An Asymptotic Expansion Approach to Pricing Financial Contingent Claims

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Akihiko Takahashi †

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Abstract

We propose a new methodology for the valuation problem of financial contingent claims when the underlying asset prices follow a general class of continuous Itô processes. Our method can be applicable to a wide range of valuation problems including contingent claims associated with stocks, foreign exchange rates, the term structure of interest rates, and even their combinations. We illustrate our method by discussing the Black-Scholes economy when the underlying asset prices follow the continuous diffusion processes, which are not necessarily time homogeneous. The standard Black-Scholes model on stocks and the Cox-Ingersol-Ross model on the spot interest rate are simple examples. Then we shall give a series of examples on the valuation formulae including plain vanilla options, average options, and other contingent claims. We shall also give some numerical evidence of the accuracy of the approximations we have obtained for practical purposes. Our approach can be rigorously justified by an infinite dimensional mathematics, the Malliavin-Watanabe-Yoshida theory recently developed in stochastic analysis.

Key Words

Asymptotic Expansion, Black-Scholes Economy, Small Disturbance Asymptotics, Options and Derivatives, Malliavin-Watanabe Calculus

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†Graduate School of Mathematical Sciences, the University of Tokyo

1 Introduction

This paper proposes a new approach in the valuation problems of financial contingent claims based on the *small disturbance asymptotics*. We shall extensively develop an asymptotic expansion technique for financial asset pricing problems in the continuous time stochastic process framework. This approach is general enough to be applicable to a broad class of continuous Itô processes for asset prices and their functionals and it is possible to evaluate the fair values of complicated payoffs such as the average options under the general class of asset processes. Contrary to other approaches, we can derive explicit formulae for many valuation problems of contingent claims when the underlying asset prices follow the general Itô stochastic processes.

The asymptotic expansion approach was first proposed and developed by Kunitomo and Takahashi (1995), which mainly have dealt with the valuation problems of interest rates contingent claims in the HJM framework. (See Heath, Jarrow, and Morton (1992).) Our approach is based on the key empirical observation that the observed and estimated volatilities of financial asset prices may very over time, but they are not very large in comparison with the observed level of asset prices. Then we can develop an asymptotic expansion method in which the continuous Itô process can be expanded around the corresponding deterministic process. In the asymptotic expansions of the continuous stochastic processes we shall derive, the first term is a Gaussian random variable and the following terms are some adjustment terms. Thus we can derive explicit approximations for the valuation problems of contingent claims based on the asymptotic expansions of the stochastic processes. This method was first introduced by an intuitive reasoning but can be justified in a rigorous mathematical fashion. However, since the general Itô processes involve a space of Wiener measure, the mathematical validity of our method is far from standard one even in probability theory.

The main purpose of this paper is to develop our asymptotic expansion approach for various problems of contingent claim evaluation. For this purpose we mainly concentrate on the specific situation when the stochastic processes of asset prices follow a general class of the diffusion processes, that is, the continuous Markovian processes. The Black-Sholes economy for stocks and foreign exchange rates and the Cox-Ingersol-Ross economy for spot interest rate are special cases of this framework. Since this is the simplest situation in our framwork, our method can be rather explained in a straightforward way and the resulting formulae are relatively simple. Also it is relatively straightforward to examine the accuracy of the resulting approximations in the present case. We shall demonstrate that they are accurate for many practical purposes. In a companion of this paper, Kunitomo and Takahashi (1995) have applied the asymptotic expansion approach to the valuation problem of interest rates based contingent claims. Since they have directly dealt with the stochastic processes for forward rates which are not necessarily Markovian as we usually define in the standard stochastic analysis, they need some discussions on the special features of the approach to obtain a rigorous mathematical validity. However, non-Markovian processes can be represented by multi-dimensional Markovian processes in most cases and hence the method discussed in this paper could be also applied to the HJM framework.

The organization of this paper is as follows. In Section 2 we explain the basic method in the Black-Scholes economy where there is a constant risk-free interest rate and the underlying asset price follow a one-factor stochastic process of the general Markovian type. Then in Section 3 we shall apply our approach to several problems including basket options, average options, options with stochastic volatilities, and illustrate the general applicability of our approach to many other problems. In Section 4 we shall investigate some numerical examples. We report the accuracy of numerical pricing values when the underlying asset prices are the square−root process and the log−normal process as examples in particular. In Section 5, we shall give some concluding remarks on the asymptotic expansion approach and discuss some generalizations. Finally the mathematical validity of our approach will be discussed in the Appendix.

2 The Asymptotic Expansion Approach

The stochastic processes we mainly consider in this paper can be described by the stochastic differential equation of Markovian type. More explicitly, in the Black-Scholes economy, each underlying asset process $\{S^{(\delta)}(t)\}$ follows

$$dS^{(\delta)} = rS^{(\delta)}dt + \delta \sum_{i=1}^{N} \sigma_i(S^{(\delta)}, t)d\tilde{w}_{it} \tag{2.1}$$

under the equivalent martingale measure, where $0 < \delta \leq 1$, $\tilde{w}_{it} = (\tilde{w}_i(t))$ are independent standard Brownian motions, and r is a positive constant. In the Black-Scholes economy the interest rate of the riskless asset is constant and the risky assets follow the continuous Markov processes. There is a strong restriction imposed on the drift coefficients because we shall use the no-arbitrage theory which has been standard in financial economics while the volatility functions depend on both time and the underlying asset price. We emphasize that our method can be applicable to more general situations and we shall discuss this issue briefly in Section 5.

In this section, in order to illustrate and clarify the basic method of the asymptotic expansion approach, we take the Black-Scholes economy when $N = 1$. More general Markovian cases will be explained in Section 3 and the validity of our approach will be discussed in the Appendix. Thus the derivations in the next two sections are formal in the proper mathematical sense. When $N = 1$, there is one factor and the volatility function depends on time and the underlying asset price in the economy. Hence it includes the standard Black-Scholes model, the CEV (Constant Elasticity of Variance) processes, and the processes derived as the continuous limit of the implied trees in our framework. For instance, the Cox-Ingersol-Ross model for the spot interest rate is a special case of the CEV processes.

The asymptotic expansion approach we are proposing consists of two steps. First, we make the detailed derivation of the asymptotic expansion for the density function of the normalized price of the risky asset. Next, by making use of the approximated density function, we can derive new formulas for the valuation of the contingent claims when the underlying asset follows an approximated stochastic Markov process. In this section we shall illustrate these steps by deriving some explicit formulas for the plain vanila options.

2.1 The Asymptotic Expansion Approach to the Black-Sholes Economy

We consider an economy where there is one risky asset and a riskless asset. The volatility function in the risky asset process depends on the current level of the asset and the current time. We assume that the process of the risky asset is described by

$$dS^{(\delta)} = rS^{(\delta)}dt + \delta\sigma(S^{(\delta)}, t)d\tilde{w}_t \tag{2.2}$$

and the process of the riskless bond price $\{B_t\}$ is determined by

$$dB = rBdt \ , \tag{2.3}$$

where $0 < \delta \leq 1$, $\tilde{w}_t = (\tilde{w}(t))$ is a one-dimensional standard Brownian motion, and r is a positive constant. Alternatively, the integral form of the risky asset process can be expressed as

$$S^{(\delta)}(t) = S(0) + r\int_0^t S^{(\delta)}(s)ds + \delta\int_0^t \sigma(S^{(\delta)}, s)d\tilde{w}(s) \ . \tag{2.4}$$

Our first task is to obtain an asymptotic expansion of the random variable $S^{(\delta)}(t)$ around $\delta = 0$. By inserting $\delta = 0$ in (2.4), we have a deterministic process described by the differential equation

$$S^{(0)}(t) \equiv \lim_{\delta \to 0} S^{(\delta)}(t) = S(0) + r \int_0^t S^{(0)} ds.$$

By solving the above equation, we have the solution as

$$S^{(0)}(t) = e^{rt} S(0) \ . \tag{2.5}$$

Next, we shall expand the integral equation (2.4) with respect to δ in a formal way. In order to calculate the first order coefficient of δ, let

$$A(t) = \frac{\partial S^{(\delta)}(t)}{\partial \delta}|_{\delta=0}.$$

By differentiating (2.4) with respect to δ, we obtain the stochastic differential equation which $A(t)$ must follow :

$$dA(t) = rA(t)dt + \sigma(S^{(0)}, t) d\tilde{w}(t) \ .$$

This stochastic differential equation can be solved as

$$A(t) = \int_0^t e^{r(t-s)} \sigma(S^{(0)}, s) d\tilde{w}(s) \ . \tag{2.6}$$

By expanding (2.4) twice and three times with respect to δ, we can also obtain the second and third order coefficients of δ. For this purpose, let

$$B(t) = \frac{\partial^2 S^{(\delta)}(t)}{\partial \delta^2}|_{\delta=0}.$$

Then we obtain the stochastic differential equation of $B(t)$:

$$dB(t) = rB(t)dt + 2\partial\sigma(S^{(0)}, t) A(t) d\tilde{w}(t) \ ,$$

where we have used the notation for convenience

$$\partial\sigma(S^{(0)}, t) \equiv \frac{\partial \sigma(S^{(\delta)}, t)}{\partial S^{(\delta)}}|_{S^{(\delta)}=S^{(0)}} \ .$$

We can solve the above stochastic differential equation and $B(t)$ can be expressed as

$$B(t) = 2 \int_0^t e^{r(t-s)} \partial\sigma(S^{(0)}, s) A(s) d\tilde{w}(s) \ . \tag{2.7}$$

We repeat the above construction and let also

$$C(t) = \frac{\partial^3 S^{(\delta)}(t)}{\partial \delta^3}|_{\delta=0} \ .$$

Then we have the stochastic differential equation :

$$dC(t) = rC(t)dt + 3\partial^2\sigma(S^{(0)}, t)A(t)^2 d\tilde{w}(t) + 3\partial\sigma(S^{(0)}, t)B(t)d\tilde{w}(t) \ .$$

Hence $C(t)$ can be expressed as

$$C(t) = 3\int_0^t e^{r(t-s)}\partial^2\sigma(S^{(0)}, s)A(s)^2 d\tilde{w}(s) + 3\int_0^t e^{r(t-s)}\partial\sigma(S^{(0)}, s)B(s)d\tilde{w}(s). \tag{2.8}$$

By summarizing the above terms up to $O(\delta^3)$, we have a formal asymptotic expansion of the random variable $S^{(\delta)}(t)$. We state our reslut as the following proposition.

Proposition 2.1 *An asymptotic expansion of the price of the risky asset, $S^{(\delta)}(t)$ at any particular time point t, as $\delta \to 0$ is given by*

$$S^{(\delta)}(t) = S^{(0)}(t) + \delta A(t) + \delta^2\frac{B(t)}{2} + \delta^3\frac{C(t)}{6} + o(\delta^3) \ , \tag{2.9}$$

where $A(t), B(t)$, and $C(t)$ are defined by (2.6), (2.7) and (2.8), respectively.

Since S^0 is a deterministic function of time, $A(t)$ in the above expression follows a normal distribusion. By a simple calculation

$$A(t) \sim N(0, \Sigma_{A_t}) \ , \tag{2.10}$$

where

$$\Sigma_{A_t} = \int_0^t e^{2r(t-s)}\sigma(S^{(0)}, s)^2 ds \ . \tag{2.11}$$

Next we shall derive an asymptotic expansion of the density function of the random variable $X_t^{(\delta)}$. For this purpose we normalize $X_t^{(\delta)}$ around the deterministic process S^0 and let

$$\begin{aligned} X_t^{(\delta)} = \{\frac{S^{(\delta)}(t) - S^{(0)}(t)}{\delta}\} &= A(t) + \delta\frac{B(t)}{2} + \delta^2\frac{C(t)}{6} + \cdots \\ &\equiv g_1 + \delta g_2 + \delta^2 g_3 + \cdots \ , \end{aligned} \tag{2.12}$$

where we have implicitly defined g_i $(i = 1, 2, 3)$. Since we know that

$$g_1 \sim N(0, \Sigma_{A_t}) = N(0, \Sigma_{g_1}) \ , \tag{2.13}$$

an asymptotic expansion of the density function of $X_t^{(\delta)}$ can be obtained as the normal density function combined with the adjusted terms as $\delta \to 0$. In order to obtain the explicit functional forms of the adjusted terms, we shall use the inversion method based on the characteristic function which will be explained below. First, we make the following assumption in all the subsequent analyses of this paper.

Assumption I : The variance of the random variable $X_t^{(\delta)}$, Σ_{g_1}, is positive for any t.

