigma\right]\right) \\ &+\epsilon^2\,e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y + f\,d + c\,y\left(\frac{1}{2}\frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma}\frac{\partial \Sigma}{\partial s_0}\right)\right)n\left[y;0,\Sigma\right],\end{aligned} \tag{7}$$

and the coefficents are given by

$$y = \frac{s_0 e^{\mu\,T} - K}{\epsilon},$$

$$
\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\,dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\,dt,
\end{aligned}
$$

$$
c = \frac{1}{\Sigma^2}\int_0^T \int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds
$$

and

$$
f = -c\Sigma.
$$

3 The Delta of Average Call Option

Utilizing the asymptotic expansion of S_t, Takahashi [1999] derived

$$
\widetilde{S}_t \sim \widetilde{A}_{0t} + \epsilon \widetilde{A}_{1t} + \frac{\epsilon^2}{2}\,\widetilde{A}_{2t} + o(\epsilon^2), \tag{8}
$$

where $\widetilde{S}_t = \frac{1}{t}\int_0^t S_u\,du$, $\widetilde{A}_{0t} = \frac{1}{t}\int_0^t A_{0u}\,du$ and so on. Now we put $\widetilde{X}_t = \left(\widetilde{S}_t - \widetilde{A}_{0t}\right)/\epsilon$, $g_1 = \widetilde{A}_{1T}$ and $g_2 = \widetilde{A}_{2T}\,/\,2$. Then,

$$
\widetilde{X}_T \sim g_1 + \epsilon\, g_2 + o(\epsilon^2).
$$

The asymptotic expantion of the average call option $e^{-rT}E\left[\left(\widetilde{S}_T - K\right)_+\right]$ is given by

$$
\begin{aligned}
e^{-rT}E\left[\left(\widetilde{S}_T - K\right)_+\right] &= \epsilon\, e^{-rT}E\left[\left(y + \widetilde{X}_T\right)_+\right] \\
&= \epsilon\, e^{-rT}\,E\left[\left(y + g_1 + \epsilon\, g_2 + \cdots\right)\,1_{\{g_1+\epsilon\, g_2+\cdots \geq -y\}}\right],
\end{aligned}
$$

where $y = \left(\widetilde{A}_{0T} - K\right)/\epsilon$. We obtain its approximated value

$$
\begin{aligned}
& e^{-rT}E\left[\left(\widetilde{S}_T - K\right)_+\right] \\
= \;& \epsilon\, e^{-rT}\,E\left[\left(y + g_1\right)\,1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\,E\left[g_2\,1_{\{g_1 \geq -y\}}\right] \\
& + \epsilon^2\, e^{-rT}\,E\left[\left(y + g_1\right)\,g_2\,\left(\partial 1\right)_{\{g_1 \geq -y\}}\right] + o\left(\epsilon^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma &= E\left[\left(\frac{1}{T}\int_0^T \int_0^t e^{\mu\,(t-s)}\,\sigma\left(A_{0s}\right)\,dW_s\,dt\right)^2\right] \\
&= \frac{1}{T^2}\,E\left[\left(\int_0^T \int_s^T e^{\mu\,(t-s)}\,\sigma\left(A_{0s}\right)\,dt\,dW_s\right)^2\right] \\
&= \frac{1}{T^2}\,E\left[\left(\int_0^T \frac{1}{\mu}\,\left(e^{\mu(T-s)} - 1\right)\,\sigma\left(A_{0s}\right)\,dW_s\right)^2\right] \\
&= \frac{1}{\mu^2 T^2}\int_0^T \left(e^{\mu(T-s)} - 1\right)^2\,\sigma\left(A_{0s}\right)^2\,ds.
\end{aligned} \tag{9}
$$

In addition, the coefficients of the conditional expectation $E\left[g_2|g_1=x\right]=cx^2+f$ are given by

$$\begin{aligned} c &= \frac{1}{\Sigma^2\,T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\ &\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \end{aligned}$$

and $f=-c\,\Sigma$. Accordingly,

$$\begin{aligned} &e^{-rT}E\left[\left(\widetilde{S}_T-K\right)_+\right] \\ =\ &\epsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\,n\left[x;0,\Sigma\right]\,dx+\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx+o\left(\epsilon^2\right). \end{aligned}$$

Now we define the approximated value as

$$\begin{aligned} C\left(0,T\right) &= \epsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\,n\left[x;0,\Sigma\right]\,dx+\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx \\ &= \epsilon\,e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n\left[y;0,\Sigma\right]\right)+\epsilon^2\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right], \end{aligned} \tag{10}$$

and consider its differentiation by s_0:

$$\begin{aligned} &\frac{\partial}{\partial s_0}C\left(0,T\right) \\ =\ &\epsilon\,e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\left(y\left(d-\frac{y}{2\,\Sigma}\frac{\partial\Sigma}{\partial s_0}\right)+\frac{\partial\Sigma}{\partial s_0}+\Sigma\left(-\frac{1}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}-\frac{yd}{\Sigma}+\frac{y^2}{2\Sigma^2}\frac{\partial\Sigma}{\partial s_0}\right)\right)n\left[y;0,\Sigma\right]\right) \\ &+\epsilon^2\,e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y+f\,d+f\,y\left(-\frac{1}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}-\frac{yd}{\Sigma}+\frac{y^2}{2\Sigma^2}\frac{\partial\Sigma}{\partial s_0}\right)\right)n\left[y;0,\Sigma\right] \\ =\ &\epsilon\,e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{1}{2}\frac{\partial\Sigma}{\partial s_0}n\left[y;0,\Sigma\right]\right) \\ &+\epsilon^2\,e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y+f\,d+c\,y\left(\frac{1}{2}\frac{\partial\Sigma}{\partial s_0}+yd-\frac{y^2}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}\right)\right)n\left[y;0,\Sigma\right], \end{aligned}$$

where $d=\frac{\partial y}{\partial s_0}=\left(e^{\mu T}-1\right)/(\mu T\epsilon)$. The derivatives of coefficients are calculated as

$$\begin{aligned} \frac{\partial\Sigma}{\partial s_0} &= \frac{1}{\mu^2T^2}\frac{\partial}{\partial s_0}\int_0^T\left(e^{\mu(T-s)}-1\right)^2\sigma\left(A_{0s}\right)^2\,ds \\ &= \frac{2}{\mu^2T^2}\int_0^T\left(e^{\mu(T-s)}-1\right)^2 e^{\mu\,s}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,ds \\ &= \frac{2}{\mu^2T^2}\int_0^T\left(e^{2\mu T-\mu s}-2e^{\mu T}+e^{\mu\,s}\right)\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,ds, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial c}{\partial s_0} &= \frac{\partial}{\partial s_0}\{\frac{1}{\Sigma^2 T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt\} \\
&= -\frac{2}{\Sigma^3}\frac{\partial\Sigma}{\partial s_0}\frac{1}{T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{1}{\Sigma^2 T^3}\int_0^T\int_0^t e^{\mu\,t}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma'\left(A_{0s}\right)^2 \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{1}{\Sigma^2 T^3}\int_0^T\int_0^t e^{\mu\,t}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma''\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{2}{\Sigma^2 T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)\,\sigma'\left(A_{0v}\right)\,dv\,ds\,dt,
\end{aligned}
$$

and

$$
\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0}\left(c\Sigma\right) = -\frac{\partial c}{\partial s_0}\Sigma - c\frac{\partial\Sigma}{\partial s_0}.
$$

Then we sum up the result as a theorem.

Theorem 2. *The asymptotic expantion of the Delta of an average call option, whose payoff function with maturity time T is expressed as $\left(\widetilde{S}_T - K\right)_+$ is given by*

$$
D\left(0,T\right) + o\left(\epsilon^2\right),
$$

where

$$
\begin{aligned}
&D\left(0,T\right) \\
=\;& \epsilon\, e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{1}{2}\frac{\partial\Sigma}{\partial s_0}n\left[y;0,\Sigma\right]\right) \\
&+\epsilon^2\, e^{-rT}\left(\frac{\partial f}{\partial s_0}\,y + f\,d + c\,y\left(\frac{1}{2}\frac{\partial\Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma}\frac{\partial\Sigma}{\partial s_0}\right)\right)n\left[y;0,\Sigma\right],
\end{aligned} \tag{11}
$$

and the coefficents are given by

$$
y = \frac{s_0\left(e^{\mu T}-1\right)/\mu T - K}{\epsilon},
$$

$$d = \frac{e^{\mu T} - 1}{\epsilon \mu T},$$

$$\begin{aligned}\Sigma &= E\left[\left(\frac{1}{T}\int_0^T \int_0^t e^{\mu\,(t-s)}\,\sigma\left(A_{0s}\right)\,dW_s\,dt\right)^2\right] \\ &= \frac{1}{\mu^2 T^2}\int_0^T \left(e^{\mu(T-s)} - 1\right)^2 \sigma\left(A_{0s}\right)^2\,ds,\end{aligned}$$

$$\begin{aligned}c &= \frac{1}{\Sigma^2\,T^3}\int_0^T \int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\ &\quad \times \int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)} - 1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt,\end{aligned}$$

and

$$f = -c\Sigma.$$

4 Variance Reduction Technique for Delta of Call Option

Previous section showed the validity of the asymptotic expansion technique for the Delta, but we sometimes need more precise values, especially when ϵ is quite large or when the true value is quite small. For this purpose, we propose a variance reduction method, which is an extension of Takahashi and Yoshida[2004].

Although the value of Delta is obtained by calculating $E\left[e^{-rT}\,Y_T\,1_{\{s_T \geq K\}}\right]$, it may be impossible to calculate it analytically. In the case, Monte Carlo simulation based on Euler-Maruyama Scheme is a standard method, which is called ”crude Monte Calro”in this paper. However, crude Monte Carlo is usually quite time-consuming because of its low convergence speed, and hence, Takahashi and Yoshida [2004] proposed the method based on an asymptotic approach to accelerate the simulation.

The convergence speed depends on the variance of sample paths. The smaller the variance, the higher its convergence speed. Let X be a random variable. We take another random variable Y whose expectation is 0. Then, the random variable $X - Y$ has the same expectation as X, and its variance is given by

$$\begin{aligned}Var\left(X - Y\right) &= E\left[\left(X - E\left[X\right] - Y\right)^2\right] \\ &= E\left[\left(X - E\left[X\right]\right)^2\right] - 2\,E\left[\left(X - E\left[X\right]\right)Y\right] + E\left[Y^2\right] \\ &= Var\left(X\right) + Var\left(Y\right) - 2\,Corr\left(X,Y\right)\sqrt{Var\left(X\right)Var\left(Y\right)} \\ &= Var\left(X\right) - \sqrt{Var\left(Y\right)}\left(2\,Corr\left(X,Y\right)\sqrt{Var\left(X\right)} - \sqrt{Var\left(Y\right)}\right),\end{aligned}$$

where $Corr\left(X,Y\right)$ denotes the correlation between X and Y. Let $a > 0$, and we use aY instead of Y. Then, $Var\left(X - aY\right)$ is minimized when $a = Corr\left(X,Y\right)\sqrt{\frac{Var(X)}{Var(Y)}}$, and we obtain

$$\begin{aligned}Var\left(X - aY\right) &= Var\left(X\right) - Corr\left(X,Y\right)^2\,Var\left(X\right) \\ &= \left(1 - Corr\left(X,Y\right)^2\right)Var\left(X\right).\end{aligned}$$

Thus, we have to find Y whose correlation with X is close to one.

