On the Term Structure of Interest Rates with Basis Spreads, Collateral and Multiple Currencies *

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   - Term Structure Model with Basis spreads and Collateral

3. **Pricing of Vanilla Products under the Collateralization**

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5. **Conclusions**
This talk is based on the following three works:


General Needs

- **Single Currency products**
  - Swap
  - Swaption, Cancellable Swap, Cap/Floor.
  - TARN/Callable/Range Accrual of CMS, CMS-spread, Inverse Floater, etc.

- **Multi-Currency products**
  - Short term Vanilla FX products
  - Cross Currency Swap
  - TARN/Callable/Knockout of PRDC, PRDC with chooser option
  - Quantoed CMS, CMS-spread, etc.

- We need a term structure model in multi-currency environment.
## Textbook-style Implementation

### Basic Setup (an Example)

- **Complete probability space** $(\Omega, \mathcal{F}, P)$ on which $d$-dimensional Brownian motion $W$ is defined. $\{\mathcal{F}_t\}_{0 \leq t \leq \bar{T}}$: Augmented Brownian filtration
- $r = \{r(t); 0 \leq t \leq \bar{T}\}$: The instantaneous risk-free short rate following an Itô process
- $\beta(t)$: Price of the money market account at time $t$, where
  $$\beta(t) = \exp\left(\int_0^t r(s)ds\right).$$
- $P_{t,T}$: Price of the risk-free zero-coupon bond with maturity $T$ at time $t$
  $$P_{t,T} = \mathbb{E}_t^Q \left[ \exp\left(-\int_t^T r(s)\,ds\right) \right] \quad (\mathbb{E}_t^Q[\cdot] \equiv \mathbb{E}^Q[\cdot|\mathcal{F}_t])$$
- $Q$: The Equivalent Martingale Measure (EMM) or the risk-neutral measure, where the numeraire is the money market account.
- Asset prices as well as all the factors and indexes affecting the asset prices are assumed to follow Itô processes.
Textbook-style Implementation

- **Curve Construction**
  - Most Standard IRS: Fixed-vs-Libor

\[
\text{IRS}_M(t) = \sum_{m=1}^{M} \Delta m P_{t,T_m} = \sum_{m=1}^{M} \delta_m P_{t,T_m} E_t^{T_m}[L(T_m-1, T_m)]
\]

Here, \(P_{t,T}\) is the price of risk-free zero-coupon bond, and \(E_t^{T_m}[\cdot]\) denotes the forward expectation with \(P_{t,T_m}\) as the numeraire.

- Exchange Fixed Rate (Swap Rate) with Libor
- Market Quotes are the Swap Rates for various terms
Textbook-style Implementation

- Treat Libor as risk-free rate
  \[ E^T_m [L(T_{m-1}, T_m)] = \frac{1}{\delta_m} \left( \frac{P_{t,T_{m-1}}}{P_{t,T_m}} - 1 \right) \]

- Determine \( \{P_{t,T}\} \) by Bootstrap
  \[ P_{t,T_M} = \frac{P_{t,T_0} - IRS_M(t) \sum_{m=1}^{M-1} \Delta_m P_{t,T_m}}{1 + IRS_M(t) \Delta_M} \]

- Repeat the bootstrap for each currency ⇒ Discounting rates (and hence, forward Libors) for each currency
Textbook-style Implementation

- **Implementation in Multi-Currency Environment**
  - **Initial conditions**
    - spot FXs
    - discounting rate for each currency
  - **Simulation based on**
    - Arbitrage-free dynamics of risk-free rate
    - Arbitrage-free dynamics of spot FX
  - Derive reference rates (such as Libor, Swap rate, etc) from simulated discounting rate.
  - Calibration on volatilities and correlations

⇒ Basically, we are done..
Problems in Textbook-style Implementation

- Is there any problem for the textbook-style implementation?
  ⇒ Yes, ..., and in Very Critical way...

- The potential loss can be a few percentage points of notional outstanding for IR/FX related trades...
Problems in Textbook-style Implementation

We will see...

List of problems:

- **wrong discounting** for secured (collateralized) trades
- **mispricing** of various important swaps
  - Tenor Swap (TS)
  - Cross Currency Swap (CCS)
    - \( \Rightarrow \) FX Forward
  - Overnight Index Swap (OIS)
Problems in Textbook-style Implementation

- **Tenor Swap (TS)**
  - Libor (short tenor) +spread
  - Libor (long tenor)

  ![Diagram of Tenor Swap](image)

- **Textbook-style Implementation** ⇒ **Zero spread.**
- **Market:** Spread is quite significant and volatile since late 2007.

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1It is also common that payment of short-tenor Leg is compounded and paid at the same time with the other Leg.
Problems in Textbook-style Implementation

Historical data for JPY 3m/6m Tenor Swap Spread

Figure: Source:Bloomberg
Problems in Textbook-style Implementation

Historical data for USD 3m/6m Tenor Swap Spread

Figure: Source: Bloomberg
Problems in Textbook-style Implementation

- Cross Currency Swap (CCS)

![Graph showing Cross Currency Swap (CCS)]

- Textbook-style Implementation ⇒ Zero spread.
- Market:
  - Spread is quite significant and volatile for long time.
  - Drastic/Rapid change in recent years.
Problems in Textbook-style Implementation

Historical data for USDJPY CCS spread

Figure: Source:Bloomberg
Problems in Textbook-style Implementation

- **Unsecured Funding/Contract (old picture)**

  ![Diagram](image)

  - Libor is unsecured offer rate in the interbank market.
  - Libor discounting is appropriate for unsecured trades between financial firms with Libor credit quality.
  - Libor discounting makes the present value of Loan zero.
Problems in Textbook-style Implementation

- **Collateralized (Secured) Contract (current picture)**

  ![Diagram](#)

  - No outright cash flow (collateral = PV)
  - No external funding is needed.
  - Funding is provided by collateral agreement
    \[ \Rightarrow \text{Libor discounting is inappropriate.} \]
Problems in Textbook-style Implementation

Collateralized OTC Derivatives Market

- **30% (2003) → 65% (2009)** in terms of trade volume (ISDA)
- Fixed income trade among financial firms are mostly collateralized
- **84%** of collateral (received) is Cash
  (USD = 49.4%, EUR = 29.5%)
- Collateral rate on cash is the overnight rate controlled by the central bank: (Fed-Fund Rate, EONIA, MUTAN...)
- Remainings are mostly government securities
Problems in Textbook-style Implementation

Overnight Index Swap (OIS)

- **Overnight Index Swap (OIS)**
  - OIS rate
  - Compounded ON

Floating side: Daily compounded ON rate
Usually, there is only one payment for < 1yr.
Market Quote: fixed rate, called OIS rate.
Problems in Textbook-style Implementation

Historical data for USD&JPY Libor-OIS spread

Figure: Source: Bloomberg
Implications for Financial Firms

- **Wrong discounting for secured trades**
  - Mispricing of future cash flow (over/under-estimate)
  - Significant impact on multi-currency trades where there usually exist final notional exchanges.
  - Change $\sim$ Notional $\times$ Duration $\times$ (Diff. of discounting rate)

- **Inconsistency with CCS**
  - Implied FX forwards are off the market
  - Mispricing of foreign Libors
  - Long-dated FX products are most severely affected.

