Modeling Term Structure of Default Correlation

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Motivation
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• Multiname credit modeling:
  ➢ Marginal Probability of Default (PD)
  ➢ Default Correlation

• PD has a “term structure”

<table>
<thead>
<tr>
<th></th>
<th>1yr</th>
<th>2yr</th>
<th>3yr</th>
<th>4yr</th>
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<td>Aaa</td>
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<td>.01</td>
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<td>.04</td>
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<td>Aa</td>
<td>.02</td>
<td>.06</td>
<td>.09</td>
<td>.16</td>
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<td>.17</td>
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<td>.72</td>
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<td>Baa</td>
<td>.18</td>
<td>.49</td>
<td>.91</td>
<td>1.4</td>
<td>1.9</td>
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</table>
Motivation

\[ P(\tau \leq T) \]
Motivation

• Extracting term structure of PD...
  ➢ Single-name products (i.e., CDSs) of different maturities
• What about “term structure of correlation”?...
• I.e., how correlation of \( \tau^{(i)} < t \) and \( \tau^{(j)} < t \) varies with \( t \)
• Extracted using different information:
  ➢ “Correlation products” such as index tranches
  ➢ Comovement between CDSs of different maturities
Motivation from CDS Market

- 3-yr or 10-yr CDSs also imply different levels of correlation
- Q: How to reconcile?
Motivation from Basket Credit Derivatives

- Q: Term structure of correlation might be key to correlation skew?
Objective

Develop a model that:

• Allows correlation structure to **vary with maturity**
• Imposes correlation structure **on top of** term structure of PD
• Consistent with **single-name & correlation products**
• **Tractable**
Agenda:

- Existing models and challenges
- Model description
- Correlation Term Structure
- Calibration Example
Challenges in Existing Models

• Copula
• Merton’s
• First-Passage
• Intensity-based Conditionally Independent Default (CID)
Challenges in Existing Models

• Copula

$$\tau^{(i)} < t \iff X^{(i)} > \Phi^{-1}(F(t))$$

• Where $X^{(i)}$ is standard normal and $F$ is cdf of $\tau^{(i)}$

• Cannot specify correlation structure that varies with $t$

• Attempt to turn $X^{(i)}$ into a process
Challenges in Existing Models

• Merton’s (1974) model:

\[ \tau^{(i)} < t \iff X_t^{(i)} > B^{(i)} \]

• \( X_t^{(i)} \) usually interpreted as firm’s net liability

• Correlation among \( X_t^{(i)} \) can be made vary with \( t \)

• Schlosser and Zagst (2009), Brunlid (2006)

• But can result in “multiple defaults”
Challenges in Existing Models

• First-passage model (Hull and White (2001), Zhou (2001))

\[ \tau^{(i)} < t \iff \sup_{s \leq t} X^{(i)}_t > B^{(i)} \]

• First time process \( X^{(i)}_t \) crosses barrier \( B^{(i)} \)

• Time-varying correlation: Metzler (2008), Hull et al. (2010), ...

• \( \sup_{s \leq t} X^{(i)}_t \) loosely interpreted as how close we’ve come to default

• This “maximum-to-date” process makes the model intractable
Challenges in Existing Models

• Intensity-based CID models:

\[ \tau^{(i)} < t \iff \int_0^t \lambda_s^{(i)} ds > E^{(i)} \]

• Where \( \lambda_t^{(i)} \) is the default intensity, \( E^{(i)} \sim \) exponential

• Correlation introduced through factor structure among \( \lambda_t^{(i)} \)'s

• Tractable, and allow time-varying correlation

• But factor structure usually affects marginal distribution of \( \tau \)
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Model Description

- Default by time $t$

$$\tau^{(i)} < t \iff \int_{0}^{t} \delta_s^{(i)} dN_s^{(i)} > b_t^{(i)}$$

- $\tau^{(i)} = \text{first passage of a pure jump process across } b_t^{(i)}$

- $N_t^{(i)}$ is a Cox process with intensity $\lambda_t^{(i)}$

- $\delta_t^{(i)}$ is the jump size
Model Description

• Default by time $t$

$$\tau^{(i)} < t \iff \int_0^t \delta_s^{(i)} \, dN_s^{(i)} > b_t^{(i)}$$

• Approximate traditional first-passage time model

• Pure-jump process approximates maximum-to-date process

• Able to calibrate to any marginal distribution of $\tau^{(i)}$
Model Description

• Default by time $t$

$$\tau^{(i)} < t \iff \int_0^t \delta_s^{(i)} dN_s^{(i)} > b_t^{(i)}$$

• How to introduce correlation?