From (2.11) this condition is satisfied if $\sigma(S^0, t) > 0$ for some t. Next, we define the characteristic function of the random variable $X^{(\delta)}(t)$ by

$$\psi(\xi) = \mathbf{E}[e^{i\xi X_t^{(\delta)}}] ,$$

where $\mathbf{E}[\cdot]$ is the expectation operator. Then, the characteristic function $\psi(\xi)$ itself can be formally expanded along the polynomial orders of δ as $\delta \to 0$:

$$\begin{aligned}
\psi(\xi) &= \mathbf{E}[e^{i\xi(g_1+\delta g_2+\delta^2 g_3+\cdots)}] \\
&= \mathbf{E}[e^{i\xi g_1}\{1+\delta(i\xi)g_2+\delta^2(i\xi)g_3+\frac{\delta^2}{2}(i\xi)^2 g_2^2+\cdots\}] \\
&= \mathbf{E}[e^{i\xi g_1}]+\delta(i\xi)\mathbf{E}[e^{i\xi g_1}g_2]+\delta^2(i\xi)\mathbf{E}[e^{i\xi g_1}g_3] \\
&+ \frac{\delta^2}{2}(i\xi)^2\mathbf{E}[e^{i\xi g_1}g_2^2]+\cdots \\
&= e^{\frac{(i\xi)^2\Sigma_{g_1}}{2}}+\delta(i\xi)\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_2|g_1=x]\right]+\delta^2(i\xi)\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_3|g_1=x]\right] \\
&+ \frac{1}{2}\delta^2(i\xi)^2\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_2^2|g_1=x]\right]+\cdots ,
\end{aligned}$$

where $\mathbf{E}[\cdot|g_1]$ is the conditional expectation operator given g_1. Then we can explicitly evaluate each term in this expansion of the characteristic function. First, we shall show that $\mathbf{E}[g_2|g_1 = x]$, $\mathbf{E}[g_3|g_1 = x]$, and $\mathbf{E}[g_2^2|g_1 = x]$ are some polynomial functions of x, which will be denoted as $h_2(x)$,$h_3(x)$, and $h_{22}(x)$, respectively. To do this, we present useful formulae to evaluate those conditional expectations which will be used repeatedly in this paper.

Lemma 2.1 (1) *Let $\vec{w}_t$ be an N dimensional Brownian motion. Let $\vec{x}$ be a k dimensional vector. Suppose $\underline{q_1}(t)$ be a $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\underline{q_2}(t)$ and $\underline{q_3}(t)$ be $R^1 \mapsto R^{m\times N}$ non-stochastic functions. Then,*

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t \left[\int_0^s \underline{q_2}(u)d\vec{w}_u\right]^\top \underline{q_3}(s)d\vec{w}_s | \int_0^T \underline{q_1}(u)d\vec{w}_u = \vec{x}\right] \\
&= \mathit{trace}\int_0^t\int_0^s \underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\underline{q_3}(s)^\top\underline{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top-\underline{\Sigma}_{g_1}\right]duds.
\end{aligned}$$

(2) *Let* $\vec{\tilde{w}}_t$ *be an* N *dimensional Brownian motion. Let* $\vec{x}$ *be a* k *dimensional vector. Suppose* $\underline{q_1}(t)$ *be a* $R^1 \mapsto R^{k\times N}$ *non-stochastic function. Suppose also* $\vec{q_2}(t)$ *and* $\vec{q_3}(t)$ *be* $R^1 \mapsto R^N$ *non-stochastic functions.*
Then,

$$\begin{aligned}
& \mathbf{E}\left[\left[\int_0^t \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\left[\int_0^t \vec{q_3}(s)d\vec{\tilde{w}}_s\right] \mid \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= & \int_0^t \vec{q_2}(u)\vec{q_3}(u)^\top du + \left[\int_0^t \vec{q_2}(u)\underline{q_1}(u)^\top du\right]\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\left[\int_0^t \underline{q_1}(s)\vec{q_3}(s)^\top ds\right].
\end{aligned}$$

(3) *Let* $\vec{\tilde{w}}_t$ *be an* N *dimensional Brownian motion. Let* $\vec{x}$ *be a* k *dimensional vector. Suppose* $\underline{q_1}(t)$ *be a* $R^1 \mapsto R^{k\times N}$ *non-stochastic function. Suppose also* $\vec{q_2}(t)$, $\vec{q_3}(t)$ *and* $\vec{q_4}(t)$ *be* $R^1 \mapsto R^N$ *non-stochastic functions.*
Then,

$$\begin{aligned}
& \mathbf{E}\left[\int_0^t \left[\int_0^s \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\left[\int_0^s \vec{q_3}(u)d\vec{\tilde{w}}_u\right]\vec{q_4}(s)d\vec{\tilde{w}}_s \mid \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= & \int_0^t\int_0^s\int_0^s \left[\vec{q_2}(v)\underline{q_1}(v)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_3}(u)^\top\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right. \\
- & \left.\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\left[\vec{q_3}(v)^\top\vec{q_2}(u) + \vec{q_2}(v)^\top\vec{q_3}(u)\right]\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right] dudvds \\
+ & \int_0^t\int_0^s \vec{q_2}(u)\vec{q_3}(u)^\top\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}duds
\end{aligned}$$

(4) *Let* $\vec{\tilde{w}}_t$ *be an* N *dimensional Brownian motion. Let* $\vec{x}$ *be a* k *dimensional vector. Suppose* $\underline{q_1}(t)$ *be a* $R^1 \mapsto R^{k\times N}$ *non-stochastic function. Suppose also* $\vec{q_2}(t)$, $\vec{q_3}(t)$ *and* $\vec{q_4}(t)$ *be* $R^1 \mapsto R^N$ *non-stochastic functions.*
Then,

$$\begin{aligned}
& \mathbf{E}\left[\int_0^t\int_0^s \left[\int_0^v \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\vec{q_3}(v)d\vec{\tilde{w}}_v\vec{q_4}(s)d\vec{\tilde{w}}_s \mid \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= & \int_0^t\int_0^s\int_0^v \left[\vec{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right]\left[\vec{x}^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_3}(v)^\top\right]\left[\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right]dudvds \\
- & \int_0^t\int_0^s\int_0^v \left[\vec{q_2}(u)\underline{q_1}(u)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_3}(v)^\top\right]\left[\vec{q_4}(s)\underline{q_1}(s)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds \\
- & \int_0^t\int_0^s\int_0^v \left[\vec{q_3}(v)\underline{q_1}(v)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_4}(s)^\top\right]\left[\vec{q_2}(u)\underline{q_1}(u)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds \\
- & \int_0^t\int_0^s\int_0^v \left[\vec{q_4}(s)\underline{q_1}(s)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_2}(u)^\top\right]\left[\vec{q_3}(v)\underline{q_1}(v)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds
\end{aligned}$$

(5) *Let* $\vec{\tilde{w}}_t$ *be an* N *dimensional Brownian motion. Let* $\vec{x}$ *be a* k *dimensional*

vector. Suppose $\underline{q_1}(t)$ *be a* $R^1 \mapsto R^{k\times N}$ *non-stochastic function. Suppose also* $\vec{q_2}(t)$, $\vec{q_3}(t)$ $\vec{q_4}(t)$, *and* $\vec{q_5}(t)$ *be* $R^1 \mapsto R^N$ *non-stochastic functions.*
Then,

$$
\begin{aligned}
& \mathbf{E}\left[\left[\int_0^t \left[\int_0^s \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\vec{q_3}(s)d\vec{\tilde{w}}_s\right]\left[\int_0^t \left[\int_0^s \vec{q_4}(u)d\vec{\tilde{w}}_u\right]\vec{q_5}(s)d\vec{\tilde{w}}_s\right] \mid \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= \; & \int_0^t\int_0^t\int_0^s\int_0^v \Big[\vec{q_2}(u)\underline{q_1}(u)^\top \underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_3}(s)^\top \times \\
& \vec{q_4}(u^{'})\underline{q_1}(u^{'})^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_5}(v)^\top \\
- \; & \vec{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\left[\underline{q_1}(v)\vec{q_5}(v)^\top\vec{q_4}(u^{'})\underline{q_1}(u^{'}) + \underline{q_1}(v)\vec{q_4}(v)^\top\vec{q_5}(u^{'})\underline{q_1}(u^{'})^\top\right] \times \\
& \underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_3}(s)^\top \\
- \; & \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\underline{q_1}(v)\vec{q_5}(v)^\top\vec{q_4}(u^{'})\underline{q_1}^\top(u^{'}) + \underline{q_1}(u^{'})\vec{q_4}(u^{'})^\top\vec{q_5}(v)\underline{q_1}(v)^\top\right] \times \\
& \underline{\Sigma}_{g_1}^{-1}\vec{x}\vec{x}^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_2}(u)^\top\Big]du^{'}dudvds \\
+ \; & \int_0^t\int_0^s\int_0^u \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u^{'})\vec{q_4}(u^{'})^\top\vec{q_5}(u)\vec{q_2}(u)^\top du^{'}duds \\
+ \; & \int_0^t\int_0^s\int_0^u \vec{q_2}(u')\underline{q_1}(u')^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_5}(s)^\top\vec{q_3}(u)\vec{q_4}(u)^\top du^{'}duds \\
+ \; & \int_0^t \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\left[\int_0^s \underline{q_1}(v)\vec{q_5}(v)^\top\int_0^v \vec{q_2}(u)\vec{q_4}(u)^\top dudv\right]ds \\
+ \; & \int_0^t \left[\int_0^v\int_0^s \vec{q_2}(u)\vec{q_4}(u)^\top du\vec{q_3}(s)\underline{q_1}(s)^\top ds\right]\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_5}(v)^\top dv \\
+ \; & \int_0^t \left[\int_0^s \vec{q_2}(u)\underline{q_1}(u)^\top du\right]\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\left[\int_0^s \underline{q_1}(u)\vec{q_4}(u)^\top du\right]\vec{q_3}(s)\vec{q_5}(s)^\top ds \\
+ \; & \int_0^t\int_0^s \vec{q_3}(s)\vec{q_5}(s)^\top\vec{q_2}(u)\vec{q_4}(u)^\top duds
\end{aligned}
$$

Formulas (1) and (2) are slight generalizations of Lemma 5.7 of Yoshida (1992a), which are already reported as Lemma 6.1 of Kunitomo and Takahashi (1995). Formulas (3), (4) and (5) are the direct results of calculations by utilizing the Gaussianity of continuous processes involved. Since they are quite tedious but straightforward to be done, we omit their derivations.
By using *Lemma 2.1*, we can evaluate the conditional expectations appeared in the asymptotic expansion of the characteristic function. First, by applying $formula$(1) to $\mathbf{E}[g_2|g_1 = x]$, we have

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{2.14}$$

where

$$c = \frac{1}{\Sigma_{g_1}^2}\int_0^t e^{r(t-s)}\sigma(S^{(0)}, s)\partial\sigma(S^{(0)}, s)\int_0^s e^{2r(t-v)}\sigma(S^{(0)}, v)^2 dvds\ ,$$

and

$$f = -c\Sigma_{g_1} \ .$$

Second, we evaluate $\mathbf{E}[g_3|g_1 = x]$ by using $formula(3)$ and $formula(4)$. We note that

$$\begin{aligned} g_3 &= \frac{C(t)}{6} \\ &= \frac{1}{2}e^{rt}\int_0^t \left[e^{rs}\partial^2\sigma(S^{(0)},s)\right]\left[\int_0^s e^{-rv}\sigma(S^{(0)},v)d\tilde{w}(v)\right]^2 d\tilde{w}(s) \\ &+ e^{rt}\int_0^t \partial\sigma(S^{(0)},s)\int_0^s \partial\sigma(S^{(0)},v)\int_0^v e^{-ru}\sigma(S^{(0)},u)d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s) \ . \end{aligned}$$