- **Wrong forward Libors**
  - Overestimation of forward Libors with short tenors
  - Accumulation of receiver position of Libors with short tenor
  $\Rightarrow$ See next page example.
Implications for Financial Firms

Example: Funding legs of structured products

- Tenor of Libor is usually 3m, or 6m.
- Simulated discounting rate is based on 6m Libor. (Standard IRS convention for JPY)
- Overestimation of value of receipt of 3m Libor
- It is easy to make deals with 3m Libors.

\[
\text{Loss} \sim \text{Notional}(3m \text{ Funding}) \times PV01(\text{or Annuity}) \times TS \text{ spread}
\]
Review of Recent Works and Un-addressed Issues

- **M. Johannes and S. Sundaresan (2007)**
  - Point out the importance of collateralization on swap rates.
  - Provide a pricing formula for a collateralized contract.
  - Introducing an unobservable "convenience yield" and put more emphasis on the empirical study for the dynamics of the swap rate and the convenience yield in the US market.

- **V. Piterbarg (2010)**
  - Pricing formula for the collateralized stock options similar to the one given in M. Johannes et.al. (2007).
  - Treating partially collateralized case, but the counter-party default risk is neglected.
Review of Recent Works and Un-addressed Issues

- Kijima et.al. (2009)
  - Consistent CCS pricing by separating JPY discounting and Libor curves, while assuming USD Libor as risk-free rate, and a numerical demonstration using a simple short-rate based model.
  - No discussion for collateralization and tenor spreads.
  - As for curve construction, it is a conventional method being used by US financial firms for many years where USD Libor is their funding cost.

\[
IRS_N \sum_{n=1}^{N} \Delta_n P_{t,T_n} = \sum_{n=1}^{N} \delta_n E_t^{T_n} [L(T_{n-1}, T_n; \tau)] P_{t,T_n}
\]

\[
NP_{JPY} \left\{ -P_{t,T_0} + \sum_{n=1}^{N} \delta_n \left( E_t^{T_n} [L(T_{n-1}, T_n; \tau)] + b_N \right) P_{t,T_n} + P_{t,T_N} \right\} = f_x(t) \left\{ -P_{t,T_0} + \sum_{n=1}^{N} \delta_n E_t^{T_n} [L(S(T_{n-1}, T_n; \tau))] P_{t,T_n} + P_{t,T_N} \right\} (= 0)
\]

\[
\Rightarrow \sum_{n=1}^{N} (\Delta_n IRS_N + \delta_n b_N) P_{t,T_n} = P_{t,T_0} - P_{t,T_N}
\]
Review of Recent Works and Un-addressed Issues

- Ametrano and Bianchetti (2009)
  - Bootstrapping the swap quotes within each tenor separately, assuming "segmentation" of the market.
  - Arbitrage possibility due to multiple discounting curves within single currency.

- Bianchetti (2008)
  - Using FX analogy to remove arbitrage possibility in multi-curve setup.
  - Calibration of basis spreads needs to be done by quanto correction.
  - Curve construction cannot be separated from option calibration.
  - No guarantee that one can recover the observed basis spreads with reasonable size of volatility and correlation.
Review of Recent Works and Un-addressed Issues

- **F. Mercurio (2008)**
  - Introducing an efficient simulation scheme with multiple curves in Libor Market Model in single currency environment.
  - Referring to the work of Ametrano et al. (2009) for details of curve construction and assuming the existence of constructed yield curves.

- **F. Mercurio (March 2010)**
  - Adopting the OIS-based curve construction in single currency, which is equivalent to a result given in one of our works ("A note on construction of multiple swap curves... (Jul. 2009)").
  - Assuming the independence between OIS and Libor-OIS spreads to get analytical tractability.
  - Proposing the specific form of volatility functions for the OIS process to retain the consistency among simple rates with different tenors in OIS.
Un-addressed Issues in Existing Works

- Un-addressed issues in these works
  - No discussion for the term structure construction under the collateralization.
    - It is crucial for the model to handle the situation where the payment currency and the collateral currency are different.
    - Example: CCS. Moreover, US financial firms to prefer USD cash collateral even for the JPY single currency products.
  - In addition to the collateralization issues, existing models can work only in single currency environment.
    - If financial firms really adopt these models, they are forced to have a set of curves for each currency desk, but they are inconsistent with cross currency markets.
    - It makes impossible to carry out the consistent risk-management for all currencies across the desks, which is crucial for most of the financial institutions.
Criteria for the New Model

- **Consistent discounting/forward curve construction**
  - Price all types of IR swaps correctly: (OIS, IRS, TS)
  - Take collateralization into account.
  - Maintain consistency in multi-currency environment (FX forward, CCS, MtMCCS)

- **Stochastic Modeling of Basis spreads**
  - Systematic calibration procedures
  - Allow volatility risk management
Contributions of Our Works

- "A note on construction of multiple swap curves...(Jul. 2009)"
  - Extend the formula in M. Johannes et al. (2007) to the contract whose payment and collateral currencies are different.
  - Provide consistent term structure construction in the presence of basis spreads and collateral in a multi-currency environment for the first time.

- "A market model of interest rates...(Dec. 2009)"
  - Making the multi-currency curve construction more tractable by a simplifying assumption while maintaining the flexibility of calibration to the observable market quotes.
  - General multi-currency HJM framework in the presence of collateral and stochastic basis spreads.

