A Market Model of Interest Rate with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies *

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Outlines

OTC Market and Collateralization

- **Collateralization**
  - **The most important credit risk mitigation tool.**
    - **margin call, settlement and associated procedures.**
    - **legal specifications are provided by CSA (Credit Support Annex).**
  - **Dramatic increase in recent years (ISDA [4])**
    - **30%(2003) $\rightarrow$ 70%(2009) in terms of trade volume for all OTC.**
    - **Coverage goes up to 78% (for all OTC) and 84% (for fixed income) among major financial institutions.**
    - **More than 80% of collateral is Cash.**
    - **About half of the cash collateral is USD.**
    - **Almost all the credit derivatives are collateralized.**

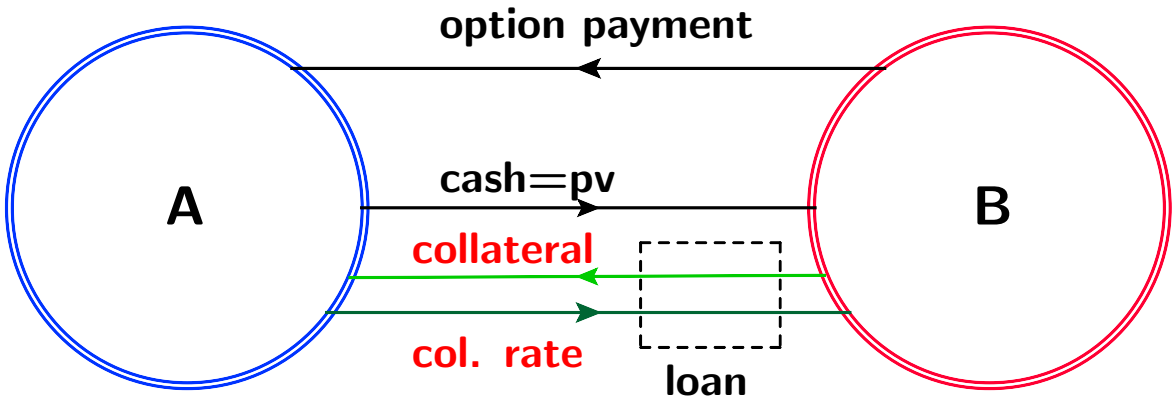
Impact of Collateralization

Impact of collateralization :

- **Reduction of Counter-party Exposure.**
  - **Associated change in CVA has been actively studied.**
- **Change of Funding Cost (topic of this talk)**
  - **Require new term structure model to distinguish discounting and reference rates.**
  - **Cost of collateral is differ from currency to currency.**
  - **"cheapest-to-deliver" option.**
  - **Significant impact on derivative pricing and risk management.**

Source of Funding Cost Difference

- **Collateralized (Secured) Contract (current picture)**



  - **No outright cash flow (collateral=PV)**
  - **No external funding is needed.**
  - **Funding is determined by over-night (ON) rate.**
    $\Rightarrow$ **Libor discounting is inappropriate.**

Fundamental Market Instruments

Historical behavior of IRS (1Y)-OIS (1Y) spreads (bps)

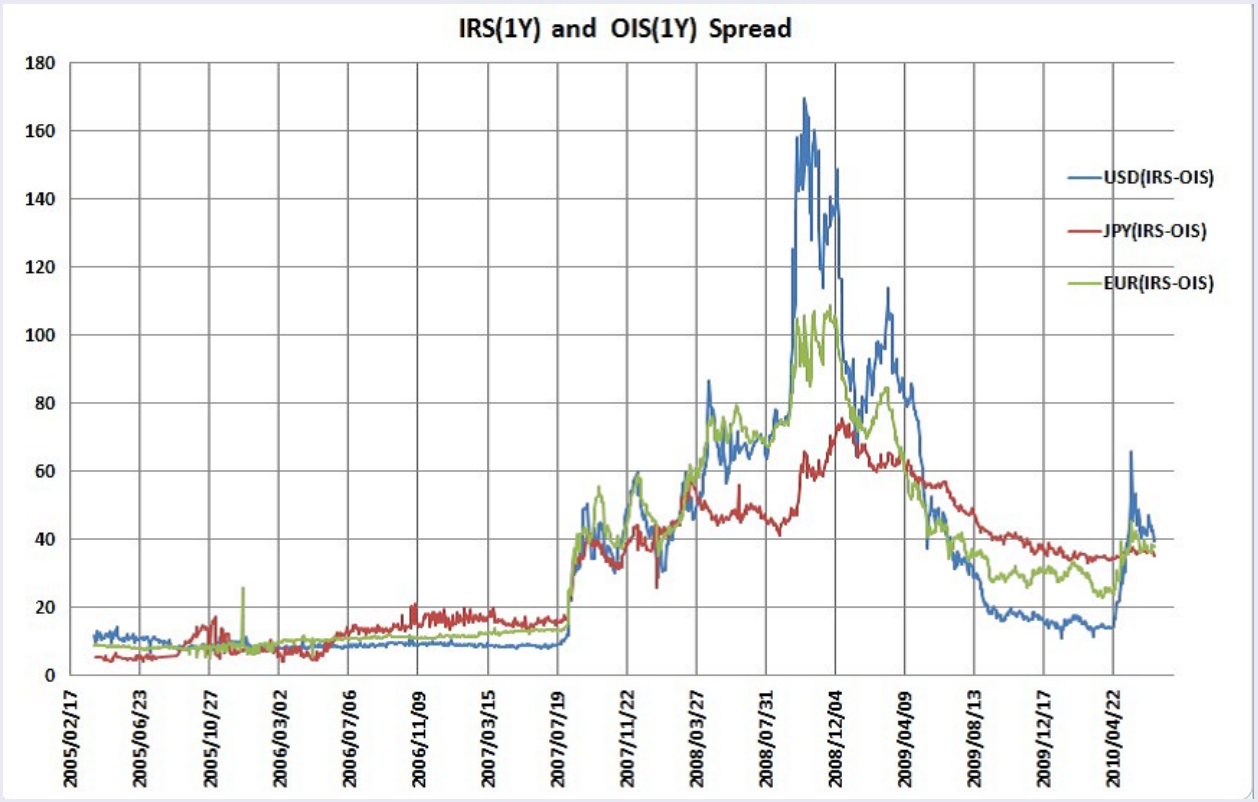


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of JPY TS spreads (bps)