By applying $formula(3)$ and $formula(4)$ to the first and second term of $\mathbf{E}[g_3|g_1 = x]$, respectively, we obtain

$$\mathbf{E}\left[g_3|g_1 = x\right] = \frac{1}{2}e^{rt}[x^3c_{11} + xf_{11}] + e^{rt}[x^3c_{12} + xf_{12}] \tag{2.15}$$

where

$$c_{11} = \frac{1}{\Sigma_{g_1}^3}e^{3rt}\int_0^t \left[\int_0^s e^{-2rv}\sigma(S^{(0)},v)^2dv\right]^2 \partial^2\sigma(S^{(0)},s)\sigma(S^{(0)},s)ds \ ,$$

$$f_{11} = \frac{1}{\Sigma_{g_1}}e^{rt}\int_0^t \left[\int_0^s e^{-2rv}\sigma(S^{(0)},v)^2dv\right]\partial^2\sigma(S^{(0)},s)\sigma(S^{(0)},s)ds - 3\Sigma_{g_1}c_{11} \ ,$$

$$\begin{aligned} c_{12} &= \frac{1}{\Sigma_{g_1}^3}e^{3rt}\int_0^t \left[e^{-rs}\sigma(S^{(0)},s)\partial\sigma(S^{(0)},s)\right]\int_0^s \left[e^{-rv}\sigma(S^{(0)},v)\partial\sigma(S^{(0)},v)\right] \\ &\quad \int_0^v \left[e^{-2ru}\sigma(S^{(0)},u)^2\right]dudsdt \end{aligned}$$

and

$$f_{12} = -3\Sigma_{g_1}c_{12} \ .$$

Alternatively, we can write

$$\mathbf{E}\left[g_3|g_1 = x\right] = c_1x^3 + f_1x \tag{2.16}$$

where

$$c_1 = \frac{1}{2}e^{rt}c_{11} + e^{rt}c_{12}$$

and

$$f_1 = \frac{1}{2}e^{rt}f_{11} + e^{rt}f_{12} \ .$$

Similarly, by using $formula(5)$, we can show that by the use of the constants c_2, f_2 and k_2,

$$\mathbf{E}\left[g_2^2|g_1 = x\right] = c_2x^4 + f_2x^2 + k_2 \ , \tag{2.17}$$

where

$$
\begin{aligned}
c_2 &= e^{2rt}\frac{1}{\Sigma_{g_1}^4}\times l \\
f_2 &= e^{2rt}\left[\frac{-6}{\Sigma_{g_1}^3}\times l+\frac{1}{\Sigma_{g_1}^2}\times(2m+n+o)\right] \\
k_2 &= e^{2rt}\left[\frac{3}{\Sigma_{g_1}^2}\times l+\frac{1}{\Sigma_{g_1}}\times(-2m-n-o)\right] ,
\end{aligned}
$$

and we have defined l, m, n and o as follows;

$$
\begin{aligned}
l &= e^{4rt}\left[\int_0^t e^{-rs}\sigma(S^{(0)},s)\partial\sigma(S^{(0)},s)\int_0^s e^{-2ru}\sigma(S^{(0)},u)^2 duds\right]^2 \\
m &= e^{2rt}\int_0^t e^{-rs}\sigma(S^{(0)},s)\partial\sigma(S^{(0)},s)\int_0^s e^{-rv}\sigma(S^{(0)},v)\partial\sigma(S^{(0)},v)\int_0^v e^{-2ru}\sigma(S^{(0)},u)^2 dudvds \\
n &= 2e^{2rt}\left[\int_0^t e^{-rs}\sigma(S^{(0)},s)\partial\sigma(S^{(0)},s)\int_0^s e^{-rv}\sigma(S^{(0)},v)\partial\sigma(S^{(0)},v)\int_0^v e^{-2ru}\sigma(S^{(0)},u)^2 dudvds\right] \\
o &= e^{2rt}\int_0^t[\partial\sigma(S^{(0)},s)]^2\left[\int_0^s e^{-2ru}\sigma(S^{(0)},u)^2du\right]^2 ds
\end{aligned}
$$

By collecting terms of the asymptotic expansion, we obtain a simpler form of the characteristic function as

$$
\begin{aligned}
\psi(\xi) &= e^{\frac{(i\xi)^2\Sigma_{g_1}}{2}}+\delta(i\xi)\mathbf{E}\left[e^{i\xi x}h_2(x)\right]+\delta^2(i\xi)\mathbf{E}\left[e^{i\xi x}h_3(x)\right] \\
&+ \frac{\delta^2}{2}(i\xi)^2\mathbf{E}\left[e^{i\xi x}h_{22}(x)\right]+\cdots .
\end{aligned}
$$

The explicit formulas of the expectations such as $\mathbf{E}\left[e^{i\xi x}h_2(x)\right]$, $\mathbf{E}\left[e^{i\xi x}h_3(x)\right]$, and $\mathbf{E}\left[e^{i\xi x}h_{22}(x)\right]$ in $\psi(\xi)$ can be easily obtained due to the Gaussianity of the leading term.

As the final step to obtain the asymptotic expansion of the density function of $X^{(\delta)}(t)$, we need to invert $\psi(\xi)$ (i.e. the inverse Fourier transformation). We make use of the following formula, which has been given by *Fujikoshi etal.(1982)*, to summarize both steps of evaluating characteristic function and implementing the inverse-Fourier transformation.

Lemma 2.2 *Suppose that $\vec{x}$ follows N-dimensional normal distribution with mean $\vec{0}$ and variance-covariance matrix $\underline{\Sigma}$. Then, for any polynomial functions $h(\cdot)$ and $g(\cdot)$,*

(2.18) $$\mathcal{F}^{-1}\left[g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]\right]_{<\vec{\omega}>}=g\left[\frac{\partial}{\partial\vec{\omega}}\right]h(\vec{\omega})n[\vec{\omega};\vec{0},\underline{\Sigma}],$$

where

$$\mathcal{F}^{-1}\left[g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]\right]_{<\vec{\omega}>}=(\frac{1}{2\pi})^N\int_{R^N}e^{-i\xi^\top\vec{\omega}}g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]d\vec{\xi},$$

the expectation $\mathbf{E}[\cdot]$ *is taken over* x, *and* $\mathcal{F}^{-1}[\cdot]_{<\vec{\omega}>}$ *denotes* $\mathcal{F}^{-1}[\cdot]$ *being evaluated at* $\vec{\omega}$.

The proof of this lemma is simple. If we notice that

$$(\frac{1}{2\pi})^N \int_{R^N} e^{-i\xi^\top \vec{\omega}} \mathbf{E}\left[h(\vec{x}) e^{i\xi^\top \vec{x}}\right] d\vec{\xi} = h(\vec{\omega}) n[\omega; \vec{0}, \underline{\Sigma}] \ ,$$

then by differenciating both sides with respect to the elements of $\vec{\omega}$, we can obtain the result.

As the final step, by applying each term in the asymptotic expansion of the characteristic function, we can obtain the corresponding asymptotic expansion of the density function of $X_t^{(\delta)}$, which is denoted by $f_{X_t^{(\delta)}}$, as

$$\begin{aligned} f_{X_t^{(\delta)}} &\sim n[x; 0, \Sigma_{g_1}] + \delta\left[-\frac{\partial}{\partial x}\{h_2(x) n[x; 0, \Sigma_{g_1}]\}\right] \\ &+ \delta^2\left[-\frac{\partial}{\partial x}\{h_3(x) n[x; 0, \Sigma_{g_1}]\}\right] + \frac{1}{2}\delta^2\left[\frac{\partial^2}{\partial x^2}\{h_{22}(x) n[x; 0, \Sigma_{g_1}]\}\right] + \cdots . \end{aligned}$$

where

$$X_t^{(\delta)} = \frac{S^{(\delta)}(t) - S^{(0)}(t)}{\delta},$$

and the density function of the Gaussian distribution is given by

$$n[x; 0, \Sigma_{g_1}] = \frac{1}{\sqrt{2\pi\Sigma_{g_1}}} \exp\left[-\frac{x^2}{2\Sigma_{g_1}}\right].$$

Using the polynomial functions of $h_2(x)$,$h_3(x)$, and $h_{22}(x)$, we can obtain more explicit form of the density function. For later use, we state our result as the following theorem.

Theorem 2.1 *An asymptotic expansion of the density function of* $X_t^{(\delta)} = [S^{(\delta)}(t) - S^{(0)}(t)]/\delta$ *as* $\delta \to 0$ *is given by*

$$\begin{aligned} (2.19) \quad f_{X_t^{(\delta)}} &= n[x; 0, \Sigma_{g_1}] \\ &+ \delta\left[\{\frac{c}{\Sigma_{g_1}} x^3 + (\frac{f}{\Sigma_{g_1}} - 2c) x\} n[x; 0, \Sigma_{g_1}]\right] \\ &+ \delta^2\left[\{\frac{c_2}{2\Sigma_{g_1}^2} x^6 + (\frac{f_2}{2\Sigma_{g_1}^2} - \frac{9c_2}{2\Sigma_{g_1}} + \frac{c_1}{\Sigma_{g_1}}) x^4\right. \\ &+ \left.(\frac{k_2}{2\Sigma_{g_1}^2} - \frac{5f_2}{2\Sigma_{g_1}} + \frac{f_1}{\Sigma_{g_1}} - 3c_1 + 6c_2) x^2 + (-f_1 - \frac{k_2}{2\Sigma_{g_1}} + f_2)\} n[x; 0, \Sigma_{g_1}]\right] \\ &+ o(\delta^2) . \end{aligned}$$

We again note that the asymptotic expansion we have obtained is formal in the proper mathematical sense and it may not be a valid expansion of the density function. However, we can give a rigorous mathematical validity as we shall discussed in Section 6. The mathematical devices to justify our approach are far from standard ones mainly because the continuous diffusion processes are involved.

2.2 A New Computational Method for Options

We next show how to evaluate plain vanilla options with the general volatility function by using an asymptotic expansion of the density function of $X_t^{(\delta)}$ obtained previously. This is the simplest case which illustrates our approach to the valuation problems of more complicated contingent claims. The payoffs of plain vanilla options are defined as

$$V(T) = (S(T) - K)^+$$

or

$$V(T) = (K - S(T))^+ \ ,$$

where $(X)^+ = \max(X, 0)$. We shall use the martingale technique which has been standard in financial economics and the value at the initial date is given by

$$V(0) = e^{-rT}\mathbf{E}^*[V(T)] \tag{2.20}$$

where the expectation is taken under the equivalent martingale measure. Because we have imposed a restriction on the drift function in the Black-Scholes economy, this expectation is the same as the expectation operator $E(\cdot)$ we have already used.

In the following, we only consider the asymptotic expansion of a call option because that of a put option is obtained in the similar manner. We note that by using $X_T^{(\delta)}$, the $V(T)$ can be expressed as

$$V(T) = \delta\left[\frac{S^{(0)}(T) - K}{\delta} + X_T^{(\delta)}\right]^+ = \delta\left[y^{(\delta)} + X_T^{(\delta)}\right]^+ \ , \tag{2.21}$$

where

$$y^{(\delta)} = \frac{S^{(0)}(T) - K}{\delta} \ .$$

In order to evaluate the terminal payoff function at the initial period, we need an assumption.

Assumption II : There exists a constant y such that

$$y^{(\delta)} = y + O(\delta).$$

The above condition means that we are considering the situation where the strike price is near $S^{(0)}(T) = e^{rT}S_0$. It corresponds to that we are considering the valuation of call options when the strike price is near the forward value of the underlying asset at the contracting period, that is, $K_\delta = S^{(0)}(T) - \delta y + O(\delta^2)$. For the notational convenience, we omit δ of the strike price K_δ and will use the notation K as before. This assumption could be relaxed to a certain extent, but then there could be some more complications in the following analyses. We shall use the formulas $\mathbf{E}[g_2|g_1 = x] = cx^2 + f$, $\mathbf{E}[g_3|g_1 = x] = c_1x^3 + f_1x$, and $\mathbf{E}[g_2^2|g_1 = x] = c_2x^4 + f_2x^2 + k_2$, where c, f, c_1, f_1, c_2, f_2 and k_2 are defined in the previous subsection. By substituting these formulas into the terminal payoff of the call options, we can obtain the initial value of the call option. The resulting expression is given by

$$\begin{aligned}
V(0) &= e^{-rT}\delta\mathbf{E}[(y^{(\delta)} + X_T^{(\delta)})^+] \\
&= e^{-rT}\delta\left[y^{(\delta)}\int_{-y^{(\delta)}}^{\infty} f_{X_T^{(\delta)}}(x)dx + \int_{-y^{(\delta)}}^{\infty} x f_{X_T^{(\delta)}}(x)dx\right] \\
&\sim e^{-rT}\delta\Bigg[(y + a_1\delta + a_2\delta^2)\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} n[x;0,\Sigma_{g_1}]dx \\
&+ \delta(y + a_1\delta + a_2\delta^2)\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} \frac{-\partial\{(cx^2+f)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\
&+ \delta^2(y + a_1\delta + a_2\delta^2)\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} \frac{-\partial\{(c_1x^3+f_1x)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\
&+ \frac{1}{2}\delta^2(y + a_1\delta + a_2\delta^2)\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} \frac{\partial^2\{(c_2x^4+f_2x^2+k_2)n[x;0,\Sigma_{g_1}]\}}{\partial x^2}dx \\
&+ \int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} xn[x;0,\Sigma_{g_1}]dx + \delta\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} x\frac{-\partial\{(cx^2+f)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\
&+ \delta^2\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} x\frac{-\partial\{(c_1x^3+f_1x)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\
&+ \frac{1}{2}\delta^2\int_{-(y+a_1\delta+a_2\delta^2)}^{\infty} x\frac{\partial^2\{(c_2x^4+f_2x^2+k_2)n[x;0,\Sigma_{g_1}]\}}{\partial x^2}dx\Bigg]
\end{aligned}$$

where we use $y^{(\delta)} \sim y + a_1\delta + a_2\delta^2$ for some constants y, a_1, and a_2. In the following theorem, we present a more explicit formula upto the third order which may be convenient to evaluate the value of the call option. [1]

[1] In numerical computation of section 4, given $S^{(0)}(T)$, δ and K, we substitute actual $y^{(\delta)}$ for y and set $a_1 = a_2 = 0$ in the formula, which shows enough accuracy for practical purpose.