First framework ready to be implemented for consistent pricing and risk-management in global derivatives business.
Pricing under the Collateralization

**Assumption**
- Continuous adjustment of collateral amount
- Perfect collateralization by Cash
- No threshold

**Comments**
- Daily margin call is the market best practice.
- By making use of Repo / Reverse-Repo, other collateral assets can be converted into the equivalent amount of cash collateral.
- General Collateral (GC) repo rate closely tracks overnight rate.
Proposition

$T$-maturing European option under the collateralization is given by

$$h^{(i)}(t) = E_t^{Q_i} \left[ e^{-\int_t^T r^{(i)}(s) ds} \left( e^{\int_t^T y^{(j)}(s) ds} \right) h^{(i)}(T) \right]$$

where,

$$y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$$

- $h^{(i)}(T)$: option payoff at time $T$ in currency $i$
- collateral is posted in currency $j$
- $c^{(j)}(s)$: instantaneous collateral rate of currency $j$ at time $s$
- $r^{(j)}(s)$: instantaneous risk-free rate of currency $j$ at time $s$
- $Q_i$: Money-Market measure of currency $i$
Collateral amount in currency $j$ at time $s$ is given by \( \frac{h(i)(s)}{f_{x(i,j)}(s)} \), which is invested at the rate of $y(j)(s)$:

\[
h(i)(t) = E_t^{Q_i} \left[ e^{-\int_t^T r(i)(s) ds} h(i)(T) \right] + f_{x(i,j)}(t) E_t^{Q_j} \left[ \int_t^T e^{-\int_t^s r(j)(u) du} y(j)(s) \left( \frac{h(i)(s)}{f_{x(i,j)}(s)} \right) ds \right]
\]

\[
= E_t^{Q_i} \left[ e^{-\int_t^T r(i)(s) ds} h(i)(T) + \int_t^T e^{-\int_t^s r(i)(u) du} y(j)(s) h(i)(s) ds \right] .
\]

- $f_{x(i,j)}(t)$: Foreign exchange rate at time $t$ representing the price of the unit amount of currency "$j" in terms of currency "$i".  

Note that $X(t) = e^{-\int_0^t r(i)(s) ds} h(i)(t) + \int_0^t e^{-\int_0^s r(i)(u) du} y(j)(s) h(i)(s) ds$ is a $Q_i$-martingale. Then, the process of the option value is written by  

\[
dh(i)(t) = \left( r(i)(t) - y(j)(t) \right) h(i)(t) dt + dM(t)
\]

with some $Q_i$-martingale $M$. This establishes the proposition.
Remark : Derivation using the collateral account

- **Dynamics of a collateral account in currency** $j$, ”$V(j)$”, **is given by**

$$dV(j)(s) = y(j)(s)V(j)(s)ds + a(s)d\left[h(i)(s)/f_x^{i,j}(s)\right],$$

where $a(s)$ **is the number of positions of the derivative**. We have

$$V(j)(T) = e^{\int_t^T y(j)(s)ds}V(j)(t) + \int_t^T e^{\int_s^T y(j)(u)du}a(s)d\left[h(i)(s)/f_x^{i,j}(s)\right].$$

- **Adopt a trading strategy:**

  - $V(j)(t) = h(i)(t)/f_x^{i,j}(t)$, $a(s) = \exp\left(\int_t^s y(j)(u)du\right)$,

  which yields $V(j)(T) = e^{\int_t^T y(j)(s)ds}\left(h(i)(T)/f_x^{i,j}(T)\right)$.

- **Hence**, we obtain

$$h(i)(t) = V(j)(t)f_x^{i,j}(t) = E_{Q_i}^t \left[e^{-\int_t^T r(i)(s)ds}V(j)(T)f_x^{i,j}(T)\right]$$

$$= E_{Q_i}^t \left[e^{-\int_t^T r(i)(s)ds}\left(e^{\int_t^T y(j)(s)ds}\right)h(i)(T)\right].$$
Corollary

If payment and collateral currencies are the same, the option value is given by

\[ h(t) = E_t^Q \left[ e^{-\int_t^T c(s) ds} h(T) \right] \]

\[ = D(t, T) E_t^{T_c} [h(T)] . \]

In the 2nd line, we have defined collateralized forward measure \( T^c \), where the collateralized zero coupon bond

\[ D(t, T) = E_t^Q \left[ e^{-\int_t^T c(s) ds} \right] \]

is used as the numeraire.
Curve Construction in Single Currency

- **Collateralized Overnight Index Swap**
  - payment and collateral currencies are the same
  - collateral rate is given by the overnight rate
- **Condition for the length-** $N$ **OIS rate:**
  \[
  \text{OIS}_N(t) \sum_{n=1}^{N} \Delta_n E^Q_t \left[ e^{- \int_{t}^{T_n} c(s) \, ds} \right] = \sum_{n=1}^{N} E^Q_t \left[ e^{- \int_{t}^{T_n} c(s) \, ds} \left( e^{\int_{T_n}^{T_{n-1}} c(s) \, ds} - 1 \right) \right]
  \]
  or, equivalently,
  \[
  \text{OIS}_N(t) \sum_{n=1}^{N} \Delta_n D(t, T_n) = D(t, T_0) - D(t, T_N).
  \]
  Then, the collateralized zero coupon bond price can be bootstrapped as
  \[
  D(t, T_N) = \frac{D(t, T_0) - \text{OIS}_N(t) \sum_{n=1}^{N-1} \Delta_n D(t, T_n)}{1 + \text{OIS}_N(t) \Delta_N}.
  \]
Curve Construction in Single Currency

- **Collateralized IRS**

\[
\text{IRS}_M(t) \sum_{m=1}^{M} \Delta_m D(t, T_m) = \sum_{m=1}^{M} \delta_m D(t, T_m) E_t^{T_c} [L(T_{m-1}, T_m; \tau)]
\]

- **Collateralized TS\(^2\)**

\[
\sum_{n=1}^{N} \delta_n D(t, T_n) \left( E_t^{T_c} [L(T_{n-1}, T_n; \tau_S)] + TS_N(t) \right) = \sum_{m=1}^{M} \delta_m D(t, T_m) E_t^{T_c} [L(T_{m-1}, T_m; \tau_L)]
\]

Market quotes of collateralized OIS, IRS, TS, and proper spline method allow us to determine

\[
\{ D(t, T) \}, \quad \{ E_t^{T_c} [L(T_{m-1}, T_m, \tau)] \}
\]

for all the relevant \(T, \ T_m\) and tenor \(\tau\) of Libor.

\(^2\)The impact from the possible compounding of the short-tenor Leg is negligible.
Curve Construction in Multiple Currencies

- **OIS, IRS, TS**
  - Repeat the same procedures in the previous section.
  - Obtain \( \{D^{(i)}(t, T)\}, \{E_t^{\tau_m,(i)}[L^{(i)}(T_{m-1}, T_m, \tau)]\} \) for each currency "i"

- **CCS and FX forwards**
  - Various practical issues under the most generic setup.
    - Forward expectation involves covariance between \( y(t) = r(t) - c(t) \) and other variables.
    - No separate market quote is available for each collateral currency.
    - ...