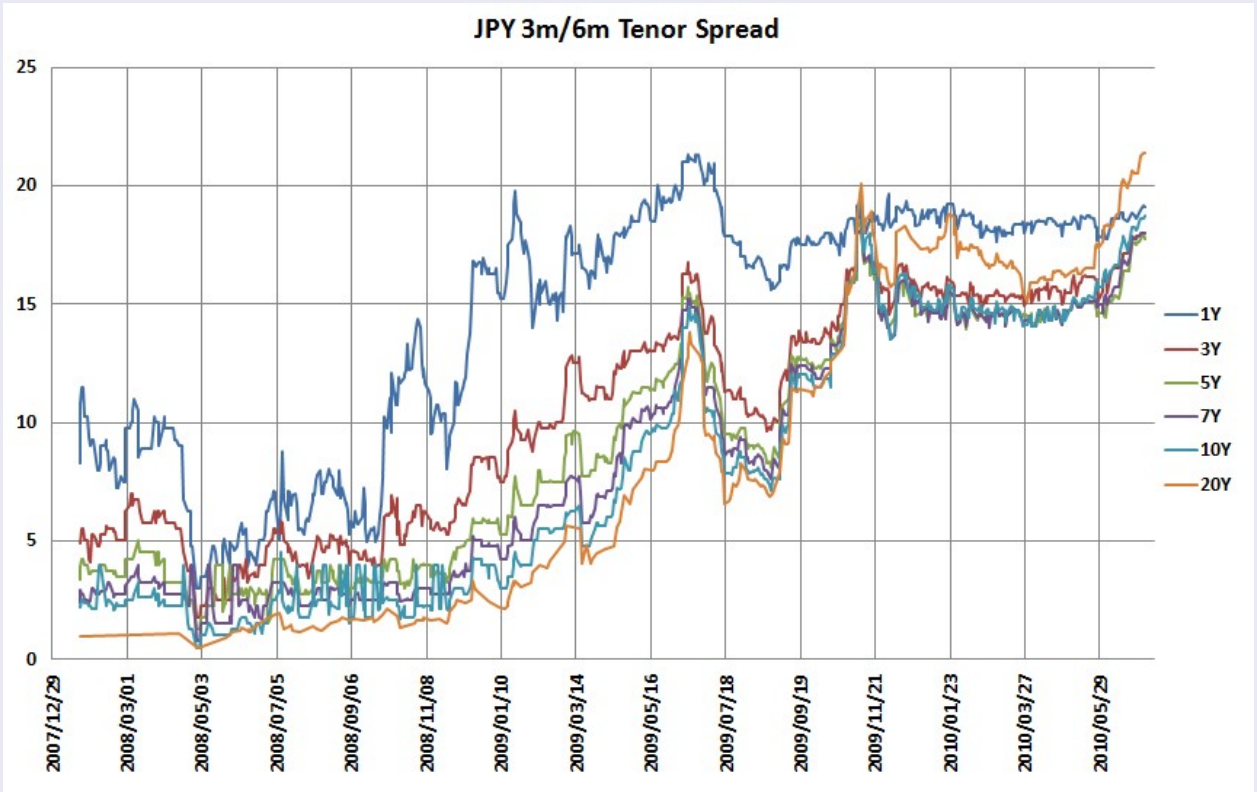


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of USD TS spreads (bps)

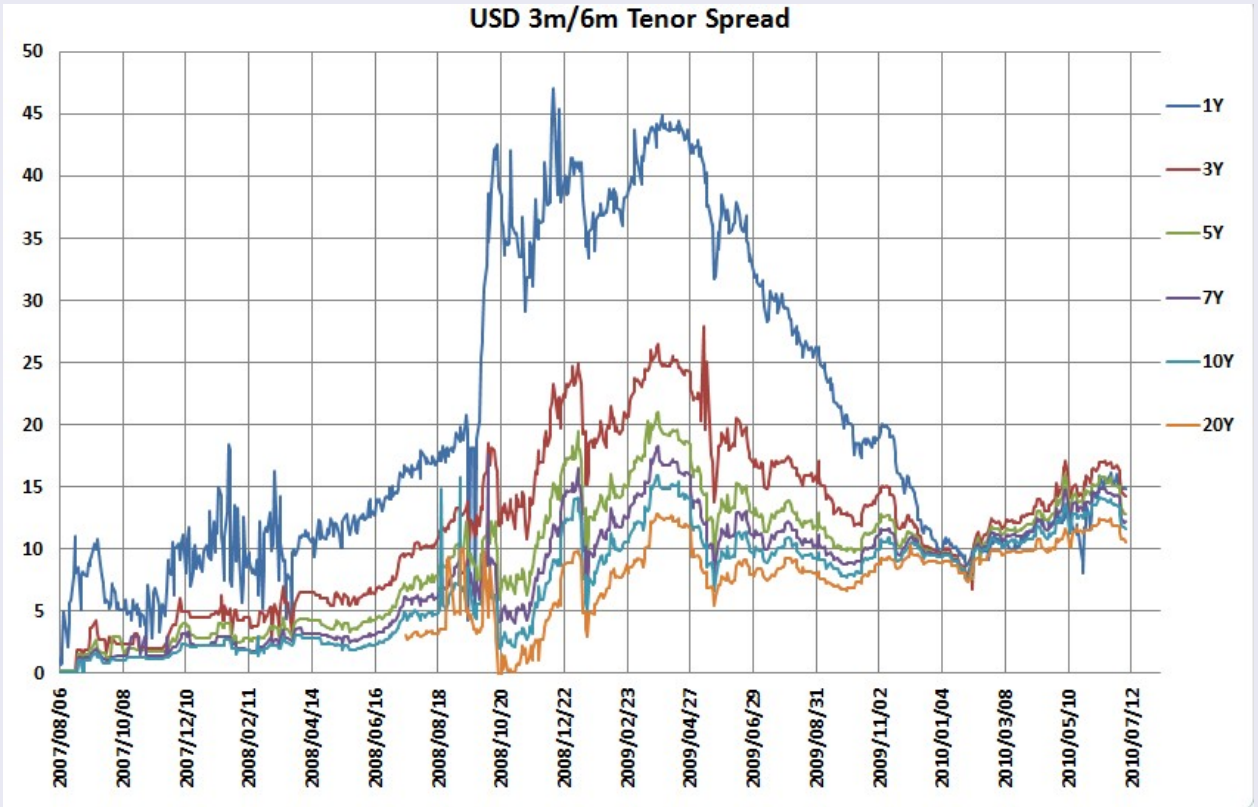


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of EUR TS spreads (bps)