We utilize the approximated Delta denoted by $D(0,T)$. In the derivation of the theorem 1 (or theorem 2), we obtain the approximated option price in an integral representation:

$$C(0,T) = \epsilon\, e^{-rT} \int_{-y}^{\infty} (y+x)\; n[x;0,\Sigma]\; dx + \epsilon^2\, e^{-rT} \int_{-y}^{\infty} \left(cx^2+f\right)\; n[x;0,\Sigma]\; dx.$$

The candidate for a random variable Y is obtained by differentiating it with respect to s_0.

Next, we prepare two lemmas for the ease of calculation.

Lemma 1. *Let $\varphi(x)$ be a C_0^∞ class function. Then*

$$\int_{-\infty}^{\infty} (\partial 1)_{\{x\geq -y\}}\; \varphi(x)\; dx = \varphi(-y)\,. \tag{12}$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} (\partial 1)_{\{x\geq -y\}}\; \varphi(x)\; dx &= -\int_{-\infty}^{\infty} 1_{\{x\geq -y\}}\, \varphi'(x)\; dx \\
&= \varphi(-y)
\end{aligned}$$

■

Lemma 2. *The differentiation of $n[x;0,\Sigma]$ with respect to s_0 is*

$$\frac{\partial}{\partial s_0} n[x;0,\Sigma] = \frac{1}{2}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial \Sigma}{\partial s_0} n[x;0,\Sigma] \tag{13}$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial s_0} n[x;0,\Sigma] &= \frac{\partial}{\partial s_0}\left\{\frac{1}{\sqrt{2\pi}}\Sigma^{-1/2}\,\exp\left(-\frac{1}{2}x^2\,\Sigma^{-1}\right)\right\} \\
&= -\frac{1}{2}\frac{1}{\sqrt{2\pi}}\Sigma^{-3/2}\,\frac{\partial \Sigma}{\partial s_0}\exp\left(-\frac{1}{2}x^2\,\Sigma^{-1}\right) + \frac{1}{\sqrt{2\pi}}\Sigma^{-1/2}\,\exp\left(-\frac{1}{2}x^2\,\Sigma^{-1}\right)\frac{1}{2}x^2\,\Sigma^{-2}\frac{\partial \Sigma}{\partial s_0} \\
&= \frac{1}{2}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial \Sigma}{\partial s_0}\frac{1}{\sqrt{2\pi}}\Sigma^{-1/2}\,\exp\left(-\frac{1}{2}x^2\,\Sigma^{-1}\right) \\
&= \frac{1}{2}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial \Sigma}{\partial s_0} n[x;0,\Sigma]
\end{aligned}$$

■

Using lemmas 1 and 2, the approximation of Delta is obtained as

$$\begin{aligned}
& D\left(0,T\right) \\
= \;& \epsilon\, e^{-rT}\,\frac{\partial}{\partial s_0}\int_{-y}^{\infty}\left(y+x\right)\,n\left[x;0,\Sigma\right]\,dx+\epsilon^2\,e^{-rT}\,\frac{\partial}{\partial s_0}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx \\
= \;& \epsilon\, e^{-rT}\int_{-y}^{\infty}\left(d+\frac{1}{2}\left(y+x\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\right)\,n\left[x;0,\Sigma\right]\,dx \\
& +\epsilon^2\,e^{-rT}\,d\,\left(cy^2+f\right)\,n\left[y;0,\Sigma\right] \\
& +\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(\frac{\partial c}{\partial s_0}x^2+\frac{\partial f}{\partial s_0}\right)\,n\left[x;0,\Sigma\right]\,dx \\
& +\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\frac{1}{2}\left(cx^2+f\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\,n\left[x;0,\Sigma\right]\,dx \\
= \;& \epsilon\, e^{-rT}\int_{-y}^{\infty}\left(d+\frac{1}{2}\left(y+x\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\right)\,n\left[x;0,\Sigma\right]\,dx \\
& +\epsilon^2\,e^{-rT}\,d\,\left(cy^2+f\right)\,n\left[y;0,\Sigma\right] \\
& +\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(\frac{\partial c}{\partial s_0}x^2+\frac{\partial f}{\partial s_0}\right)\,n\left[x;0,\Sigma\right]\,dx \\
& +\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\frac{1}{2}\left(cx^2+f\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\,n\left[x;0,\Sigma\right]\,dx.
\end{aligned}$$

On the other hand, $D\left(0,T\right)$ is expressed as

$$\begin{aligned}
& D\left(0,T\right) \\
= \;& \epsilon\, e^{-rT}\,E[d+\frac{1}{2}\left(y+g_1\right)\left(\frac{g_1^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0} \\
& +\epsilon\left(d\,\left(cy^2+f\right)\,n\left[y;0,\Sigma\right]+\left(\frac{\partial c}{\partial s_0}g_1^2+\frac{\partial f}{\partial s_0}\right)+\frac{1}{2}\left(cg_1^2+f\right)\left(\frac{g_1^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\right))1_{\{g_1\geq -y\}}] \\
= \;& E\left[\varphi\left(g_1\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\varphi(x) \;=\;& \epsilon\, e^{-rT}((d+\frac{1}{2}\left(y+x\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0} \\
& +\epsilon\left(\left(\frac{\partial c}{\partial s_0}x^2+\frac{\partial f}{\partial s_0}\right)+\frac{1}{2}\left(cx^2+f\right)\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\frac{\partial\Sigma}{\partial s_0}\right))\,1_{\{x\geq -y\}}+\epsilon d\,\left(cy^2+f\right)\,n\left[y;0,\Sigma\right].
\end{aligned} \tag{14}$$

Finally, we set $Y=\varphi\left(g_1\right)-D\left(0,T\right)$. We call this new method "Hybrid Monte Carlo".

5 The Vega of Plain Vanilla Call Option

In the previous section, we obtained an approximated value of Plain Vanilla Call Option price as

$$\begin{aligned}
C\left(0,T\right) \;=\;& \epsilon\, e^{-rT}\int_{-y}^{\infty}\left(y+x\right)\,n\left[x;0,\Sigma\right]\,dx+\epsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx \\
=\;& \epsilon\, e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n\left[y;0,\Sigma\right]\right)+\epsilon^2\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right],
\end{aligned}$$

and consider its differentiation by ϵ to obtain an approximation of the Vega.

$$\begin{aligned}
& \frac{\partial}{\partial \epsilon} C(0,T) \\
= \; & \frac{\partial}{\partial \epsilon}\left(\epsilon\, e^{-rT}\left(y\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \Sigma\, n[y;0,\Sigma]\right) + \epsilon^2\, e^{-rT}\, f\, y\, n[y;0,\Sigma]\right) \\
= \; & e^{-rT}\left(y\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \Sigma\, n[y;0,\Sigma] - y\, N\left(\frac{y}{\sqrt{\Sigma}}\right) - y^2\, n[y;0,\Sigma] + y^2\, n[y;0,\Sigma]\right) \\
& + 2\epsilon\, e^{-rT}\, f\, y\, n[y;0,\Sigma] - \epsilon\, e^{-rT}\, f\, y\, n[y;0,\Sigma] + \epsilon\, \frac{f\, y^3}{\Sigma}\, n[y;0,\Sigma] \\
= \; & e^{-rT}\left(\Sigma + \epsilon\left(f\, y + \frac{f\, y^3}{\Sigma}\right)\right) n[y;0,\Sigma].
\end{aligned}$$

Then, we summarize the result as a theorem.

Theorem 3. *The asymptotic expantion of the Vega of Plain Vanilla Call option of which payoff function with maturity time T is expressed as $(S_T - K)_+$ is given by*

$$V(0,T) + o(\epsilon),$$

where

$$\begin{aligned}
& V(0,T) \\
= \; & e^{-rT}\left(\Sigma + \epsilon\left(f\, y + \frac{f\, y^3}{\Sigma}\right)\right) n[y;0,\Sigma],
\end{aligned} \tag{15}$$

and the coefficents are given by

$$y = \frac{s_0 e^{\mu T} - K}{\epsilon},$$

$$\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu(T-t)}\, \sigma(A_{0t})\, dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu(T-t)}\, \sigma(A_{0t})^2\, dt,
\end{aligned}$$

$$c = \frac{1}{\Sigma^2}\int_0^T \int_0^s e^{\mu(T-s)}\, \sigma(A_{0s})\, \sigma'(A_{0s})\, e^{2\mu(T-v)}\, \sigma(A_{0v})^2\, dv\, ds,$$

and

$$f = -c\Sigma.$$

6 The Vega of Average Call Option

Repeating the similar argument used in the previous sections, we can obtain the approximated value of the Vega of Average Call Option. Hence, we only state the result as a theorem.

Theorem 4. *The asymptotic expantion of the Vega of Average Call option of which payoff function with maturity time T is expressed as $(S_T - K)_+$, is given by*

$$V(0,T) + o(\epsilon),$$

where

$$\begin{aligned} & V(0,T) \\ = \; & e^{-rT}\left(\Sigma + \epsilon\left(f\,y + \frac{f\,y^3}{\Sigma}\right)\right) n\left[y; 0, \Sigma\right], \end{aligned} \tag{16}$$

and the coefficents are given by

$$y = \frac{s_0\left(e^{\mu T} - 1\right)/\mu T - K}{\epsilon},$$

$$d = \frac{e^{\mu T} - 1}{\epsilon \mu T},$$

$$\begin{aligned} \Sigma &= E\left[\left(\frac{1}{T}\int_0^T \int_0^t e^{\mu(t-s)}\,\sigma(A_{0s})\,dW_s\,dt\right)^2\right] \\ &= \frac{1}{\mu^2 T^2}\int_0^T \left(e^{\mu(T-s)} - 1\right)^2 \sigma(A_{0s})^2\,ds, \end{aligned}$$

$$\begin{aligned} c \;=\; & \frac{1}{\Sigma^2\,T^3}\int_0^T \int_0^t e^{\mu(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma(A_{0s})\,\sigma'(A_{0s}) \\ & \times \int_0^s e^{\mu(s-v)}\left[\frac{e^{\mu(T-v)} - 1}{\mu}\right]\sigma(A_{0v})^2\,dv\,ds\,dt, \end{aligned}$$

and

$$f = -c\Sigma.$$

7 Variance Reduction Technique for Vega of Call Option

Previous section showed the validity of the asymptotic expansion technique for the Vega. However, we sometimes need more precise values, especially when ϵ is quite large or when the true value is quite small. For this purpose, we propose a variance reduction method.

Although this method is quite similar to the case of Delta, we need some adjustments. We define the "differentiation with respect to the volatility coefficient ϵ" of S_t satisfying the following SDE:

$$\begin{cases} dS_t = \mu\,S_t\,dt + \epsilon\sigma(S_t)\,dW_t \\ S_0 = s_0 \;(>0) \end{cases}$$

Definition 3. *We define Z_t as a stocastic process which satisfies the SDE:*

$$\begin{cases} dZ_t = \mu Z_t \, dt + \left[\sigma\left(S_t\right) + \epsilon \sigma'\left(S_t\right) \, Z_t\right] \, dW_t \\ Z_0 = 0 \end{cases} \tag{17}$$

The value of Vega is obtained by calculating $E\left[e^{-rT} \, Z_T \, 1_{\{s_T \geq K\}}\right]$, but it is sometimes impossible to calculate it analytically. Although we utilize Monte Carlo simulations by using Euler-Maruyama Scheme, it is sometimes time-consuming. In a similar way as Variance Reduction Technique for Delta, let X be the sample path and we introduce a new random variable Y.