Curve Construction in Multiple Currencies

- Collateralized FX Forward: $f_{x}^{(i,j)}(t, T)$
  - Amount of currency $i$ to be exchanged by the unit amount of currency $j$, assuming collateralization is done by currency $k$:
    \[
    0 = f_{x}^{(i,j)}(t, T) E^{Q_{i}}_{t} \left[ e^{-\int_{t}^{T} r^{(i)}(s) ds} e^{\int_{t}^{T} y^{(k)}(s) ds} \right] \\
    \quad - f_{x}^{(i,j)}(t) E^{Q_{j}}_{t} \left[ e^{-\int_{t}^{T} r^{(j)}(s) ds} e^{\int_{t}^{T} y^{(k)}(s) ds} \right]
    \]
    and then,
    \[
    f_{x}^{(i,j)}(t, T) = f_{x}^{(i,j)}(t) \frac{P^{(j)}(t, T)}{P^{(i)}(t, T)} \left( \frac{E^{T^{(j)}}_{t} \left[ e^{\int_{t}^{T} y^{(k)}(s) ds} \right]}{E^{T^{(i)}}_{t} \left[ e^{\int_{t}^{T} y^{(k)}(s) ds} \right]} \right).
    \]

- If the spread $y$ is stochastic, the currency triangle, such as $\text{USD/JPY} \times \text{EUR/USD} = \text{EUR/JPY}$, holds only among those with the same collateral currency.
Curve Construction in Multiple Currencies

Assumption

Spread between the risk-free and collateral rate of each currency

\[ y^{(i)}(t) = r^{(i)}(t) - c^{(i)}(t) \]

is a deterministic function of time.

We will achieve:

- Enough flexibility to fit available market quotes.
- Currency triangle relation holds among FX forwards.
Remark:

Option in currency $i$ collateralized with currency $j$ under the assumption of deterministic spread:

$$ h^{(i)}(t) = E^Q_t \left[ e^{- \int_t^T r^{(i)}(s) ds} \left( e^{\int_t^T y^{(j)}(s) ds} h^{(i)}(T) \right) \right] $$

$$ = P^{(i)}(t, T) e^{\int_t^T y^{(j)}(s) ds} E^{\tau(i)}_t \left[ h^{(i)}(T) \right] $$

$$ = D^{(i)}(t, T) e^{\int_t^T y^{(j,i)}(s) ds} E^{\tau(i)}_t \left[ h^{(i)}(T) \right] , $$

where $y^{(j,i)}(s) = y^{(j)}(s) - y^{(i)}(s)$. On the other hand,

$$ h^{(i)}(t) = E^Q_t \left[ e^{- \int_t^T c^{(i)}(s) ds} e^{\int_t^T y^{(j,i)}(s) ds} h^{(i)}(T) \right] $$

$$ = D^{(i)}(t, T) e^{\int_t^T y^{(j,i)}(s) ds} E^{\tau^c(i)}_t \left[ h^{(i)}(T) \right] , $$

and thus,

$$ E^{\tau^c(i)}_t \left[ h^{(i)}(T) \right] = E^{\tau(i)}_t \left[ h^{(i)}(T) \right] . $$
Curve Construction in Multiple Currencies

- When spread $y$ is deterministic, the previous forward FX becomes

$$f_{x}^{(i,j)}(t, T) = f_{x}^{(i,j)}(t) \frac{P^{(j)}(t, T)}{P^{(i)}(t, T)} = f_{x}^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)} e^{\int_{t}^{T} y^{(i,j)}(s) ds},$$

which is independent from the choice of collateral currency.

- Fitting to FX forward
  - Bootstrap the spread \{y^{(i,j)}(s)\} using the relation:

$$f_{x}^{(i,j)}(t, T) = f_{x}^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)} e^{\int_{t}^{T} y^{(i,j)}(s) ds}$$

- Except the $y^{(i,j)}$, all the variables are already fixed or observable in the market.

- It can be used only for relatively short maturities due to liquidity reason.
Curve Construction in Multiple Currencies

- **Fitting to CCS with Constant Notional**

  "j" Libor + spread
  
  \[ N_j = 1 \]

  "i" Libor
  
  \[ N_i (\text{set by spot fx}) \]

  \[ \text{col. currency} \]

  \[
  PV_i(t) = -E_t^Q_i \left[ e^{-\int_t^{T_0} c^{(i)}(s)ds} \right] + E_t^Q_i \left[ e^{-\int_t^{T_N} c^{(i)}(s)ds} \right] \\
  + \sum_{n=1}^{N} \delta_n^{(i)} E_t^{Q_i} \left[ e^{-\int_t^{T_n} c^{(i)}(s)ds} L^{(i)}(T_{n-1}, T_n; \tau) \right]
  
  PV_j(t) = -E_t^Q_j \left[ e^{-\int_t^{T_0} (r^{(j)}(s)-y^{(i)}(s))ds} \right] + E_t^Q_j \left[ e^{-\int_t^{T_N} (r^{(j)}(s)-y^{(i)}(s))ds} \right] \\
  + \sum_{n=1}^{N} \delta_n^{(j)} E_t^{Q_j} \left[ e^{-\int_t^{T_n} (r^{(j)}(s)-y^{(i)}(s))ds} \left( L^{(j)}(T_{n-1}, T_n; \tau) + B^{\text{CCS}}_N(t) \right) \right]
  \]


After simplification,

\[ PV_i(t) = -D^{(i)}(t, T_0) + D^{(i)}(t, T_N) \]
\[ + \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t, T_n) E_t^{\tau_{n,(i)}} \left[ L^{(i)}(T_{n-1}, T_n; \tau) \right] \]

\[ PV_j(t) = -D^{(j)}(t, T_0) e^{\int_t^{T_0} y^{(i,j)}(s) ds} + D^{(j)}(t, T_N) e^{\int_t^{T_N} y^{(i,j)}(s) ds} \]
\[ + \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_t^{T_n} y^{(i,j)}(s) ds} \left( E_t^{\tau_{n,(j)}} \left[ L^{(j)}(T_{n-1}, T_n; \tau) \right] + B_N^{CCS}(t) \right), \]

where \( y^{(i,j)} \) is the only unknown.

**Bootstrap** \( \{ y^{(i,j)}(s) \} \) using \( N_i PV_i(t) = f_{x}^{(i,j)}(t) PV_j(t) \).
Curve Construction in Multiple Currencies

Interpretation of the spread between the risk-free and collateral rates

- The ON rate controlled by the central bank is not necessarily equal to the risk-free rate.
- Imposing $r = c$ leaves us no freedom to calibrate FX forwards and CCS.
- $y^{(i,j)}$ may be reflecting the difference of the stance between the two central banks of currency "i" and "j".
There are two different types of Cross Currency Swap.

- Constant Notional Cross Currency Swap (CNCCS) CCS which we have already explained
- Mark-to-Market Cross Currency Swap (MtMCCS)

Both of the instruments play critical roles in long-dated FX market.