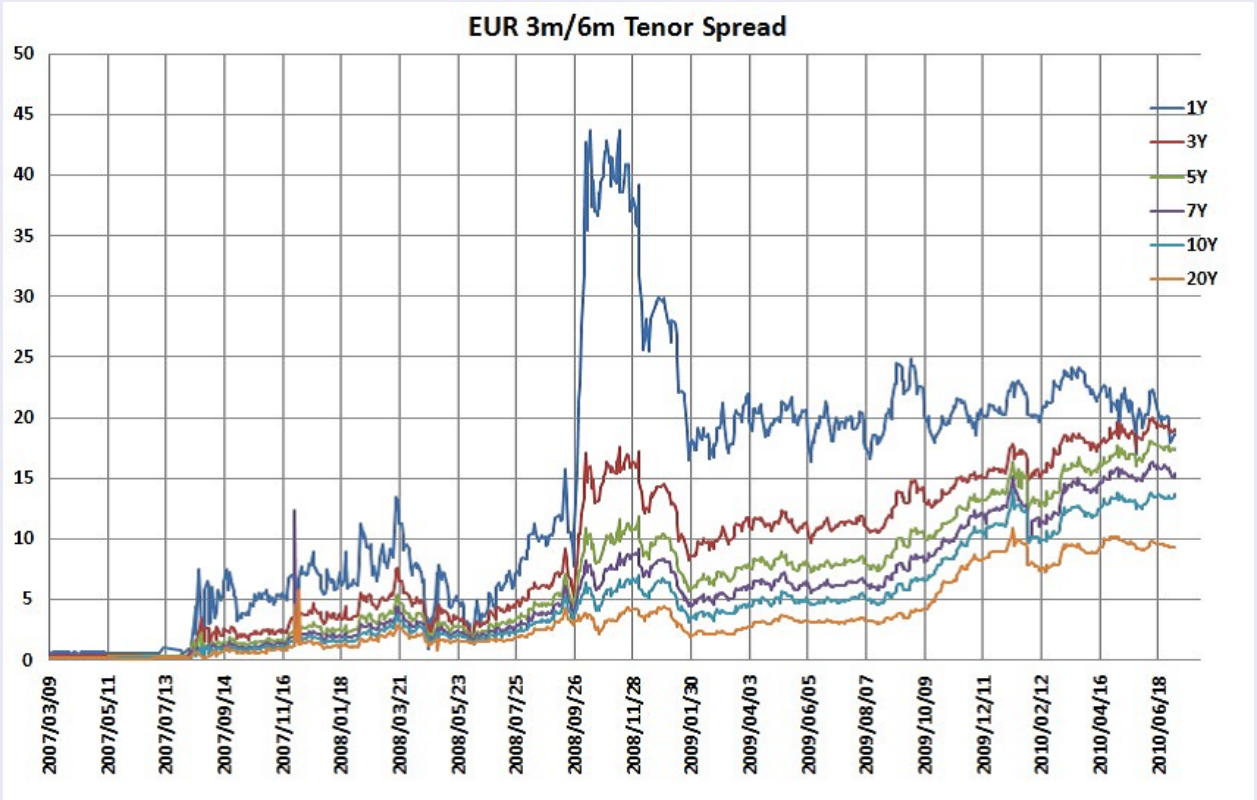


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of USDJPY CCS spreads (bps)

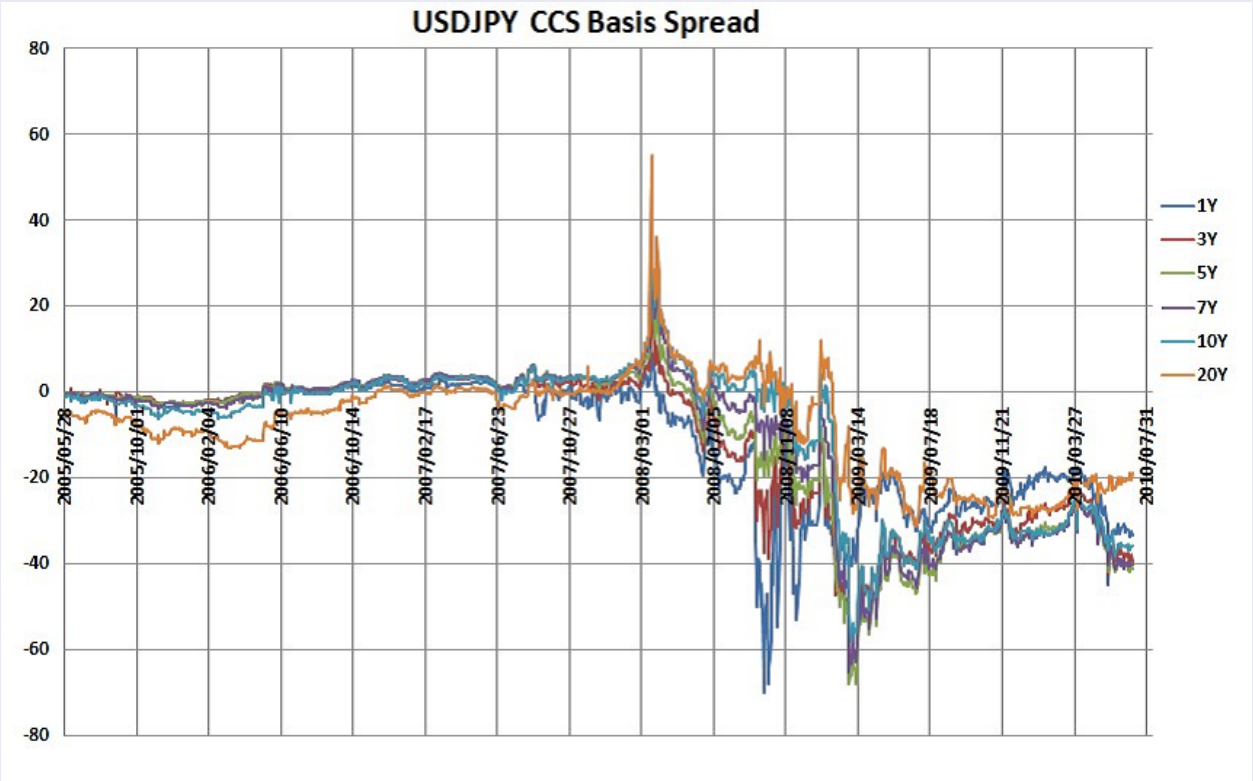


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of EURUSD CCS spreads (bps)