• Let’s first assume homogeneity and constant jump size...

$$\tau^{(i)} < t \iff \delta N_t^{(i)} > b_t$$

• Correlation introduced by letting $N_t^{(i)} = M_t^{(i)} + M_t^* \quad (1 - a_t)\lambda_t + a_t\lambda_t$
Model Description

• Assuming fixed jump size $\delta$

$$\tau^{(i)} < t \iff \delta M^{(i)}_t + \delta M^*_t > b^{(i)}_t \quad (1 - a_t)\lambda_t \quad a_t\lambda_t$$

• **Note:** $a_t$ used to allocate *intensity*, not *magnitude*

• Instead of one Cox process with factor structure in intensity, we’ve two Cox processes whose intensities are fraction of $\lambda_t$

• $a_t$ does not affect marginal distribution of $\tau^{(i)}$
Model Description

- Assuming fixed jump size $\delta$
  \[ \tau^{(i)} < t \iff \delta M^{(i)}_t + \delta M^*_t > b^{(i)}_t \]

- Conditional independence

- Condition on value of $M^*_t$, not on its intensity
Model Description

• Joint default probability

\[ P(\tau^{(i)} < t, \tau^{(j)} < t) \]

\[ = P_{m_t^*}(b_t/\delta) + \sum_{k=0}^{\lfloor b_t/\delta \rfloor} p_{m_t^*}(k)[P_{m_t}(b_t/\delta - k)]^2 \]

• where \( p_\nu(x) = e^{-\nu} \nu^x / x! \) and \( P_\nu(x) = 1 - \sum_{k=0}^{\lfloor x \rfloor} p_\nu(k) \)

• \( m_t^* = \int_0^t a_s \lambda_s ds \) and \( m_t = \int_0^t (1 - a_s) \lambda_s ds \)
Model Description

• For portfolio of credit instruments
  - Laplace transform of its aggregate loss is available
  - Use inverse transform to compute loss distribution
Agenda:

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Modeling Term Structure of Correlation

- How correlation between \( \{\tau^{(i)} \leq t\} \) and \( \{\tau^{(j)} \leq t\} \) depends on \( t \)
- This term structure of correlation is controlled by \( a_t \)

Tail Dependence

\[
P(\tau^{(i)} \leq t \mid \tau^{(j)} \leq t)
\]
Base Correlation Curve

- Tranche pricing, detached at 3%, 6%, 9%, 12%, 22%
- Use correlation term structure to control base correlation curve

Figure 2: Base correlation curves using three different term structures of correlation. The leftmost curve uses \( \{g(t)\}_{t=1,...,5} = [0.00 \ 0.04 \ 0.10 \ 0.17 \ 0.26] \), the middle \([0.00 \ 0.02 \ 0.08 \ 0.17 \ 0.28]\), and the rightmost \([0.00 \ 0.01 \ 0.06 \ 0.16 \ 0.29]\)
Extension

• Correlation between recovery rate and default
• Random correlation
• Random jump size to take care of clustering
• Randomizing the jump size equivalent to multifactor model
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Calibration Example

• 5-year iTraxx Europe monthly fixings, Mar ’08 – Jan ’09

<table>
<thead>
<tr>
<th></th>
<th>0–3%</th>
<th>3–6%</th>
<th>6–9%</th>
<th>9–12%</th>
<th>12–22%</th>
<th>Index Level</th>
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<tr>
<td>Nov ’08</td>
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<tr>
<td>Jan ’09</td>
<td>64%</td>
<td>1186</td>
<td>607</td>
<td>316</td>
<td>97</td>
<td>165</td>
</tr>
</tbody>
</table>

• Calibrate the correlation, assuming

\[ P(\tau < t) = 1 - e^{-st/(1-R)} \]
Calibration Example

- Base correlation:
Calibration Example

• Can produce the skew by calibrating only the correlation...

• **Note**: compared with “implied copula” (Hull 2006)
Conclusion

• Motivate the concept of correlation term structure
  ➢ Implied from market data
  ➢ Important role in explaining correlation skew

• Develop a model of correlated default that:
  ➢ Imposes term structure of correlation on top of PD
  ➢ Easy to calibration to single-name & correlation products
  ➢ Tractable, even under generalizations