Theorem 2.2 *Under Assumption II, an asymptotic expansion of the price of a call option with the general volatility function is given by*

$$
\begin{aligned}
(2.22)\qquad V(0) = e^{-rT} \Bigg[\delta \left(yN(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) + \Sigma_{g_1} n[y;0,\Sigma_{g_1}] \right) \\
+ \quad \delta^2 \left((c\Sigma_{g_1} + f + a_1)N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - cy\Sigma_{g_1}\ n[y;0,\Sigma_{g_1}] \right) \\
+ \quad \delta^3 \Bigg(a_2 N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) + \{c_1(2\Sigma_{g_1}^2 + y^2\Sigma_{g_1}) + f_1\Sigma_{g_1} + \frac{1}{2}(c_2y^4 + f_2y^2 + k_2) \\
+ \quad (\frac{1}{2}a_1^2 + ca_1y^2 + a_1 f)\} n[y;0,\Sigma_{g_1}] \Bigg) \Bigg] + o(\delta^3).
\end{aligned}
$$

The proof is a result of the straightforward calculation from the previous equation for V(0). In order to derive (2.22) and to evaluate the integrals which would appear in the coefficients of $\delta^k, k \geq 4$, the following formulae are useful. We omit the proofs because they are easily obtained by the repeated applications of integration by parts operations.

$$
\begin{aligned}
\int_{-y}^{\infty} xn[x;0,\Sigma_{g_1}]dx &= \Sigma_{g_1} n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^2 n[x;0,\Sigma_{g_1}]dx &= \Sigma_{g_1} N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - y\Sigma_{g_1} n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^3 n[x;0,\Sigma_{g_1}]dx &= (2\Sigma_{g_1}^2 + \Sigma_{g_1} y^2) n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^4 n[x;0,\Sigma_{g_1}]dx &= 3\Sigma_{g_1}^2 N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - (3\Sigma_{g_1}^2 y + \Sigma_{g_1} y^3) n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^5 n[x;0,\Sigma_{g_1}]dx &= (8\Sigma_{g_1}^3 + 4\Sigma_{g_1}^2 y^2 + \Sigma_{g_1} y^4) n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^6 n[x;0,\Sigma_{g_1}]dx &= 15\Sigma_{g_1}^3 N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - (15\Sigma_{g_1}^3 y + 5\Sigma_{g_1}^2 y^3 + \Sigma_{g_1} y^5) n[y;0,\Sigma_{g_1}]\ , \\
\int_{-y}^{\infty} x^7 n[x;0,\Sigma_{g_1}]dx &= (48\Sigma_{g_1}^4 + 24\Sigma_{g_1}^3 y^2 + 6\Sigma_{g_1}^2 y^4 + \Sigma_{g_1} y^6) n[y;0,\Sigma_{g_1}]\ .
\end{aligned}
$$

3 Applications

Our new computational method previously presented is so general that it can be applicable to various problems in the valuation of contingent claims in the unified manner. To demonstrate this, we show three examples in this section, the pricing of basket options, average options, and options with stochastic volatilities. It has been known that the explicit formulas are hardly obtainable for these problems even when the underlying asset price follows the geometric Brownian motion case as in the original Black-Scholes model. On the other hand, all examples in this section are evaluated under the assumption that underlying assets and factors follow the general class of Markov processes, which are not necessarily time homogeneous.

3.1 Basket Options

We consider the pricing of basket options (including so called "spread" options) which is a natural extension of plain vanilla options by using our method. We formally define "basket" $I(t)$ as

$$I(t) = \sum_{j=1}^{N} \alpha_j S_j(t) \tag{3.23}$$

where $S_j(t)$ denotes the the price of the j th risky asset which is a component of the basket. We note that, as a special case, the "spread" is defined by $\alpha_{j1} = 1$, $\alpha_{j2} = -1$, $j2 \neq j1$ and $j = 0$ for $j \neq j1, j2$ in $I(t)$. That is,

$$I(t) = S_{j1}(t) - S_{j2}(t).$$

Then the payoffs of the basket options are expressed as

$$V(T) = (I(T) - K)^{+}$$

or

$$V(T) = (K - I(T))^{+}$$

. In what follows, we consider call options and we set $V(T) = (I(T) - K)^{+}$. For the present pricing problem, we consider the Black-Sholes economy where there are risky assets, each of which may depend on N independent Brownian motions.

$$dS_j^{(\delta)}(t) = rS_j^{(\delta)}(t)dt + \delta \sum_{i=1}^{N} \sigma_i(S_j^{(\delta)}(t), t)d\tilde{w}_i(t) \ , \tag{3.24}$$

where

$$dB = rBdt \ ,$$

r is a positive constant, and $0 < \delta \leq 1$.
In this formulation we note that δ can differ in j and can be δ_i, but then we redefine δ such that $\delta = min[\delta_i]_i$ and we have the same expression of the processes as above. Also we shall use the notations, $\sigma_{ij}^{\delta}(t)$ and $\partial_j \sigma_{ij}^{0}(t)$ for

$\sigma_i(S_j^{(\delta)}(t),t)$, and $\frac{\partial \sigma_i(S_j^{(\delta)},t)}{\partial S_j^{(\delta)}}|_{S_j^{(\delta)}=S_j^{(0)}}$, respectively. Then, following the steps in the previous subsection, for each j we define

$$
\begin{aligned}
S_j^{(0)}(t) &\equiv \lim_{\delta\to 0} S_j^{(\delta)}(t) = e^{rt}S_j(0) \\
A_j(t) &\equiv \frac{\partial S_j^{(\delta)}(t)}{\partial \delta}|_{\delta=0} = \int_0^t e^{r(t-s)} \sum_{i=1}^N \sigma_{ij}^0(t) d\tilde{w}_i(s) \\
B_j(t) &\equiv \frac{\partial^2 S_j^{(\delta)}(t)}{\partial \delta^2}|_{\delta=0} = 2\int_0^t e^{r(t-s)} \sum_{i=1}^N \partial_j \sigma_{ij}^0(s) A_j(s) d\tilde{w}_i(s)
\end{aligned}
$$

and

$$
\begin{aligned}
C_j(t) &\equiv \frac{\partial^3 S_j^{(\delta)}(t)}{\partial \delta^3}|_{\delta=0} \\
&= 3\int_0^t e^{r(t-s)} \sum_{i=1}^N \partial_j^2 \sigma_{ij}^0(s) A_j(s)^2 d\tilde{w}_i(s) + 3\int_0^t e^{r(t-s)} \sum_{i=1}^N \partial_j \sigma_{ij}^0(s) B_j(s) d\tilde{w}_i(s) .
\end{aligned}
$$

Then, as in the previous subsection, we can obtain the asymptotic expansion of each risky asset as $S_j^{(\delta)}(t)$ $(j = 1, 2, \cdots, N)$.

$$
S_j^{(\delta)}(t) = S_j^{(0)}(t) + \delta A_j(t) + \delta^2 \frac{B_j(t)}{2} + \delta^3 \frac{C_j(t)}{6} \cdots .
$$

Because the "basket" $I^{(\delta)}(t)$ is a linear combination of finite number of risky assets $S_j^{(\delta)}(t)$ $(j = 1, 2, \cdots, N)$, we can easily obtain the asymptotic expansion of the random variable of the basket :

$$
I^{(\delta)}(t) \sim \sum_{j=1}^N \alpha_j S_j^{(0)}(t) + \delta \sum_{j=1}^N \alpha_j A_j(t) + \frac{\delta^2}{2} \sum_{j=1}^N \alpha_j B_j(t) + \frac{\delta^3}{6} \sum_{j=1}^N \alpha_j C_j(t) + \cdots .
$$

Next, as in the previous subsection, we define $X^{(\delta)}(t)$ for which we can explicitly obtain the density function. Let

$$
X^{(\delta)}(t) \equiv \frac{I^{(\delta)}(t) - I^{(0)}(t)}{\delta} \sim g_1 + \delta g_2 + \delta^2 g_3 + \cdots \tag{3.25}
$$

where

$$
\begin{aligned}
g_1 &= \int_0^t e^{r(t-s)} \vec{\sigma}_I^{(0)}(s)^\top d\vec{\tilde{w}}(s), \\
g_2 &= e^{rt} \sum_{j=1}^N \alpha_j \int_0^t \left[\int_0^s e^{-rv} \vec{\sigma}_j^0(v)^\top d\vec{\tilde{w}}(v)\right] \partial_j \vec{\sigma}_j^0(s)^\top d\vec{\tilde{w}}(s) .
\end{aligned}
$$

$\vec{\sigma}_I^{(0)}(s)$, $\vec{\sigma}_j^0(s)$ and $\partial_j \vec{\sigma}_j^0(s)$ denote N dimensional vectors of which the i-th elements are $\sum_{j=1}^N \alpha_j \sigma_{ij}^0(s)$, $\sigma_{ij}^0(s)$ and $\partial_j \sigma_{ij}(s)$ respectively. Because the integrand of g_1 is a deterministic function, g_1 follows the Gaussian distribution and we can write

$$
g_1 \sim N(0, \Sigma_{g_1}) ,
$$

where

$$\Sigma_{g_1} \equiv \int_0^t e^{2r(t-s)} \vec{\sigma}_I^{(0)}(s)^\top \vec{\sigma}_I^{(0)}(s) ds \ .$$

For the pricing problem of the call basket options, we need to evaluate the conditional expectations such as $\mathbf{E}[g_2|g_1 = x]$, $\mathbf{E}[g_3|g_1 = x]$, and $\mathbf{E}[g_2^2|g_1 = x]$. By using formulae in *Lemma 2.1*, we can evaluate those expectations. For example, by applying $formula(1)$, we obtain

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \ , \tag{3.26}$$

where

$$\begin{aligned}
c &= e^{rt} \sum_{j=1}^N \alpha_j c_j \ , \\
f &= e^{rt} \sum_{j=1}^N \alpha_j f_j \ , \\
c_j &= \frac{1}{\Sigma_{g_1}^2} e^{2rt} \int_0^t \left[\int_0^s e^{-2rv} \vec{\sigma}_I^{(0)}(v)^\top \vec{\sigma}_j^0(v) dv \right] e^{-rs} \vec{\sigma}_I^{(0)}(s)^\top \partial_j \vec{\sigma}_j^0(s) ds \ , \\
&\text{and} \\
f_j &= -\Sigma_{g_1} c_j \ .
\end{aligned}$$

Therefore, by applying the pricing formula in *Theorem 2.2* and replacing $y^{(\delta)}$ by $y^{(\delta)} \equiv [I^{(0)}(T) - K]/\delta$, we can obtain the initial value of the basket (call) option.

