Derivatives Pricing under Collateralization *

Masaaki Fujii[†], Kenichiro Shiraya[‡], Akihiko Takahashi [§]

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[†] Graduate School of Economics, The University of Tokyo

[‡] Graduate School of Economics, The University of Tokyo

[§] Speaker, Graduate School of Economics, The University of Tokyo

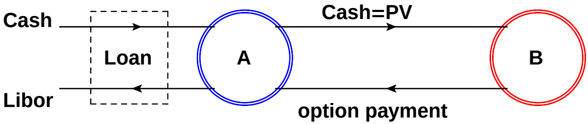
Introduction

New market realties after the Financial Crisis

- **Wide use of collateralization in OTC**
  Dramatic increase in recent years (ISDA Margin Survey 2011)
  - **30%(2003) → 70%(2010) in terms of trade volume for all OTC.**
  - **Coverage goes up to 79% (for all OTC) and 88% (for fixed income) among major financial institutions.**
  - **More than 80% of collateral is Cash.**
    (About half of the cash collateral is USD.)
- **Persistently wide basis spreads:**
  Much more volatile Cross Currency Swap(CCS) basis spread.
  Non-negligible basis spreads even in the single currency market. (e.g. Tenor swap spread, Libor-OIS spread)

Source of Funding Cost

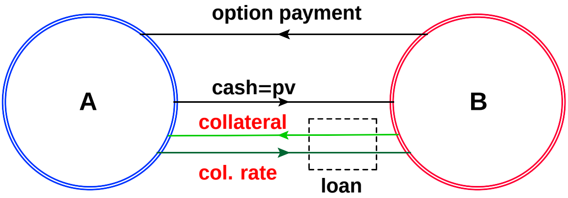
- **Unsecured Funding and Contract (old picture)**



  - **Libor is unsecured offer rate in the interbank market.**
  - **Libor discounting is appropriate for unsecured trades between financial firms with Libor credit quality.**

Source of Funding Cost

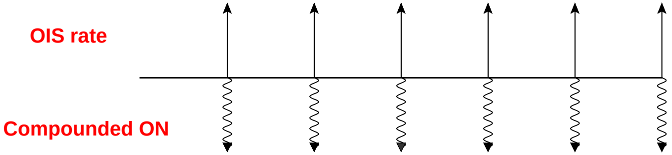
- **Collateralized (Secured) Contract (current picture)**



  - **No outright cash flow (collateral=PV)**
  - **No external funding is needed.**
  - **Funding is determined by over-night (ON) rate.**
    ⇒ Libor discounting seems inappropriate.

Important Instruments and Market Realities

- **Overnight Index Swap (OIS)** [1]



  - **Floating side: Daily compounded ON rate**
  - **Market Quote : fixed rate, called OIS rate**

[1] Usually, there is only one payment for $< 1yr$.

Important Instruments and Market Realities

Historical behavior of IRS (1Y)-OIS (1Y) spreads (bps)

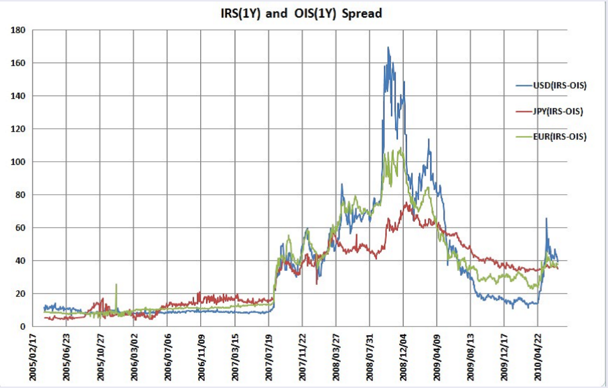
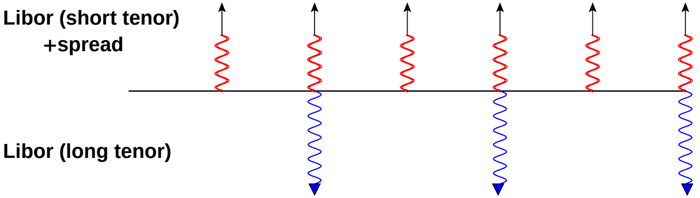


Figure: Source:Bloomberg

Important Instruments and Market Realities

- **Tenor Swap (TS)**[2]



- **Spread is quite significant and volatile since late 2007.**

[2] It is also common that payment of short-tenor Leg is compounded and paid at the same time with the other Leg.

Important Instruments and Market Realities

Historical behavior of JPY TS spreads (bps)

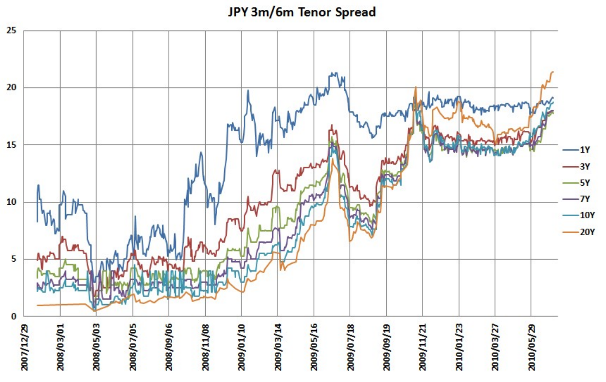
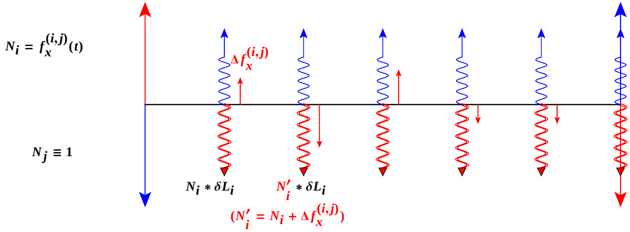


Figure: Source:Bloomberg

Important Instruments and Market Realities

- **Mark-to-Market Cross Currency Swap**



- **USD Libor is exchanged by Libor +spread of the other currency.**
- **USD leg notional is reset every start of accrual period.**
- **Spread is quite significant and volatile for long time. (it has been changing drastically and rapidly since the financial crisis.)**

Important Instruments and Market Realities

Historical behavior of USDJPY CCS spreads (bps)

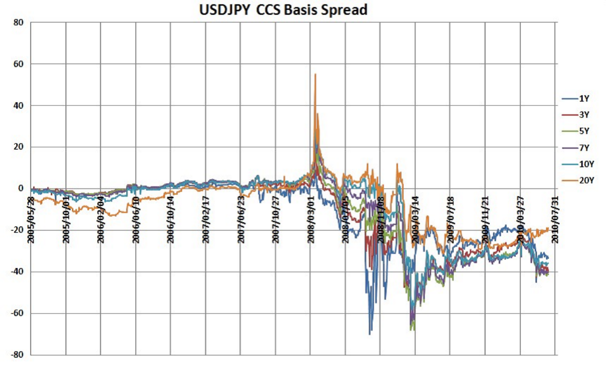


Figure: Source:Bloomberg

Important Instruments and Market Realities

Historical behavior of EURUSD CCS spreads (bps)

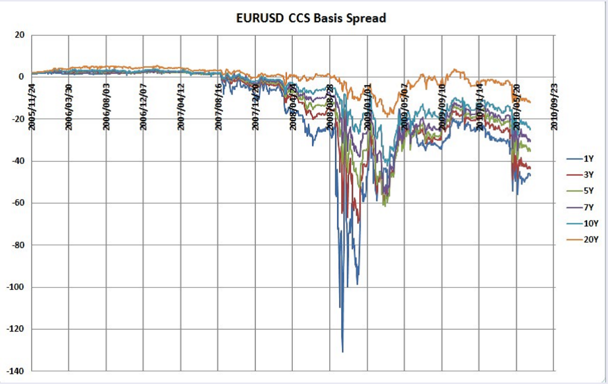


Figure: Source:Bloomberg

Impact of Collateralization

Impact of collateralization :

- **Reduction of Counter party Exposure**
  - **Associated change in CVA has been actively studied. (e.g. CVA is charged for a contract with imperfect collateralization.)**
- **Change of Funding Cost**
  - **Require new term structure model to distinguish discounting and reference rates.**
  - **Significant impact on derivative pricing.**

Topics of this talk

- **Valuation framework under collateralization**
  - **Perfect collateralization**
  - **Asymmetric collateralization**
  - **Imperfect collateralization and CVA**
- **New approximation scheme for FBSDEs[3] (it seems useful for pricing securities under asymmetric/imperfect collateralization.)**
  - **Perturbation scheme**
  - **Perturbation with interacting particle method**
- **Numerical example for CVA and imperfect collateralization**

[3] forward backward stochastic differential equations

Simple Case: Pricing under Full Collateralization

- **Assumption**
  - **Continuous adjustment of collateral amount**
  - **Symmetric/Perfect collateralization by Cash**
  - **Zero minimum transfer amount**

(Daily cash margin call/settlement is becoming popular.)

Simple Case: Pricing under Full Collateralization

Suppose the payment currency of a European derivative is i, while its collateral currency is j. Then, under full collateralization the time-t price of the derivative with payoff $h^{(i)}(T)$ at maturity T is obtained as follows:

$$h^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right],$$

where

$$y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s). \tag{2.1}$$

- Q_i: **risk-neutral measure of currency i**
- $h^{(i)}(T)$: **derivative payoff at time T in currency i**
- **collateral is posted in currency j**
- $c^{(j)}(s)$: **instantaneous collateral rate of currency j at time s**
- $r^{(i)}(s)$ ($r^{(j)}(s)$): **instantaneous risk-free rate of currency i (j) at time s**

Simple Case: Pricing under Full Collateralization

- Collateral amount in currency j at time s is given by $\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}$, which is invested at the rate of $y^{(j)}(s)$:

$$
\begin{aligned}
h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}h^{(i)}(T)\right] \\
&\qquad + f_x^{(i,j)}(t)E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du}y^{(j)}(s)\left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right)ds\right] \\
&= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du}y^{(j)}(s)h^{(i)}(s)ds\right].
\end{aligned}
$$

Note that $X(t) = e^{-\int_0^t r^{(i)}(s)ds}h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)h^{(i)}(s)ds$ is a Q_i-martingale. Then, the process of the derivative value is written by

$$
dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right)h^{(i)}(t)dt + dM(t)
$$

with some Q_i-martingale M. This establishes the proposition.

$f_x^{(i,j)}(t)$: Foreign exchange rate at time t(the price of the unit amount of currency "j" in terms of currency "i").