We will see quite different risk characteristics between the twos.
Two different types of Cross Currency Swap

- **Mark-to-Market Cross Currency Swap (MtMCCS)**
  - One of the most liquid long-dated cross currency product
  - Smaller FX exposure than CCS with constant notional
  - The notional of the Leg which pays Libor flat (usually USD) is reset at the every start of the Libor calculation period based on the spot FX at that time.
    \[
    (\rightarrow \Delta f_{x}^{(i,j)} = f_{x}^{(i,j)}(t + \tau) - f_{x}^{(i,j)}(t))
    \]
  - The notional and spread of the other leg is kept constant throughout the contract period.

\[
N_{i} = f_{x}^{(i,j)}(t) \\
N_{j} \equiv 1
\]

\[
N_{i} \ast \delta L_{i} \quad N_{i}^{'} \ast \delta L_{i} \\
(N_{i}^{'} = N_{i} + \Delta f_{x}^{(i,j)})
\]
Two different types of Cross Currency Swap

Definition: Libor-OIS (with single payment) spread

\[ B(t, T_k; \tau) = L^c(t, T_{k-1}, T_k; \tau) - L^{OIS}(t, T_{k-1}, T_k) \]

where

\[ L^c(t, T_{k-1}, T_k; \tau) = E^{T_k}_t [L(T_{k-1}, T_k; \tau)] \]

\[ L^{OIS}(t, T_{k-1}, T_k) = E^{T_k}_t \left[ \frac{1}{\delta_k} \left( \frac{1}{D(T_{k-1}, T_k)} - 1 \right) \right] \]

\[ = \frac{1}{\delta_k} \left( \frac{D(t, T_{k-1})}{D(t, T_k)} - 1 \right) \]
Two different types of Cross Currency Swap

- j-Leg of $T_0$-start $T_N$-maturing MtMCCS collateralized by currency $i$

The notional is fixed, and hence exactly the same with the j-Leg of CNCCS.

$$PV_j(t) = -E_t^{Q_j} \left[ e^{-\int_t^{T_0} (r(j)(s)-y(i)(s))ds} + E_t^{Q_j} \left[ e^{-\int_t^{T_N} (r(j)(s)-y(i)(s))ds} \right] \right]$$

$$+ \sum_{n=1}^{N} \delta_n^{(j)} E_t^{Q_j} \left[ e^{-\int_t^{T_n} (r(j)(s)-y(i)(s))ds} \left( L(j)(T_{n-1}, T_n; \tau) + B_{N^M^{MtM}}(t) \right) \right]$$

$$= \sum_{n=1}^{N} \delta_n^{(j)} D(j)(t, T_n)e^{\int_t^{T_n} y(i,j)(s)ds} \left( B(j)(t, T_n; \tau) + B_{N^M^{MtM}}(t) \right)$$

$$+ \sum_{n=1}^{N} D(j)(t, T_{n-1})e^{\int_t^{T_{n-1}} y(i,j)(s)ds} \left( e^{\int_T^{T_{n-1}} y(i,j)(s)ds} - 1 \right)$$
Two different types of Cross Currency Swap

- i-Leg of $T_0$-start $T_N$-maturing MtMCCS collateralized by currency $i$

\[
N_i = f^{(i,j)}(t) \\
N'_i = f^{(i,j)}(t + \tau)
\]

\[
PV_i(t) = - \sum_{n=1}^{N} E_t^{Q_i} \left[ e^{-\int_t^{T_n-1} c^{(i)}(s) ds} f_x^{(i,j)}(T_{n-1}) \right] \\
+ \sum_{n=1}^{N} E_t^{Q_i} \left[ e^{-\int_t^{T_n} c^{(i)}(s) ds} f_x^{(i,j)}(T_{n-1}) \left( 1 + \delta_n^{(i)} L^{(i)}(T_{n-1}, T_n; \tau) \right) \right] \\
= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t, T_n) E_t^{T_n^{c,(i)}} \left[ f_x^{(i,j)}(T_{n-1}) B^{(i)}(T_{n-1}, T_n; \tau) \right]
\]
Two different types of Cross Currency Swap

- **i-Leg of** $T_0$-start $T_N$-maturing CNCCS collateralized by currency $i$ (Revisited)

  \[
  PV_i(t) = N_i \left\{ -E_t^{Q_i} \left[ e^{-\int_{T_0}^T c^{(i)}(s) ds} \right] + E_t^{Q_i} \left[ e^{-\int_{T_N}^T c^{(i)}(s) ds} \right] + \sum_{n=1}^N \delta_n^{(i)} E_t^{Q_i} \left[ e^{-\int_{T_0}^{T_n} c^{(i)}(s) ds} L^{(i)}(T_n-1, T_n; \tau) \right] \right\}
  \]

  \[
  = N_i \sum_{n=1}^N \delta_n^{(i)} D^{(i)}(t, T_n) B^{(i)}(t, T_n; \tau)
  \]

- $N_i$ is set by the spot FX at the trade inception, and kept constant.

Par CCS basis spread is obtained by

\[
PV_i(t) / f_x^{(i,j)}(t) = PV_j(t)
\]
Two different types of Cross Currency Swap

\[ T_0 \text{-start } T_N \text{-maturing } (i, j) \text{-MtMCCS par spread} \]

\[ \text{(collateralized by currency } i) \]

\[
B_{N}^{\text{MtM}}(t, T_0, T_N; \tau) = \\
\left[ \sum_{n=1}^{N} \delta_{n}^{(j)} D(j)(t, T_n) e^{\int_{t}^{T_n} y(i,j)(s) ds} \right. \times \\
\left\{ \frac{\delta_{n}^{(i)}}{\delta_{n}^{(j)}} E_{t}^{T_n, (i)} \left[ \frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(t, T_n)} B(i)(T_{n-1}, T_n; \tau) \right] - B(j)(t, T_n; \tau) \right\} \\
- \sum_{n=1}^{N} D(j)(t, T_{n-1}) e^{\int_{t}^{T_{n-1}} y(i,j)(s) ds} \left( e^{\int_{T_{n-1}}^{T_n} y(i,j)(s) ds} - 1 \right) \right] \\
/ \sum_{n=1}^{N} \delta_{n}^{(j)} D(j)(t, T_n) e^{\int_{t}^{T_n} y(i,j)(s) ds}
\]

- MtMCCS spread observable in the market can be calibrated by adjusting correlation between \( f_{x}^{(i,j)} \) and Libor-OIS spread \( B(i) \).
Two different types of Cross Currency Swap