3.2 Average Options

We next consider as a more complicated example the average options commonly known as "Asian Options" in the simple Black-Scholes economy with the general volatility function. The payoffs of the average options are defined as

$$V(T) = (Z^{(\delta)}(T) - K)^+ \tag{3.27}$$

or

$$V(T) = (K - Z^{(\delta)}(T))^+ \ ,$$

where

$$Z^{(\delta)}(T) = \frac{1}{T} \int_0^T S^{(\delta)}(t) dt \ .$$

Then the value of average options at the initial date can be again expressed as

$$V(0) = e^{-rT} \mathbf{E}[V(T)].$$

In what follows, we evaluate the call options as an example. In the average option case, we consider the asymptotic expansion of the functional of the risky asset $Z^{(\delta)}(T)$ and we have

$$Z^{(\delta)}(T) = Z^{(0)}(T) + \frac{1}{T}\int_0^T A(t)dt + \delta\frac{1}{T}\int_0^T \frac{B(t)}{2}dt\int_0^T \frac{C(t)}{6}dt + \cdots \ , \tag{3.28}$$

where

$$Z^{(0)}(T) \equiv \lim_{\delta\to 0} Z^{(\delta)}(T) = \frac{1}{T}\int_0^T S^{(0)}(t)dt \ .$$

By defining $X_T^{(\delta)}$ for which we obtain the density function as

$$X_T^{(\delta)} = \frac{Z^{(\delta)}(T) - Z^{(0)}(T)}{\delta},$$

the the stochastic expansion of $X_T^{(\delta)}$ is given by

$$X_T^{(\delta)} = g_1 + \delta g_2 + \delta^2 g_3 + \cdots \ ,$$

where

$$g_1 = \int_0^T A(t)dt,$$

$$g_2 = \int_0^T \frac{B(t)}{2}dt,$$

and

$$g_3 = \int_0^T \frac{C(t)}{6}dt \ .$$

We note that the leading term in the average options case also can be written as

$$g_1 = \int_0^T \frac{1}{T}\left[\frac{e^{r(T-s)}-1}{r}\right]\sigma(S^{(0)},s)d\tilde{w}(s) \ . \tag{3.29}$$

Because of the same reason as before, g_1 follows a normal distribution. The variance of the Gaussian random variable can be calculated as

$$g_1 \sim N(0, \Sigma_{g_1}) \ , \tag{3.30}$$

where

$$\Sigma_{g_1} = \int_0^T \frac{1}{T^2}\left[\frac{e^{r(T-s)}-1}{r}\right]^2 \sigma(S^{(0)},s)^2 ds \ .$$

By following the same method as in the previous sebsection, we can express the asymptotic expansion of the density function as

$$\begin{aligned} f_{X_T^{(\delta)}} \ \sim \ & n[x;0,\Sigma_{g_1}] + \delta\frac{-\partial\{\mathbf{E}[g_2|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x} \\ + \ & \delta^2\frac{-\partial\{\mathbf{E}[g_3|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x} + \frac{1}{2}\delta^2\frac{\partial^2\{\mathbf{E}[g_2^2|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x^2} + \cdots \ . \end{aligned}$$

Then, we need to evaluate the conditional expectations such as $\mathbf{E}[g_2|g_1 = x]$, $\mathbf{E}[g_3|g_1 = x]$, and $\mathbf{E}[g_2^2|g_1 = x]$.
For instance, noting $g_2 = \frac{1}{T}\int_0^T \int_0^t e^{rt}\partial\sigma(S^{(0)}, s) \int_0^s e^{-rv}\sigma(S^{(0)}, v)d\tilde{w}(v)d\tilde{w}(s)dt$, and using the $formula(1)$ in *Lemma 2.1*, we can show

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \ , \tag{3.31}$$

where

$$\begin{aligned} c &= \frac{1}{\Sigma_{g_1}^2}\frac{1}{T^3}\int_0^T \int_0^t e^{rt}\left[\frac{e^{r(T-s)}-1}{r}\right]\sigma(S^{(0)}, s)\partial\sigma(S^{(0)}, s) \\ &\times \int_0^s e^{-rv}\left[\frac{e^{r(T-v)}-1}{r}\right]\sigma(S^{(0)}, v)^2 dvdsdt \ , \end{aligned} \tag{3.32}$$

and

$$f = -c\Sigma_{g_1} \ .$$

3.3 Options with a Stochastic Volatility

The third example in this section is the evaluation problem of options with stochastic volatilities which He (1992) has developed in an equilibrium framework. We assume that there exists a risk-free rate, r which is a positive constant. In general, under the equivalent martingale measure, the processes of the underlying asset and a state variable in He (1992) are defined by

$$dS_1^{(\delta)}(t) = \mu_1(S_1^{(\delta)}, Y^{(\delta)}, t)dt + \delta\vec{\sigma_1}(S_1^{(\delta)}, Y^{(\delta)}, t)^{\top}d\vec{\tilde{w}}_t \tag{3.33}$$

and

$$dY^{(\delta)}(t) = \mu_2(S_1^{(\delta)}, Y^{(\delta)}, t)dt + \delta\vec{\sigma_2}(S_1^{(\delta)}, Y^{(\delta)}, t)^{\top}d\vec{\tilde{w}}_t \ , \tag{3.34}$$

where $0 < \delta \leq 1$, $\vec{\tilde{w}}_t$ is the standard two dimensional Brownian motion, and $\vec{\sigma_1}(S_1^{(\delta)}, Y^{(\delta)}, t)$ and $\vec{\sigma_2}(S_1^{(\delta)}, Y^{(\delta)}, t)$ denote two dimensional vectors. By using vector representation, two equations can be rewritten as

$$d\vec{S}_t^{(\delta)} = \vec{\mu}(S_1^{(\delta)}, Y^{(\delta)}, t)dt + \delta\underline{\Sigma}(S^{(\delta)}, Y^{(\delta)}, t)d\vec{\tilde{w}}_t \ , \tag{3.35}$$

where

$$\vec{S}_t^{(\delta)} = \begin{bmatrix} S_{1t}^{(\delta)} \\ Y_t^{(\delta)} \end{bmatrix},$$

$$\vec{\mu}(S_1^{(\delta)}, Y^{(\delta)}, t) = \begin{bmatrix} \mu_1(S_1^{(\delta)}, Y^{(\delta)}, t) \\ \mu_2(S_1^{(\delta)}, Y^{(\delta)}, t) \end{bmatrix},$$

and

$$\underline{\Sigma}(S^{(\delta)}, Y^{(\delta)}, t) = \left[\begin{array}{c} \vec{\sigma_1}(S^{(\delta)}, Y^{(\delta)}, t)^\top \\ \vec{\sigma_2}(S^{(\delta)}, Y^{(\delta)}, t)^\top \end{array} \right].$$

In fact, we know that

$$\mu_1(S_1^{(\delta)}, Y^{(\delta)}, t) = rS_{1t}^{(\delta)} .$$

Next, we define a non-singular matrix, $\underline{G}_t$ which satisfies a (deterministic) differential equation:

(3.36) $$d\underline{G}_t = \partial\underline{\mu}^0 \underline{G}_t dt ,$$

where

$$\partial\underline{\mu}^0 = \left[\begin{array}{cc} \partial_1\mu_1^0 & \partial_2\mu_1^0 \\ \partial_1\mu_2^0 & \partial_2\mu_2^0 \end{array} \right],$$

$$\partial_i\mu_j^0 \equiv \frac{\partial\mu_j^\delta}{\partial S_i^{(\delta)}}|_{\delta=0}, \text{ and } S_2^{(\delta)} \equiv Y^{(\delta)}.$$

In order to derive the asymptotic expansion of $\vec{S}_t^{(\delta)}$, first $\vec{S}_t^{(0)}$ is defined so that this solves the differential equation :

(3.37) $$d\vec{S}_t^{(0)} = \vec{\mu}(S^{(0)}, Y^{(0)}, t)dt .$$

Next, we define $\vec{A}_t$ as

$$\vec{A}_t = \frac{\partial\vec{S}_t^{(\delta)}}{\partial\delta}|_{\delta=0}$$

$\vec{A}_t$ must satisfy the following stochastic differential equation :

$$d\vec{A}_t = \partial\underline{\mu}^0 \vec{A}_t dt + \underline{\Sigma}^0 d\vec{\tilde{w}}_t .$$

This stochastic differential equation can be solved as

(3.38) $$\vec{A}_t = \underline{G}_t \int_0^t \underline{G}_s^{-1} \underline{\Sigma}^0 d\vec{\tilde{w}}_t.$$

Third, we define $\vec{B}_t$ as

$$\vec{B}_t = \frac{\partial^2\vec{S}_t^{(\delta)}}{\partial\delta^2}|_{\delta=0}.$$

Then we can show that $\vec{B}_t$ must satisfy the following stochastic differential equation :

$$d\vec{B}_t = \left[\sum_{i=1}^2 \sum_{j=1}^2 \partial_i\partial_j\vec{\mu^0} A_{it}A_{jt} + \sum_{i=1}^2 \partial_i\vec{\mu^0} B_{it} \right] dt + 2\sum_{i=1}^2 \partial_i\underline{\Sigma}^0 A_{it} d\vec{\tilde{w}}_t .$$

This equation also can be solved as

$$\vec{B}_t = \int_0^t \underline{G}_t \underline{G}_s^{-1} \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \partial_i \partial_j \vec{\mu}^0 A_{is} A_{js} \right] ds$$
$$+2 \int_0^t \underline{G}_t \underline{G}_s^{-1} \left[\sum_{i=1}^{2} \partial_i \underline{\Sigma}^0 A_{is} \right] d\vec{\tilde{w}}_s \, .$$

Hence, as in the previous sections, the stochastic expansion of $\vec{S}_t^{(\delta)}$ is given by

$$\vec{S}_t^{(\delta)} = \vec{S^{(0)}} + \delta \vec{g_1} + \delta^2 \vec{g_2} + \cdots \, , \tag{3.39}$$

where $\vec{g_1} = \vec{A}_t$, and $\vec{g_2} = \frac{1}{2}\vec{B}_t$. In this case the two dimensional leading term follows the two dimensional Gaussian distribution with

$$\vec{g_1} \sim N(\vec{0}, \underline{\Sigma}_{g_1}) \, , \tag{3.40}$$

$$\underline{\Sigma}_{g_1} \equiv \underline{G}_t \int_0^t \left[\underline{G}_s^{-1} \underline{\Sigma}^0 \underline{\Sigma}^{0\top} \underline{G}_s^{-1\top} \right] ds \underline{G}_t^{\top} \, .$$

To evaluate a call option, we first define $\vec{X_t^{(\delta)}} \equiv \frac{\vec{S}_t^{(\delta)} - \vec{S}_t^{(0)}}{\delta}$, and derive the asymptotic expansion of the density function of $\vec{X_T^{(\delta)}}$ by utilizing the formulas of *Lemma 2.1* and *Lemma 2.2*, and then we compute $e^{-rT}\delta \mathbf{E}[(X_{1T}^{(\delta)} + y^{(\delta)})^+]$ where $X_{1T}^{(\delta)} = \frac{S_{1T}^{(\delta)} - S_{1T}^{(0)}}{\delta}$ and $y^{(\delta)} = \frac{S_{1T}^{(0)} - K}{\delta}$. The detail is omitted since the similar argument can be applied as in the previous examples.

4 Numerical Examples

In this section, we will present several numerical investigations of the approximations derived by the asymptotic expansion method introduced and explained in the previous sections. In the Black-Scholes economy, as the first numerical example we will present the numerical examples of plain vanilla call options for the square-root process [2] of the underlying asset. As the second example, we will give numerical results on of average call options for the square-root process of the underlying asset as well as for the log-normal process of the underlying asset. In the Black Scholes economy the latter process has been commonly used in

[2] We have used this because it is a typical time-homogenous diffusion example. However, the volatility function is not smooth at the origin and we need to use a smoothed version of the square root process at the origin for the mathmatical point of view. (See the conditions (6.48) in Section 6, for instance.) However, we can show that the smoothing does not make significant differences and the effects are negligible in the *small disturbance asymptotic theory*.

practice. Under the equivalent martingale measure, we assume that the processes of the underlying asset are given by

$$dS^{(\delta)} = (r-q)S^{(\delta)}dt + \delta(S^{(\delta)})^{\frac{1}{2}}d\tilde{w}_t \tag{4.41}$$

or

$$dS^{(\delta)} = (r-q)S^{(\delta)}dt + \delta S^{(\delta)}d\tilde{w}_t \ , \tag{4.42}$$

where r and q denote the risk-free interest rate and a dividend yield, respectively, both of which are assumed to be positive constants in the Black-Scholes economy, and $\tilde{w}_t$ denotes the one dimensional Brownian motion.

Tables 1-3 show the numerical values of plain vanilla call options for the square-root processes of the underlying asset which represents an equity index with no dividend. The values obtained by the stochastic expansions upto the first and second order are given respectively. For the comparative purpose, the values by the Monte Carlo simulations are also given, which are based on 500,000 trials implemented in each case. We note that all the "difference" or "difference rate" appearing in Tables 1-14 are those from, or those relative to, the corresponding values by the Monte Carlo simulations. The spot prices and the risk-free interest rate are assumed to be 40.00 and 5 % respectively, and the term to expiry is assumed to be one year. The volatilities δ are set so that the instantaneous variances at time 0 are equivalent to those of the log-normal process whose volatilities are 10 % in Table 1, 20 % in Table 2, and 30 % in Table 3. The values of out-of-the money (strike price K= 45), at-the-money (K=40), and in-the-money (K=35) are given. We observe that the values obtained by the stochastic expansions upto the second order are improved and more accurate than those with the first order.