Simple Case: Pricing under Full Collateralization

Special Case

- **If payment and collateral currencies are the same, the option value is given by**

$$h(t) = E_t^Q\left[e^{-\int_t^T c(s)ds} h(T)\right]$$

- **The discounting is determined by "collateral rate", which is consistent with the schematic picture seen before.**

Setup

Pricing Framework ([17])

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F}$ contains all the market information including defaults.
- Consider two firms, $i \in \{1, 2\}$, whose default time is $\tau^i \in [0, \infty]$, and $\tau = \tau^1 \wedge \tau^2$.
- τ^i (and hence τ) is assumed to be totally-inaccessible $\mathbb{F}$-stopping time. (i.e. a default event is modeled as a jump process.)
- Default indicator functions: $H_t^i = 1_{\{\tau^i \le t\}} (i = 1, 2)$, $H_t = 1_{\{\tau \le t\}}$
- Assume the existence of absolutely continuous compensator for H^i:

$$A_t^i = \int_0^t h_s^i 1_{\{\tau^i > s\}} ds, \quad t \ge 0$$

- Assume no simultaneous defaults, and hence the hazard rate of H is

$$h_t = h_t^1 + h_t^2 .$$

- Money market account: $\beta_t = \exp\left(\int_0^t r_u du\right)$

Collateralization

- When party $i \in \{1,2\}$ has negative mark-to-market, it has to post cash collateral to party $j(\neq i)$, and it is assumed to be done continuously.
- collateral coverage ratio is $\delta^i_t \in \mathbb{R}_+$, and the amount of collateral at time t is given by $\delta^i_t(-V^i_t)$ when party i posts collateral. (V^i_t denotes the mark-to-market value of the contract from the view point of party i.)
  - δ^i_t effectively takes into account under- as well as over-collateralization. Thus, $\delta^i_t < 1$ and $\delta^i_t > 1$ are possible.
- party j has to pay the collateral rate c^i_t on the posted cash continuously.
- c^i_t is determined by the currency posted by party i.
  - market convention is to use overnight (O/N) rate at time t of corresponding currency.
    $\Rightarrow$ Traded through OIS (overnight index swap), which is also collateralized.
  - In general, $c^i_t \neq r^i_t$. (r^i_t is the risk-free interest rate of the same currency.) This is necessary to explain CCS basis spread consistently .

Counter party Exposure and Recovery Scheme

- Counter party exposure to party j at time t from the view point of party i is given as:

$$\max(1-\delta_t^j, 0)\max(V_t^i, 0) + \max(\delta_t^i - 1, 0)\max(-V_t^i, 0).$$

- Assume party-j's recovery rate at time t as $R_t^j \in [0, 1]$.
- Then, the recovery value at the time of j's default is given as:

$$R_t^j\left([1-\delta_t^j]^+[V_t^i]^+ + [\delta_t^i - 1]^+[-V_t^i]^+\right),$$

$x^+ \equiv \max(x, 0)$.

Pricing Formula

- **Pricing from the view point of party 1.**

$$S_t = \beta_t E^Q \Bigg[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \Big\{ dD_u + (y_u^1 \delta_u^1 1_{\{S_u < 0\}} + y_u^2 \delta_u^2 1_{\{S_u \geq 0\}}) S_u du \Big\} + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \Big(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \Big) \Bigg| \mathcal{F}_t \Bigg]$$

- **D: cumulative dividend to party 1.**
- **Default payoff: Z^i when party i defaults.**

$$Z^1(t, v) = \Big(1 - l_t^1 (1 - \delta_t^1)^+\Big) v 1_{\{v<0\}} + \Big(1 + l_t^1 (\delta_t^2 - 1)^+\Big) v 1_{\{v \geq 0\}}$$

$$Z^2(t, v) = \Big(1 - l_t^2 (1 - \delta_t^2)^+\Big) v 1_{\{v \geq 0\}} + \Big(1 + l_t^2 (\delta_t^1 - 1)^+\Big) v 1_{\{v<0\}},$$

$l_t^i \equiv (1 - R_t^i), i = 1, 2$

- **$y_t^i = r_t^i - c_t^i$, $(i \in \{1, 2\})$ denotes the instantaneous return for j or funding cost for i at time t from the cash collateral posted by party i.**

Pricing Formula

- **(Remark) The return from risky investments, or the borrowing cost from the external market can be quite different from the risk-free rate, of course.**
However, if one wants to treat this fact directly, an explicit modeling of the associated risks is required.
Here, we use the risk-free rate as net return/cost after hedging these risks.
As we shall see, under full collateralization the final formula does not require any knowledge of the risk-free rate, and hence there is no need of its estimation.

Pricing Formula

Following the method in Duffie&Huang (1996), the pre-default value of the contract V_t such that $V_t 1_{\{\tau>t\}} = S_t$ is given by

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))du\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T,$$

where

$$\begin{aligned} \mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \text{ (adjusted term of the discount rate)} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-\delta_t^i)^+(l_t^i h_t^i) + (\delta_t^i - 1)^+(l_t^j h_t^j), \end{aligned}$$

if some technical condition(so called *no jump* condition for V at default) [4] is satisfied, which is assumed hereafter.(for instance, r, D, y^i, δ^i, l^i and h^i, ($i = 1, 2$) are adapted to background filtration. In particular, we limit our attention to the intensities conditional on no-default.)

[4] This technical condition ($\Delta V_\tau = 0$) becomes important when we consider credit derivatives: the condition is violated in general when the contagious effects induce jumps to variables contained in pre-default value process. (e.g. Schönbucher(2000), Collin-Dufresne-Goldstein-Hugonnier(2004), Brigo-Capponi(2009), [19])

Symmetric Case

Effective discount factor is non-linear:

$$r_t - \mu(t, v) = r_t - (\tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}}),$$

which makes the portfolio value non-additive.

If $\tilde{y}_t^1 = \tilde{y}_t^2 = \tilde{y}_t$, then we have

$$\mu(t, v) = \tilde{y}_t \ .$$

Further, if $\tilde{y}$ is not explicitly dependent on V, we can recover the linearity.

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

Portfolio valuation can be decomposed into that of each payment.

$\Downarrow$

A good characteristic for market benchmark price.

Symmetric Perfect Collateralization

Special Cases

Case 1 : Benchmark for single currency product

- **bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)**
- **both parties use the same currency (i) as collateral, which is also the payment (evaluation) currency.**

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

The valuation method for single currency swap adopted by LCH Swapclear (2010) is the same with this equation. [5]

[5] See also Piterbarg (2010) for other derivation of this equation.

Symmetric Perfect Collateralization

Special Cases

Case 2 : Collateral in a Foreign Currency

- **bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)**
- **both parties use the same currency (k) as collateral, which is different from the payment (evaluation) currency (i)**

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

Funding Spread between the two currencies

$$y^{(i,k)} = y^{(i)} - y^{(k)} = \left(r^{(i)} - c^{(i)}\right) - \left(r^{(k)} - c^{(k)}\right)$$

- **This is necessary to explain CCS basis spreads consistently.**

Collateral Rate

Overnight Index Swap (OIS)

- exchange fixed rate(F) with compounded overnight rate periodically.
- collateralized by domestic currency
- Par rate at t for T_0 ($> t$)-start T_N-maturing OIS with currency (i):

$$\mathrm{OIS}_N(t) = F^{par}(t) = \frac{D^{(i)}(t,T_0) - D^{(i)}(t,T_N)}{\sum_{n=1}^{N} \Delta_n D^{(i)}(t,T_n)},$$

  (Δ_n : daycount fraction).
- $D^{(i)}(t,T) = E^{Q^{(i)}}\left[e^{-\int_t^T c_u^{(i)} du} \middle| \mathcal{F}_t \right]$ is a value of domestically collateralized zero-coupon bond.

Funding Spread

(i, j) Mark-to-Market Cross Currency OIS:
The funding spread(the difference of collateral costs) is directly linked to the corresponding CCOIS, though it seems not liquid in the current market.

- **compounded O/N rate of currency i is exchanged by that of currency j with additional spread periodically.**
- **notional of currency j is kept constant while that of currency i is refreshed at every reset time with the spot FX rate. (currency i is usually USD.)**
- **collateralized by currency i .**

Funding Spread

- Define
$$D^{(j,i)}(t,T) = E^{Q^{(j)}}\left[e^{-\int_t^T (c_u^{(j)}+y_u^{(j,i)})du}\middle|\mathcal{F}_t\right] = D^{(j)}(t,T)e^{-\int_t^T y^{(j,i)}(t,s)ds}.$$
$$y^{(j,i)}(t,T) = -\frac{\partial}{\partial T}\ln E^{T^{(j)}}\left[e^{-\int_t^T y_u^{(j,i)}du}\middle|\mathcal{F}_t\right].$$ (instantaneous fwd rate of the funding spread)
- $D^{(j,i)}(t,T)$: the zero coupon bond of currency j collateralized by currency i.
  $E^{T^{(j)}}[\cdot|\mathcal{F}_t]$: conditional expectation under the fwd measure associated with $D^{(j)}(t,T)$.
  Then, under a simplifying assumption such as independence between $c^{(j)}$ and $y^{(j,i)}$[6],
- MtMCCOIS basis spread is obtained by:
$$\begin{aligned} B_N &= \frac{\sum_{n=1}^N D^{(j,i)}(t,T_{n-1})\left(1-e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right)}{\sum_{n=1}^N \delta_n D^{(j,i)}(t,T_n)} \\ &\sim \frac{1}{T_N-T_0}\int_{T_0}^{T_N} y^{(j,i)}(t,u)du. \end{aligned}$$

[6] The assumption seems reasonable for the recent data studied in [15].

Modeling framework of Interest rates

Symmetric perfectly collateralized price is becoming the market benchmark, at least for standardized products.

"Term structure construction procedures": [7]

- **(1), OIS $\Rightarrow c^{(i)}(0,T)$(T-maturity instantaneous fwd rate at time 0)**
- **(2), results of (1) + IRS + TS $\Rightarrow B^{(i)}(0,T;\tau)$ (i-currency forward Libor-OIS spread with tenor τ)**
- **(3), results of (1),(2) +CCS $\Rightarrow y^{(i,j)}(0,T)$(funding spread)**

Given the initial term structures, no-arbitrage dynamics of $c^{(i)}(t,T)$,$B^{(i)}(t,T;\tau)$ and $y^{(i,j)}(t,T)$ in HJM-framework can be constructed. (For the detail, please see our paper [13], [22]. For other approaches, see Bianchetti(2010), Mercurio(2009), Morini(2009), for instance.)

[7] Assume collateralization in domestic currency for OIS, IRS and TS. Assume collateralization in USD for CCS (USD crosses).