According to Kunitomo and Takahashi [2003b], the asymptotic expantion of S_t is obtained by

$$S_t \quad \sim \quad A_{0t} + \epsilon A_{1t} + \frac{\epsilon^2}{2} \, A_{2t} + \cdots \,,$$

where

$$\begin{aligned} dA_{0t} &= \mu \, A_{0t} \, dt, \qquad A_{00} = s_0 \\ dA_{1t} &= \mu \, A_{1t} \, dt + \sigma\left(A_{0t}\right) \, dW_t, \qquad A_{10} = 0 \\ dA_{2t} &= \mu \, A_{2t} \, dt + 2 \, \sigma'\left(\, A_{0t}\right) \, A_{1t} \, dW_t, \qquad A_{20} = 0. \end{aligned}$$

The asympotic expansion of Z_t is also given by

$$Z_t \; \sim \; A_{1t} + \epsilon \, A_{2t} + \cdots \,.$$

Now we put $X_t = \left(S_t - A_{0t}\right)/\epsilon$, $g_1 = A_{1T}$ and $g_2 = A_{2T} \, / \, 2$. Then, the asymptotic expantion of the Vega $E\left[e^{-rT} \, Z_T \, 1_{\{s_T \geq K\}}\right]$ is given by

$$E\left[e^{-rT} \, Z_T \, 1_{\{s_T \geq K\}}\right] = e^{-rT} E\left[\left(g_1 + 2\epsilon \, g_2 + \cdots\right) \, 1_{\{g_1 + \epsilon \, g_2 + \cdots \geq -y\}}\right],$$

where $y = \left(A_{0T} - K\right)/\epsilon$. Thus, we obtain its approximated value as

$$\begin{aligned} & E\left[e^{-rT} \, Z_T \, 1_{\{s_T \geq K\}}\right] \\ = \; & e^{-rT} \, E\left[g_1 \, 1_{\{g_1 \geq -y\}}\right] + \epsilon \, e^{-rT} \, E\left[2 g_2 \, 1_{\{g_1 \geq -y\}}\right] \\ & + \epsilon \, e^{-rT} \, E\left[g_1 \, g_2 \, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] + o\left(\epsilon\right). \end{aligned}$$

Finally, the approximated value of Vega $V\left(0, T\right)$ is expressed as

$$\begin{aligned} & V\left(0, T\right) \\ = \; & e^{-rT} \, E\left[\left(g_1 + 2\epsilon \, g_2\right) \, 1_{\{g_1 \geq -y\}}\right] + \epsilon \, e^{-rT} \, E\left[g_1 \, g_2 \, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\ = \; & e^{-rT} \, E\left[\left(g_1 + 2\epsilon \, E\left[g_2 | g_1\right]\right) \, 1_{\{g_1 \geq -y\}}\right] + \epsilon \, e^{-rT} \, E\left[g_1 \, E\left[g_2 | g_1\right] \, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\ = \; & e^{-rT} \, E\left[\left(g_1 + 2\epsilon \, \left(c \, g_1^2 + f\right)\right) \, 1_{\{g_1 \geq -y\}}\right] + \epsilon \, e^{-rT} \, E\left[g_1 \, \left(c \, g_1^2 + f\right) \, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\ = \; & E\left[\varphi\left(g_1\right)\right], \end{aligned}$$

where

$$\varphi\left(x\right) = e^{-rT} \, \left(x + 2\epsilon \, \left(c \, x^2 + f\right)\right) \, 1_{\{x \geq -y\}} - e^{-rT} \, y \, \left(c \, y^2 + f\right) \, n\left[y; 0, \Sigma\right]. \tag{18}$$

Finally, we introduce a new random variable $Y = \varphi\left(g_1\right) - V\left(0, T\right)$, and implement the variance reduction technique described in section 4. We call this method ”Hybrid Monte Carlo” as in section 4.

8 The Case of CEV processes

8.1 Another Deriviation of Approximated Value of Delta

In this sections we assume the price processes of the underlying asset to be CEV processes:

$$\begin{cases} dS_t = \mu\, S_t\, dt + \epsilon S_t^{\gamma}\, dW_t \quad ,0 < \gamma \leq 1 \\ S_0 = s_0\ (> 0) \end{cases} \tag{19}$$

Then, we derive the approximation of the Delta in a more straightforward manner. We define Y_t as $Y_t = \frac{\partial S_t}{\partial s_0}$. Note that Y_t the SDE:

$$\begin{cases} dY_t = \mu\, Y_t\, dt + \epsilon\, \gamma\, S_t^{\gamma-1}\, Y_t\ dW_t \\ \quad Y_0 = 1 \end{cases} , \tag{20}$$

and the Delta of Plain Vanilla Call Option is obtained by

$$e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right]. \tag{21}$$

See Imamura, Takahashi and Uchida [2005] for the derivation for instance. According to Kunitomo and Takahashi [2003b], the asymptotic expantion of S_t is given by $S_t\ \sim\ A_{0t} + \epsilon A_{1t} + \frac{\epsilon^2}{2}\, A_{2t} + \cdots$, where

$$\begin{aligned} A_{0t} &= s_0\, e^{\mu\, t} \\ A_{1t} &= s_0^{\gamma} \int_0^t e^{\mu t}\, e^{-\mu(1-\gamma)s}\, dW_s \\ A_{2t} &= 2\, \gamma\, s_0^{\gamma-1} \int_0^t e^{\mu t}\, e^{\mu(\gamma-2)s}\, A_{1s}\, dW_s. \end{aligned}$$

Similarly, the asymptotic expantion of Y_t is obtained by $Y_t\ \sim\ B_{0t} + \epsilon B_{1t} + \frac{\epsilon^2}{2}\, B_{2t} + \cdots$, where B_{0t}, B_{1t} and B_{2t} satisfy SDEs:

$$\begin{aligned} dB_{0t} &= \mu\, B_{0t}\, dt, \qquad B_{00} = 1 \\ dB_{1t} &= \mu\, B_{1t}\, dt + \gamma\, A_{0t}^{\gamma-1}\, B_{0t}\, dW_t, \qquad B_{10} = 0 \\ dB_{2t} &= \mu\, B_{2t}\, dt + \left[2\, \gamma\, (\gamma - 1)\ A_{0t}^{\gamma-2}\, A_{1t}\, B_{0t} + 2\, \gamma\, A_{0t}^{\gamma-1}\, B_{1t}\right]\, dW_t, \qquad B_{20} = 0 \end{aligned}$$

Then, we can easily obtain B_{0t}, B_{1t}, B_{2t} as

$$\begin{aligned} B_{0t} &= \frac{1}{s_0} A_{0t} \\ B_{1t} &= \frac{\gamma}{s_0} A_{1t} \\ B_{2t} &= \frac{2\, \gamma - 1}{s_0} A_{2t}. \end{aligned}$$

Now we put $X_t = (S_t - A_{0t})\, / \epsilon,\ g_1 = A_{1T}$ and $g_2 = A_{2T}\, /\, 2$, then

$$X_T \sim\ g_1 + \epsilon\, g_2 + \cdots.$$

Therefore,

$$e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right] = \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1 + \epsilon\, \frac{2\, \gamma - 1}{s_0} g_2 + \cdots\right)\ 1_{\{g_1 + \epsilon\, g_2 + \cdots \geq -y\}}\right],$$

where $y = \left(s_0 e^{\mu T} - K\right)/\epsilon$ and $d = \frac{\partial y}{\partial s_0} = e^{\mu T}/\epsilon$. We obtain its approximated value as

$$\begin{aligned}
& e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right] \\
= \;& \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, 1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\, E\left[\frac{2\,\gamma - 1}{s_0} g_2\, 1_{\{g_1 \geq -y\}}\right] \\
& + \epsilon^2\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, g_2\, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] + o\left(\epsilon^2\right),
\end{aligned}$$

where $\left(\partial 1\right)_{\{\cdot \geq -y\}}$ means the derivation of $1_{\{\cdot \geq -y\}}$ in the sense of distribution. By the property of Brownian Motion, we know the distribution of $g_1 \sim N\left(0, \Sigma\right)$, where

$$\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\, \sigma\left(A_{0t}\right)\, dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu\,(T-t)}\, \sigma\left(A_{0t}\right)^2\, dt \\
&= \begin{cases} \frac{s_0^{2\gamma}}{2\mu\,(1-\gamma)}\, \left(e^{2\mu T} - e^{2\mu\gamma T}\right) & \text{if } 0 < \gamma < 1 \\ s_0^2\, T\, e^{2\mu T} & \text{if } \gamma = 1. \end{cases}
\end{aligned}$$

In addition, the conditional expectation $E\left[g_2 | g_1 = x\right] = c x^2 + f$,where

$$\begin{aligned}
c &= \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu\,(T-s)}\, \sigma\left(A_{0s}\right)\, \sigma'\left(A_{0s}\right)\, e^{2\mu\,(T-v)}\, \sigma\left(A_{0v}\right)^2\, dv\, ds \\
&= \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\, e^{3\mu T}\, \frac{\left(1 - e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \text{if } 0 < \gamma < 1 \\ \frac{s_0^3}{\Sigma^2}\, \frac{T^2}{2}\, e^{3\mu T} & \text{if } \gamma = 1, \end{cases}
\end{aligned}$$

and $f = -c\,\Sigma$. Accordingly,

$$\begin{aligned}
& e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right] \\
= \;& \epsilon\, e^{-rT} \int_{-y}^{\infty} \left(d + \frac{\gamma}{s_0} x\right)\, n\left[x; 0, \Sigma\right]\, dx + \epsilon^2\, e^{-rT}\, \frac{2\,\gamma - 1}{s_0} \int_{-y}^{\infty} \left(c x^2 + f\right)\, n\left[x; 0, \Sigma\right]\, dx \\
& + \epsilon^2\, e^{-rT}\, \left(d + \frac{\gamma}{s_0}\left(-y\right)\right)\left(c y^2 + f\right)\, n\left[-y; 0, \Sigma\right] + o\left(\epsilon^2\right).
\end{aligned}$$

Thus we obtain the approximation as

$$\epsilon\, e^{-rT}\, \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \left(\frac{\gamma}{s_0}\Sigma + \epsilon\left(d - \frac{\gamma}{s_0} y\right)\left(c y^2 + f\right) + \epsilon\, \frac{2\gamma - 1}{s_0} f\, y\right) n\left[y; 0, \Sigma\right]\right).$$

where $N\left(x\right)$ denotes a cumulative distribution function of standard normal distribution $N\left(0, 1\right)$.

Finally we see the agreement of the two approximated values derived in different manners. By the theorem 1, the approximated value of the Delta is given by

$$\begin{aligned}
& D\left(0, T\right) \\
= \;& \epsilon\, e^{-rT}\, \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{1}{2}\frac{\partial \Sigma}{\partial s_0} n\left[y; 0, \Sigma\right]\right) \\
& + \epsilon^2\, e^{-rT}\, \left(\frac{\partial f}{\partial s_0}\, y + f\, d + c\, y\left(\frac{1}{2}\frac{\partial \Sigma}{\partial s_0} + y d - \frac{y^2}{2\Sigma}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\left[y; 0, \Sigma\right].
\end{aligned}$$

In addition, the coefficients $\Sigma,\ c,\ f$ are monomials of s_0, and their differentiation with respect to s_0 are

$$\frac{\partial \Sigma}{\partial s_0}=\frac{2\gamma\,\Sigma}{s_0},\quad \frac{\partial c}{\partial s_0}=-\frac{c}{s_0},\quad \frac{\partial f}{\partial s_0}=\frac{(2\gamma-1)\ f}{s_0}.$$

Substituting them, we obtain

$$\begin{aligned} & D\left(0,T\right) \\ = \ & \epsilon\, e^{-rT}\left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{\gamma\,\Sigma}{s_0}n\left[y;0,\Sigma\right]\right) \\ & +\epsilon^2\, e^{-rT}\ \left(\frac{(2\gamma-1)}{s_0}f\, y+f\, d+c\, y\left(\frac{\gamma\,\Sigma}{s_0}+yd-\frac{\gamma\, y^2}{s_0}\right)\right) n\left[y;0,\Sigma\right] \\ = \ & \epsilon\, e^{-rT}\left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{\gamma\,\Sigma}{s_0}n\left[y;0,\Sigma\right]\right) \\ & +\epsilon^2\, e^{-rT}\ \left(\frac{(2\gamma-1)}{s_0}f\, y+\left(c\, y^2+f\right)\left(d-\frac{\gamma}{s_0}y\right)\right) n\left[y;0,\Sigma\right]. \end{aligned}$$

Hence, the two approximated values agree.