Tables 4-10 show the numerical values of the average call options when the underlying assets follow square-root processes, where the underlying asset is an equity index with no dividend (that is, $q = 0$) in Tables 4-6 and it is the foreign exchange rate of Japanese yen and US dollar (that is, q is a US Interest rate) in Table 7-10. The results given by the stochastic expansion are those from the computation upto the second order. For the comparative purpose, the values by the Monte Carlo simulations are also shown, which are based on 500,000 trials implemented in each case.

In Tables 4-6, the spot prices and the risk-free interest rate are assumed to be 40.00 and 5 %, respectively, and the volatilities (δ) are set so that the instantaneous variances at time 0 are equivalent to those of log-normal process where the volatilities are 30 %. The vales of out-of-the money (strike price K= 45), at-the-money (K=40), and in-the-money (K=35) are shown for each of the time to maturities : three months, six months and one year.

In Tables 7-10, the spot prices, the risk-free Japanese interest rate, and the US interest rate are assumed to be 100.00, 3 %, and 5 % respectively. The volatilities(δ) are set so that the instantaneous variances at time 0 are equivalent to those of the log-normal process where the volatilities are 10 % in Tables 7-9 and 30 % in Table 10. The values of out-of-the money (K= 105 for Tables 7-9 and K=110 for Table 10), at-the-money (K=100 for Tables 7-10), and in-the-money (K=95 for Tables 7-9 and K=90 for Table 10) are shown for each of the time to maturities : three months, six months, and one year.

Tables 11-14 show the numerical values of average call options when the underlying assets follow log-normal processes, where the underlying asset is the foreign exchange rate of Japanese yen and US dollar. The assumptions for the spot prices, the risk-free Japanese, and US interest rates are same as in Tables 7-10. The volatilities are assumed to be 10 % in Tables 11-13, and 30 % in Table 14. The values of out-of-the money (K= 105 for Tables 11-13 and K=110 for Table 14), at-the-money (K=100 for Tables 11-14), and in-the-money (K=95 for Tables 11-13 and K=90 for Table 14) are shown for each of the time to maturities : three months, six months, and one year. The results given by the asymptotic expansion are those from the approximations upto the second order as well as from computation upto the first order. We observe that the values from the asymptotic expansion upto the second order are much more improved than those upto the first order. Figure 1 shows the difference of the distributions of the $X_T^{\delta}/\Sigma_{g_1}^{0.5}$ obtained by the asymptotic expansions from those obtained by the Monte Carlo simulations. We can observe that the difference is significantly smaller in the asymptotic expansions upto the second order than those upto the first order, which leads to the much improved values of the option prices. For the comparative purpose, the values by the Monte Carlo simulations are shown, which are based on 500,000 trials implemented in each case, and moreover, the values obtained by the **PDE** method developed in He and Takahashi (1996) are given.

5 Concluding Remarks

In this paper we have proposed a new methodology for the valuation problems of financial contingent claims when the underlying asset prices follow the general class of continuous Markov processes. The method is applicable to a wide range of the valuation problems of contingent claims associated with stocks, foreign exchange rates, and those in a stochastic interest rate enviroment. We illustrate the method by giving a series of examples when the processes of underlying assets and economic factors are described by the general stochastic differential equations of Markovian type. In a companion of this paper, Kunitomo and Takahashi (1995) have systematically presented the results of our asymptotic expansion approach in a non-Markovian setting of term structure of interest rates, whic was originally developed by Heath, Jarrow, and Morton (1992). Also it is even possible to extend our approach to the valuation of the average option on foreign exchange rates in the stochastic interest rates economy. [3]

As we have seen in Section 3, the formulae for the various pricing problems of contingent claims are simple analytic functions based on the Gaussian kernel, which can be evaluated quite easily. As we have shown in Section 4, the approximations we have obtained are numerically accurate for practical purpose in many cases. Also as we shall discuss in Section 6, our method is not an *ad hoc* approximation because we have developed a rigorous mathematical theory for the validity of our asymptotic expansion method, which is basically along the line of the Malliavin-Watanabe-Yoshida theory. The latter theory has been recently investigated by probabilists and our asymptotic theory can be regarded as their natural application. Therefore *the Small Disturbance Asymptotic Theory* and the resulting asymptotic expansions we are proposing in this paper does have not only practical usefulness but also may have some theoretical interest in finance.

6 Mathematical Appendix: Validity of the Asymptotic Expansion Approach

The mathematical validity of the asymptotic expansion approach in this paper can be given along the line based on the remarkable work by Watanabe (1987) on the Malliavin calculus in stochastic analysis. Yoshida (1992a,b) have utilized the results and method originally developed by Watanabe (1987) and given some useful results on the validity of the asymptotic expansions of some functionals on continuous time homogenous diffusion processes. The validity of our method can be obtained by the similar arguments used by Yoshida (1992a,b, 1997) and Chapter V of Ikeda and Watanabe (1989) but with some modifications. Since the rigorous proofs of our claims in this section can be quite lengthy but some parts are straightforward extensions of the existing results in stochastic analysis, we shall only give their essentials and the modifications we need for our applications

[3] Takahashi (1995) has dealt with this case for the multi-countries economy with interest rates and derive some useful formulae in the framwork of Heath-Jarrow-Morton interest model.

in the previous sections. [4] The main aim in the following steps will be to check the truncated version of the non-degeneracy condition for the Malliavin-covariance in our situation and show the sufficient conditions of Theorem 2.2 of Yoshida (1992b).

First, we shall prepare some notations. For this purpose, we shall freely use the notations by Ikeda and Watanabe (1989) as a standard textbook. We shall only discuss the validity of the asymptotic expansion approach based on the one-dimensional Wiener space without loss of generality. We only need more complicated notations in the general case. (See Ikeda and Watanabe (1989) for the details.) Let $(\boldsymbol{W}, P)$ be the $1-$dimensional Wiener space and let $\boldsymbol{H}$ be the Cameron-Martin subspace of $\boldsymbol{W}$ endowed with the norm

$$|h|_H^2 = \int_0^T |\dot{h}(t)|^2 dt \tag{6.43}$$

for $h \in \boldsymbol{H}$. The norm of $\boldsymbol{R}-$valued Wiener functional g for any $s \in \boldsymbol{R}$, and $p \in (1, \infty)$ is defined by

$$\|g\|_{p,s} = \|(I - \mathcal{L})^{s/2} g\|_p \ , \tag{6.44}$$

where $\mathcal{L}$ is the Ornstein-Uhlenbeck operator and $\|\cdot\|_p$ is the L_p-norm in the standard stochastic analysis. An $\boldsymbol{R}-$valued function $g : \boldsymbol{W} \mapsto \boldsymbol{R}$ is called an $\boldsymbol{R}-$valued polynomial functional if $g = p([h_1](w), \cdots, [h_n](w))$, where $n \in \boldsymbol{Z}^+, h_i \in \boldsymbol{H}$, $p(x_1, \cdots, x_n)$ is a polynomial, and

$$[h](w) = \int_0^T \dot{h}(t) dw(t)$$

for $h \in \boldsymbol{H}$ are defined in the sense of stochastic integrals.

Let $\boldsymbol{P}(\boldsymbol{R})$ denote the totality of $\boldsymbol{R}-$valued polynomials on the Wiener space $(\boldsymbol{W}, P)$. Then $\boldsymbol{P}(\boldsymbol{R})$ is dense in $L_p(\boldsymbol{R})$. The Banach space $\boldsymbol{D}_p^s$ is the completion of $\boldsymbol{P}(\boldsymbol{R})$ with respect to $\|\cdot\|_{p,s}$. The dual space of $\boldsymbol{D}_p^s$ is the $\boldsymbol{D}_q^{-s}$, where $s \in \boldsymbol{R}, p > 1$, and $1/p + 1/q = 1$. The space $\boldsymbol{D}^\infty = \cap_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^s$ is the set of Wiener functionals and $\tilde{\boldsymbol{D}}^{-\infty} = \cup_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^{-s}$ is a space of generalized Wiener functionals. For $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$, the derivative of F in the direction of h is defined by

$$D_h F(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(w + \varepsilon h) - F(w)\} \ . \tag{6.45}$$

[4] For complete discussions on mathematical validity of our approach, see Kunitomo and Takahashi (1998).

Then for $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$ there exists $DF \in \boldsymbol{P}(\boldsymbol{H})$ such that $D_hF(w) =< DF(w), h >_H$, where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$ and DF is called the $H-$derivative of F. It is known that the norm $\| \cdot \|_{p,s}$ is equivalent to the norm $\sum_{k=0}^{s} \|D^k \cdot \|_p$. For $F \in \boldsymbol{D}^{\infty}$, we can define the Malliavin-covariance by

$$\sigma(F) =< DF(w), DF(w) >_H \ , \tag{6.46}$$

where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$. It is known that the operator D can be well-defined in $\boldsymbol{D}^{\infty}$. (See Chapter V of Ikeda and Watanabe (1989) for the details.)

Now we give the proof of validity of our method. For the ease of exposition, we consider a one dimensional stochastic differentail equation and the validity of the multidimentional case could be obtained by the similar arguments with more complicated notations. For the fixed $T < 0$ and $\delta \in (0,1]$, we consider a stochastic differential equation :

$$S_T^{(\delta)} = S_0 + \int_0^T \mu(S_s^{(\delta)}, s)ds + \int_0^T \delta\sigma(S_s^{(\delta)}, s)d\tilde{w}_s \ , \tag{6.47}$$

where $\mu(S_s^{(\delta)}, s)$ and $\sigma(S_s^{(\delta)}, s)$ are $R \times [0,T] \to R$ and Borel measurable in $(S^{(\delta)}, s)$. We assume that they are $\boldsymbol{C}^{\infty}(R \to R)$ for $s \in [0,T]$ with bounded derivatives of any orders in the first arguments. That is, for the first argument there exists $M > 0$ such that

$$\begin{aligned} \sup_{S \in R, 0 \le s \le T} \left|\frac{\partial^k \mu(S_s, s)}{\partial S^k}\right| &< M \ , \\ \sup_{S \in R, 0 \le s \le T} \left|\frac{\partial^k \sigma(S_s, s)}{\partial S^k}\right| &< M \end{aligned} \tag{6.48}$$

for any $k = 1, 2, 3, \cdots$. We further assume that there exists a positve M' such that

$$\sup_{0 \le s \le T} [|\mu(0,s)| + |\sigma(0,s)|] < M'.$$

These conditions imply that there exists some positve K such that for all $s \in [0,T]$,

$$|\mu(S^{(\delta)}, s)| + |\sigma(S^{(\delta)}, s)| < K(1 + |S_s^{(\delta)}|), \tag{6.49}$$

$$|\mu(S1^{(\delta)}, s) - \mu(S2^{(\delta)}, s)| + |\sigma(S1^{(\delta)}, s) - \sigma(S2^{(\delta)}, s)| < K|S1^{(\delta)} - S2^{(\delta)}|. \tag{6.50}$$

By the standard argument (e.g. Ikeda and Watanabe (1989)) we have the existence of the unique strong solution which has continuous sample paths and is in

L_p for any $1 \le p < \infty$. In the remaining of the section, we will discuss the validity of the asymptotic expansion of $\phi(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)})$ where $X_T^{(\delta)}$ is defined by

$$X_T^{(\delta)} = \frac{S_T^{(\delta)} - S_T^{(0)}}{\delta}$$

and $\mathcal{B}$ is a Borel set. In the typical examples of European call options and put options, we take $\phi(x) = (x+y)$ and $\mathcal{B} = [-y, \infty)$ for call options, and $\phi(x) = (-x-y)$ and $\mathcal{B} = (-\infty, -y]$ for put options, where y is a constant.

Lemma 6.1 *Under the assumptions we have stated, $S_T^{(\delta)}$ is in $\boldsymbol{D}^\infty$ and has an asymptotic expansion*

$$S_T^{(\delta)} \sim S_T^{(0)} + \delta g_{1T} + \delta^2 g_{2T} + \cdots \tag{6.51}$$

in $\boldsymbol{D}^\infty$ as $\delta \downarrow 0$ with $g_{1T}, g_{2T}, \cdots \in \boldsymbol{D}^\infty$.