Curve Construction

- **Collateralized OIS**

$$\mathrm{OIS}_N(0)\sum_{n=1}^{N}\Delta_n D(0,T_n) = D(0,T_0) - D(0,T_N)$$

- **Collateralized IRS**

$$\mathrm{IRS}_M(0)\sum_{m=1}^{M}\Delta_m D(0,T_m) = \sum_{m=1}^{M}\delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau)]$$

- **Collateralized TS**[8]

$$\sum_{n=1}^{N}\delta_n D(0,T_n)\left(E^{T_n}[L(T_{n-1},T_n;\tau_S)] + \mathrm{TS}_N(0)\right)$$

$$= \sum_{m=1}^{M}\delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau_L)]$$

(Δ_m, Δ_n, δ_m, δ_n: daycount fractions)

Market quotes of collateralized OIS, IRS, TS, (and a proper spline method) allow us to determine all the relevant $\{D(0,T)\}$, and forward Libors $\{E^{T_m}[L(T_{m-1},T_m,\tau)]\}$.

[8] The short-tenor leg may be compounded and then, additional small corrections exist.

Curve Construction

Collateralized FX Forward: USD/JPY

- **Suppose USD= i, JPY= j and collateral currency is USD.**
- **Current time: t. Maturity: T**
- **At T , one unit of i is exchanged for K (fixed at t) units of j.**
- **FX forward is the break-even value of K.**

$$KE_t^{Q^{(j)}}\left[e^{-\int_t^T (c_s^{(j)}+y_s^{(i,i)})ds}\right] = f_x^{(j,i)}(t)E_t^{Q^{(i)}}\left[e^{-\int_t^T c_s^{(i)}ds}1\right].$$ [9]

$$\begin{aligned} f_x^{(j,i)}(t,T;(i)) &= f_x^{(j,i)}(t)\frac{D^{(i)}(t,T)}{D^{(j)}(t,T)}\exp\left(\int_t^T {\color{red}y^{(j,i)}(t,u)}du\right), \\ {\color{red}y^{(j,i)}(t,T)} &= -\frac{\partial}{\partial T}\ln E_t^{T^{(j)}}\left[e^{-\int_t^T y_s^{(j,i)}ds}\right]. \end{aligned}$$

- **FX Forward → Forward curve of funding spread (${\color{red}\{y^{(j,i)}(t,T)\}}$)**
- **CCS for longer maturities.**

[9] $f_x^{(j,i)}(t)$ denotes spot FX rate at t that is, the price of the unit amount of currency i in terms of currency j.

Curve Construction

Remark: Constant Notional CCS vs MtM-CCS

(USD-LIBOR) is exchanged for (X-currency LIBOR + basis spread).

- **Constant Notional CCS (CNCCS)**
  - **Notional of both legs are kept constant.**
- **Mark-to-Market CCS (MtMCCS)**
  - **Notional of currency X is kept constant at N_X.**
  - **Notional of USD is readjusted to $f_x^{(USD;X)} \times N_X$ at every start of LIBOR accrual period.**

Curve Construction

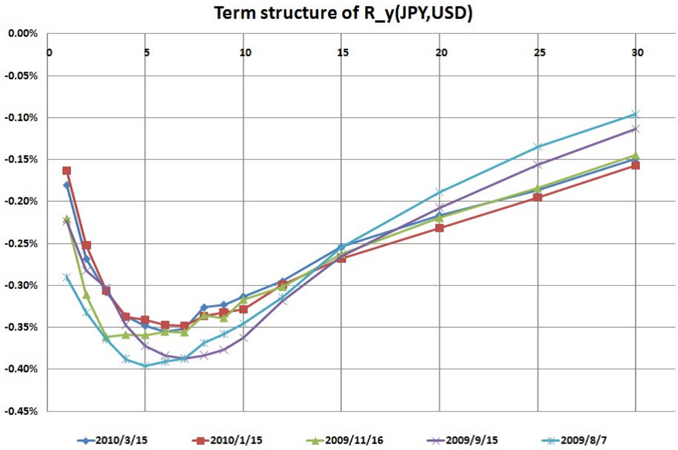
Remark: the difference between MtM and Constant notional basis spreads:

$$B_N^{MtM} - B_N^{CN} = \frac{\sum_{n=1}^N \delta_n^{(i)} D^{(i)}(0,T_n) E^{T_n^{(i)}}\left[\left(\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} - 1\right) B^{(i)}(T_{n-1},T_n)\right]}{\sum_{n=1}^N \delta_n^{(j)} D^{(j,i)}(0,T_n)},$$

where $B^{(i)}(T_{n-1},T_n)$ stands for the Libor-OIS spread of the currency i at T_{n-1}. This spread is not zero in general.

For the two USDJPY CCSs, the two swaps should have the same basis spreads if USD LIBOR-OIS spreads are all zero. This held approximately well before the Lehman crisis but the spread has been far from zero since then. If USD interest rate level is higher than JPY, as is usually the case, the equation tells us that the spread for MtMCCS is quite likely to be higher than that of CNCCS, $B_N^{MtM} > B_N^{CN}$. The size of spread may not be negligible dependent on situations, and hence it is worthwhile paying enough attention to the difference in this post crisis era.

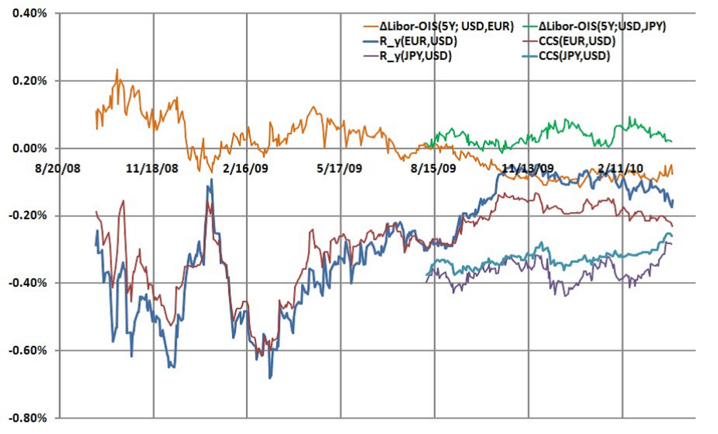
Curve Construction



$R_y(j,i) = \left(\int_0^T y^{j,i}(0,u)du\right)/T$(funding spread curve): posting USD as collateral tends to be expensive for collateral payers.

Curve Construction

Close relationship - CCS Basis and Funding Spread -



A significant portion of CCS spreads movement stems from the change in the funding spreads. Libor-OIS spread seems to have minor effect.

HJM-framework under full collateralization

$$
\begin{aligned}
dc^{(i)}(t,s) &= \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW_t^{Q^{(i)}} \\
dy^{(i,k)}(t,s) &= \sigma_y^{(i,k)}(t,s)\cdot\left(\int_t^s (\sigma_y^{(i,k)}(t,u)+\sigma_c^{i}(t,u))du\right)dt + \sigma_y^{(i,k)}(t,s)\cdot dW_t^{Q^{(i)}} \\
\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} &= \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW_t^{Q^{(i)}} \\
\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} &= \left(c^{(i)}(t)-c^{(j)}(t)+y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW_t^{Q^{(i)}},
\end{aligned}
$$

$$
B^{(i)}(t,T_k;\tau) = E_t^{T_k^{(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)}-1\right)
$$

is forward LIBOR-OIS spread.

Choice of Collateral Currency

Special Cases

Case 3 : Multiple Eligible Collaterals

- bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)
- both parties choose the optimal currency from the eligible collateral set C. Currency (i) is used as the evaluation currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k\in C}[y_u^{(i,k)}])du\right) dD_s \middle| \mathcal{F}_t\right]$$

- The party who needs to post collateral has optionality.
- The cheapest collateral currency is chosen based on CCS information.
  To choose ”strong” currency, such as USD,
  is expensive for the collateral payer.

Choice of Collateral Currency

Role of $y^{(j,i)}$

Optimal behavior of collateral payer can significantly change the derivative value.

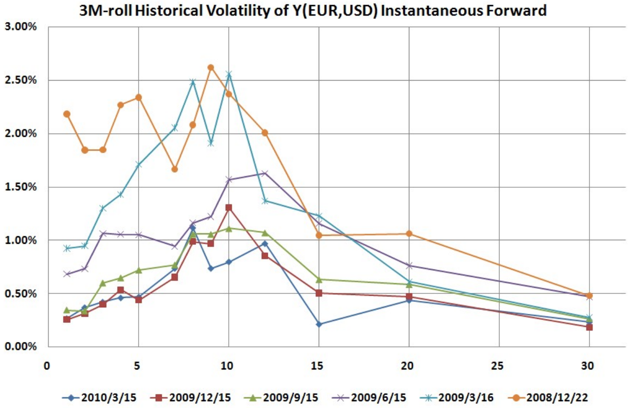
- **Payment currency and USD as eligible collateral is relatively common. Then, the effective discounting factor becomes**

$$D^{(j)}(t,T) \Rightarrow E_t^{T^{(j)}}\left[e^{-\int_t^T \max\{y^{(j,USD)}(s),0\}ds}\right] D^{(j)}(t,T)$$

  except correlation effects.

- **Volatility of $y^{(j,USD)}$ is an important determinant. (Embedded option change effective discounting factor, which crucially depends on the volatility of funding spread.)**

Choice of Collateral Currency



vols tend to be 50 bps in a calm market, but they were more than a percentage point just after the market crisis, which reflects a significant widening of the CCS basis to seek USD cash in the low liquidity market.

Choice of Collateral Currency

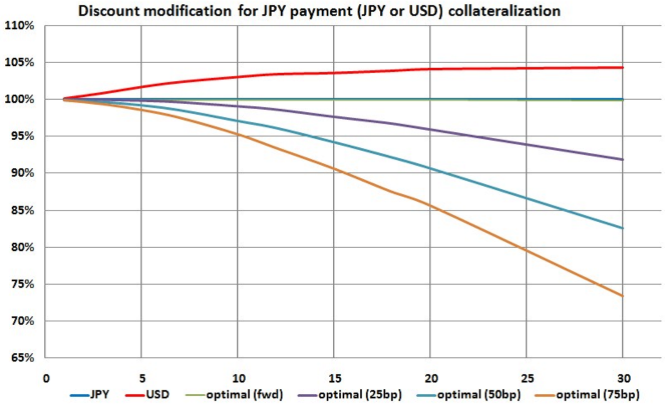


Figure: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16.

the effective discounting rate is increased by around 50 bps annually even when the annualized vol. of $y^{(JPY,USD)}$ is 50 bps.