$T_0$-start $T_N$-maturing $(i, j)$-CNCCS par spread (collateralized by currency $i$)

$$B_{N}^{CCS}(t, T_0, T_N; \tau) =$$

$$\left[ \sum_{n=1}^{N} \delta_n^{(j)} D(j)(t, T_n) e^{\int_{t}^{T_n} y^{(i,j)}(s) ds} \times \right.$$  

$$\left\{ \frac{\delta_n^{(i)}}{\delta_n^{(j)}} \frac{N(i)}{f_x^{(i,j)}(t, T_n)} B^{(i)}(t, T_n; \tau) - B^{(j)}(t, T_n; \tau) \right\}$$

$$- \sum_{n=1}^{N} D^{(j)}(t, T_{n-1}) e^{\int_{t}^{T_{n-1}} y^{(i,j)}(s) ds} \left( e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s) ds} - 1 \right) \right]$$

$$/ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_{t}^{T_n} y^{(i,j)}(s) ds}$$

- If Libor-OIS spread of $i$-Leg (USD) is zero, par basis spreads of the two CCSs are exactly the same.
Two different types of Cross Currency Swap

- Difference of FX exposure between the two CCSs
- Suppose that we are now at time \( T \) after the inception of trade at time \( t \)
- Label the next closest payment time as \( T_S \).
- j-Leg (eg. JPY) has the same value both for the CNCCS and MtMCCS

i-Leg value of CNCCS at time \( T \):

\[
P V_i(T) = N_i \left\{ D_{T,T_S}^{(i)} \delta_S^{(i)} L(T_{S-1}, T_S; \tau) + D_{T,T_N}^{(i)} \right\} + N_i \sum_{n=S+1}^{N} \delta_n^{(i)} E_T^{Q_i} \left[ e^{-\int_T^{T_n} c^{(i)}(s)ds} L^{(i)}(T_{n-1}, T_n; \tau) \right]
\]

\[
= N_i \left\{ D_{T,T_S}^{(i)} \left( 1 + \delta_S^{(i)} L^{(i)}(T_{S-1}, T_S; \tau) \right) + \sum_{n=S+1}^{N} D_{T,T_n}^{(i)} \delta_n^{(i)} B^{(i)}(T, T_n; \tau) \right\}
\]
Two different types of Cross Currency Swap

i-Leg value of MtMCCS at time $T$:

$$PV_i(T) = f_{x,i}^{(i,j)}(T_{S-1})D^{(i)}_{T,T_S} \left( 1 + \delta^{(i)}_S L^{(i)}(T_{S-1}, T_S; \tau) \right)$$

$$- \sum_{n=S+1}^{N} E^{Q_i}_T \left[ e^{-\int_{T}^{T_{n-1}} c^{(i)}(s) ds} f^{(i,j)}_{x}(T_{n-1}) \right]$$

$$+ \sum_{n=S+1}^{N} E^{Q_i}_T \left[ e^{-\int_{T}^{T_{n}} c^{(i)}(s) ds} f^{(i,j)}_{x}(T_{n-1}) \left( 1 + \delta_n^{(i)} L^{(i)}(T_{n-1}, T_n; \tau) \right) \right]$$

$$= f_{x,i}^{(i,j)}(T_{S-1})D^{(i)}_{T,T_S} \left( 1 + \delta^{(i)}_S L^{(i)}(T_{S-1}, T_S; \tau) \right)$$

$$+ \sum_{n=S+1}^{N} D^{(i)}_{T,T_n} \delta_n^{(i)} E^{T^{(i)}_n}_T \left[ f^{(i,j)}_{x}(T_{n-1}) B^{(i)}(T_{n-1}, T_n; \tau) \right]$$
Two different types of Cross Currency Swap

The value of j-Legs are the same between the twos, and remains \( \sim 1 \), in terms of currency \( j \).

**i-Leg value in terms of currency \( j \)**

- **CNCCS**

\[
\frac{PV_i(T)}{f_x^{(i,j)}(T)} = \frac{N_i}{f_x^{(i,j)}(T)} \left\{ D_{T,T_S}^{(i)} \left( 1 + \delta_S^{(i)} L^{(i)}(T_{S-1}, T_S; \tau) \right) + \sum_{n=S+1}^{N} D_{T,T_n}^{(i)} \delta_n^{(i)} B^{(i)}(T, T_n; \tau) \right\}
\]

- **MtMCCS**

\[
\frac{PV_i(T)}{f_x^{(i,j)}(T)} = \frac{f_x^{(i,j)}(T_{S-1})}{f_x^{(i,j)}(T)} D_{T,T_S}^{(i)} \left( 1 + \delta_S^{(i)} L^{(i)}(T_{S-1}, T_S; \tau) \right) + \sum_{n=S+1}^{N} D_{T,T_n}^{(i)} \delta_n^{(i)} E_{T_n}^{c, (i)} \left[ \frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(T)} B^{(i)}(T_{n-1}, T_n; \tau) \right]
\]
Two different types of Cross Currency Swap

Summary of CNCCS and MtMCCS

- Both CCSs have the same par basis spread if $B(i)$ (or, USD Libor-OIS spread) is zero.
- Potentially significant mis-pricing.
- CNCCS has significant FX exposure.
- MtMCCS has only a limited size of FX exposure.
Term Structure Model with Single Currency

Make the multiple reference rates stochastic consistently with no-arbitrage conditions in an HJM-type framework.

- **Definition: Instantaneous Forward Collateral Rate**
  \[ c(t, T) = -\frac{\partial}{\partial T} \ln D(t, T) \]
  or
  \[ D(t, T) = \exp \left( -\int_t^T c(t, s) ds \right) \]

**Proposition**

The SDE of the forward collateral rate under the Money-Market measure \( Q \) is given by

\[
dc(t, s) = \sigma_c(t, s) \cdot \left( \int_t^s \sigma_c(t, u) du \right) dt + \sigma_c(t, s) \cdot dW^Q(t),
\]

where \( W^Q \) is the \( d \)-dimensional Brownian motion under the measure \( Q \).
Term Structure Model with Single Currency

Write the dynamics of $c(t, s)$ as

$$dc(t, s) = \alpha(t, s)dt + \sigma_c(t, s) \cdot dW^Q(t).$$

Applying Itô’s formula,

$$
\frac{dD(t, T)}{D(t, T)} = \left\{ c(t) - \int_t^T \alpha(t, s)ds + \frac{1}{2} \left\| \int_t^T \sigma_c(t, s)ds \right\|^2 \right\} dt
- \left( \int_t^T \sigma_c(t, s)ds \right) \cdot dW^Q_t.
$$

Imposing the fact that the drift rate of $D(t, T)$ is $c(t)$:

$$
\alpha(t, s) = \sum_{j=1}^d [\sigma_c(t, s)]_j \left( \int_t^s \sigma_c(t, u)du \right)_j
= \sigma_c(t, s) \cdot \left( \int_t^s \sigma_c(t, u)du \right).
$$
Proposition