Proof: First, we shall prove $S_T^{(\delta)}$ in $\boldsymbol{D}^\infty$. Let us define $Y^{(\delta)}$ by

$$dY^{(\delta)} = \partial\mu(S^{(\delta)}, t)Y^{(\delta)}dt + \delta\partial\sigma(S^{(\delta)}, t)Y^{(\delta)}dw_t, Y_0^{(\delta)} = 1 \ ,$$

where $\partial\mu$ and $\partial\sigma$ denotes the $\frac{\partial\mu}{\partial S^{(\delta)}}$ and $\frac{\partial\sigma}{\partial S^{(\delta)}}$, respectively. Then we see that $Y^{(\delta)}$ has the unique strong solution and $Y^{(\delta)} \in L_p$. Let $W_t^\delta = Y_t^{(\delta)-1}$. Then, by using It$\hat{o}$'s Lemma, W_t^δ satisfies the stochastic differential equation

$$dW^\delta = -\{\partial\mu(S^{(\delta)}, t) - \delta^2\partial\sigma(S^{(\delta)}, t)^2\}W^\delta dt - \delta\partial\sigma(S^{(\delta)}, t)W^\delta dw_t, W_0^\delta = 1 \ .$$

Hence W_t^δ has also the unique strong solution and $Y^{(\delta)-1} \in L_p$.

In order to show our assertion, we calculate the first order H-derivative of $S_T^{(\delta)}$. For any $h \in H$, we note that $D_h S_T^{(\delta)}$ satisfies an stochastic integral equation :

$$D_h S_T^{(\delta)} = \int_0^T \delta\partial\sigma(S^{(\delta)}, s)D_h S_s^{(\delta)}dw(s) + \int_0^T \partial\mu(S^{(\delta)}, s)D_h S_s^{(\delta)}ds + \int_0^T \delta\sigma(S^{(\delta)}, s)\dot{h}_s ds.$$

Then for $h \in H$,

$$D_h S_T^{(\delta)} = \int_0^T Y_T^{(\delta)}Y_s^{(\delta)-1}\delta\sigma(S^{(\delta)}, s)\dot{h}_s ds.$$

Hence for the first order H-derivative we have

$$|DS_T^{(\delta)}|_H^2 = \int_0^T |Y_T^{(\delta)}Y_s^{(\delta)-1}\delta\sigma(S^{(\delta)}, s)|^2 ds.$$

We note

$$|DS_T^{(\delta)}|_H^2 \le \delta^2|Y_T^{(\delta)}|^2\left[\int_0^T |Y_s^{(\delta)-1}|^2K^2(1 + |S_s^{(\delta)}|)^2 ds\right].$$

Thus we have an inequality

$$\mathbf{E}\left[|DS_T^{(\delta)}|_H^2\right] \leq \delta^2 \mathbf{E}\left[|Y_T^{(\delta)}|^2 \{\int_0^T |Y_s^{(\delta)-1}|^2 K^2 (1+|S_s^{(\delta)}|)^2 ds\}\right] .$$

Likewise for any $2 < p < \infty$, we can show an inequality :

$$\mathbf{E}\left[|DS_T^{(\delta)}|_H^p\right] \leq (\delta K)^p T^{(\frac{p-2}{2})} \mathbf{E}\left[|Y_T^{(\delta)}|^p \{\int_0^T |Y^{(\delta)-1}|^p (1+|S_s^{(\delta)}|)^p ds\}\right].$$

In our evaluation we often use Hölder inequality for expectations:

$$\mathbf{E}[|x_s y_s|] \leq \mathbf{E}[|x_s|^p]^{\frac{1}{p}} \mathbf{E}[|y_s|^q]^{\frac{1}{q}}$$

for $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. By using an inequality

$$(|x| + |y|)^p \leq 2^{(p-1)}(|x|^p + |y|^p)$$

for $p \geq 1$ and Fubini's theorem, we can evaluate the right hand side of the last equation as

$$\begin{aligned}
& \mathbf{E}\left[|Y_T^{(\delta)}|^p \{\int_0^T |Y^{(\delta)-1}|^p (1+|S_s^{(\delta)}|)^p ds\}\right] \\
\leq \; & \mathbf{E}\left[|Y_T^{(\delta)}|^{2p}\right]^{\frac{1}{2}} \mathbf{E}\left[(\int_0^T \{|Y^{(\delta)-1}|(1+|S_s^{(\delta)}|)\}^p ds)^2\}\right]^{\frac{1}{2}} \\
\leq \; & \mathbf{E}\left[|Y_T^{(\delta)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \mathbf{E}\left[\int_0^T |Y^{(\delta)-1}|^{2p} (1+|S_s^{(\delta)}|)^{2p} ds\right]^{\frac{1}{2}} \\
\leq \; & \mathbf{E}\left[|Y_T^{(\delta)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \{\int_0^T \mathbf{E}\left[|Y^{(\delta)-1}|^{4p}\right]^{\frac{1}{2}} \mathbf{E}\left[(1+|S_s^{(\delta)}|)^{4p}\right]^{\frac{1}{2}} ds\}^{\frac{1}{2}} \\
\leq \; & \mathbf{E}\left[|Y_T^{(\delta)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \{\int_0^T \mathbf{E}\left[\{|Y^{(\delta)-1}|^{4p}\right]^{\frac{1}{2}} \mathbf{E}\left[2^{(4p-1)}(1+|S_s^{(\delta)}|^{4p})\right]^{\frac{1}{2}}\}^{\frac{1}{2}}.
\end{aligned}$$

Because $S_s^{(\delta)}, Y_s^{(\delta)}, Y_s^{(\delta)-1} \in L_p$ for $s \in [0,T]$ and any $1 < p < \infty$, we have $\mathbf{E}\left[|DS_T^{(\delta)}|_H^p\right] < \infty$ for any $p > 1$. Therefore, we conclude $S_T^{(\delta)} \in \cap_{1<p<\infty} \boldsymbol{D}_p^1$.

Repeating the similar arguments as for the first derivative, we can show the boundedness of higher order H-derivatives [5] with L_p estimates of $S_T^{(\delta)}$. Then we conclude $S^{(\delta)} \in \boldsymbol{D}^\infty$.

Next, we shall prove the second part of the lemma. The coefficients of the asymptotic expansion of $S_T^{(\delta)}$ is given by the Taylor formula. For instance,

$$g_{1T} \;=\; \int_0^T Y_T Y_s^{-1} \sigma(S^{(0)}, s) dw_s \, ,$$

[5] To be rigorous mathematically, we have to use Lemma 2.1 of Kusuoka and Strook (1982) and some related results for abstract spaces. See the proof of Theorem 3.1 of Kunitomo and Takahashi (1998) for the detail.

$$
\begin{aligned}
g_{2T} &= \int_0^T \frac{1}{2} Y_T Y_s^{-1}\{\partial^2\mu(S^{(0)},s)g_{1s}^2 ds + 2\partial\sigma(S^{(0)},s)g_{1s}dw_s\}\ , \\
\textit{and} & \\
g_{3T} &= \int_0^T Y_T Y_s^{-1}\{\partial^2\mu(S^{(0)},s)g_{1s}g_{2s}ds + \frac{1}{6}\partial^3\mu(S^{(0)},s)g_{1s}^3 ds \\
&+ \frac{1}{2}\partial^2\sigma(S^{(0)},s)g_{1s}^2 ds + \partial\sigma(S^{(0)},s)g_{2s}dw_s\}\ ,
\end{aligned}
$$

where $Y_t = Y_t^{(0)}$ is the solution of the differential equation

$$
dY = \partial\mu(S^{(0)},t)Ydt, Y_0 = 1.
$$

Thus we have $Y_t = \exp(\int_0^t \mu(S^{(0)},s)ds)$. By the boundedness of $Y_T, Y_s^{-1}, \sigma(S^{(0)},s)$ on $[0,T]$, it is easily seen $\mathbf{E}[|g_{1s}|^p] < \infty, s \in [0,T]$ for any $1 < p < \infty$. Given $g_{1s} \in L_p$, we can easily see by Burkholder's inequality (or local martingale inequality in *Theorem III-3.1* of Ikeda and Watanabe (1989)), $\mathbf{E}[|g_{2s}|^p] < \infty$ for any $1 < p < \infty$. Likewise, $g_{ks} \in L_p$ is obtained recursively given $g_{js} \in L_p, j = 1, 2, \cdots k-1$. Hence $g_{1T}, g_{2T}, \cdots \in \cap_{1<p<\infty} \boldsymbol{D}_p^1$.
Next, we note $D_h g_{1T} = Y_T \int_0^T Y_s^{-1}\sigma(S^{(0)},s)\dot{h}_s ds$ and $D^k_{h_1,\cdots,h_k} g_1 = 0$ for $k = 2, 3, \cdots$. Thus we can show $g_{1T} \in \boldsymbol{D}^\infty$. We also have

$$
\begin{aligned}
D_h g_{2T} &= Y_T\{\int_0^T Y_s^{-1}\partial^2\mu(S^{(0)},s)g_{1s}D_h g_{1s}ds + \int_0^T Y_s^{-1}\partial\sigma(S_s^{(0)},s)D_h g_{1s}dw_s \\
&+ \int_0^T Y_s^{-1}\partial\sigma(S^{(0)},s)\dot{h}_s ds\}, \\
D^2_{h_1,h_2} g_{2T} &= Y_T\{\int_0^T Y_s^{-1}\partial^2\mu(S^{(0)},s)D_{h_1}g_{1s}D_{h_2}g_{1s}ds + \int_0^T Y_s^{-1}\partial\sigma(S^{(0)},s)D_{h_1}g_{1s}\dot{h}_{2s}ds\}.
\end{aligned}
$$

and $D^k_{h_1,\cdots,h_k} g_{2T} = 0$ for $k = 3, 4, \cdots$. Then, given $g_{1s} \in \boldsymbol{D}^\infty$ for any $s \in [0,T]$, we can conclude $g_{2T} \in \boldsymbol{D}^\infty$.

By using similar arguments, recursively we can show the L_p-boundedness of any order H-derivatives of $g_{kT}, k = 3, 4, \cdots$. Therefore, we have proven the second part. $Q.E.D.$

Next, we consider the normalized random variable $X_T^{(\delta)}$ as $X_T^{(\delta)} = \frac{S_T^{(\delta)} - S_T^{(0)}}{\delta}$. By Lemma 6.1, we see $X_T^{(\delta)}$ is in $\boldsymbol{D}^\infty$ and has the asymptotic expansion :

$$
X_T^{(\delta)} \sim g_{1T} + \delta g_{2T} + \cdots
$$

in $\boldsymbol{D}^\infty$ as $\delta \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^\infty$. We also have the first order H-derivative as

$$
D_h X_T = \int_0^T Y_T^{(\delta)} Y_s^{(\delta)-1}\sigma(S^{(\delta)},s)\dot{h}_s ds.
$$

Hence the Malliavin covariance $\sigma(X_T^{(\delta)}) =< DX_T^{(\delta)}, DX_T^{(\delta)} >_H$ is explicitly given by

$$\int_0^T \{Y_T^{(\delta)} Y_s^{(\delta)-1} \sigma(S^{(\delta)}, s)\}^2 ds. \tag{6.52}$$

Note

$$\sigma(X_T^{(\delta)}) \to \Sigma_{g_1} = \int_0^T \{Y_T Y_s^{-1} \sigma(S^{(0)}, s)\}^2 ds \tag{6.53}$$

as $\delta \downarrow 0$ where Σ_{g_1} denotes the variance of g_{1T}, which is the limiting random variable.

We consider the uniform non-degeneracy of Malliavin covarince, which is the important step of the application of *Theorem 2.2* of Yoshida (1992b). In order to do this application, we make the following assumption.

Assumption I' :

$$\Sigma_{g_1} = \int_0^T \{Y_T Y_s^{-1} \sigma(S^{(0)}, s)\}^2 ds > 0 \ . \tag{6.54}$$

Next, we define η_c^δ by for any $c > 0$,

$$\eta_c^\delta = c \int_0^T |Y_T^{(\delta)} (Y_s^{(\delta)})^{-1} \sigma(S_s^{(\delta)}) - Y_T Y_s^{-1} \sigma(S_s^{(0)})|^2 ds.$$

Then we have the following lemma.