More generic situations: marginal impact of asymmetry

$$
\begin{aligned}
V_t &= E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))du\right) dD_s \middle| \mathcal{F}_t\right] \\
\mu(t,v) &= \tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}} \\
\tilde{y}_t^i &= \delta_t^i y_t^i - (1-\delta_t^i)^+ (l_t^i h_t^i) + (\delta_t^i - 1)^+ (l_t^j h_t^j)
\end{aligned}
$$

- **Make use of Gateaux derivative(GD) as the first-order Approximation [10]:**

$$
\lim_{\epsilon \downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta}; \eta) - \frac{V_t(\bar{\eta} + \epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0,
$$
$(\eta, \bar{\eta}$: bounded and predictable)

- **We want to expand the price around a symmetric benchmark price.**

$$
\mu(t,v) = y_t + \Delta\tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \Delta\tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}}, \ (\Delta\tilde{y}_t^i = \tilde{y}_t^i - y_t)
$$

- **Calculate GD at symmetric $\mu = y$ point.**

$$
V_t(\mu) \simeq V_t(y) + \nabla V_t(y, \mu - y)
$$

[10] Duffie&Skiadas (1994), Duffie&Huang (1996)

Asymmetric Collateralization(marginal impact of asymmetry)

- Then, V_t is decomposed as $V_t = \overline{V}_t + \nabla V_t$, where

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\nabla V_t = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\overline{V}_s \left(\Delta\tilde{y}_s^1 \mathbf{1}_{\{\overline{V}_s<0\}} + \Delta\tilde{y}_s^2 \mathbf{1}_{\{\overline{V}_s\geq 0\}}\right) ds \middle| \mathcal{F}_t\right]$$

If y is chosen in such a way that it reflects the funding cost of the standard collateral agreements, $\overline{V}$ turns out to be the market benchmark price, and ∇V represents the correction for it.

Asymmetric Collateralization(marginal impact of asymmetry)

An example of asymmetric perfect collateralization

- **party 1 choose optimal currency from the eligible collateral set C, but the party 2 can only use currency (i) as collateral, either due to the asymmetric CSA or lack of easy access to foreign currency pool. The evaluation (payment) currency is (i).**

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du \right) dD_s \middle| \mathcal{F}_t \right]$$

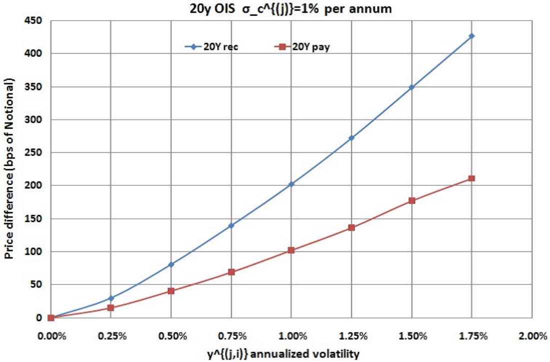
$$\nabla V_t = E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s c_u^{(i)} du \right) [-\overline{V}_s]^+ \max_{k \in C}[y_s^{(i,k)}] \middle| \mathcal{F}_t \right]$$

$$V_t \simeq \overline{V}_t + \color{red}{\nabla V_t}$$

$\Rightarrow$ Expansion around the symmetric collateralization with currency (i).

Asymmetric Collateralization(marginal impact of asymmetry)

- **Numerical Example of ∇V for JPY-OIS** [11].
- **Eligible collateral are USD and JPY for party-1 but only JPY for party-2.**



- **OIS rate is set to make $\overline{V} = 0$.**
- **Difference between Receiver and Payer comes from up-ward sloping term structure. (the receiver's mark-to-market value tends to be negative in the long end of the contract, which makes the optionality larger.)**

[11] based on the data in early 2010, see [17] for the detail.

Imperfect Collateralization

CVA as the Deviation from the Perfect Collateralization

- **Assume the both parties use the same currency for simplicity, and hence $y^1 = y^2 = y$.**

$$\mu(t,v) = y_t - \Big\{\Big((1-\delta_t^1)y_t + (1-\delta_t^1)^+(l_t^1 h_t^1) - (\delta_t^1 - 1)^+(l_t^2 h_t^2)\Big)\mathbb{1}_{\{v<0\}} + \Big((1-\delta_t^2)y_t + (1-\delta_t^2)^+(l_t^2 h_t^2) - (\delta_t^2 - 1)^+(l_t^1 h_t^1)\Big)\mathbb{1}_{\{v\geq 0\}}\Big\}$$

- **GD(Gateaux derivative) around $\mu = y$ decomposes the price into three parts:**
  - **Symmetric perfectly collateralized benchmark price**
  - **$(1-\delta^i)y\mathbb{1}_{\{v\lessgtr 0\}} \Rightarrow$ Collateral Cost Adjustment (CCA)**
  - **Remaining h dependent terms $\Rightarrow$ Credit Value Adjustment (CVA)**

$$\begin{aligned} V_t &\simeq \overline{V}_t + \nabla V_t \\ &= \overline{V}_t + \mathrm{CCA} + \mathrm{CVA} \end{aligned}$$

Imperfect Collateralization

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du} y_s \left((1-\delta_s^1)[-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$

$$E^Q\Bigg[\int_t^T e^{-\int_t^s (r_u - y_u)du}(l_s^1 h_s^1)\left[(1-\delta_s^1)^+[-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right] ds$$

$$-\int_t^T e^{-\int_t^s (r_u - y_u)du}(l_s^2 h_s^2)\left[(1-\delta_s^2)^+[\overline{V}_s]^+ + (\delta_s^1 - 1)^+[-\overline{V}_s]^+\right] ds \Bigg| \mathcal{F}_t\Bigg]$$

- **The discounting rate is different from the risk-free rate and reflects the funding cost of collateral, while the terms in CVA are pretty similar to the usual result of bilateral CVA.**
- **Dependence among y, δ and other variables such as $\overline{V}, h^i$ is particularly important. $\Rightarrow$ New type of Wrong (Right)-way Risk. (e.g. y is closely related to the CCS basis spread. Hence, y is expected to be highly sensitive to the market liquidity, and is also strongly affected by the overall market credit conditions.)**

Collateral Thresholds

- **Thresholds: $\Gamma^i > 0$ for party-i: A threshold is a level of exposure below which collateral will not be called, and hence it represents an amount of uncollateralized exposure. Only the incremental exposure will be collateralized if the exposure is above the threshold.**

Case of perfect collateralization above the thresholds

$$S_t = \beta_t E^Q \left[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \{dD_u + q(u, S_u) S_u du\} \right.$$
$$\left. + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \left\{ Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right\} \middle| \mathcal{F}_t \right]$$

$$q(t, S_t) = y_t^1 \left(1 + \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + y_t^2 \left(1 - \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t > \Gamma_t^2\}}$$

$$Z^1(t, S_t) = S_t \left[\left(1 + l_t^1 \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + R_t^1 1_{\{-\Gamma_t^1 \leq S_t < 0\}} + 1_{\{S_t \geq 0\}} \right]$$

$$Z^2(t, S_t) = S_t \left[\left(1 - l_t^2 \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t \geq \Gamma_t^2\}} + R_t^2 1_{\{0 \leq S_t < \Gamma_t^2\}} + 1_{\{S_t < 0\}} \right]$$

Collateral Thresholds

Assume the domestic currency as collateral $y^1 = y^2 = y$.

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s c_u du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = -E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \overline{V}_s \mathbf{1}_{\{-\Gamma_s^1 \le \overline{V}_s < \Gamma_s^2\}} ds \middle| \mathcal{F}_t\right]$$

$$+E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \left\{\Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}} - \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s \ge \Gamma_s^2\}}\right\} ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$

$$E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{(l_s^1 h_s^1)[-\overline{V}_s \mathbf{1}_{\{-\Gamma_s^1 \le \overline{V}_s < 0\}} + \Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}}]\right\} ds \middle| \mathcal{F}_t\right]$$

$$-E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{(l_s^2 h_s^2)[\overline{V}_s \mathbf{1}_{\{0 < \overline{V}_s \le \Gamma_s^2\}} + \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s > \Gamma_s^2\}}]\right\} ds \middle| \mathcal{F}_t\right]$$

The terms in CCA reflect the fact that no collateral is posted in the range $\{-\Gamma_t^1 \le V_t \le \Gamma_t^2\}$, and that the posted amount of collateral is smaller than $|V|$ by the size of threshold.

The terms in CVA represent bilateral uncollateralized credit exposure, which is capped by each threshold.

FBSDE Approximation Scheme

([19])

- **The forward backward stochastic differential equations (FBSDEs) have been found particularly relevant for various valuation problems (e.g. pricing securities under asymmetric/imperfect collateralization, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets).**
- **Their financial applications are discussed in details for example, El Karoui, Peng and Quenez [1997], Ma and Yong [2000], a recent book edited by Carmona [2009], Crépey [2012(a,b)], [44], and references therein.**
- **We will present a simple analytical approximation with perturbation scheme for the non-linear FBSDEs.**

FBSDE Approximation Scheme - Setup-

- **We consider the following FBSDE:**

$$\begin{aligned} dV_t &= -f(X_t, V_t, Z_t)dt + Z_t \cdot dW_t && (6.1) \\ V_T &= \Phi(X_T), && (6.2) \end{aligned}$$

where V takes the value in $\mathbb{R}$, W is a r-dimensional Brownian motion, and $X_t \in \mathbb{R}^d$ is assumed to follow a diffusion which is the solution to the (forward) SDE:

$$dX_t = \gamma_0(X_t)dt + \gamma(X_t) \cdot dW_t; \;\; X_0 = x\,. \qquad (6.3)$$

- **We assume that the appropriate regularity conditions are satisfied for the necessary treatments. See Takahashi-Yamada [44] for the mathematical validity based on Malliavin calculus.**

Perturbative Expansion for Non-linear Generator

- In order to solve the pair of (V_t, Z_t) in terms of X_t, we extract the linear term from the generator f and treat the residual **non-linear term as the perturbation to the linear FBSDE**.
- We introduce the perturbation parameter ϵ, and then write the equation as

$$\begin{aligned} dV_t^{(\epsilon)} &= c(X_t)V_t^{(\epsilon)}dt - \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \\ V_T^{(\epsilon)} &= \Phi(X_T) \,, \end{aligned} \tag{6.4}$$

where $\epsilon = 1$ corresponds to the original model by [12]

$$f(X_t, V_t, Z_t) = -c(X_t)V_t + g(X_t, V_t, Z_t) \,. \tag{6.5}$$

[12] Or, one can consider $\epsilon = 1$ as simply a parameter convenient to count the approximation order. The actual quantity that should be small for the approximation is the residual part g.