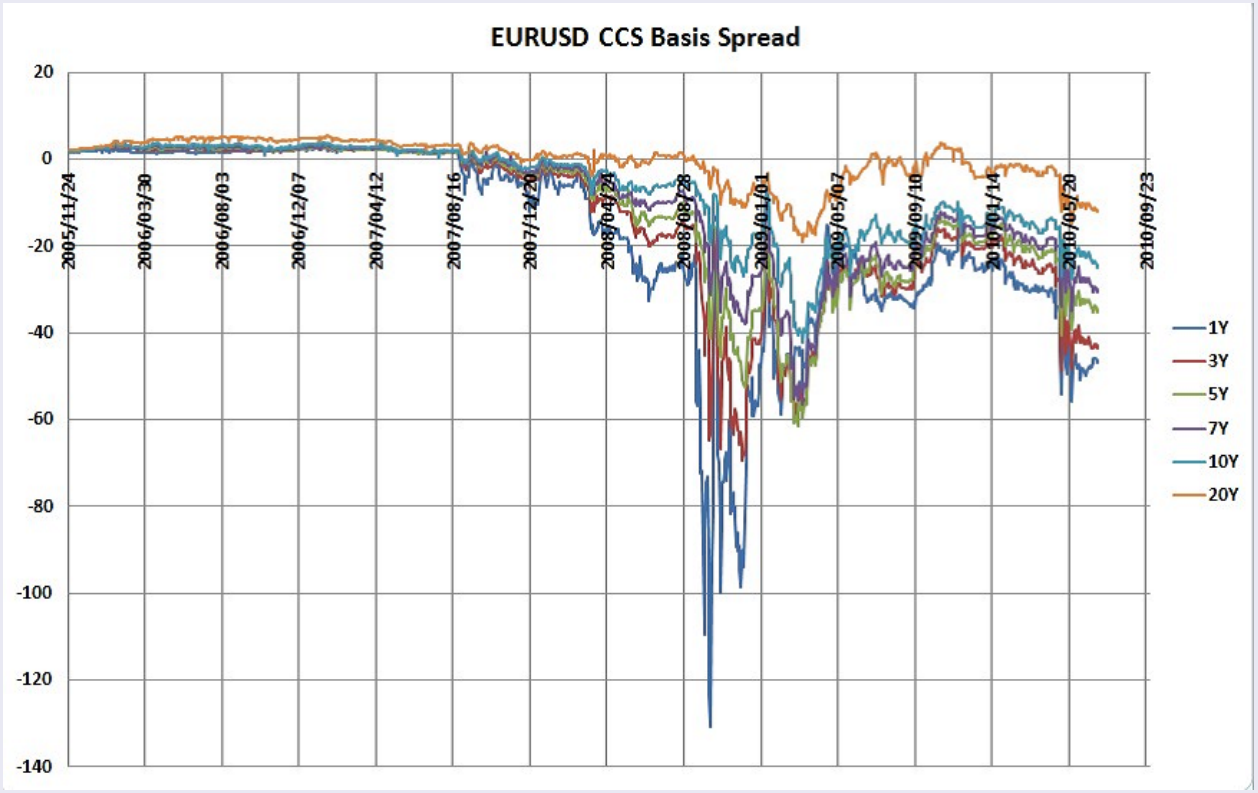


Figure: Source:Bloomberg

Fundamental Market Instruments

Traditional IR model (such as LMM) has become ill-suited for actual derivative business, because...

- **Impossible to calibrate fundamental instruments, such as:**
  - **Tenor Swap (TS) (or IRS with different tenor/frequency)**
  - **Cross Currency Swap (CCS) $\Rightarrow$ useless for long-dated FX products**
  - **Overnight Index Swap (OIS)**
- **Unable to recognize the important delta exposure, such as to Libor-OIS spread.**
  - **Proper control of risk exposure is impossible.**

Criteria for Models Workable in Real Business

Criteria

- **Consistent discounting/forward curve construction**
  - **Price all types of IR swaps correctly:**
    - **OIS, IRS and TS**
  - **Take collateralization into account.**
  - **Maintain consistency in multi-currency environment**
    - **CCS basis spreads need to be recovered.**
    - **Cost of cash collateral and its difference among major currencies should be taken into account.**
- **Stochastic Modeling of Basis spreads**
  - **Allow systematic calibration procedures**
  - **Flexible enough to allow non-trivial term structure of spreads.**

Pricing under the Collateralization

- **Assumption**
  - **Continuous adjustment of collateral amount**
  - **Perfect collateralization by Cash**
  - **Zero minimum transfer amount**

- **Comments**
  - **Daily margin call/settlement is becoming popular.**
  - **By making use of Repo / Reverse-Repo, other collateral assets can be converted into the equivalent amount of cash collateral.**
  - **General Collateral (GC) repo rate closely tracks overnight rate.**

Pricing under the Collateralization

Proposition:

T-maturing European option under the collateralization is given by [a]

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] \\ &= D^{(i)}(t,T)E_t^{\mathcal{T}_{(i)}^c}\left[\left(e^{-\int_t^T y^{(i,j)}(s)ds}\right)h^{(i)}(T)\right] \end{aligned}$$

where,

$$\begin{aligned} y^{(j)}(s) &= r^{(j)}(s) - c^{(j)}(s) \quad , \quad y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \\ D^{(i)}(t,T) &= E_t^{Q_i}\left[e^{-\int_t^T c^{(i)}(s)ds}\right] \end{aligned}$$

- $h^{(i)}(T)$: **option payoff at time T in currency i**
- **collateral is posted in currency j**
- $c^{(j)}(s)$: **instantaneous collateral rate of currency j at time s**
- $r^{(j)}(s)$: **instantaneous risk-free rate of currency j at time s**
- $E^{\mathcal{T}_{(i)}^c}[\cdot]$: **expectation under the fwd measure associated with $D^{(i)}(\cdot,T)$**

[a] Fujii,Shimada,Takahashi (2009) [1]

Pricing under the Collateralization

- **Collateral amount in currency j at time s is given by $\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}$, which is invested at the rate of $y^{(j)}(s)$:**

$$h^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T)\right]$$
$$+ f_x^{(i,j)}(t) E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s)\left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right)ds\right]$$
$$= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds\right].$$

Note that $X(t) = e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds$ is a Q_i-martingale. Then, the process of the option value is written by

$$dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t)dt + dM(t)$$

with some Q_i-martingale M. This establishes the proposition.

$f_x^{(i,j)}(t)$: **Foreign exchange rate at time t representing the price of the unit amount of currency "j" in terms of currency "i".**

Pricing under the Collateralization

Corollary

- **If payment and collateral currencies are the same, the option value is given by**

$$\begin{aligned} h(t) &= E_t^Q \left[e^{-\int_t^T c(s)ds} h(T) \right] \\ &= D(t,T) E_t^{\mathcal{T}^c} \left[h(T) \right] \ . \end{aligned}$$

- **The discounting is determined by "collateral rate", which is consistent with the schematic picture seen before.**

Building Blocks for IR Term Structure Model

Building Blocks

$$
\begin{aligned}
c^{(i)}(t,T) &= -\frac{\partial}{\partial T}\ln D^{(i)}(t,T) \\
B^{(i)}(t,T_k;\tau) &= E_t^{\mathcal{T}^c_{k,(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)} - 1\right) \\
y^{(i,k)}(t,T) &= -\frac{\partial}{\partial T}\ln\left(E_t^{Q_i}\left[e^{-\int_t^T y^{(i,k)}(s)ds}\right]\right)
\end{aligned}
$$

- **These building blocks are enough to calibrate all the relevant OIS, IRS, TS and CCS.**

Construction of Term Structure

Term structure construction procedures:[1]