Then we sum up the fact as a theorem.

Theorem 5. *The asymptotic expantion of the Delta of Plain Vanilla Call option, whose payoff function with maturity time T is expressed as $(S_T-K)_+$, is given by*

$$D\left(0,T\right)+o\left(\epsilon^2\right),$$

where

$$\begin{aligned} & D\left(0,T\right) \\ = \ & \epsilon\, e^{-rT}\left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{\gamma\,\Sigma}{s_0}n\left[y;0,\Sigma\right]\right) \\ & +\epsilon^2\, e^{-rT}\ \left(\frac{(2\gamma-1)}{s_0}f\, y+\left(c\, y^2+f\right)\left(d-\frac{\gamma}{s_0}y\right)\right) n\left[y;0,\Sigma\right], \end{aligned} \tag{22}$$

and the coefficents are given by

$$y=\frac{s_0 e^{\mu\, T}-K}{\epsilon},$$

$$\begin{aligned} \Sigma \ &= \ E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\, dW_t\right)^2\right] \\ &= \ \int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\ dt \\ &= \ \begin{cases} \frac{s_0^{2\gamma}}{2\mu\,(1-\gamma)}\ \left(e^{2\mu T}-e^{2\mu\gamma T}\right) & \textit{if}\ 0<\gamma<1 \\ s_0^2\, T\, e^{2\mu T} & \textit{if}\ \gamma=1, \end{cases} \end{aligned}$$

$$\begin{aligned} c \ &= \ \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\ \sigma'\left(A_{0s}\right)\ e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\ dv\, ds \\ &= \ \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\ e^{3\mu T}\ \frac{\left(1-e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \textit{if}\ 0<\gamma<1 \\ \frac{s_0^3}{\Sigma^2}\ \frac{T^2}{2}\ e^{3\mu T} & \textit{if}\ \gamma=1, \end{cases} \end{aligned}$$

and

$$f = -c\Sigma.$$

Similarly, we gain the approximation formula for the Delta of the average option.

Theorem 6. *The asymptotic expantion of the Delta of Average Call option, whose payoff function with maturity time T is expressed as $\left(\widetilde{S}_T - K\right)_+$, is given by*

$$D\left(0,T\right) + o\left(\epsilon^2\right),$$

where

$$\begin{aligned} & D\left(0,T\right) \\ = \; & \epsilon\, e^{-rT}\left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{\gamma\,\Sigma}{s_0} n\left[y;0,\Sigma\right]\right) \\ & + \epsilon^2\, e^{-rT}\;\left(\frac{(2\gamma-1)}{s_0} f\, y + \left(c\, y^2 + f\right)\left(d - \frac{\gamma}{s_0} y\right)\right) n\left[y;0,\Sigma\right], \end{aligned} \tag{23}$$

and the coefficents are given by

$$y = \frac{s_0\left(e^{\mu T}-1\right)/\mu T - K}{\epsilon},$$

$$d = \frac{e^{\mu T}-1}{\epsilon \mu T},$$

$$\begin{aligned} \Sigma \;&=\; \int_0^T \left(\frac{1}{T}\,\frac{e^{\mu\,(T-t)}-1}{\mu}\,\sigma\left(A_{0t}\right)\right)^2 dt \\ &=\; \begin{cases} \frac{s_0^{2\gamma}}{\mu^2\,T^2}\left\{\frac{e^{2\mu T}}{2\mu(1-\gamma)} - \frac{2e^{\mu T}}{\mu(1-2\gamma)} - \frac{1}{2\mu T} + \frac{e^{\mu\gamma T}}{2\mu\gamma(1-\gamma)(1-2\gamma)}\right\} & \textit{if } 0<\gamma<1,\ \gamma \neq \frac{1}{2} \\ \frac{s_0}{\mu^2 T^2}\left(\frac{e^{2\mu T}}{\mu} - 2T\, e^{\mu T} - \frac{1}{\mu}\right) & \textit{if } \gamma = \frac{1}{2} \\ \left(\frac{s_0}{\mu T}\right)^2\left\{e^{2\mu T}\left(T - \frac{3}{2\mu}\right) + \frac{2e^{\mu T}}{\mu} - \frac{1}{2\mu}\right\} & \textit{if } \gamma = 1, \end{cases} \end{aligned}$$

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2\, T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\,\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\,\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&= \begin{cases}
\frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2\,\mu^5\,T^3}\begin{pmatrix}
\left(\left(-1+e^{\mu T}\right)\left(2-2\gamma+e^{\mu T}\left(-1+2\gamma\right)\right)^2\right. \\
+(e^{\mu(4\gamma-1)T}\gamma\left(-7+10\gamma\right) \\
+4\left(-1+\gamma\right)^2\left(3-13+12\gamma^2\right) \\
+e^{\mu(1+2\gamma)T}\left(6-44\gamma+96\gamma^2-64\gamma^3\right) \\
+4e^{2\mu\gamma T}\left(-3+19\gamma-32\gamma^2+16\gamma^3\right) \\
+e^{2\mu T}\gamma\left(-5+36-76\gamma^2+48\gamma^3\right) \\
-2e^{\mu T}\left(3-28\gamma+89\gamma^2-112\gamma^3+48\gamma^4\right)) \\
/\gamma\left(3-16\gamma+16\gamma^2\right))/8\left(1-3\gamma+2\gamma^2\right)^2
\end{pmatrix} & \textit{if}\, 0<\gamma<1,\,\gamma\neq\frac{1}{2} \\
\frac{s_0}{2\Sigma^2\mu^2T^3}\left\{\frac{e^{3\mu T}}{2\mu^3}+e^{2\mu T}\left(\frac{1}{\mu^3}-\frac{2T}{\mu^2}\right)+e^{\mu T}\left(-\frac{1}{2\mu^3}-\frac{T}{\mu^2}+\frac{T^2}{\mu}\right)-\frac{1}{\mu^3}\right\} & \textit{if}\,\gamma=\frac{1}{2} \\
\frac{s_0^3}{2\mu^2\mu^5T^3}\left\{e^{3\mu T}\left(\mu^2T^2-3\mu T+\frac{17}{6}\right)+e^{2\mu T}\left(2\mu T-5\right)+\frac{5}{2}e^{\mu T}-\frac{1}{3}\right\} & \textit{if}\,\gamma=1,
\end{cases}
\end{aligned}
$$

and

$$
f=-c\Sigma.
$$

8.2 The Gamma of Call Option

The approximated value of Delta of Plain Vanilla (Average) Call Option is given by

$$
D\left(0,T\right)+o\left(\epsilon^2\right),
$$

where

$$
\begin{aligned}
& D\left(0,T\right) \\
={}& \epsilon\, e^{-rT}\left(d\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\frac{\gamma\,\Sigma}{s_0}n\left[y;0,\Sigma\right]\right) \\
&+\epsilon^2\, e^{-rT}\,\left(\frac{(2\gamma-1)}{s_0}f\,y+\left(c\,y^2+f\right)\left(d-\frac{\gamma}{s_0}y\right)\right)n\left[y;0,\Sigma\right].
\end{aligned}
$$

Then, we can differentiate it with respect to s_0 again. We get an approximated value of the Gamma.

Theorem 7. *The asymptotic expantion of the Gamma of Plain Vanilla Call option, whose payoff function with maturity time T is expressed as $\left(S_T-K\right)_+$, is given by*

$$
G\left(0,T\right)+o\left(\epsilon^2\right),
$$

where

$$
\begin{aligned}
&G(0,T) \\
=\;& \epsilon\, e^{-rT}\left(d^2 - \frac{2\gamma}{s_0} d\, y - \frac{\gamma}{s_0^2}\Sigma + \frac{\gamma^2}{s_0^2}\Sigma + \frac{\gamma^2}{s_0^2}\, y\right) n\left[y;0,\Sigma\right] \qquad (24)\\
&-\epsilon^2\, e^{-rT}\left(\left(d - \frac{\gamma}{s_0}y\right)\left(cy^2+f\right) + \frac{2\gamma-1}{s_0} f\, y\right)\left(\frac{\gamma}{s_0} + \left(y\, d - \frac{\gamma}{s_0}y^2\right)\frac{1}{\Sigma}\right) n\left[y;0,\Sigma\right] \\
&+\epsilon^2\, e^{-rT}\,(\left(\frac{\gamma}{s_0^2}y - \frac{\gamma}{s_0}d\right)\left(cy^2+f\right) + \left(d - \frac{\gamma}{s_0}y\right)\left(-\frac{c}{s_0}y^2 + 2c\, y\, d + \frac{2\gamma-1}{s_0}f\right) \\
&+\frac{(2\gamma-1)^2}{s_0^2} f\, y + \frac{2\gamma-1}{s_0} f\, d) n\left[y;0,\Sigma\right],
\end{aligned}
$$

and the coefficents are given by

$$
y = \frac{s_0 e^{\mu T} - K}{\epsilon},
$$

$$
\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\, dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\, dt \\
&= \begin{cases} \frac{s_0^{2\gamma}}{2\mu\,(1-\gamma)}\left(e^{2\mu T} - e^{2\mu\gamma T}\right) & \textit{if } 0<\gamma<1 \\ s_0^2\, T\, e^{2\mu T} & \textit{if } \gamma = 1, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\, e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\, dv\, ds \\
&= \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\, e^{3\mu T}\,\frac{\left(1-e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \textit{if } 0<\gamma<1 \\ \frac{s_0^3}{\Sigma^2}\,\frac{T^2}{2}\, e^{3\mu T} & \textit{if } \gamma = 1, \end{cases}
\end{aligned}
$$

and

$$
f = -c\Sigma.
$$

8.3 The Delta of Digital Option

We consider the Digital Option whose payoff function at maturity time T is given by

$$
1_{\{K_1 \le S_T \le K_2\}}.
$$

Before considering the Delta of Digital Option, we note that its price is represented by $e^{-rT}\, E\left[1_{\{K_1\le S_T\le K_2\}}\right]$. As above, $X_T \sim\, g_1 + \epsilon\, g_2 + \cdots$, where $X_t = \left(S_t - A_{0t}\right)/\epsilon$, and

$$
\begin{aligned}
e^{-rT}\, E\left[1_{\{K_1\le S_T\le K_2\}}\right] &= e^{-rT}\, E\left[1_{\{-y_1\le X_T\le -y_2\}}\right] \\
&= e^{-rT}\, E\left[1_{\{-y_1\le g_1+\epsilon\, g_2+\cdots\le -y_2\}}\right],
\end{aligned}
$$

where $y_1 = (A_{0T} - K_1)/\epsilon$ and $y_2 = (A_{0T} - K_2)/\epsilon$. We obtain its approximated value as

$$
\begin{aligned}
& e^{-rT} E\left[1_{\{K_1 \le S_T \le K_2\}}\right] \\
= \; & e^{-rT} E\left[1_{\{-y_1 \le g_1 \le -y_2\}}\right] + \epsilon\, e^{-rT} E\left[g_2\ (\partial 1)_{\{-y_1 \le g_1 \le -y_2\}}\right] + o(\epsilon) \\
= \; & e^{-rT} \left\{ N\left(\frac{y_1}{\sqrt{\Sigma}}\right) - N\left(\frac{y_2}{\sqrt{\Sigma}}\right) \right\} \\
& + \epsilon\ \left(c y_1^2 + f\right)\ n\left[y_1; 0, \Sigma\right] - \epsilon\ \left(c y_2^2 + f\right)\ n\left[y_2; 0, \Sigma\right].
\end{aligned}
$$

Its differentiation corresponds to the Delta of the Digital option.