Lemma 6.2 *Under Assumption I', the Malliavin covariance* $\sigma(X_T^{(\delta)})$ *is uniformly non-degenerate. That is, there exists* $c_0 > 0$ *such that for* $c > c_0$ *and any* $p > 1$,

$$\sup_{\delta \in (0,1]} \mathbf{E}\left[1_{\{\eta_c^\delta \le 1\}} \{det\ \sigma(X_T^{(\delta)})\}^{-p}\right] < \infty. \tag{6.55}$$

Proof : Let $\xi_{s,t}^\delta = Y_t^{(\delta)} (Y_s^{(\delta)})^{-1} \sigma(S_s^{(\delta)})$ and $\xi_{s,t} = Y_t Y_s^{-1} \sigma(S_s^{(0)})$.
Then, $|\eta_c^\delta| \le 1$ is equivalent to $\int_0^T |\xi_{s,T}^\delta - \xi_{s,T}|^2 ds \le \frac{1}{c}$.
Note

$$\begin{aligned} |\sigma(X_T^{(\delta)}) - \Sigma_{g_1}| &= |\int_0^T (\xi_{s,T}^\delta)^2 - (\xi_{s,T})^2 ds| \\ &\le \int_0^T |\xi_{s,T}^\delta - \xi_{s,T}|^2 ds + 2\int_0^T |\xi_{s,T}||\xi_{s,T}^\delta - \xi_{s,T}| ds \\ &\le \frac{1}{c} + 2\Sigma_{g_1}^{\frac{1}{2}} (\frac{1}{c})^{\frac{1}{2}}. \end{aligned}$$

Hence we can take $c_0 > 0$ such that for any $c > c_0 > 0$,

$$0 < \Sigma_{g_1} - |\sigma(X_T^{(\delta)}) - \Sigma_{g_1}| < \sigma(X_T^{(\delta)})$$

holds uniformly for $\delta \in (0,1]$. Thus, we obtain the result. *Q.E.D.*

Next, we present two inequalities which are useful to show the truncation by $\psi(\eta_c^\delta)$ is negligible in the asymptotic expansions. We omit their proofs because they are quite lengthy. [6]

Lemma 6.3 *(1) There exist positive constants a_i $(i=1,2)$ independent of δ such that*

$$P(\sup_{0\le s\le T}|S_s^{(\delta)} - S_s^{(0)}| > a_0) \le \frac{a_1}{a_0}(a_0+C)\exp(-\frac{a_2a_0^2}{(a_0+C)^2}\delta^{-2}) \tag{6.56}$$

for all $a_0 > 0$.
(2) There exist positive constants a_i $(i=1,2)$ independent of δ such that

$$P(\sup_{0\le s\le T}|Y_s^{(\delta)} - Y_s| > a_0) \le \frac{a_1}{a_0}(a_0+C)\exp(-\frac{a_2a_0^2}{(a_0+C)^2}\delta^{-2}) \tag{6.57}$$

for all $a_0 > 0$.

By using Lemma 6.3, we now can show the truncation is negligible in probability by utilizing the above large deviation inequalities. We present this result as the next lemma, but omit its proof because it is straightforward but quite lengthy.[7]

Lemma 6.4 *For $c > 0$, η_c^δ is $O(1)$ in $\boldsymbol{D}^\infty$ and for $c_0 > 0$, there exist some constants $c_i,\ i=1,2,3$, such that*

$$P(\{|\eta_c^\delta| > c_0\}) \le c_1\exp(-c_2\delta^{-c_3}) \tag{6.58}$$

Then all conditions stated in *Theorem 2.2* of Yoshida (1992b) are satisfied and we have the desired result as its direct consequence.

Proposition 6.1 *Under the assumptions we have made, for a smooth function $\phi^\delta(x)$ with all derivatives of polynomial growth orders, $\psi(\eta_c^\delta)\phi^\delta(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)})$ has an asymptotic expansion;:*

$$\psi(\eta_c^\delta)\phi^\delta(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)}) \sim \Phi_0 + \delta\Phi_1 + \cdots \tag{6.59}$$

in $\tilde{\boldsymbol{D}}^{-\infty}$ as $\delta \downarrow 0$ where $\mathcal{B}$ is a Borel set, $\psi(x)$ is a smooth function such that $0 \le \psi(x) \le 1$ for $x \in R$,$\psi(x) = 1$ for $|x| \le 1/2$ and $\psi = 0$ for $|x| \ge 1$, and $\Phi_0, \Phi_1, \cdots$ are determined by the formal Taylor expansion.

[6] See Lemma 3.5 of Kunitomo and Takahashi (1998) for the proofs.
[7] See Lemma 3.6 of Kunitomo and Takahashi (1998) for the proof.

Finally, we obtain an asymptotic expansion of the expectation of $\phi^\delta(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)})$ as $\delta \to 0$, which is summarized in the next theorem.

Theorem 6.1 *Under the assumptions we have made, an asymptotic expansion of* $\mathbf{E}[\phi^\delta(X^\delta)I_{\mathcal{B}}(X_T^{(\delta)})]$ *is given by*

$$\begin{aligned}\mathbf{E}[\phi^\delta(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)})] &\sim \mathbf{E}[\psi(\eta_c^\delta)\phi^\delta(X_T^{(\delta)})I_{\mathcal{B}}(X_T^{(\delta)})] \\ &\sim \mathbf{E}[\Phi_0] + \delta\mathbf{E}[\Phi_1] + \cdots \end{aligned} \tag{6.60}$$

as $\delta \downarrow 0$.

As a final concluding remark, we should mention that the inversion technique we have used is different from the one used by Yoshida (1992a,b). He has used the Schwartz's type distribution theory for the generalized Wiener functionals while our method is based on the simple inversion technique for the characteristic functions of random variables, which has been standard in the statistical asymptotic theory. Hence what we need to show is that the resulting formulae by our method are equivalent to his final formulae. Let take $\phi^\delta(x) = 1$ in Proposition 6.1 as an illustration. [8] Then Yoshida (1992a,b) used the notation

$$p_1^{'}(x) = (-1)\frac{d}{dx}\mathbf{E}\left[\phi(f_0)f_1\partial I_A(f_0)|f_0 = x\right],$$

and

$$p_1^{''}(x) = (-1)\frac{d}{dx}\mathbf{E}\left[\{f_1\partial\phi(f_0)\}I_A(f_0)|f_0 = x\right],$$

where $I_A(f_0)$ is the indicator function and f_0 corresponds to the random variable of the order $O_p(1)$, which are the same as $I_{\mathcal{B}}(\cdot)$ and g_1, respectively, in our notations. The differentiation of indicator functions in the above has a proper mathematical meaning in the sense of differentiation on the generalized Wiener functionals. (See Watanabe (1987) and Yoshida (1992a,b) for its details.) By the use of the pull-back operation of the generalized Wiener functionals, Yoshida (1992a) has obtained the explicit expansion form of the density function for a particular functional in his problem as

$$p_1(x) = p_1^{'}(x) + p_1^{''}(x).$$

In our framework it is straightforward to show that

$$p_1^{'}(x) = (-1)\frac{d}{dx}\left[\mathbf{E}(g_2|g_1 = x)n(x; 0, \Sigma_{g_1})\right]$$

[8] Theorem 3.7 of Kunitomo and Takahashi (1998) shows the proof of the equivalence up to the third order term.

and $p_1''(x) = 0$ since $\partial\phi(\cdot) = 0$. Then we notice that $p_1(x)$ is exactly what the inversion formula gives as the second order term in the asymptotic expansion of the density function of the normalized random variable $X_T^{(\delta)}$. We have the similar arguments for higher order terms in the asymptotic expansions.

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Figure 1: Errors in the Expansion around the Normal distribution

Table 1: Plain Vanilla Call Options-Square root process -(vol. = 10%)

Strike price	45	40	35
(1)Monte Carlo	0.5771	2.7226	6.7676
(2)Stochastic Expansion(second)	0.5763	2.7228	6.7640
Diff. Rate%	-0.144	0.007	-0.053
(3)Stochastic Expansion(first)	0.5548	2.7398	6.7796
Diff. Rate%	-3.865	0.632	0.178

Table 2: Plain Vanilla Call Options-Square root process -(vol. = 20%)

Strike price	45	40	35
(1)Monte Carlo	2.0005	4.1841	7.4802
(2)Stochastic Expansion(second)	1.9979	4.1858	7.4855
Diff. Rate%	-0.130	0.041	0.071
(3)Stochastic Expansion(first)	1.9460	4.2231	7.5776
Diff. Rate%	-2.724	0.932	1.303

Table 3: Plain Vanilla Call Options-Square root process -(vol. = 30%)

Strike price	45	40	35
(1)Monte Carlo	3.5347	5.7069	8.6453
(2)Stochastic Expansion(second)	3.5379	5.7105	8.6502
Diff. Rate%	0.091	0.064	0.057
(3)Stochastic Expansion(first)	3.4573	5.7674	8.8191
Diff. Rate%	-2.189	1.067	2.010

Table 4: Average Call Options on Equity -Square root process -(T=0.25y)

Strike price	45	40	35
(1)Monte Carlo	0.1559	1.4985	5.2659
(2)Stochastic Expansion	0.1562	1.4983	5.2679
Difference	0.00029	-0.00020	0.00210
Diff. Rate%	0.18	-0.01	0.04

Table 5: Average Call Options on Equity-Square root process-(T=0.50y)

Strike price	45	40	35
(1)Monte Carlo	0.5221	2.1758	5.6468
(2)Stochastic Expansion	0.5228	2.1788	5.6516
Difference	0.00078	0.00301	0.00482
Diff. Rate %	0.15	0.14	0.09

Table 6: Average Call Options on Equity-Square root process-(T=1.0y)

Strike price	45	40	35
(1)Monte Carlo	1.2802	3.1848	6.3845
(2)Stochastic Expansion	1.2813	3.1873	6.3881
Difference	0.00112	0.00255	0.00362
Diff. Rate %	0.09	0.08	0.06

Table 7: Average Call Options on FX-Square root process- (T=0.25y)

Strike price	105	100	95
(1)Monte Carlo	0.0416	1.0217	4.7672
(2)Stochastic Expansion	0.0419	1.0215	4.7698
Difference	0.00031	-0.00025	0.00254
Diff. Rate %	0.75	-0.02	0.05

Table 8: Average Call Options on FX-Square root process-(T=0.50y)

Strike price	105	100	95
(1)Monte Carlo	0.1721	1.3625	4.6858
(2)Stochastic Expansion	0.1730	1.3654	4.6931
Difference	0.00090	0.00286	0.00730
Diff. Rate %	0.52	0.21	0.16

Table 9: Average Call Options on FX-Square root process-(T=1.0y,Vol.=10%)

Strike price	105	100	95
(1)Monte Carlo	0.4443	1.7700	4.6525
(2)Stochastic Expansion	0.4426	1.7709	4.6585
Difference	-0.00166	0.00090	0.00600
Diff. Rate %	-0.37	0.05	0.13

Table 10: Average Call Options on FX-Square root process(T=1.0y,Vol.=30%)

Strike price	110	100	90
(1)Monte Carlo	2.7995	6.18088	11.7334
(2)Stochastic Expansion	2.8045	6.1881	11.7464
Difference	0.00502	0.007221	0.00130
Diff. Rate %	0.18	0.12	0.11

Table 11: Average Options on FX -Log-normal process- (T=0.25y)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.0384	1.0199	4.7738
Diff. Rate %	-15.97	-0.19	0.16
(2)Stochastic Expansion(normal)(2nd)	0.0452	1.0220	4.7650
Diff. Rate %	-1.09	-0.02	-0.02
(3)Finite difference(Crank-Nicholson method)	0.0457	1.0216	4.7659
Diff. Rate %	0.01	-0.02	-0.00
(4)Monte Carlo simulation method	0.0457	1.0218	4.7660

Table 12: Average Options on FX -Log-normal process- (T=0.50y)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.1620	1.3610	4.7040
Diff. Rate %	-11.96	-0.53	0.53
(2)Stochastic Expansion(normal)(2nd)	0.1830	1.3660	4.6800
Diff. Rate %	-0.54	-0.16	0.01
(4)Finite difference(Crank-Nicholson method)	0.1831	1.3656	4.6788
Diff. Rate %	-0.49	-0.19	-0.01
(5)Monte Carlo simulation method	0.1840	1.3682	4.6793

Table 13: Average Options on FX -Log-normal process- (T=1.00y,Vol.=10%)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.4180	1.7590	4.6750
Diff. Rate %	-10.30	-0.61	0.81
(2)Stochastic Expansion(normal)(2nd)	0.4640	1.7720	4.6410
Diff. Rate %	-0.43	-0.12	0.08
(3)Finite difference(Crank-Nicholson method)	0.4640	1.7715	4.6315
Diff. Rate %	-0.43	-0.09	-0.13
(4)Monte Carlo simulation method	0.4660	1.7699	4.6375

Table 14: Average Options on FX -Log-normal process- (T=1.00y,Vol.30%)

Strike price	110	100	90
(1)Stochastic Expansion(normal)(1st)	2.6107	6.1516	11.8900
Diff. Rate %	-12.22	-0.76	2.61
(2)Stochastic Expansion(normal)(2nd)	2.9699	6.1910	11.5751
Diff. Rate %	-0.14	-0.12	-0.11
(3)Monte Carlo simulation method	2.9740	6.1985	11.5874