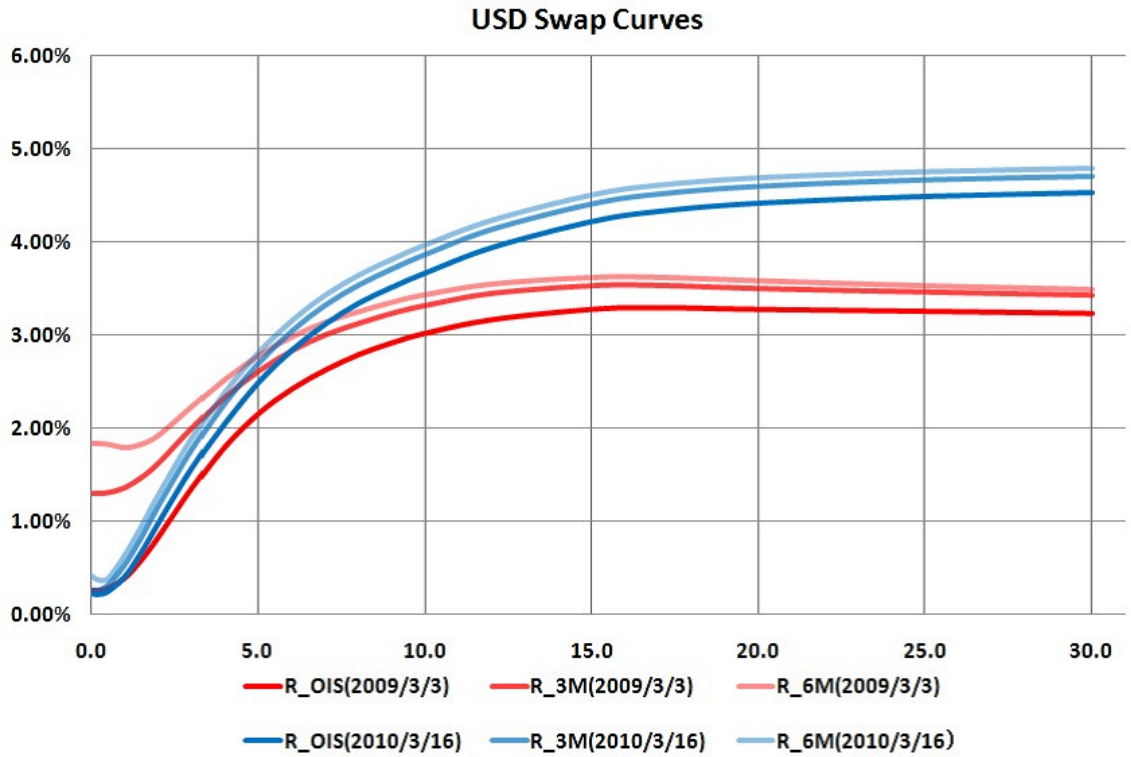
- **(1), OIS** $\Rightarrow c^{(i)}(t,s)$
- **(2), results of (1) + IRS + TS** $\Rightarrow B^{(i)}(t,s;\tau)$
- **(3), results of (1,2) +CCS** $\Rightarrow y^{(i,j)}(t,s)$

- **Assume collateralization in domestic currency for OIS, IRS and TS** [2]**.**
- **Assume collateralization in USD for CCS (USD crosses).**

[1] See, (Fujii, Shimada, Takahashi 2009) [1] for details.

[2] Assumption on collateral currency has only minor impact on the market par quotes.