Theorem 8. *The asymptotic expantion of the Delta of Digital option, whose payoff function of with maturity time T is expressed as $1_{\{K_1 \le S_T \le K_2\}}$, is given by*

$$
D(0,T) + o\left(\epsilon^2\right),
$$

where

$$
\begin{aligned}
D(0,T) = \; & (\left(d - \frac{\gamma}{s_0} y_1\right) \\
& + \epsilon \left(-\frac{c}{s_0} y_1^2 + 2c\, y_1\, d + \frac{2\gamma - 1}{s_0}\ f + \left(c y_1^2 + f\right) \left(-\frac{\gamma}{s_0} - \frac{y_1 d}{\Sigma} + \frac{y_1^2 \gamma}{s_0 \Sigma} \right) \right))\, n\left[y_1; 0, \Sigma\right] \\
& - (\left(d - \frac{\gamma}{s_0} y_2\right) \\
& + \epsilon \left(-\frac{c}{s_0} y_2^2 + 2c\, y_2\, d + \frac{2\gamma - 1}{s_0}\ f + \left(c y_2^2 + f\right) \left(-\frac{\gamma}{s_0} - \frac{y_2 d}{\Sigma} + \frac{y_2^2 \gamma}{s_0 \Sigma} \right) \right))\, n\left[y_2; 0, \Sigma\right],
\end{aligned} \tag{25}
$$

and the coefficents are given by

$$
y_1 = \frac{s_0 e^{\mu T} - K_1}{\epsilon},\ y_2 = \frac{s_0 e^{\mu T} - K_2}{\epsilon},
$$

$$
\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu(T-t)}\, \sigma(A_{0t})\ dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu(T-t)}\, \sigma(A_{0t})^2\ dt \\
&= \begin{cases} \frac{s_0^{2\gamma}}{2\mu(1-\gamma)} \left(e^{2\mu T} - e^{2\mu\gamma T}\right) & \textit{if } 0 < \gamma < 1 \\ s_0^2\, T\, e^{2\mu T} & \textit{if } \gamma = 1, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu(T-s)}\, \sigma(A_{0s})\ \sigma'(A_{0s})\ e^{2\mu(T-v)}\, \sigma(A_{0v})^2\ dv\, ds \\
&= \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\, e^{3\mu T}\, \frac{\left(1 - e^{2\mu(1-\gamma)T}\right)^2}{8\mu^2 (1-\gamma)^2} & \textit{if } 0 < \gamma < 1 \\ \frac{s_0^3}{\Sigma^2}\ \frac{T^2}{2}\, e^{3\mu T} & \textit{if } \gamma = 1, \end{cases}
\end{aligned}
$$

and

$$
f = -c\Sigma.
$$

9 Numerical Examples

The BS(Black-Sholes) model is represented as

$$\begin{cases} dS_t = \mu\, S_t\, dt + \epsilon S_t\, dW_t \\ S_0 = s_0\ (>0)\,, \end{cases} \tag{26}$$

and the CEV (Constant Elasticity of Variance) model is represented as

$$\begin{cases} dS_t = \mu\, S_t\, dt + \epsilon S_t^{\gamma}\, dW_t \quad ,0<\gamma<1 \\ S_0 = s_0\ (>0)\,. \end{cases}$$

Hence, the volatility function of the models are expressed as $\sigma(S_t) = S_t^{\gamma}\ (0<\gamma\le 1)$.

9.1 Approximation for the Delta of Plain Vanilla Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

$$\begin{aligned} \Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma(A_{0t})\ dW_t\right)^2\right] \\ &= \int_0^T e^{2\mu\,(T-t)}\,\sigma(A_{0t})^2\ dt \\ &= \begin{cases} \frac{s_0^{2\gamma}}{2\mu\,(1-\gamma)}\left(e^{2\mu T} - e^{2\mu\gamma T}\right) & \text{if } 0<\gamma<1 \\ s_0^2\, T\, e^{2\mu T} & \text{if } \gamma=1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} c &= \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma(A_{0s})\ \sigma'(A_{0s})\ e^{2\mu\,(T-v)}\,\sigma(A_{0v})^2\ dv\, ds \\ &= \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\, e^{3\mu T}\,\frac{\left(1-e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \text{if } 0<\gamma<1 \\ \frac{s_0^3}{\Sigma^2}\,\frac{T^2}{2}\, e^{3\mu T} & \text{if } \gamma=1. \end{cases} \end{aligned}$$

The coefficients Σ, c, and f are the function of s_0, T, γ and μ. In particular, paying attention to γ, we represent $\Sigma = \Sigma(\gamma)\,,\ c = c(\gamma)$. We can easily check the continuity at $\gamma = 1$, i.e. $\Sigma(\gamma) \to \Sigma(1)$ and $c(\gamma) \to c(1)$,as $\gamma \to 1$.

We apply the theorem 5 and obtain the approximated value of the Delta.

We show some numerical examples (figure 1). The initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model ($\gamma = 1$), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model ($\gamma < 1$). The error ratio is calculated as $(\text{approximate value} - \text{true value})\,/\text{true value}$.

We can observe that the accuracy of this approximation is valid.

9.2 Approximation for the Delta of Average Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

$$
\begin{aligned}
\Sigma &= \int_0^T \left(\frac{1}{T}\,\frac{e^{\mu\,(T-t)}-1}{\mu}\,\sigma\left(A_{0t}\right)\right)^2 dt \\
&= \begin{cases}
\frac{s_0^{2\gamma}}{\mu^2\,T^2}\left\{\frac{e^{2\mu T}}{2\mu(1-\gamma)}-\frac{2e^{\mu T}}{\mu(1-2\gamma)}-\frac{1}{2\mu T}+\frac{e^{\mu\gamma T}}{2\mu\gamma(1-\gamma)(1-2\gamma)}\right\} & \text{if } 0<\gamma<1,\ \gamma\neq\frac{1}{2} \\
\frac{s_0}{\mu^2 T^2}\left(\frac{e^{2\mu T}}{\mu}-2T\,e^{\mu T}-\frac{1}{\mu}\right) & \text{if } \gamma=\frac{1}{2} \\
\left(\frac{s_0}{\mu\,T}\right)^2\left\{e^{2\mu T}\left(T-\frac{3}{2\mu}\right)+\frac{2e^{\mu T}}{\mu}-\frac{1}{2\mu}\right\} & \text{if } \gamma=1,
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2\,T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&= \begin{cases}
\frac{\gamma\,s_0^{4\gamma-1}}{\Sigma^2\,\mu^5\,T^3}\left(\begin{array}{c}
\left(\left(-1+e^{\mu T}\right)\left(2-2\gamma+e^{\mu T}\left(-1+2\gamma\right)\right)^2\right. \\
+(e^{\mu(4\gamma-1)T}\gamma\left(-7+10\gamma\right) \\
+4\left(-1+\gamma\right)^2\left(3-13+12\gamma^2\right) \\
+e^{\mu(1+2\gamma)T}\left(6-44\gamma+96\gamma^2-64\gamma^3\right) \\
+4e^{2\mu\gamma T}\left(-3+19\gamma-32\gamma^2+16\gamma^3\right) \\
+e^{2\mu T}\gamma\left(-5+36-76\gamma^2+48\gamma^3\right) \\
-2e^{\mu T}\left(3-28\gamma+89\gamma^2-112\gamma^3+48\gamma^4\right)) \\
/\gamma\left(3-16\gamma+16\gamma^2\right))/8\left(1-3\gamma+2\gamma^2\right)^2
\end{array}\right) & \text{if } 0<\gamma<1,\ \gamma\neq\frac{1}{2} \\
\frac{s_0}{2\Sigma^2\mu^2 T^3}\left\{\frac{e^{3\mu T}}{2\mu^3}+e^{2\mu T}\left(\frac{1}{\mu^3}-\frac{2T}{\mu^2}\right)+e^{\mu T}\left(-\frac{1}{2\mu^3}-\frac{T}{\mu^2}+\frac{T^2}{\mu}\right)-\frac{1}{\mu^3}\right\} & \text{if } \gamma=\frac{1}{2} \\
\frac{s_0^3}{2\mu^2\mu^5 T^3}\left\{e^{3\mu T}\left(\mu^2T^2-3\mu T+\frac{17}{6}\right)+e^{2\mu T}\left(2\mu T-5\right)+\frac{5}{2}e^{\mu T}-\frac{1}{3}\right\} & \text{if } \gamma=1.
\end{cases}
\end{aligned}
$$

As with the Plain Vanilla case, the coefficients Σ, c, and f are the function of s_0, T, γ and μ. We can easily check the continuity at $\gamma = 1$, i.e. $\Sigma\left(\gamma\right) \to \Sigma\left(1\right)$ and $c\left(\gamma\right) \to c\left(1\right)$,as $\gamma \to 1$. We apply the theorem 6 and obtain the approximated value of the Delta.

We show some numerical examples (figure 2). The initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model ($\gamma = 1$), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model ($\gamma < 1$). The error ratio is calculated as $\left(\text{approximate value} - \text{true value}\right)/\text{true value}$.

We can observe that the accuracy of this approximation is valid in general.

9.3 Computaion of Delta by the Variance Reduction Technique

In this section, we show some numerical examples of variance reduction method. We use the BS and CEV models as the dynamics of stock prices. The result of Plain Vanilla Delta is exhibited in figure 3. In theorem 5, the Delta of Plain Vanilla Call Option is obtained by

$$e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right],$$

and its asymptotic expansion is expressed as

$$e^{-rT}\, E\left[Y_T\, 1_{\{s_T \geq K\}}\right] = \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1 + \epsilon\, \frac{2\,\gamma - 1}{s_0} g_2 + \cdots\right)\, 1_{\{g_1 + \epsilon\, g_2 + \cdots \geq -y\}}\right],$$

where $y = \left(s_0 e^{\mu\, T} - K\right)/\epsilon$ and $d = \frac{\partial y}{\partial s_0} = e^{\mu\, T}/\epsilon$. Then, we obtained its approximated value

$$\begin{aligned}
& D\,(0,T) \\
= \;& \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, 1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\, E\left[\frac{2\,\gamma - 1}{s_0} g_2\, 1_{\{g_1 \geq -y\}}\right] \\
& + \epsilon^2\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, g_2\, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\
= \;& \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, 1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\, E\left[\frac{2\,\gamma - 1}{s_0} E\left[g_2 | g_1\right]\, 1_{\{g_1 \geq -y\}}\right] \\
& + \epsilon^2\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, E\left[g_2 | g_1\right]\, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\
= \;& \epsilon\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, 1_{\{g_1 \geq -y\}}\right] + \epsilon^2\, e^{-rT}\, E\left[\frac{2\,\gamma - 1}{s_0}\, \left(c\, g_1^2 + f\right)\, 1_{\{g_1 \geq -y\}}\right] \\
& + \epsilon^2\, e^{-rT}\, E\left[\left(d + \frac{\gamma}{s_0} g_1\right)\, \left(c\, g_1^2 + f\right)\, \left(\partial 1\right)_{\{g_1 \geq -y\}}\right] \\
= \;& E\left[\varphi\,(g_1)\right],
\end{aligned}$$

where

$$\varphi\,(x) = \epsilon\, e^{-rT}\, \left(d + \frac{\gamma}{s_0} x + \frac{2\,\gamma - 1}{s_0} \epsilon\, \left(c\, x^2 + f\right)\right)\, 1_{\{x \geq -y\}} + \epsilon^2\, e^{-rT}\, \left(d - \frac{\gamma}{s_0} y\right)\, \left(c\, y^2 + f\right)\, n\,[y; 0, \Sigma]\,. \tag{27}$$

Next, we define Y as $Y = \varphi\,(g_1) - D\,(0,T)$, and call this Y “Attendant” random variable of X generated by “Crude” Monte Carlo.