The SDE of Libor-OIS spread in Money-Market measure is given by

\[
\frac{dB(t, T; \tau)}{B(t, T; \tau)} = \sigma_B(t, T; \tau) \cdot \left( \int_t^T \sigma_c(t, s) ds \right) dt + \sigma_B(t, T; \tau) \cdot dW^Q(t) .
\]

\(B(\cdot, T; \tau)\) is a martingale under the collateralized forward measure \(T^c\):

\[
dB(t, T; \tau) = B(t, T; \tau) \sigma_B(t, T; \tau) \cdot dW^{T^c}(t)
\]

Maruyama-Girsanov’s theorem indicates

\[
dW^{T^c}(t) = \left( \int_t^T \sigma_c(t, s) ds \right) dt + dW^Q(t) .
\]
Term Structure Model with Single Currency

**Summary in Single Currency Environment**

- Bootstrap \( \{D(t, T)\} \) and \( \{E_t^{T_c}[L(T_{m-1}, T_m; \tau)]\} \) from OIS, IRS and TS.

- Construct continuous curves for the forward collateral rate and Libor-OIS spread of each tenor.
  - Initial conditions: \( \{c(t, s)\}, \{B(t, T; \tau)\} \).

- Simulation based on SDEs:
  
  \[
  dc(t, s) = \sigma_c(t, s) \cdot \left( \int_t^s \sigma_c(t, u) du \right) dt + \sigma_c(t, s) \cdot dW^Q(t)
  \]
  
  \[
  dB(t, T; \tau)/B(t, T; \tau) = \sigma_B(t, T; \tau) \cdot \left( \int_t^T \sigma_c(t, s) ds \right) dt + \sigma_B(t, T; \tau) \cdot dW^Q(t)
  \]

- Calibration to the Option Market.
**Term Structure Model with Multiple Currencies**

**SDE for the spot FX process**

\[
\begin{align*}
\frac{df^{(i,j)}(t)}{f^{(i,j)}(t)} &= \left(r^{(i)}(t) - r^{(j)}(t)\right) dt + \sigma^{(i,j)}(t) \cdot dW^{Q_i}(t) \\
&= \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(s)\right) dt + \sigma^{(i,j)}(t) \cdot dW^{Q_i}(t)
\end{align*}
\]

The Maruyama-Girsanov’s theorem indicates

\[
dW^{Q_i}(t) = \sigma^{(i,j)}(t) dt + dW^{Q_j}(t)
\]

which determines the SDEs of the foreign interest rates.
Term Structure Model with Multiple Currencies

Set of SDEs in Multi-Currency Environment

\[
\frac{df_{x}^{(i,j)}(t)}{f_x^{(i,j)}(t)} = \left( c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(s) \right) dt + \sigma_X^{(i,j)}(t) \cdot dW^{Q_i}(t)
\]

\[
dc^{(i)}(t, s) = \sigma_c^{(i)}(t, s) \cdot \left( \int_t^s \sigma_c^{(i)}(t, u) du \right) dt + \sigma_c^{(i)}(t, s) \cdot dW^{Q_i}(t)
\]

\[
dB^{(i)}(t, T; \tau) = \sigma_B^{(i)}(t, T; \tau) \cdot \left( \int_t^T \sigma_c^{(i)}(t, s) ds \right) dt + \sigma_B^{(i)}(t, T; \tau) \cdot dW^{Q_i}(t)
\]

\[
dc^{(j)}(t, s) = \sigma_c^{(j)}(t, s) \cdot \left[ \left( \int_t^s \sigma_c^{(j)}(t, u) du \right) - \sigma_X^{(i,j)}(t) \right] dt + \sigma_c^{(j)}(t, s) \cdot dW^{Q_i}(t)
\]

\[
dB^{(j)}(t, T; \tau) = \sigma_B^{(j)}(t, T; \tau) \cdot \left[ \left( \int_t^T \sigma_c^{(j)}(t, s) ds \right) - \sigma_X^{(i,j)}(t) \right] dt + \sigma_B^{(j)}(t, T; \tau) \cdot dW^{Q_i}(t)
\]
Term Structure Model with Multiple Currencies

Summary in Multi-Currency Environment

- **Curve Construction**
  - Bootstrap \( \{ D^{(i)}(t, T) \} \) and \( \{ E_t^{T_m,(i)} [L^{(i)}(T_{m-1}, T_m; \tau)] \} \) from OIS, IRS and TS of each currency "i".
  - Bootstrap \( \{ y^{(i,j)}(s) \} \) for all the relevant currency pairs from FX forwards and CCS.
  - Construct continuous curves using appropriate spline techniques.
    \( \Rightarrow \) Initial conditions: \( \{ c^{(i)}(t, s) \} \) and \( \{ B^{(i)}(t, T; \tau) \} \) for each currency, and \( \{ y^{(i,j)}(s) \} \) for all the relevant currency pairs.

- Simulation based on the SDEs in the previous page.
- Calibration to the Option Market and MtMCCS.
Pricing of Single Currency Products

- **Collateralized Swaption on OIS**
  - $T_0$-start $T_N$-maturing forward OIS rate:
    \[
    \text{OIS}(t, T_0, T_N) = \frac{D(t, T_0) - D(t, T_N)}{A(t, T_0, T_N)}
    \]
    \[
    A(t, T_0, T_N) = \sum_{n=1}^{N} \Delta_n D(t, T_n)
    \]
  - Define annuity measure $A$, where $A(\cdot, T_0, T_N)$ is the numeraire.

Collateralized payer swaption on $T_0$-start $T_N$-maturing OIS with strike $K$

\[
PV(t) = A(t, T_0, T_N) E_t^A \left[ (\text{OIS}(T_0, T_0, T_N) - K)^+ \right]
\]
Pricing of Single Currency Products

Maruyma-Girsanov’s theorem indicates

\[ dW^A(t) = dW^Q(t) + \frac{1}{A(t, T_0, T_N)} \sum_{n=1}^{N} \Delta_n D(t, T_n) \left( \int_{t}^{T_n} \sigma_c(t, s) ds \right) dt \]

and the SDE of the forward OIS rate is given by

\[ dOIS(t, T_0, T_N) = OIS(t, T_0, T_N) \left\{ \frac{D(t, T_N)}{D(t, T_0) - D(t, T_N)} \left( \int_{T_0}^{T_N} \sigma_c(t, s) ds \right) \right. \]
\[ + \left. \frac{1}{A(t, T_0, T_N)} \sum_{n=1}^{N} \Delta_n D(t, T_n) \left( \int_{T_0}^{T_n} \sigma_c(t, s) ds \right) \right\} \cdot dW^A(t) \]
Pricing of Single Currency Products