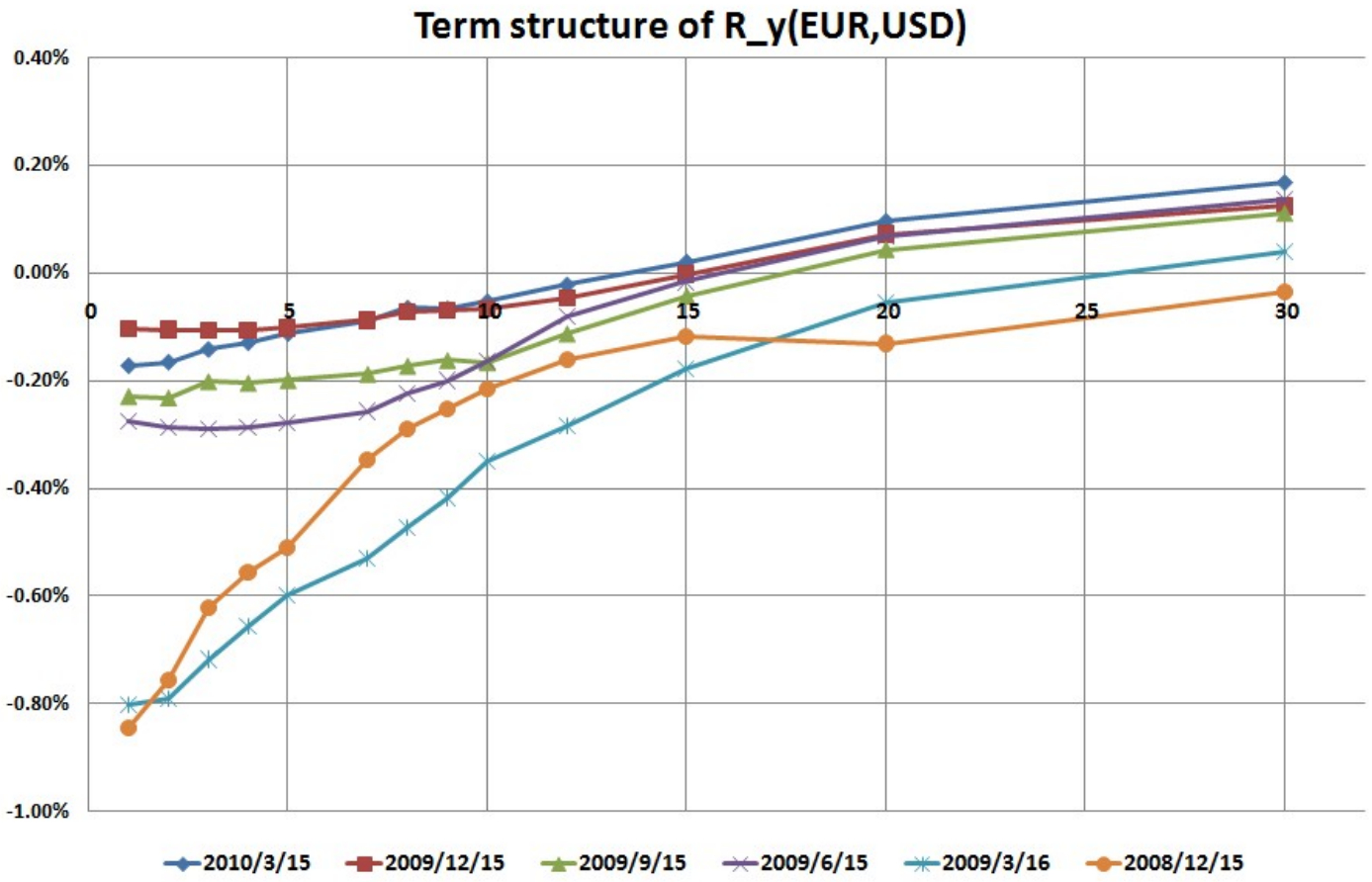
Construction of Term Structure



$$R_{\mathrm{OIS}}(T) = -\ln(D(0,T))/T$$

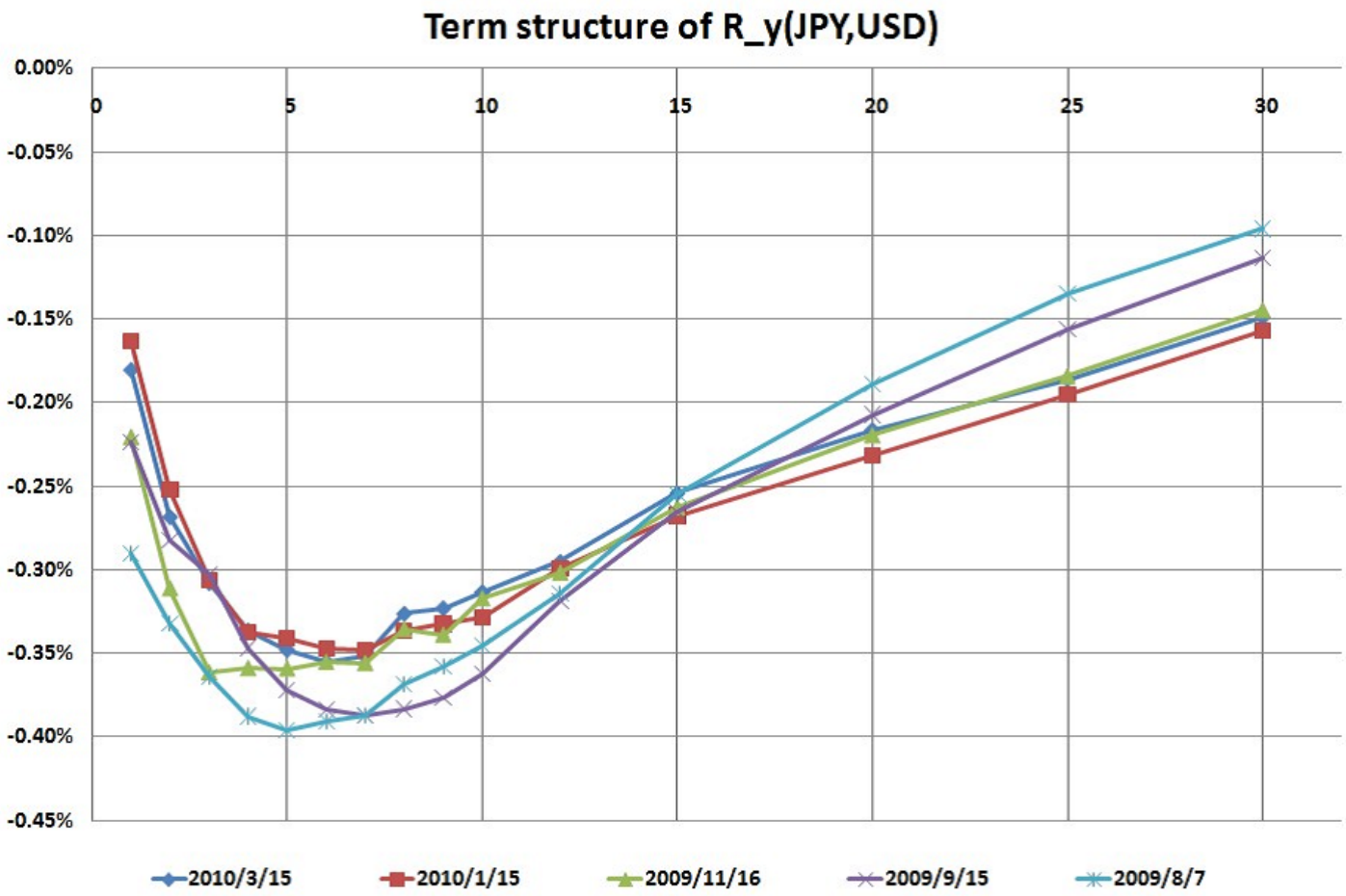
$$E^{\mathcal{T}_m^c}[L(T_{m-1},T_m;\tau)] = \frac{1}{\delta_m}\left(\frac{e^{-R_\tau(T_{m-1})T_{m-1}}}{e^{-R_\tau(T_m)T_m}} - 1\right)$$

Construction of Term Structure



$$R_{y(i,j)}(T) = -\frac{1}{T}\ln\left(E^{Q_i}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right]\right) = \frac{1}{T}\int_0^T y^{(i,j)}(0,s)ds$$

Construction of Term Structures



HJM-framework under the collateralization

SDEs for HJM-framework

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s) \cdot \left(\int_t^s \sigma_c^{(i)}(t,u)du \right) dt + \sigma_c^{(i)}(t,s) \cdot dW_t^{Q_i}$$

$$dy^{(i,k)}(t,s) = \sigma_y^{(i,k)}(t,s) \cdot \left(\int_t^s \sigma_y^{(i,k)}(t,u)du \right) dt + \sigma_y^{(i,k)}(t,s) \cdot dW_t^{Q_i}$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau) \cdot \left(\int_t^T \sigma_c^{(i)}(t,s)ds \right) dt + \sigma_B^{(i)}(t,T;\tau) \cdot dW_t^{Q_i}$$

$$\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t) \right) dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{Q_i}$$

- For construction of swap curves, the independence of y is useful assumption.
- See Fujii, Shimada, Takahashi (2009,2010) [2, 3].

Choice of Collateral Currency

Role of $y^{(i,j)}$

- **Payment currency i with Collateral currency j**

$$D^{(i)}(t,T) \Rightarrow \boldsymbol{E_t^{Q_i}}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right] D^{(i)}(t,T)$$

  after neglecting small corrections from possible non-zero correlations.

- **To choose "strong" currency, such as USD, is expensive (for the collateral payer).**

Choice of Collateral Currency

Role of $y^{(i,j)}$

Optimal behavior of collateral payer can significantly change the derivative value.

- **Payment currency i with multiple currencies as eligible collateral choice $\mathcal{C}$**

$$D^{(i)}(t,T) \Rightarrow \mathbf{E}_t^{Q_i}\left[e^{-\int_t^T \max_{j\in c}\{y^{(i,j)}(s)\}ds}\right] D^{(i)}(t,T)$$