In the simulation, the initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model $(\gamma = 1)$, and use the mean of 2,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model $(\gamma < 1)$. The errors 1,2 and 3 are defined by $(X -$ true value$)/$(true value), $Y/$(approximated value) and $(X - Y -$ true value$)/$ (true value), respectively. We generated 1000 passes and took their average, and repeated it by 100 cases. We calculated the correlation between X and Y, and the average, standard deviation, maximum and minimum of 100 cases for each of six random variables(X, Y, $(X - Y)$ and errors 1,2,3).

The example of Average Delta is exhibited in figure 4. Items on the table are the same as above. We can observe that the standard deviaition as well as maximum and minimum values of the errors of $HybridMC$(error 3) are reduced for each case relative to those of $CrudeMC$(error 1).

9.4 Approximation for the Vega of Plain Vanilla Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

$$\begin{aligned}
\Sigma &= E\left[\left(\int_0^T e^{\mu(T-t)}\,\sigma(A_{0t})\,dW_t\right)^2\right] \\
&= \int_0^T e^{2\mu(T-t)}\,\sigma(A_{0t})^2\,dt \\
&= \begin{cases} \frac{s_0^{2\gamma}}{2\mu(1-\gamma)}\left(e^{2\mu T}-e^{2\mu\gamma T}\right) & \text{if } 0<\gamma<1 \\ s_0^2\,T\,e^{2\mu T} & \text{if } \gamma=1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
c &= \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu(T-s)}\,\sigma(A_{0s})\,\sigma'(A_{0s})\,e^{2\mu(T-v)}\,\sigma(A_{0v})^2\,dv\,ds \\
&= \begin{cases} \frac{\gamma\,s_0^{4\gamma-1}}{\Sigma^2}\,e^{3\mu T}\,\frac{\left(1-e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \text{if } 0<\gamma<1 \\ \frac{s_0^3}{\Sigma^2}\,\frac{T^2}{2}\,e^{3\mu T} & \text{if } \gamma=1. \end{cases}
\end{aligned}$$

We apply the theorem 3 and get the approximated value of the Delta.

We show some numerical examples (figure 5). The initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model ($\gamma = 1$), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model ($\gamma < 1$). The error ratio is calculated as $($approximated value $-$ true value$)$ /(true value).

We can see that the accuracy of this approximation is generally valid.

9.5 Approximation for the Vega of Average Option

We apply the asymptotic expansion scheme to the BS and CEV processes. The coefficients are calculated as

$$
\begin{aligned}
\Sigma &= \int_0^T \left(\frac{1}{T} \frac{e^{\mu(T-t)}-1}{\mu} \sigma(A_{0t}) \right)^2 dt \\
&= \begin{cases}
\frac{s_0^{2\gamma}}{\mu^2 T^2} \left\{ \frac{e^{2\mu T}}{2\mu(1-\gamma)} - \frac{2e^{\mu T}}{\mu(1-2\gamma)} - \frac{1}{2\mu T} + \frac{e^{\mu\gamma T}}{2\mu\gamma(1-\gamma)(1-2\gamma)} \right\} & \text{if } 0<\gamma<1,\ \gamma\neq\frac{1}{2} \\
\frac{s_0}{\mu^2 T^2} \left(\frac{e^{2\mu T}}{\mu} - 2T e^{\mu T} - \frac{1}{\mu} \right) & \text{if } \gamma = \frac{1}{2} \\
\left(\frac{s_0}{\mu T} \right)^2 \left\{ e^{2\mu T} \left(T - \frac{3}{2\mu} \right) + \frac{2e^{\mu T}}{\mu} - \frac{1}{2\mu} \right\} & \text{if } \gamma = 1,
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[\frac{e^{\mu(T-s)-1}}{\mu} \right] \sigma(A_{0s})\, \sigma'(A_{0s}) \\
&\quad \times \int_0^s e^{\mu(s-v)} \left[\frac{e^{\mu(T-v)}-1}{\mu} \right] \sigma(A_{0v})^2\, dv\, ds\, dt \\
&= \begin{cases}
\frac{\gamma s_0^{4\gamma-1}}{\Sigma^2 \mu^5 T^3} \left(\begin{array}{c}
((-1+e^{\mu T})\left(2-2\gamma+e^{\mu T}(-1+2\gamma)\right)^2 \\
+(e^{\mu(4\gamma-1)T}\gamma(-7+10\gamma) \\
+4(-1+\gamma)^2\left(3-13+12\gamma^2\right) \\
+e^{\mu(1+2\gamma)T}\left(6-44\gamma+96\gamma^2-64\gamma^3\right) \\
+4e^{2\mu\gamma T}\left(-3+19\gamma-32\gamma^2+16\gamma^3\right) \\
+e^{2\mu T}\gamma\left(-5+36-76\gamma^2+48\gamma^3\right) \\
-2e^{\mu T}\left(3-28\gamma+89\gamma^2-112\gamma^3+48\gamma^4\right)) \\
/\gamma\left(3-16\gamma+16\gamma^2\right))/8\left(1-3\gamma+2\gamma^2\right)^2
\end{array} \right) & \text{if } 0<\gamma<1,\ \gamma\neq\frac{1}{2} \\
\frac{s_0}{2\Sigma^2\mu^2 T^3} \left\{ \frac{e^{3\mu T}}{2\mu^3} + e^{2\mu T}\left(\frac{1}{\mu^3} - \frac{2T}{\mu^2} \right) + e^{\mu T}\left(-\frac{1}{2\mu^3} - \frac{T}{\mu^2} + \frac{T^2}{\mu} \right) - \frac{1}{\mu^3} \right\} & \text{if } \gamma = \frac{1}{2} \\
\frac{s_0^3}{2\mu^2\mu^5 T^3} \left\{ e^{3\mu T}\left(\mu^2 T^2 - 3\mu T + \frac{17}{6}\right) + e^{2\mu T}(2\mu T - 5) + \frac{5}{2} e^{\mu T} - \frac{1}{3} \right\} & \text{if } \gamma = 1.
\end{cases}
\end{aligned}
$$

We apply the theorem 4 and obtained the approximated value of the Delta.

We show some numerical examples (figure 6). The initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model ($\gamma = 1$), and use the mean of 1,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model ($\gamma < 1$). The error ratio is calculated as $(\text{approximated value} - \text{true value})\,/(\text{true value})$.

As above, we can see that the accuracy of this approximation is valid.

9.6 Computation of Vega by the Variance Reduction Technique

In this section, we show some numerical examples of variance reduction method. We use the BS and CEV models as the dynamics of stock prices. The example of Plain Vanilla Vega is exhibited in figure 7. The random variables X and Y correspond to Crude_MC and Attendant, respectively. The initial value s_0 is fixed to $s_0 = 100$. For the true values, we use the analytical values for BS model $(\gamma = 1)$, and use the mean of 2,000,000 paths generated by Euler-Maruyama scheme that divides one year to 365 intervals for CEV model $(\gamma < 1)$. The error 1,2 and 3 are defined by$(X - \text{true value})$ /(true value), $Y/$(approximated value), and the error 3 is $(X - Y - \text{true value})$ /(true value), respectively. We generated 1000 passes and took their average, and repeated it by 100 cases. We calculated the correration between X and Y, and average, standard deviation, maximum and minimum of 100 cases for each of the six random variables(X, Y, $X - Y$ and errors 1,2,3).

The example of Average Vega is exhibited in figure 8. Items on the table are the same as above. We can observe that the standard deviaition as well as maximum and minimum values of the errors of $HybridMC$(error 3) are reduced for each case relative to those of $CrudeMC$(error 1).

10 Concluding Remarks

We developed an asymptotic expansion method for computing Greeks. Because it takes almost no time to compute the approximated values, it would be useful for monitoring Greeks changing rapidly in volatile markets. We also introduced so called "Hybrid Monte Carlo," a variance reduction technique based on combination of Monte Carlo simulation and the asymptotic expansion. We showed that our variance reduction technique can accelerate Monte Carlo simulations. Although we mainly treated the Delta and the Vega, our method can also be applied to computing the other Greeks such as Gamma; it is obtained by differentiating the approximated Delta by the initial price of the underlying asset.

Clearly, the same method can be applied to term structure models and credit models such as Heath-Jarrow-Morton models, Market models and Duffie-Singleton models, which are topics in the next research.

European Call Delta							
r	T	σ	γ	K	True Value	Approximation	error ratio
10%	1	10%	0.2	90	0.976116702	0.976086848	0.00%
10%	1	10%	0.2	100	0.843342612	0.843683858	0.04%
10%	1	10%	0.2	110	0.523547114	0.523263591	−0.05%
10%	1	10%	0.2	120	0.190115249	0.190523063	0.21%
10%	1	20%	0.2	90	0.843478762	0.842724364	−0.09%
10%	1	20%	0.2	100	0.698128476	0.698621588	0.07%
10%	1	20%	0.2	110	0.516581193	0.517408776	0.16%
10%	1	20%	0.2	120	0.335226623	0.335565067	0.10%
10%	1	30%	0.2	90	0.753767616	0.754598065	0.11%
10%	1	30%	0.2	100	0.641325056	0.642180446	0.13%
10%	1	30%	0.2	110	0.517235753	0.518021867	0.15%
10%	1	30%	0.2	120	0.394808905	0.394273471	−0.14%
10%	1	40%	0.2	90	0.703381045	0.704125223	0.11%
10%	1	40%	0.2	100	0.615017945	0.614636962	−0.06%
10%	1	40%	0.2	110	0.520993691	0.520252935	−0.14%
10%	1	40%	0.2	120	0.427134261	0.426404028	−0.17%
10%	0.1	20%	0.25	90	0.960879231	0.960924595	0.00%
10%	0.1	20%	0.25	100	0.565763231	0.56594217	0.03%
10%	0.1	20%	0.25	110	0.082557515	0.082217759	−0.41%
10%	0.1	20%	0.25	120	0.0018949	0.001791877	−5.44%
10%	1	20%	0.25	90	0.844504035	0.84452103	0.00%
10%	1	20%	0.25	100	0.70014607	0.700403424	0.04%
10%	1	20%	0.25	110	0.51964785	0.519331639	−0.06%
10%	1	20%	0.25	120	0.338268975	0.337990417	−0.08%
10%	1	20%	0.5	90	0.853480625	0.853018697	−0.05%
10%	1	20%	0.5	90	0.853382053	0.853018697	−0.04%
10%	1	20%	0.5	100	0.70867637	0.709151848	0.07%
10%	1	20%	0.5	100	0.708575194	0.709151848	0.08%
10%	1	20%	0.5	110	0.529413073	0.529076083	−0.06%
10%	1	20%	0.5	110	0.529207248	0.529076083	−0.02%
10%	1	20%	0.5	120	0.351056665	0.350513712	−0.15%
10%	1	20%	0.6	90	0.856948191	0.856186282	−0.09%
10%	1	20%	0.6	100	0.712330828	0.712574988	0.03%
10%	1	20%	0.6	110	0.533096525	0.533035582	−0.01%
10%	1	20%	0.6	120	0.356000643	0.355711223	−0.08%
10%	1	20%	0.7	90	0.860423398	0.859218457	−0.14%
10%	1	20%	0.7	100	0.715155518	0.715953769	0.11%
10%	1	20%	0.7	110	0.536399122	0.537030999	0.12%
10%	1	20%	0.7	120	0.361998417	0.361018332	−0.27%
10%	0.1	20%	0.75	90	0.96578897	0.965751632	0.00%
10%	0.1	20%	0.75	100	0.572099215	0.572170026	0.01%
10%	1	20%	0.75	90	0.861938725	0.860683108	−0.15%
10%	1	20%	0.75	100	0.717337906	0.717626354	0.04%
10%	1	20%	0.75	110	0.539513051	0.539042313	−0.09%
10%	1	20%	0.75	120	0.36442835	0.363713423	−0.20%
10%	1	20%	0.8	90	0.862914952	0.862113115	−0.09%
10%	1	20%	0.8	100	0.719324971	0.719287642	−0.01%
10%	1	20%	0.8	110	0.540332465	0.541062772	0.14%
10%	1	20%	0.8	120	0.366561896	0.366436441	−0.03%
1%	1	20%	0.9	90	0.746577822	0.746698594	0.02%
1%	1	20%	0.9	100	0.555316151	0.555815609	0.09%
1%	1	20%	0.9	110	0.367944108	0.367154152	−0.21%
1%	1	20%	0.9	120	0.217922263	0.219889703	0.90%
10%	1	20%	0.9	90	0.866346868	0.864868132	−0.17%
10%	1	20%	0.9	100	0.72253363	0.722576048	0.01%
10%	1	20%	0.9	110	0.544355552	0.545131345	0.14%
10%	1	20%	0.9	120	0.372811314	0.371966977	−0.23%
10%	1	20%	1	90	0.870086972	0.867481372	−0.30%
10%	1	20%	1	100	0.725746935	0.725818415	0.01%
10%	1	20%	1	110	0.549124303	0.549237168	0.02%
10%	1	20%	1	120	0.377669374	0.377611389	−0.02%
10%	1	30%	1	90	0.798010284	0.79680617	−0.15%
10%	1	30%	1	100	0.685570459	0.686543423	0.14%
10%	1	30%	1	110	0.565777013	0.566070289	0.05%
10%	1	30%	1	120	0.450497192	0.449978279	−0.12%