Perturbative Expansion for Non-linear Generator

- One should choose the linear term $c(X_t)V_t^{(\epsilon)}$ in such a way that the residual non-linear term g becomes as small as possible to achieve better convergence.
- Now, we are going to expand the solution of BSDE (6.4) in terms of ϵ: that is, suppose $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{6.6}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots . \tag{6.7}$$

Perturbative Expansion for Non-linear Generator

- **Once we obtain the solution up to the certain order, say k for example, then by putting $\epsilon = 1$,**

$$\tilde{V}_t = \sum_{i=0}^{k} V_t^{(i)}, \qquad \tilde{Z}_t = \sum_{i=0}^{k} Z_t^{(i)} \tag{6.8}$$

is expected to provide a reasonable approximation for the original model as long as the residual term g is small enough to allow the perturbative treatment.

- **$V_t^{(i)}$ and $Z_t^{(i)}$, the corrections to each order can be calculated recursively using the results of the lower order approximations.**

Recursive Approximation

Zero-th Order

- For the zero-th order of ϵ, one can easily see the following equation should be satisfied:

$$dV_t^{(0)} = c(X_t)V_t^{(0)}dt + Z_t^{(0)} \cdot dW_t \tag{6.9}$$

$$V_T^{(0)} = \Phi(X_T)\,. \tag{6.10}$$

- It can be integrated as

$$V_t^{(0)} = E\left[e^{-\int_t^T c(X_s)ds}\Phi(X_T)\middle|\mathcal{F}_t\right] \tag{6.11}$$

  which is equivalent to the pricing of a standard European contingent claim, and $V_t^{(0)}$ is a function of X_t.
- Applying Itô's formula (or Malliavin derivative), we obtain $Z_t^{(0)}$ as a function of X_t, too.

Recursive Approximation

First Order

- Now, let us consider the process $V^{(\epsilon)} - V^{(0)}$. One can see that its dynamics is governed by

$$\begin{aligned} d(V_t^{(\epsilon)} - V_t^{(0)}) &= c(X_t)(V_t^{(\epsilon)} - V_t^{(0)})dt \\ &- \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + (Z_t^{(\epsilon)} - Z_t^{(0)}) \cdot dW_t \\ V_T^{(\epsilon)} - V_T^{(0)} &= 0\,. \end{aligned} \tag{6.12}$$

- Now, by extracting the ϵ-first order term, we can once again recover the linear FBSDE

$$\begin{aligned} dV_t^{(1)} &= c(X_t)V_t^{(1)}dt - g(X_t, V_t^{(0)}, Z_t^{(0)})dt + Z_t^{(1)} \cdot dW_t \\ V_T^{(1)} &= 0\,, \end{aligned} \tag{6.13}$$

which leads to

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} g(X_u, V_u^{(0)}, Z_u^{(0)})du \,\middle|\, \mathcal{F}_t\right]. \tag{6.14}$$

Recursive Approximation

- **Because $V_u^{(0)}$ and $Z_u^{(0)}$ are some functions of X_u, we obtain $V_t^{(1)}$ as a function of X_t, and also $Z_t^{(1)}$ through Itô's formula (or Malliavin derivative).**
- **In exactly the same way, one can derive an arbitrarily higher order correction. Due to the ϵ in front of the non-linear term g, the system remains to be linear in the every order of approximation. e.g.**

$$
\begin{aligned}
dV_t^{(2)} &= c(X_t)V_t^{(2)}dt - \left(\frac{\partial}{\partial v}g(X_t, V_t^{(0)}, Z_t^{(0)})V_t^{(1)}\right. \\
&+ \left.\nabla_z g(X_t, V_t^{(0)}, Z_t^{(0)}) \cdot Z_t^{(1)}\right) dt + Z_t^{(2)} \cdot dW_t \\
V_T^{(2)} &= 0
\end{aligned}
$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Suppose we have succeeded to express backward components (V_t, Z_t) in terms of X_t up to the $(i-1)$-th order. Now, in order to proceed to a higher order approximation, we have to give the following form of expressions with some deterministic function $G(\cdot)$ in terms of the forward components X_t, in general:**

$$V_t^{(i)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} G(X_u)du \middle| \mathcal{F}_t\right] \tag{6.15}$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- Even if it is impossible to obtain the exact result, we can still obtain an analytic approximation for $(V_t^{(i)}, Z_t^{(i)})$.
- For instance, an asymptotic expansion method allows us to obtain the expression. (See [29]-[30], [38] -[45] for the detail of the asymptotic expansion method.)
  In fact, applying the method, [19] has provided some explicit approximations for $V_t^{(i)}$ and $Z_t^{(i)}$.
- Also, [20] has explicitly derived an approximation formula for the dynamic optimal portfolio in an incomplete market and confirmed its accuracy comparing with the exact result by Cole-Hopf transformation. (Zariphopoulou [2001])

Remark on Approximation of Coupled FBSDEs

- Let us consider the following generic *coupled* non-linear FBSDE:

$$\begin{aligned} dV_t &= -f(t, X_t, V_t, Z_t)dt + Z_t \cdot dW_t \\ V_T &= \Phi(X_T) \\ dX_t &= \gamma_0(t, X_t, V_t, Z_t)dt + \gamma(t, X_t, V_t, Z_t) \cdot dW_t; \;\; X_0 = x\,. \end{aligned}$$

- We can treat this case in the similar way as before(decoupled case) by introducing the following perturbation to the forward process:

$$\begin{aligned} d\tilde{V}_t &= c(t, \tilde{X}_t)\tilde{V}_t dt - \epsilon g(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)dt + \tilde{Z}_t \cdot dW_t \\ \tilde{V}_T &= \Phi(\tilde{X}_T) \\ d\tilde{X}_t &= \Big(r(t, \tilde{X}_t) + \epsilon\mu(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big)dt \\ &\qquad + \Big(\sigma(t, \tilde{X}_t) + \epsilon\eta(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big) \cdot dW_t \end{aligned}$$

- We can also apply the same method under PDE(partial differential equation) formulation based on *four step scheme* (e.g. Ma-Yong [2000]).

Please consult [19] for the details.

Forward Agreement with Bilateral Default Risk

- As the first example, we consider a toy model for a forward agreement on a stock with bilateral default risk of the contracting parties, the investor (party-1) and its counter party (party-2). The terminal payoff of the contract from the view point of the party-1 is

$$\Phi(S_T) = S_T - K \tag{6.16}$$

where T is the maturity of the contract, and K is a constant.

- We assume the underlying stock follows a simple geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{6.17}$$

where the risk-free interest rate r and the volatility σ are assumed to be positive constants.

- The default intensity of party-i, h_i is specified as

$$h_1 = \lambda, \qquad h_2 = \lambda + h \tag{6.18}$$

where λ and h are also positive constants.

Forward Agreement with Bilateral Default Risk

- In this setup, the pre-default value of the contract at time t, V_t follows

$$\begin{aligned} dV_t &= rV_t dt - h_1 \max(-V_t, 0)dt + h_2 \max(V_t, 0)dt + Z_t dW_t \\ &= (r+\lambda)V_t dt + h \max(V_t, 0)dt + Z_t dW_t \qquad (6.19) \\ V_T &= \Phi(S_T)\,. \qquad (6.20) \end{aligned}$$

- Now, following the previous arguments, let us introduce the expansion parameter ϵ, and consider the following FBSDE:

$$\begin{aligned} dV_t^{(\epsilon)} &= \mu V_t^{(\epsilon)} dt - \epsilon g(V_t^{(\epsilon)})dt + Z_t^{(\epsilon)} dW_t \qquad (6.21) \\ V_T^{(\epsilon)} &= \Phi(S_T) \qquad (6.22) \\ dS_t &= S_t(rdt + \sigma dW_t)\,, \qquad (6.23) \end{aligned}$$

where we have defined $\mu = r + \lambda$ and $g(v) = -hv\mathbf{1}_{\{v \geq 0\}}$.

Forward Agreement with Bilateral Default Risk

- **The next figure shows the numerical results of the forward contract with bilateral default risk with various maturities with the direct solution from the PDE (as in Duffie-Huang [1996]).**
- **We have used**

$$r = 0.02, \quad \lambda = 0.01, \quad h = 0.03, \tag{6.24}$$

$$\sigma = 0.2, \quad S_0 = 100\ , \tag{6.25}$$

where the strike K is chosen to make $V_0^{(0)} = 0$ for each maturity.

- **We have plot $V^{(1)}$ for the first order, and $V^{(1)} + V^{(2)}$ for the second order. (Note that we have put $\epsilon = 1$ to compare the original model.)**

Forward Agreement with Bilateral Default Risk

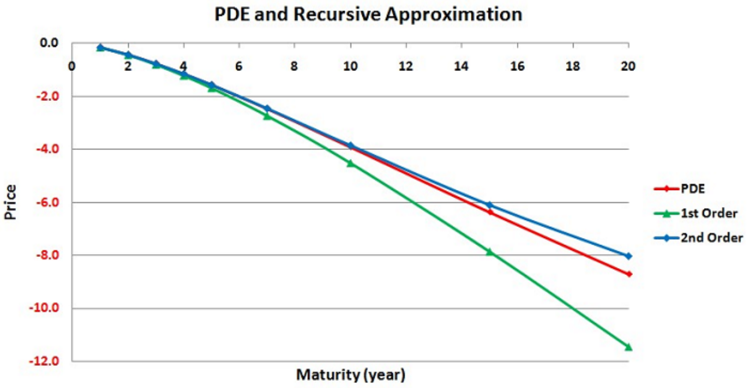


Figure: Numerical Comparison to PDE

Forward Agreement with Bilateral Default Risk

- **One can observe how the higher order correction improves the accuracy of approximation.**
- **In this example, the counter party is significantly riskier than the investor, and the underlying contract is volatile.**
- **Even in this situation, the simple approximation to the second order works quite well up to the very long maturity.**
- **In another example of [19] [13], our second order approximation has obtained a fairly close value(2.953) to the one(2.95 with std 0.01) by a regression-based Monte Carlo simulation of Gobet-Lemor-Warin[2005].**

[13] a self-financing portfolio under the situation where there exists a difference between the lending and borrowing interest rates

Perturbation Technique with Interacting Particle Method

([21], [14])

- **We will provide a straightforward simulation scheme to solve nonlinear FBSDEs at each order of perturbative approximation.**
  - **Due to the convoluted nature of the perturbative expansion, it contains multi-dimensional time integrations of expectation values, which make standard Monte Carlo too time consuming.**
  - **To avoid nested simulations, we applied the particle representation inspired by the ideas of branching diffusion models(e.g. McKean (1975), Fujita (1966), Ikeda-Nagasawa-Watanabe (1965,1966,1968), Nagasawa-Sirao (1969)).**
  - **Comparing with the direct application of the branching diffusion method, our method is expected to be less numerically intensive since the interested system is already decomposed into a set of linear problems.**

Perturbation Technique with Interacting Particle Method

- Again, let us introduce the perturbation parameter ϵ:

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)})ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T), \end{cases} \tag{7.1}$$

where $X_t \in \mathbb{R}^d$ is assumed to follow a generic Markovian forward SDE

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s; \;\; X_t = x_t. \tag{7.2}$$

- Let us fix the initial time as t. We denote the Malliavin derivative of X_u $(u \geq t)$ at time t as

$$\mathcal{D}_t X_u \in \mathbb{R}^{r \times d}. \tag{7.3}$$

Perturbation Technique with Interacting Particle Method

- **Its dynamics in terms of the future time u is specified by *stochastic flow*, $(Y_{t,u})^i_j = \partial_{x^j_t} X^i_u$**

$$\begin{aligned} d(Y_{t,u})^i_j &= \partial_k \gamma^i_0(X_u)(Y_{t,u})^k_j du + \partial_k \gamma^i_a(X_u)(Y_{t,u})^k_j dW^a_u \\ (Y_{t,t})^i_j &= \delta^i_j \end{aligned} \tag{7.4}$$

  where ∂_k denotes the differential with respect to the k-th component of X, and δ^i_j denotes Kronecker delta. Here, i and j run through $\{1, \cdots, d\}$ and $\{1, \cdots, r\}$ for a. Here, we adopt Einstein notation which assumes the summation of all the paired indexes.