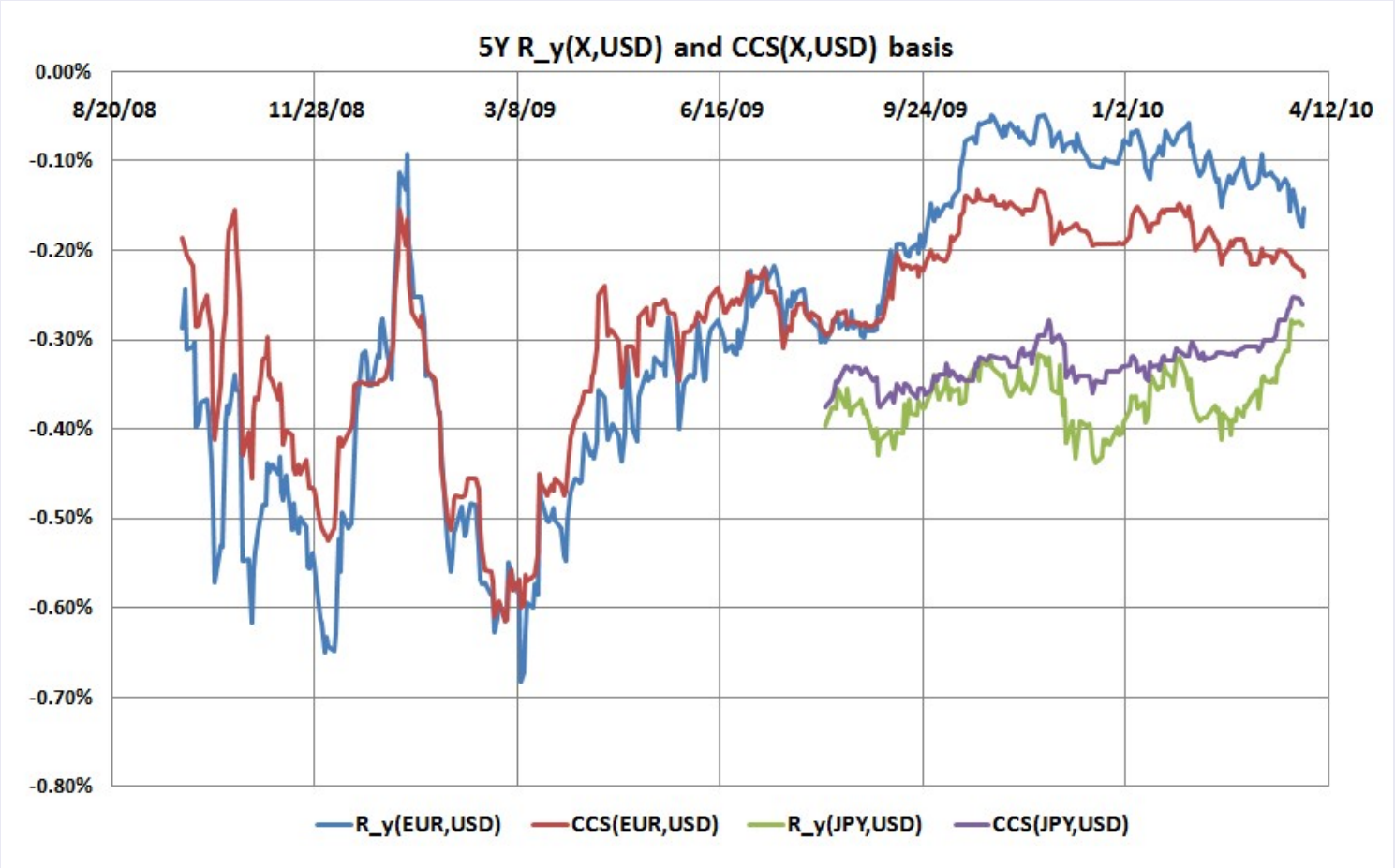
- **Payment currency and USD as eligible collateral is relatively common.**

$$D^{(i)}(t,T) \Rightarrow \mathbf{E}_t^{Q_i}\left[e^{-\int_t^T \max\{y^{(i,USD)}(s),0\}ds}\right] D^{(i)}(t,T)$$

- **Volatility of $y^{(i,j)}$ is an important determinant.**

Choice of Collateral Currency

Close relationship to CCS basis spread



Choice of Collateral Currency

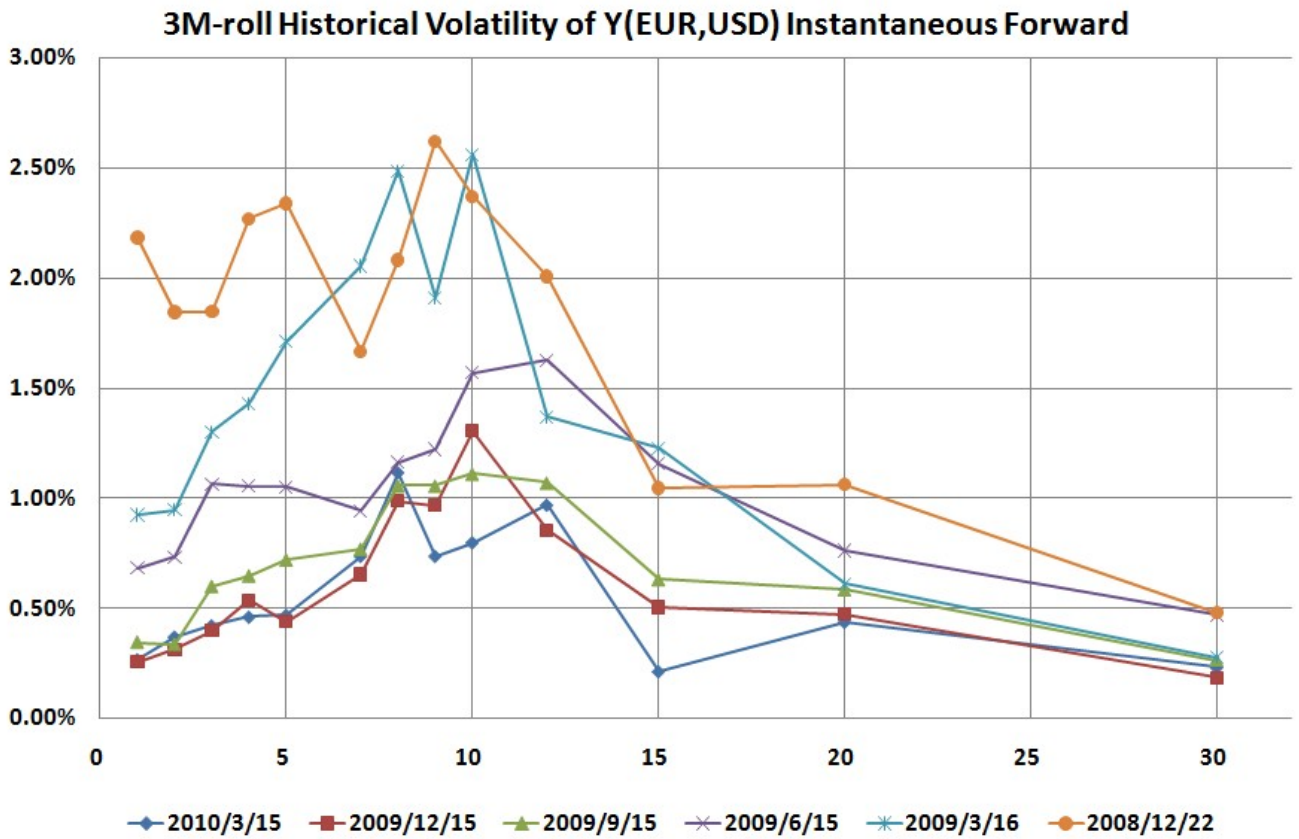


Figure: 3M-Roll historical volatility of $y^{(EUR,USD)}$ instantaneous forward. Annualized in absolute terms.