Figure 1: **Delta of Plain Vanilla Option**

Asian Call Delta							
r	T	σ	γ	K	True Value	Approximation	error ratio
10%	1	10%	0.2	90	0.946703688	0.94675927	0.01%
10%	1	10%	0.2	100	0.768201338	0.768261843	0.01%
10%	1	10%	0.2	110	0.201951045	0.203651546	0.84%
10%	1	10%	0.2	120	0.007273247	0.007444314	2.35%
10%	1	20%	0.2	90	0.858575119	0.85808018	−0.06%
10%	1	20%	0.2	100	0.637483815	0.638108348	0.10%
10%	1	20%	0.2	110	0.33028172	0.331526063	0.38%
10%	1	20%	0.2	120	0.108139027	0.109135815	0.92%
10%	1	30%	0.2	90	0.768435819	0.768741575	0.04%
10%	1	30%	0.2	100	0.588937469	0.588928763	0.00%
10%	1	30%	0.2	110	0.380452626	0.380973096	0.14%
10%	1	30%	0.2	120	0.202132734	0.203049404	0.45%
10%	1	40%	0.2	90	0.709435118	0.709919948	0.07%
10%	1	40%	0.2	100	0.563933815	0.564404437	0.08%
10%	1	40%	0.2	110	0.407408536	0.407226629	−0.04%
10%	1	40%	0.2	120	0.264824647	0.26439527	−0.16%
5%	0.5	20%	0.5	90	0.9120356	0.911511296	−0.06%
5%	0.5	20%	0.5	100	0.56037022	0.560295853	−0.01%
5%	0.5	20%	0.5	110	0.153452354	0.153316963	−0.09%
5%	0.5	20%	0.5	120	0.015937687	0.016132116	1.22%
5%	1	20%	0.5	90	0.848680745	0.847936145	−0.09%
5%	1	20%	0.5	100	0.580280908	0.580215147	−0.01%
5%	1	20%	0.5	110	0.269985302	0.269426888	−0.21%
5%	1	20%	0.5	120	0.080761156	0.081794869	1.28%
5%	1	30%	0.5	90	0.760218314	0.760158407	−0.01%
5%	1	30%	0.5	100	0.557179055	0.557292022	0.02%
5%	1	30%	0.5	110	0.345586245	0.344096243	−0.43%
5%	1	30%	0.5	120	0.178339925	0.17913761	0.45%
10%	0.5	20%	0.5	90	0.919278673	0.918786197	−0.05%
10%	0.5	20%	0.5	100	0.611276048	0.611117317	−0.03%
10%	0.5	20%	0.5	110	0.191986326	0.191253261	−0.38%
10%	0.5	20%	0.5	120	0.023628169	0.023797293	0.72%
10%	1	20%	0.5	90	0.8649266	0.864142396	−0.09%
10%	1	20%	0.5	100	0.643302812	0.643238501	−0.01%
10%	1	20%	0.5	110	0.339682441	0.338582967	−0.32%
10%	1	20%	0.5	120	0.118407109	0.119268848	0.73%
10%	1	30%	0.5	90	0.778472456	0.778295049	−0.02%
10%	1	30%	0.5	100	0.596729512	0.596891795	0.03%
10%	1	30%	0.5	110	0.391330682	0.390095722	−0.32%
10%	1	30%	0.5	120	0.214916031	0.216246178	0.62%
10%	1	20%	0.6	90	0.866320957	0.866014355	−0.04%
10%	1	20%	0.6	100	0.644645513	0.644902329	0.04%
10%	1	20%	0.6	110	0.34073559	0.340996052	0.08%
10%	1	20%	0.6	120	0.122158489	0.122857707	0.57%
10%	1	20%	0.7	90	0.868589478	0.867810116	−0.09%
10%	1	20%	0.7	100	0.646074032	0.646542615	0.07%
10%	1	20%	0.7	110	0.34351503	0.343439888	−0.02%
10%	1	20%	0.7	120	0.125609907	0.126553921	0.75%
10%	1	20%	0.8	90	0.87068924	0.869528569	−0.13%
10%	1	20%	0.8	100	0.646978648	0.648159129	0.18%
10%	1	20%	0.8	110	0.345741577	0.345914731	0.05%
10%	1	20%	0.8	120	0.12898536	0.130358551	1.06%
1%	1	20%	0.9	90	0.837577763	0.836196194	−0.16%
1%	1	20%	0.9	100	0.531381468	0.530712111	−0.13%
1%	1	20%	0.9	110	0.229461599	0.230168311	0.31%
1%	1	20%	0.9	120	0.068981823	0.070634948	2.40%
10%	1	20%	0.9	90	0.872496006	0.8711686	−0.15%
10%	1	20%	0.9	100	0.649739886	0.649751608	0.00%
10%	1	20%	0.9	110	0.348210579	0.34842081	0.06%
10%	1	20%	0.9	120	0.132583718	0.134272677	1.27%
5%	1	20%	1	90	0.860470779	0.858316977	−0.25%
5%	1	20%	1	100	0.589090363	0.589017433	−0.01%
5%	1	20%	1	110	0.28350849	0.283412971	−0.03%
5%	1	20%	1	120	0.096407229	0.099121174	2.82%
5%	0.5	30%	1	90	0.835750713	0.834073	−0.20%
5%	0.5	30%	1	100	0.553389003	0.553239205	−0.03%
5%	0.5	30%	1	110	0.265567646	0.265498309	−0.03%
5%	0.5	30%	1	120	0.093350194	0.096498241	3.37%
5%	1	30%	1	90	0.776622441	0.774935853	−0.22%
5%	1	30%	1	100	0.570638247	0.570604696	−0.01%
5%	1	30%	1	110	0.362821969	0.361812396	−0.28%
5%	1	30%	1	120	0.200898825	0.205313659	2.20%
10%	1	20%	1	90	0.874983182	0.872729081	−0.26%
10%	1	20%	1	100	0.651621365	0.651319794	−0.05%
10%	1	20%	1	110	0.351965975	0.350958365	−0.29%
10%	1	20%	1	120	0.134818069	0.13829738	2.58%
10%	0.1	30%	1	90	0.975342669	0.975121664	−0.02%
10%	0.1	30%	1	100	0.542554613	0.542373955	−0.03%
10%	0.1	30%	1	110	0.053568539	0.053789161	0.41%
10%	0.1	30%	1	120	0.000845986	0.000591309	−30.10%
10%	0.5	30%	1	90	0.847883064	0.845803231	−0.25%
10%	0.5	30%	1	100	0.585579247	0.585428875	−0.03%
10%	0.5	30%	1	110	0.296013128	0.295903652	−0.04%
10%	0.5	30%	1	120	0.110242694	0.113622975	3.07%

Figure 2: **Delta of Average Option**

Variance Reduction Method for Plain Vanilla Call Delta

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.959903129					
average	0.72557957	−0.00041	0.72599372	0.0%	−0.1%	0.0%
stdev	0.017156387	0.016684	0.00481434	2.4%	2.3%	0.7%
max	0.779884	0.051124	0.739676	7.5%	7.0%	1.9%
min	0.689382	−0.03948	0.715314	−5.0%	−5.4%	−1.4%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.956793154					
average	0.53112723	0.001849	0.52927784	0.4%	0.3%	0.1%
stdev	0.016333843	0.016663	0.00486072	3.1%	3.1%	0.9%
max	0.581436	0.053863	0.539565	10.0%	10.2%	2.0%
min	0.488625	−0.04283	0.517556	−7.6%	−8.1%	−2.1%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	110	0.5

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.977093684					
average	0.70810903	−0.00086	0.70897067	−0.1%	−0.1%	0.0%
stdev	0.016203002	0.01602	0.0034533	2.3%	2.3%	0.5%
max	0.744451	0.034014	0.715094	5.0%	4.8%	0.9%
min	0.654394	−0.05306	0.696965	−7.7%	−7.5%	−1.7%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.5

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.949576279					
average	0.6870225	0.000907	0.68611538	0.2%	0.1%	0.1%
stdev	0.017463932	0.017415	0.0055383	2.5%	2.5%	0.8%
max	0.737813	0.051122	0.698308	7.6%	7.4%	1.9%
min	0.647621	−0.03772	0.673854	−5.5%	−5.5%	−1.7%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.3	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.982425644					
average	0.70267656	0.000966	0.70171044	0.1%	0.1%	0.0%
stdev	0.014734067	0.014697	0.00275914	2.1%	2.1%	0.4%
max	0.741432	0.037448	0.707873	5.7%	5.3%	0.9%
min	0.662583	−0.04235	0.691333	−5.6%	−6.0%	−1.5%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.3

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.959868991					
average	0.71354272	−0.00175	0.7152942	−0.3%	−0.2%	−0.1%
stdev	0.013557998	0.013187	0.00380617	1.9%	1.8%	0.5%
max	0.749392	0.032719	0.726911	4.7%	4.6%	1.6%
min	0.673533	−0.04405	0.706392	−5.9%	−6.2%	−1.3%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.7

Figure 3: **Variance Reduction for Plain Vanilla Delta**

Variance Reduction Method for Average Call Delta

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.973564678					
average	0.65286561	0.002691	0.650174	0.2%	0.4%	−0.2%
stdev	0.017755937	0.017414	0.004058	2.7%	2.7%	0.6%
max	0.711461	0.06024	0.659347	9.2%	9.2%	1.2%
min	0.606232	−0.04052	0.639409	−7.0%	−6.2%	−1.9%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.972651266					
average	0.6466694	0.004066	0.642604	0.5%	0.6%	−0.1%
stdev	0.014007903	0.01441	0.003347	2.2%	2.2%	0.5%
max	0.677226	0.040301	0.649899	5.3%	6.3%	1.0%
min	0.612287	−0.03297	0.631315	−4.8%	−5.1%	−1.9%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.5