- **Collateralized Swaption on IRS**
  - $T_0$-start $T_N$-maturing forward IRS rate:
    \[
    IRS(t, T_0, T_N; \tau) = \frac{\sum_{n=1}^{N} \delta_n D(t, T_n) L^c(t, T_{n-1}, T_n; \tau)}{\sum_{n=1}^{N} \Delta_n D(t, T_n)}
    \]
    \[
    = \frac{D(t, T_0) - D(t, T_N)}{\sum_{n=1}^{N} \Delta_n D(t, T_n)} + \frac{\sum_{n=1}^{N} \delta_n D(t, T_n) B(t, T_n; \tau)}{\sum_{n=1}^{N} \Delta_n D(t, T_n)}
    \]
    \[
    = OIS(t, T_0, T_N) + Sp^{OIS}(t, T_0, T_N; \tau)
    \]

- Collateralized payer swaption on $T_0$-start $T_N$-maturing IRS with strike $K$

\[
P V(t) = A(t, T_0, T_N) E^A_t \left[(OIS(T_0, T_0, T_N) + Sp^{OIS}(T_0, T_0, T_N; \tau) - K)^+\right]
\]
**Pricing of Single Currency Products**

\[
S^\text{OIS}(t, T_0, T_N; \tau) = \frac{\sum_{n=1}^{N} \delta_n D(t, T_n) B(t, T_n; \tau)}{\sum_{n=1}^{N} \Delta_n D(t, T_n)}
\]

SDE for \( S_p \) under the \( A \)-measure is given by

\[
dS^\text{OIS}(t, T_0, T_N; \tau) = S^\text{OIS}(t) \left\{ \frac{1}{A(t)} \sum_{j=1}^{N} \Delta_j D(t, T_j) \left( \int_{T_0}^{T_j} \sigma_c(t, s) ds \right) 
+ \frac{1}{A_{sp}(t)} \sum_{n=1}^{N} \delta_n D(t, T_n) B(t, T_n; \tau) \left( \sigma_B(t, T_n; \tau) - \int_{T_0}^{T_n} \sigma_c(t, s) ds \right) \right\} \cdot dW^A(t)
\]

where

\[
A_{sp}(t) = \sum_{n=1}^{N} \delta_n D(t, T_n) B(t, T_n; \tau)
\]
Pricing of Multi-Currency Products

**FX-\((i/j)\) call option collateralized with currency "k"**

\[
P V (t) = E^Q_t \left[ e^{-\int_t^T r^{(i)}(s) ds} e^{\int_t^T y^{(k)}(s) ds} \left( f^{(i,j)}_x(T) - K \right)^+ \right]
\]

\[
= D^{(i)}(t, T) e^{\int_t^T y^{(k,i)}(s) ds} E^c_{t} T^{(i)} \left[ \left( f^{(i,j)}_x(T, T) - K \right)^+ \right]
\]

\[
\frac{d f^{(i,j)}_x(t, T)}{f^{(i,j)}_x(t, T)} = \sigma^{(i,j)}_{FX}(t, T) \cdot d W^T_{(i)}(t)
\]

\[
= \left\{ \sigma^{(i,j)}_{X}(t) + \int_t^T \sigma^{(i)}_{c}(t, s) ds - \int_t^T \sigma^{(j)}_{c}(t, s) ds \right\} \cdot d W^T_{(i)}(t)
\]
Risk Management

There are three important points:

- Hedges
- Monitoring
- Risk Reserve
Hedges

**Delta Hedge**

- The most important risk factor for all the books.
- Blipping each input of market quotes (1y, 2y, ...) separately and perform mark-to-market.
- Take the difference between the original scenario to calculate the exposure.
- Entering the relevant swap to reduce the exposure within a certain limit.
- Accurate modeling of the curve-level dependence on ATMF volatilities is important for the efficiency of the delta hedges.
Risk Management

Hedges

- **Kappa Hedge**
  - A very important risk factor for all the derivative books.
  - Although the market of basis swaps is liquid enough, derivatives on spreads and OIS are still quite rare.
  - IRS and FX kappa hedges would be enough for the daily operation.
  - Sensitivities for the change of market implied volatilities rather than the model parameters are important in practice.
  - Recalibration for each blipped scenario of implied volatility would require too much time...
A practical method of Kappa Hedge (ATMF)

- Choose $N$ hedge instruments with high liquidity.
  - Label their implied volatilities as $\{\sigma_i\}_{i=1}^N$.
  - Use the delta-neutral form of option, such as Straddle.
- Partitioning the volatility curves/surfaces into $N$ regions.
  - Label the partitions as $\{V_i\}_{i=1}^N$.
  - Make sure that $N \times N$-matrix, $\left(\frac{\partial \sigma_i}{\partial V_j}\right)$, is invertible.
- For each scenario of blipped $V_i$, calculate $\frac{\partial PV}{\partial V_i}$, $\left\{\frac{\partial \sigma_j}{\partial V_i}\right\}_{j=1}^N$.
- Calculate the exposure to the $j$-th hedge instrument as

$$\frac{\partial PV}{\partial \sigma_j} = \sum_{i=1}^N \left(\frac{\partial V_i}{\partial \sigma_j}\right) \times \left(\frac{\partial PV}{\partial V_i}\right)$$

- Hedge the exposure by using the $j$-th instrument.
Risk Management

**Monitoring**

Everyday PL decomposition is very important to check the reliability of hedges and find a signal of unexpected risk factor, or bugs of system.

- Using the change of market data and the calculated Greeks such as (Deltas, Gammas, Kappas, Thetas,...) to derive the expected PL\(^a\).
- Take the difference between the actual and the expected PLs, and check the size and dominant source of residuals.
- Understand the cause of residual if it is significant.

\(^a\)To make cross Gamma calculation feasible, it is convenient to use several principal components.
Risk Reserve

There are a lot of risk factors very difficult to hedge in practice. It requires to hold reasonable amount of risk reserve.

- Model limitation.
- Illiquidity of basis spread options.
- Illiquidity of Far OTM options.
- Exposure to correlation change.
- Stochastic correlations and their dependence on yield curve level/slope.
- etc…
Conclusions

- **Textbook-style implementation of IR model is not appropriate in the current market conditions.**
  - Existence of various basis spreads and their movements
  - Widespread use of collateral
  - Significant implication for profit/loss of financial firms and their risk management
- **We have proposed a framework of**
  - Curve Construction consistent with all the relevant swaps (and hence basis spreads)
  - IR model with stochastic basis spreads in the presence of multiple currencies and collateral agreement.