Choice of Collateral Currency

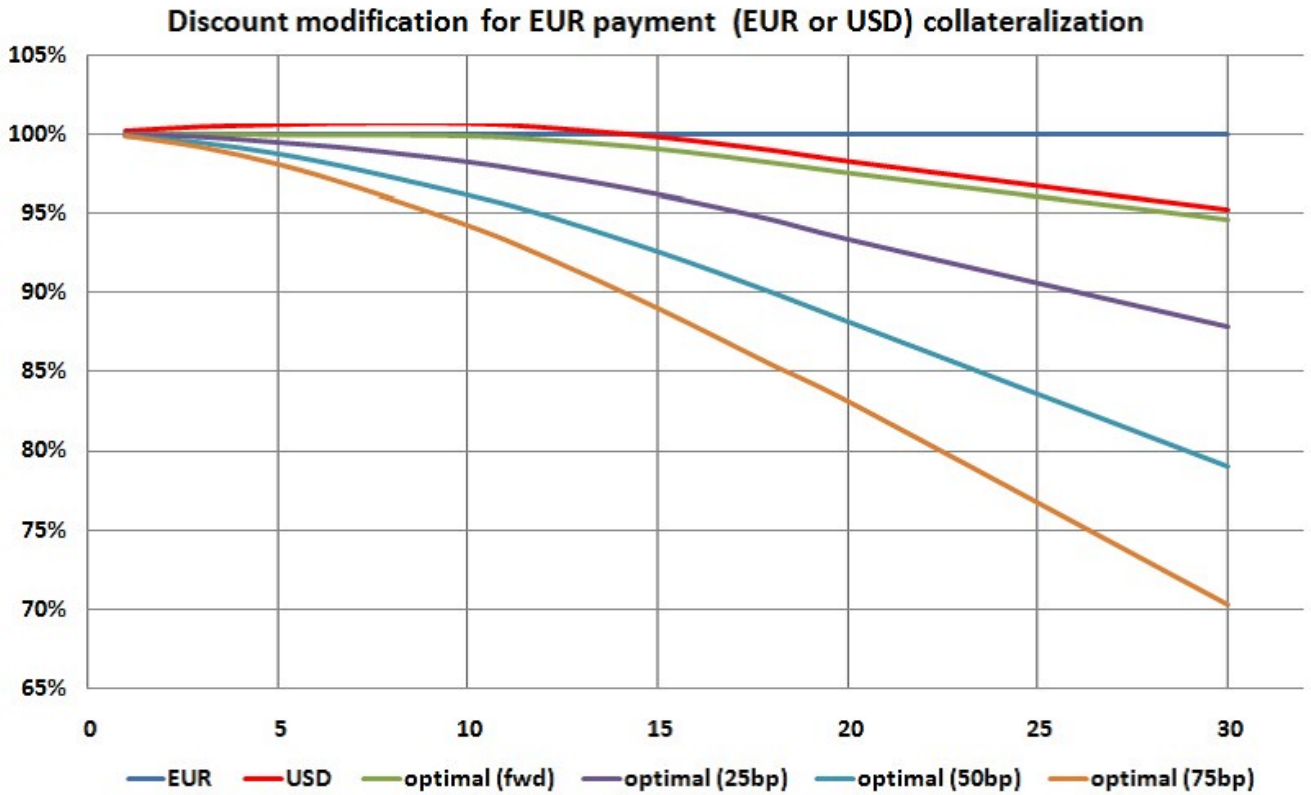


Figure: Modification of EUR discounting factors based on HW model for $y^{(EUR,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

Choice of Collateral Currency

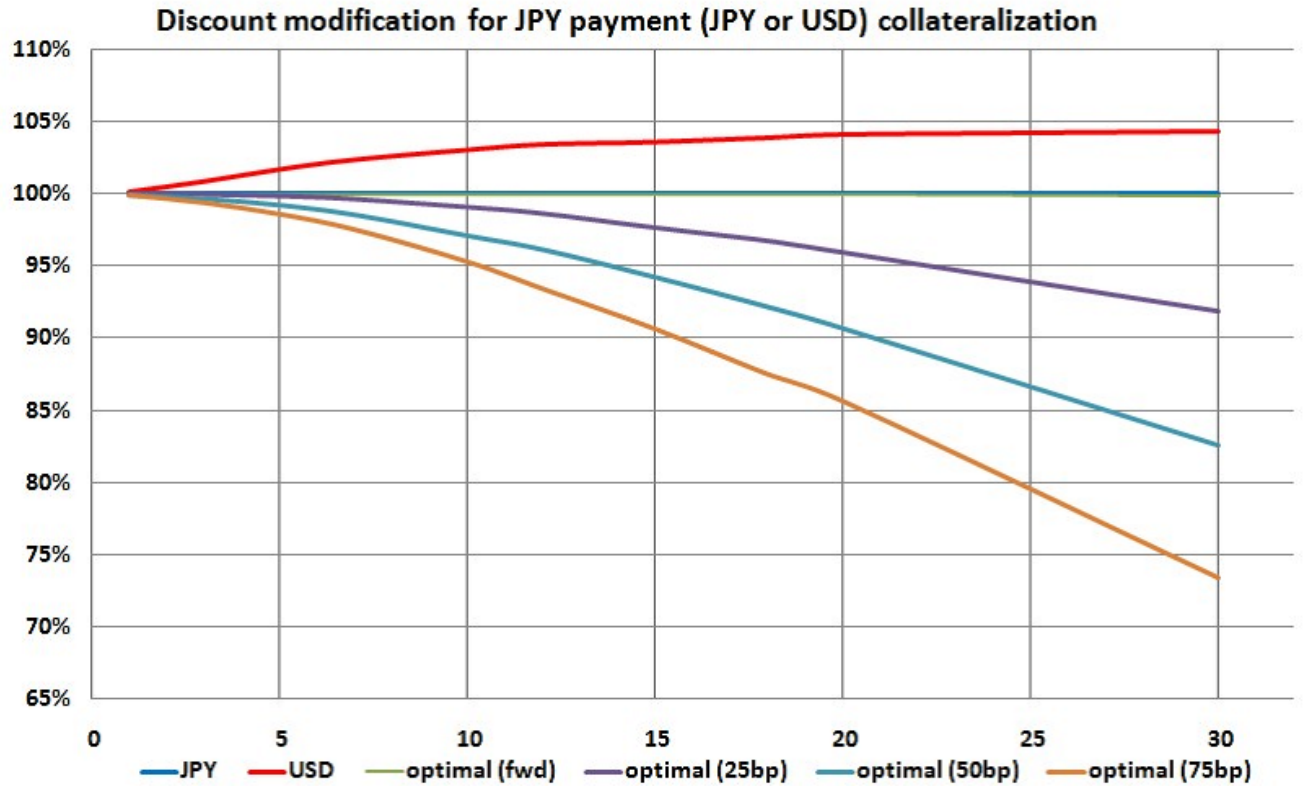


Figure: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

Conclusions

Conclusions

- **We proposed a term structure model under the collateralization, which directly relates the cost of cash-collateral to cross currency basis spreads.**
- **We pointed out**
  - **importance of choice of collateral currency.**
  - **potential impact of the embedded cheapest-to-deliver option.**

Comments:

- **Including collateral cost for modeling will be particularly important for CCP-driven derivatives markets.**

Main References

[1] Fujii, M., Shimada, Y., Takahashi, A., 2009, "A note on construction of multiple swap curves with and without collateral," CARF Working Paper Series F-154, available at http://ssrn.com/abstract=1440633.

[2] Fujii, M., Shimada, Y., Takahashi, A., 2009, "A Market Model of Interest Rates with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies", CARF Working Paper Series F-196, available at http://ssrn.com/abstract=1520618.

[3] Fujii, M., Shimada, Y., Takahashi, A., 2010, "Collateral Posting and Choice of Collateral Currency -Implications for Derivative Pricing and Risk Management-", CARF Working Paper Series F-216, available at http://ssrn.com/abstract=1601866.

[4] ISDA Margin Survey 2010, Preliminary Results
Market Review of OTC Derivative Bilateral Collateralization Practices