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.976284055					
average	0.64631725	0.000415	0.645902	0.1%	0.1%	0.0%
stdev	0.017035111	0.016832	0.003693	2.6%	2.6%	0.6%
max	0.692818	0.046845	0.656095	7.3%	7.2%	1.6%
min	0.608587	−0.04016	0.635893	−5.8%	−6.2%	−1.6%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.7

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.987282388					
average	0.64038034	0.00108	0.6393	0.1%	0.2%	0.0%
stdev	0.013700308	0.01362	0.00218	2.1%	2.1%	0.3%
max	0.673363	0.029755	0.644548	5.3%	4.7%	0.8%
min	0.608085	−0.03099	0.632347	−4.9%	−4.8%	−1.1%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.3

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.99000669					
average	0.63949469	0.00176	0.637734	0.3%	0.3%	0.0%
stdev	0.014518671	0.014709	0.002075	2.3%	2.3%	0.3%
max	0.669776	0.036076	0.641847	5.0%	5.7%	0.7%
min	0.607034	−0.03105	0.630536	−4.8%	−4.9%	−1.1%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.2

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.986044375					
average	0.33189606	−0.00178	0.333674	−0.3%	−0.5%	0.2%
stdev	0.016229574	0.016047	0.002702	4.9%	4.8%	0.8%
max	0.381386	0.048212	0.338977	14.6%	14.4%	1.8%
min	0.287723	−0.04739	0.326416	−13.6%	−14.2%	−1.9%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	110	0.3

Figure 4: **Variance Reduction for Average Delta**

European Call Vega							
r	T	σ	γ	K	True Value	Approximation	error ratio
10%	1	10%	0.2	90	0.135620063	0.136145644	0.39%
10%	1	10%	0.2	100	0.579057093	0.579043297	0.00%
10%	1	10%	0.2	110	0.960242603	0.961726804	0.15%
10%	1	10%	0.2	120	0.656887047	0.656763586	−0.02%
10%	1	20%	0.2	90	0.580709278	0.582693079	0.34%
10%	1	20%	0.2	100	0.83906647	0.842159412	0.37%
10%	1	20%	0.2	110	0.962827562	0.962582342	−0.03%
10%	1	20%	0.2	120	0.879844393	0.880976744	0.13%
10%	1	30%	0.2	90	0.756697078	0.762482701	0.76%
10%	1	30%	0.2	100	0.897587977	0.902648034	0.56%
10%	1	30%	0.2	110	0.959334662	0.962740858	0.36%
10%	1	30%	0.2	120	0.927374053	0.930217674	0.31%
10%	1	40%	0.2	90	0.828727786	0.837715386	1.08%
10%	1	40%	0.2	100	0.917847508	0.92482958	0.76%
10%	1	40%	0.2	110	0.956353045	0.962796345	0.67%
10%	1	40%	0.2	120	0.942198425	0.948094322	0.63%
10%	0.5	20%	0.5	90	1.505240303	1.517243388	0.80%
10%	0.5	20%	0.5	100	2.579305435	2.585420489	0.24%
10%	0.5	20%	0.5	110	2.675147472	2.670697783	−0.17%
10%	0.5	20%	0.5	120	1.773001909	1.787531578	0.82%
10%	1	20%	0.5	90	2.232530619	2.255920744	1.05%
10%	1	20%	0.5	100	3.332715923	3.353513757	0.62%
10%	1	20%	0.5	110	3.872463685	3.886061796	0.35%
10%	1	20%	0.5	120	3.597936065	3.623439526	0.71%
10%	1	30%	0.5	90	2.935439695	2.975313378	1.36%
10%	1	30%	0.5	100	3.551632457	3.594225029	1.20%
10%	1	30%	0.5	110	3.861935481	3.886684108	0.64%
10%	1	30%	0.5	120	3.778120297	3.814848873	0.97%
10%	1	20%	0.8	90	8.62223284	8.718743779	1.12%
10%	1	20%	0.8	100	13.26718447	13.35329575	0.65%
10%	1	20%	0.8	110	15.64510472	15.69084558	0.29%
10%	1	20%	0.8	120	14.82507253	14.89915535	0.50%
10%	1	30%	1	90	28.16294977	28.71366052	1.96%
10%	1	30%	1	100	35.49621593	35.9500663	1.28%
10%	1	30%	1	110	39.350731	39.7960357	1.13%
10%	1	30%	1	120	39.58670491	40.07273771	1.23%
10%	1	20%	1	90	21.14470728	21.44978946	1.44%
10%	1	20%	1	100	33.32246029	33.54565167	0.67%
10%	1	20%	1	110	39.59139563	39.78995782	0.50%
10%	1	20%	1	120	38.0036401	38.23501125	0.61%

Figure 5: **Vega of Plain Vanilla Option**

Asian Call Vega							
r	T	σ	γ	K	True Value	Approximation	error ratio
10%	1	10%	0.2	90	0.017620536	0.020081338	13.97%
10%	1	10%	0.2	100	0.37292946	0.374262724	0.36%
10%	1	10%	0.2	110	0.398719399	0.399866113	0.29%
10%	1	10%	0.2	120	0.028642431	0.029576347	3.26%
10%	1	20%	0.2	90	0.233997165	0.236452456	1.05%
10%	1	20%	0.2	100	0.492048592	0.494895816	0.58%
10%	1	20%	0.2	110	0.506707856	0.507429852	0.14%
10%	1	20%	0.2	120	0.265199439	0.267081362	0.71%
10%	1	30%	0.2	90	0.370148458	0.372996781	0.77%
10%	1	30%	0.2	100	0.518441131	0.521174468	0.53%
10%	1	30%	0.2	110	0.52681847	0.530316585	0.66%
10%	1	30%	0.2	120	0.397731091	0.40121822	0.88%
10%	1	40%	0.2	90	0.433479462	0.437488462	0.92%
10%	1	40%	0.2	100	0.52820917	0.530697898	0.47%
10%	1	40%	0.2	110	0.536369397	0.538568445	0.41%
10%	1	40%	0.2	120	0.458786573	0.462614756	0.83%
10%	0.5	20%	0.5	90	0.451809942	0.45590138	0.91%
10%	0.5	20%	0.5	100	1.497145322	1.504718555	0.51%
10%	0.5	20%	0.5	110	1.104185264	1.109946028	0.52%
10%	0.5	20%	0.5	120	0.228040948	0.22978291	0.76%
10%	1	20%	0.5	90	0.885201544	0.893667387	0.96%
10%	1	20%	0.5	100	1.960033268	1.966932694	0.35%
10%	1	20%	0.5	110	2.045918688	2.056901923	0.54%
10%	1	20%	0.5	120	1.133204923	1.148233844	1.33%
10%	1	30%	0.5	90	1.425729891	1.43647979	0.75%
10%	1	30%	0.5	100	2.05666194	2.07166549	0.73%
10%	1	30%	0.5	110	2.126538348	2.146753012	0.95%
10%	1	30%	0.5	120	1.663640874	1.682985408	1.16%
10%	1	20%	0.8	90	3.34324547	3.362808635	0.59%
10%	1	20%	0.8	100	7.802925684	7.81702216	0.18%
10%	1	20%	0.8	110	8.283486953	8.337629253	0.65%
10%	1	20%	0.8	120	4.858247591	4.917248513	1.21%
10%	1	30%	1	90	13.30359393	13.5367179	1.75%
10%	1	30%	1	100	20.49437224	20.66141664	0.82%
10%	1	30%	1	110	21.83397241	22.07367286	1.10%
10%	1	30%	1	120	18.01543655	18.29165237	1.53%
10%	1	20%	1	90	8.002372068	8.113176058	1.38%
10%	1	20%	1	100	19.47497484	19.61230035	0.71%
10%	1	20%	1	110	21.09200973	21.19619363	0.49%
10%	1	20%	1	120	12.79663758	12.94213589	1.14%

Figure 6: **Vega of Average Option**

Variance Reduction Method for Plain Vanilla Call Vega						
	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.993098106					
average	33.66226	0.333031	33.329227	1.0%	1.0%	0.0%
stdev	2.719191126	2.733607	0.32064568	8.2%	8.1%	1.0%
max	42.519	8.45469	34.0644	27.6%	25.2%	2.2%
min	28.0966	−5.22511	32.6062	−15.7%	−15.6%	−2.1%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.995557467					
average	3.3434785	0.005399	3.3380795	0.1%	0.2%	−0.1%
stdev	0.220618831	0.225483	0.02157966	6.6%	6.7%	0.6%
max	3.99644	0.641126	3.40658	19.7%	19.1%	2.0%
min	2.84201	−0.47944	3.29561	−14.9%	−14.3%	−1.3%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.5

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.989798005					
average	35.13766	−0.32529	35.462947	−1.0%	−0.9%	−0.1%
stdev	2.986438023	2.986035	0.42655948	8.4%	8.3%	1.2%
max	44.0392	7.91881	36.6576	24.1%	22.0%	3.3%
min	28.1204	−7.29869	34.4306	−20.8%	−20.3%	−3.0%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.3	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.994232883					
average	8.3549395	−0.0311	8.3860408	−0.4%	−0.4%	0.0%
stdev	0.619487474	0.62341	0.06685721	7.4%	7.4%	0.8%
max	9.97522	1.49157	8.59891	18.9%	17.7%	2.5%
min	6.76722	−1.66494	8.22763	−19.3%	−19.8%	−1.9%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.7

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.997097818					
average	1.332961	0.002638	1.3303235	0.2%	0.2%	0.0%
stdev	0.080399605	0.080545	0.00613282	6.0%	6.0%	0.5%
max	1.48684	0.152057	1.3474	11.7%	11.4%	1.3%
min	1.13538	−0.19802	1.31793	−14.7%	−14.8%	−0.9%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.3

Figure 7: **Variance Reduction for Plain Vanilla Vega**

Variance Reduction Method for Average Call Vega							
	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.993888106					
average	19.572025	−0.02551	19.59753	0.1%	−0.1%	0.2%
stdev	1.252353625	1.284064	0.143752	6.4%	6.5%	0.7%
max	23.4561	3.9421	19.9786	20.0%	20.1%	2.2%
min	16.0674	−3.60121	19.2102	−17.8%	−18.4%	−1.7%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	1

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.998501872					
average	0.78050641	−0.00251	0.783016	−0.2%	−0.3%	0.2%
stdev	0.040505161	0.040841	0.002252	5.2%	5.2%	0.3%
max	0.885119	0.103657	0.788773	13.2%	13.2%	0.9%
min	0.683454	−0.10238	0.777182	−12.6%	−13.1%	−0.6%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.3

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.999036023					
average	0.4969152	0.00195	0.494965	0.7%	0.4%	0.3%
stdev	0.028841021	0.028756	0.001267	5.8%	5.8%	0.3%
max	0.584236	0.086345	0.498648	18.4%	17.4%	1.1%
min	0.425687	−0.06938	0.49263	−13.7%	−14.0%	−0.1%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.2	100	0.2

	Crude_MC	Attendant	Hybrid_MC	error 1	error 2	error 3
correlation	0.987932148					
average	20.353032	−0.22172	20.57475	−0.6%	−1.1%	0.4%
stdev	1.283552187	1.274157	0.1989	6.3%	6.2%	1.0%
max	23.9281	3.34485	21.0203	16.8%	16.2%	2.6%
min	17.6652	−2.8836	20.1611	−13.8%	−14.0%	−1.6%
trial	case	r	T	σ	K	γ
1000	100	0.1	1	0.3	100	1

Figure 8: **Variance Reduction for Average Vega**

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