- **Then, it is well-known that**

$$(\mathcal{D}_t X^i_u)_a = (Y_{t,u}\gamma(x_t))^i_a,$$

  where $a \in \{1, \cdots, r\}$ is the index of r-dimensional Brownian motion.

Perturbation Technique with Interacting Particle Method

- ϵ-0th order: For the zeroth order, it is easy to see

$$V_t^{(0)} = \mathbb{E}\Big[\Psi(X_T)\Big|\mathcal{F}_t\Big] \tag{7.5}$$

$$Z_t^{a(0)} = \mathbb{E}\Big[\partial_i\Psi(X_T)(Y_{tT}\gamma(X_t))^i_a\Big|\mathcal{F}_t\Big]. \tag{7.6}$$

- It is clear that they can be evaluated by standard Monte Carlo simulation. However, for their use in higher order approximation, it is crucial to obtain explicit approximate expressions for these two quantities. (e.g. Hagan et al.[2002], asymptotic expansion technique)
- In the following, let us suppose we have obtained the solutions up to a given order of asymptotic expansion, and write each of them as a function of x_t:

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t). \end{cases} \tag{7.7}$$

Perturbation Technique with Interacting Particle Method

- **ϵ-1st order:**

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbb{E}\Big[f(X_u, V_u^{(0)}, Z_u^{(0)})\Big|\mathcal{F}_t\Big]du \\ &= \int_t^T \mathbb{E}\Big[f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\Big]du \end{aligned} \tag{7.8}$$

- **Next, define the new process for ($s > t$):**

$$\hat{V}_{ts}^{(1)} = e^{\int_t^s \lambda_u du} V_s^{(1)}, \tag{7.9}$$

where deterministic positive process λ_t (It can be a positive constant for the simplest case.).

Perturbation Technique with Interacting Particle Method

- Then, its dynamics is given by

$$d\hat{V}^{(1)}_{ts} = \lambda_s \hat{V}^{(1)}_{ts} ds - \lambda_s \hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))ds + e^{\int_t^s \lambda_u du} Z^{(1)}_s \cdot dW_s \,,$$

  where

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)).$$

- Since we have $\hat{V}^{(1)}_{tt} = V^{(1)}_t$, one can easily see the following relation holds:

$$V^{(1)}_t = \mathbb{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds} \lambda_u \hat{f}_{tu}(X_u, v^{(0)}(X_u), z^{(0)}(X_u)) du \middle| \mathcal{F}_t \right] \quad (7.10)$$

- As in credit risk modeling (e.g. Bielecki-Rutkowski [2002]), it is the present value of default payment where the default intensity is λ_s with the default payoff at $s(> t)$ as $\hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, we obtain the following proposition.

Perturbation Technique with Interacting Particle Method

Proposition

The $\mathbf{V}_t^{(1)}$ in (7.8) can be equivalently expressed as

$$\mathbf{V}_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau)\right)\middle|\mathcal{F}_t\right]. \tag{7.11}$$

Here τ is the interaction time where the interaction is drawn independently from Poisson distribution with an arbitrary deterministic positive intensity process λ_t. $\hat{f}$ is defined as

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s}e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) . \tag{7.12}$$

Perturbation Technique with Interacting Particle Method

- Now, let us consider the component $Z^{(1)}$. It can be expressed as

$$Z_t^{(1)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t f\left(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\right)\Big|\mathcal{F}_t\right] du \tag{7.13}$$

Firstly, let us observe the dynamics of Malliavin derivative of $V^{(1)}$ follows

$$\begin{aligned} d(\mathcal{D}_t V_s^{(1)}) &= -(\mathcal{D}_t X_s^i)\nabla_i(x, v^{(0)}, z^{(0)}) f(x, v^{(0)}, z^{(0)}) + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s; \\ \mathcal{D}_t V_t^{(1)} &= Z_t^{(1)}, \end{aligned} \tag{7.14}$$

where

$$\nabla_i(x, v^{(0)}, z^{(0)}) \equiv \partial_i + \partial_i v^{(0)}(x)\partial_v + \partial_i z^{a(0)}(x)\partial_{z^a}, \tag{7.15}$$

$$f(x, v^{(0)}, z^{(0)}) \equiv f(x, v^{(0)}(x), z^{(0)}(x)). \tag{7.16}$$

Perturbation Technique with Interacting Particle Method

- Define, for $(s > t)$,

$$\widehat{\mathcal{D}_t V_s^{(1)}} = e^{\int_t^s \lambda_u du}(\mathcal{D}_t V_s^{(1)}). \tag{7.17}$$

Then, its dynamics can be written as

$$\begin{aligned} d(\widehat{\mathcal{D}_t V_s^{(1)}}) &= \lambda_s(\widehat{\mathcal{D}_t V_s^{(1)}})ds - \lambda_s(\mathcal{D}_t X_s^i)\nabla_i(X_s, v^{(0)}, z^{(0)})\hat{f}_{ts}(X_s, v^{(0)}, z^{(0)})ds \\ &+ e^{\int_t^s \lambda_u du}(\mathcal{D}_t Z_s^{(0)}) \cdot dW_s. \end{aligned} \tag{7.18}$$

We again have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)}. \tag{7.19}$$

- Hence,

$$Z_t^{(1)} = \mathbb{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds}\lambda_u(\mathcal{D}_t X_u^i)\nabla_i(X_u, v^{(0)}, z^{(0)})\hat{f}_{tu}(X_u, v^{(0)}, z^{(0)})du\middle|\mathcal{F}_t\right]. \tag{7.20}$$

Perturbation Technique with Interacting Particle Method

- **Thus, following the same argument for the previous proposition, we have the result below:**

Proposition

$Z_t^{(1)}$ in (7.13) is equivalently expressed as

$$Z_t^{a(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t\tau}\gamma(X_\tau))_a^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\mathcal{F}_t\right] \quad (7.21)$$

where the definitions of random time τ and the positive deterministic process λ are the same as those in the previous proposition.

Perturbation Technique with Interacting Particle Method

Monte Carlo Method

Now, we have a new particle interpretation of $(V^{(1)}, Z^{(1)})$ as follows:

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}, z^{(0)}\right)\middle|\mathcal{F}_t\right] \quad (7.22)$$

$$Z_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\mathcal{F}_t\right] \quad (7.23)$$

which allows efficient time integration with the following Monte Carlo scheme:

- **Run the diffusion processes of X and Y**
- **Carry out Poisson draw with probability $\lambda_s\Delta s$ at each time s and if "one" is drawn, set that time as τ.**
- **Then stores the relevant quantities at τ, or in the case of $(\tau > T)$ stores 0.**
- **Repeat the above procedures and take their expectation.**

$Z_t^{(2)}$

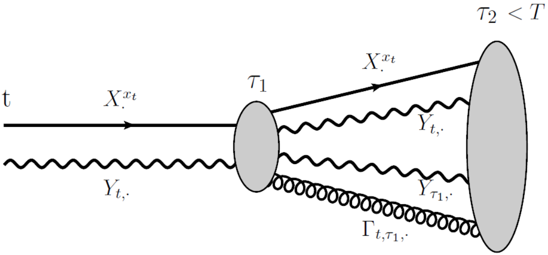


Figure 1: A particle interpretation for $Z_t^{(2)}$.

The second order stochastic flow: for $t < s < u$,

$$(\Gamma_{t,s,u})^i_{jk} \quad := \quad \frac{\partial^2}{\partial x^j_t \partial x^k_s} X^i_u; \; \left((\Gamma_{t,s,s})^i_{jk} = 0\right).$$

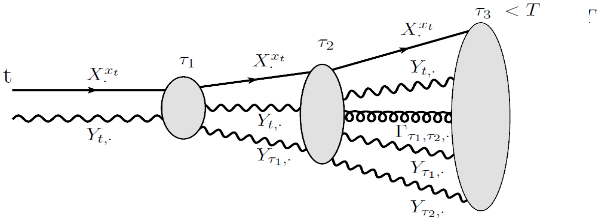


Figure 2: A particle interpretation for the first half of $V_t^{(3)}$.

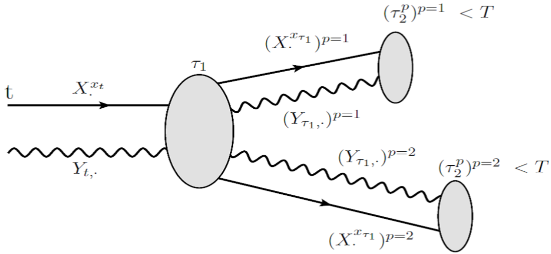


Figure 3: A particle interpretation for the second half of $V_t^{(3)}$.

Numerical Example

An example for pre-default values with imperfect collateralization [14]:

- **The counter party sells OTC European options on WTI futures. [15]**
- **For simplicity, we consider a unilateral case, where counter party is defaultable, while the investor is default-free, and the collateral is posted as the same currency as the payment currency (that is, the currency is USD).**
- **We consider the following imperfect collateral cases:**
  - **No collateral**
  - **Cash collateral with time-lag**
  - **Asset collateral with time-lag**

[14] As for an application to American option pricing, please see [11]

[15] Later, we will see a basket option on WTI and Brent futures.

Model

- **CIR model for the hazard rate process (h).**
- **SABR model for WTI futures price process (S and ν).**
- **Log-Normal model for a collateral asset price process (A).**

$$dh_t = \kappa(\theta - h_t)\,dt + \gamma\sqrt{h_t}c_1 dW_t^1;\ h_0 = h_0 \tag{8.1}$$

$$dS_t = \mu_i S_t dt + \nu_t (S_t)^\beta \Big(\sum_{\eta=1}^{2} c_{2,\eta} dW_t^\eta\Big);\ S_0 = s_0, \tag{8.2}$$

$$d\nu_t = \sigma_\nu \nu_t \Big(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta\Big);\ \nu_0 = \nu_0, \tag{8.3}$$

$$dA_t = \mu_A A_t dt + \sigma_A A_t \Big(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^\eta\Big);\ A_0 = a_0. \tag{8.4}$$

Model

- **The dynamics of pre-default value V can be described by a non-linear FBSDE:**

$$\begin{cases} dV_t = rV_t dt - f(h_t, V_t, \Gamma_t)dt + Z_t \cdot dW_t \\ V_T = (S_T - K)^+ \text{ or } (K - S_T)^+ , \end{cases} \tag{8.5}$$

where

- **Γ_t : collateral process**
  (e.g. cash collateral with a constant time lag $\Delta : \Gamma_t = V_{t-\Delta}$)
- **r(risk free rate), c(collateral rate), l(loss rate) : nonnegative constants for simplicity.** [16]

We put ϵ in front of the driver, f to apply our perturbation technique with interacting particle method.

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[16] [14] Later, we will see a more general case, where a stochastic collateral cost is taken into account.

Model

- Counter party does not post collateral or posts collateral with the constant time-lag (Δ) by cash or an asset A.
- no collateral case:

$$f(h_t, V_t, \Gamma_t) = -lh_t(V_t)^+. \tag{8.6}$$

- time-lag collateral case
  - cash collateral:

$$\begin{aligned} f(h_t, V_t, \Gamma_t) &= (r-c)V_{t-\Delta} \\ &\quad -lh_t\left(V_t - V_{t-\Delta}\right)^+, \end{aligned} \tag{8.7}$$

  - asset collateral:

$$\begin{aligned} f(h_t, V_t, \Gamma_t) &= (r-c)V_{t-\Delta}\left(\frac{A_t}{A_{t-\Delta}}\right) \\ &\quad -lh_t\left(V_t - V_{t-\Delta}\left(\frac{A_t}{A_{t-\Delta}}\right)\right)^+. \end{aligned} \tag{8.8}$$

Parameters

- **We use the data of CME WTI option and futures prices. The maturity of the underlying futures is DEC 15, and the maturity of WTI option is Nov 17, 2015.**
- **Parameters of WTI futures are obtained by calibration to the market values of futures option prices on July 10, 2012.**
- **We assume that the risk free rate r is equal to collateral rate c.**
- **The discount rate is $c = 0.295\%$ which is calculated by OIS with the same maturity as the option maturity.**
- **The recovery rate is $R = 0$ (i.e. $l = 1$).**
- **Calibrated parameters are as follows[17]:**

Table: Parameters of WTI DEC15 in SABR model

	$S(0)$	β	$\nu(0)$	σ_ν	ρ
WTI DEC15	84.48	0.5	2.117	0.410	-0.112

[17] As futures options traded in CME(WTI) are American type, we calibrate to European option prices with the implied BS(log-normal) volatilities that are obtained by a binomial method.

Parameters, Monte Carlo

- We use the results of Denault et al., 2009 [7] for the parameters of hazard rate processes.
- We calculate the pre-default value of European option whose maturity is the same as that of futures option.
- The details of Monte Carlo method simulation are as follows:
  - time step size is 1/200 years.
  - the number of trial is 10 million.
  - Hagan et al. formula [2002] is used for evaluation of default-free European options, that is $V^{(0)}$.

Analysis

- **We check the following points.**
  - **correlation effect: (S, h), (S, v), (h, v), (S, A), (v, A) and (h, A).**
  - **collateral effect: no collateral, cash collateral with constant time-lag or asset collateral with constant time-lag.**
  - **rating effect: from Aaa to B.**
  - **the second order value's effect.**
  - **maturity effect: from 2 years to 10 years.**

Correlation Effect

Firstly, we test the correlation effects among the hazard rates, the underlying asset price, its volatility and the collateral asset price.
In this example, we set the following assumptions.

- the correlations which are not explicitly specified are set to be 0.
- parameters of the hazard rate processes are those of Baa rating.
- parameters of the collateral asset are $\mu_A = 0$ and $\sigma_A = 50\%$.
- the time-lag (Δ) of collateral is 0.1.
- strike price is ATM.

Correlation Effect - No Collateral

Table: Pre-default values of call option contracts without collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.784	-0.987	-1.220	-1.465	-1.742
	2nd	0.027	0.043	0.065	0.091	0.123
	Total	13.890	13.704	13.492	13.273	13.029
S and v	0th	13.789	14.338	14.648	14.719	14.553
	1st	-1.147	-1.192	-1.220	-1.231	-1.222
	2nd	0.061	0.063	0.065	0.066	0.065
	Total	12.703	13.210	13.492	13.554	13.397
h and v	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.055	-1.134	-1.220	-1.312	-1.410
	2nd	0.050	0.057	0.065	0.074	0.085
	Total	13.642	13.570	13.492	13.410	13.322

- **When the correlation between S and h increases ($-0.7 \rightarrow +0.7$), the absolute values of the first and the second order become larger. (High correlation between S and h means that the default risk becomes high when the option value is high.)**

Correlation Effect - Cash Collateral

Table: Pre-default values of call option contracts with cash collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.116	-0.137	-0.160	-0.185	-0.211
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	14.532	14.511	14.488	14.463	14.436
S and ν	0th	13.789	14.338	14.648	14.719	14.553
	1st	-0.127	-0.144	-0.160	-0.174	-0.187
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	13.663	14.194	14.488	14.545	14.366
h and ν	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.130	-0.144	-0.160	-0.177	-0.195
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	14.518	14.503	14.488	14.471	14.452

- **The effect of the second order value seems negligible under collateralization with this level of time-lag.**

Correlation Effect - Asset Collatral

Table: Pre-default values of call option contracts with asset collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.128	-0.154	-0.183	-0.214	-0.249
	2nd	0.0001	0.0002	0.0003	0.0004	0.0006
	Total	14.520	14.494	14.465	14.433	14.399
S and v	0th	13.789	14.338	14.648	14.719	14.553
	1st	-0.154	-0.169	-0.183	-0.194	-0.204
	2nd	0.0003	0.0003	0.0003	0.0003	0.0003
	Total	13.635	14.169	14.465	14.525	14.350
h and v	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.152	-0.166	-0.183	-0.201	-0.220
	2nd	0.0002	0.0003	0.0003	0.0003	0.0004
	Total	14.496	14.481	14.465	14.447	14.428

- **The first order value with asset collateral is about 1.2 times as large as that with cash collateral.**
- **The effect of the second order value also seems negligible.**

Correlation Effect - Asset Collateral

Table: Pre-default values of call option contracts with asset collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.220	-0.202	-0.183	-0.160	-0.132
	2nd	0.0007	0.0005	0.0003	0.0001	0.0000
	Total	14.428	14.446	14.465	14.487	14.515
v and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.192	-0.188	-0.183	-0.178	-0.172
	2nd	0.0004	0.0004	0.0003	0.0002	0.0002
	Total	14.455	14.460	14.465	14.470	14.475
h and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.192	-0.188	-0.183	-0.178	-0.174
	2nd	0.0004	0.0004	0.0003	0.0002	0.0002
	Total	14.456	14.460	14.465	14.470	14.474

- **Correlation effect between the underlying asset price and the collateral asset price seems similar order as the one between the underlying asset price and the hazard rate.**
- **When the correlation between S and A is negative, the increase in the option premium and the decrease in the collateral value occur simultaneously. (That is, it requires more collateral.)**

Rating Effect - No Collateral

Table: Pre-default values of call option contracts without collateral

Strike		70	80	85	90	100
Aaa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.474	-0.351	-0.300	-0.254	-0.182
	2nd	0.005	0.004	0.003	0.003	0.002
	Total	22.189	16.450	14.036	11.928	8.564
Baa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-1.879	-1.392	-1.186	-1.007	-0.720
	2nd	0.100	0.074	0.063	0.054	0.038
	Total	20.879	15.480	13.210	11.226	8.062
B	0th	22.658	16.798	14.333	12.179	8.744
	1st	-7.877	-5.833	-4.972	-4.219	-3.017
	2nd	2.155	1.595	1.359	1.153	0.823
	Total	16.936	12.560	10.720	9.113	6.551

- **the worse is the rating, the more important the second order becomes.**
- **For the case of single B, if the second order value is not taken into account, the pre-default value is more than 10% different from the first order pre-default value.**

Rating Effect - Asset Collateral

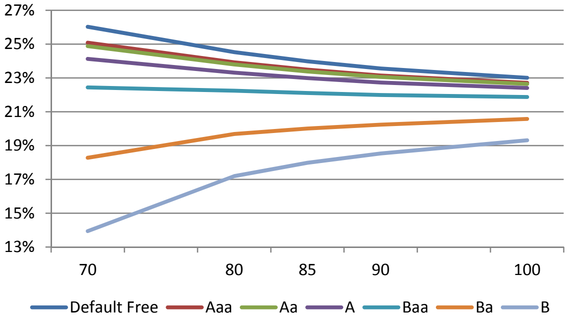
Table: Pre-default values of call option contracts with asset collateral

Strike		70	80	85	90	100
Aaa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.064	-0.051	-0.045	-0.040	-0.031
	2nd	0.00003	0.00002	0.00002	0.00001	0.00001
	Total	22.594	16.747	14.288	12.139	8.714
Baa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.250	-0.199	-0.177	-0.156	-0.120
	2nd	0.00047	0.00035	0.00030	0.00025	0.00018
	Total	22.409	16.599	14.157	12.024	8.624
B	0th	22.658	16.798	14.333	12.179	8.744
	1st	-1.029	-0.822	-0.729	-0.644	-0.497
	2nd	0.00996	0.00737	0.00628	0.00533	0.00380
	Total	21.639	15.983	13.610	11.541	8.251

- **The effect of the second order value seems negligible under collateralization with this level of time lag, even if the rating is single B.**

Rating Effect - Implied Volatility (No Collateral)

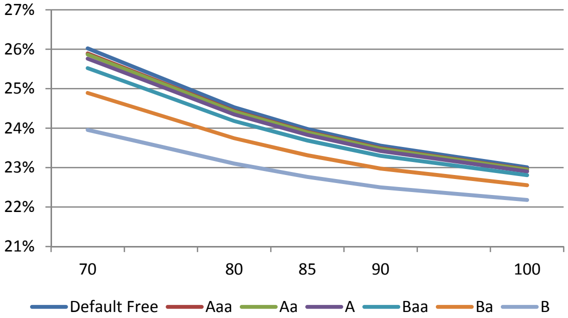
Figure: Implied volatilities of call options without collateral



- **The shape of the skew of rating B is different from that of rating Aaa. The difference of IV from the default-free case is larger for ITM, and the size of difference varies in rating.**

Rating Effect - Implied Volatility (Asset Collateral)

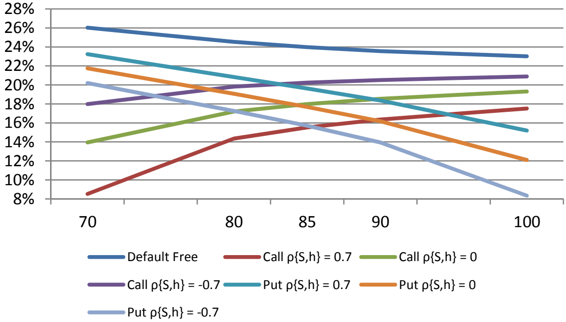
Figure: Implied volatilities of European call options with risky asset collateral



- **In this case, the shape of all ratings is similar.**
- **The level of implied volatility is different in rating.**

Correlation Effect (S, h) - Implied Volatility (Rating : B)

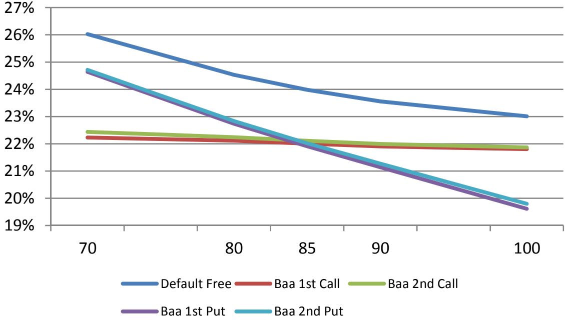
Figure: Implied volatilities of European call and put options without collateral



- **When the correlation between the underlying asset price and the hazard rate becomes high, a call option's implied volatility becomes low.**
- **This is because a default probability will increase if the price rises (that is, the option value rises).**
- **For the case of put options, the shape is reversed.**

The Second Order Effect - Implied Volatility (Baa)

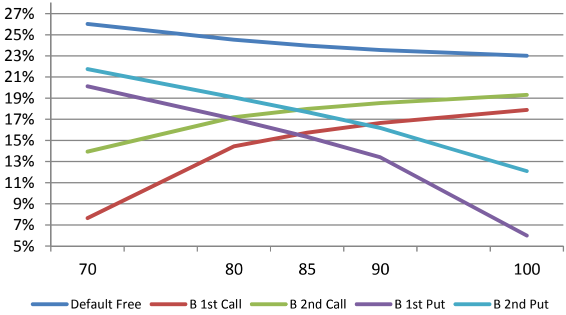
Figure: Implied volatilities of European call and put options without collateral



- **The difference between the first and the second is not so large in this case.**

The Second Order Effect - Implied Volatility (B)

Figure: Implied volatilities of European call and put options without collateral

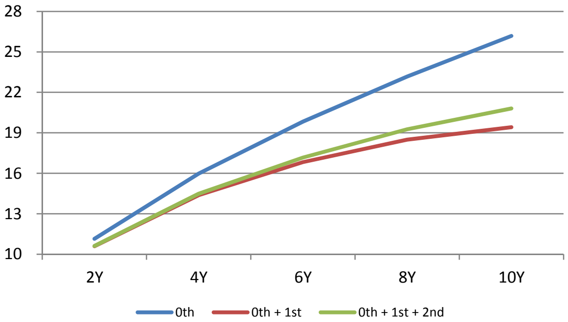


- **It seems better to take the second order value into account.**

Maturity Effect - No Collateral (Baa)

Next graph shows the values of 0th (default free), 1st and 2nd order price of the ATM option without collateral in Baa rating.

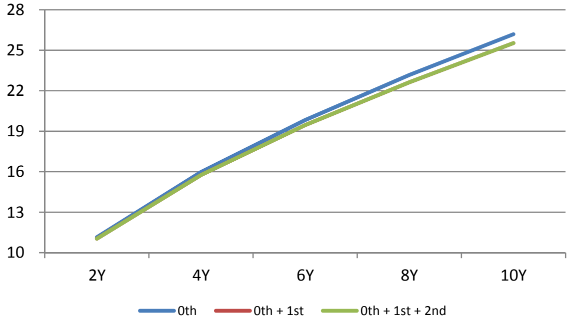
Figure: Pre-default values of call option contracts without collateral



- **For the long maturity case, the second order value has larger impact on the pre-default value.**
- **For the case of 10 years maturity, the 2nd order affects by more than 5%.**

Maturity Effect - Asset Collateral (Baa)

Figure: Pre-default values of call option contracts with asset collateral



- **When we post the collateral, the second order effect does not increase.**
- **The second order effect can be ignored even if the maturity is more than 10 years.**

Stochastic Collateral Cost

- Next, we consider a more general case:

$$dV_t = cV_t dt - f(y_t, \hat{y}_t, h_t, V_t, \Gamma_t)dt + Z \cdot dW_t, \tag{8.9}$$

where

$$f(y_t, \hat{y}_t, h_t, V_t, \Gamma_t) = \hat{y}_t\Gamma_t - y_t V_t - l h_t (V_t - \Gamma_t)^+ \tag{8.10}$$

$$y_t = r_t - c(\text{collateral cost of USD}) \tag{8.11}$$

$$\hat{y}_t = \hat{r}_t - \hat{c}_t(\text{collateral cost of } \Gamma_t) \tag{8.12}$$

- For numerical examples, we set $\hat{y} \equiv 0$ and suppose $y_t = r_t - c$ where r follows a CIR process with a nonnegative constant c. Then, we put ϵ in front of f to apply our perturbation technique with interacting particle method.

Stochastic Collateral Cost

- **CIR model for risk free rate process (r).**

$$dr_t = \kappa_r (\theta_r - r_t)\, dt + \gamma_r \sqrt{r_t} \Big(\sum_{\eta=1}^{5} c_{5,\eta} dW_t^{\eta}\Big); \; r_0 = r(0). \tag{8.13}$$

Table: Parameters of USD risk free rate process

	$r(0)$	κ_r	θ_r	γ_r
USD Risk Free Rate	1%	0.2	1%	0.05

- **The other parameters are the same as before.**
- **The rating of counter party is Baa.**
- **We check the following points.**
  - **correlation effect: (S, h), (S, y), and (h, y).**
  - **collateral effect: no collateral, cash collateral with constant time-lag 0.1.**

Correlation Effect - No Collateral

Table: Correlation Effects - No Collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.129	-1.335	-1.565	-1.821	-2.100
	2nd	0.051	0.072	0.099	0.131	0.170
	Total	13.570	13.385	13.181	12.958	12.717
S and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.418	-1.488	-1.565	-1.650	-1.742
	2nd	0.083	0.090	0.099	0.109	0.119
	Total	13.313	13.250	13.181	13.106	13.025
h and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.565	-1.565	-1.565	-1.565	-1.565
	2nd	0.093	0.096	0.099	0.102	0.105
	Total	13.176	13.179	13.181	13.184	13.187

Change in the correlation between S and y affects on the value by at most 2 %, while change in the correlation between S and h does by around 6%.

Correlation Effect - Cash Collateral

Table: Correlation Effects - Cash Collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.490	-0.511	-0.533	-0.558	-0.584
	2nd	0.007	0.007	0.007	0.008	0.008
	Total	14.164	14.144	14.121	14.097	14.072
S and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.370	-0.448	-0.533	-0.627	-0.730
	2nd	0.003	0.005	0.007	0.010	0.014
	Total	14.281	14.205	14.121	14.030	13.931
h and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.533	-0.533	-0.533	-0.534	-0.534
	2nd	0.007	0.007	0.007	0.007	0.008
	Total	14.121	14.121	14.121	14.121	14.121

Change in the correlation between S and y has a larger effect than change in the correlation between S and h, (y_t is multiplied by V_t, whereas h_t is multiplied by $V_t - V_{t-\Delta}$.)

Basket Option

Next, we consider about a basket option of WTI and Brent.

- **To calculate $V^{(0)}$ analytically, we use the asymptotic expansion method (please see [39] for the detail).**
- **The maturity of the underlying futures is DEC 15.**
- **The maturity of basket option is Nov 10, 2015.**
- **The discount rate is $c = 0.295\%$ which is calculated by OIS with the same maturity as the option maturity.**
- **The parameters of the underlying asset prices are obtained by calibration to the market values of futures options on July 10, 2012.**

Calibrated parameters are follows. [18] **:**

Table: Parameters of Brent DEC15 in SABR model

	$S(0)$	β	$\nu(0)$	σ_ν	ρ
Brent DEC15	90.14	0.5	2.184	0.446	-0.044

[18] As futures options traded in ICE(Brent) are American type, we calibrate to European option prices with the implied BS(log-normal) volatilities that are obtained by a binomial method.

Basket Option

- **The correlation between WTI futures price (or Brent futures price) and Brent volatility (or WTI volatility) is set as the same value as the correlation between WTI futures price (or Brent futures Price) and WTI volatility (or Brent volatility).**
- **The correlations between WTI futures price (or volatility) and Brent futures price (or volatility) are calculated by using logarithmic historical price changes for the 30 days before July 10, 2012.**
- **The correlation between WTI future price and Brent future price is 0.980, and the correlation between WTI volatility and Brent volatility is 0.907.**

Basket Option

Table: Pre-default values of call option contracts without collateral

Strike		140	160	170	180	200
Aaa	0th	49.798	37.475	32.224	27.590	20.083
	1st	-1.036	-0.780	-0.671	-0.575	-0.418
	2nd	0.011	0.009	0.007	0.006	0.005
	Total	48.774	36.704	31.561	27.021	19.669
Baa	0th	49.798	37.475	32.224	27.590	20.083
	1st	-4.101	-3.089	-2.657	-2.276	-1.658
	2nd	0.217	0.164	0.141	0.121	0.088
	Total	45.915	34.550	29.708	25.435	18.513
B	0th	49.798	37.475	32.224	27.590	20.083
	1st	-17.176	-12.937	-11.130	-9.534	-6.946
	2nd	4.680	3.531	3.041	2.607	1.904
	Total	37.303	28.069	24.135	20.662	15.041

- **Moreover, applying the asymptotic expansion method, we are able to calculate pre-default values of various type of basket options. (Please see [39] for the detail.)**

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