Pricing of Collateralized Derivatives *

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Outlines

Introduction : Outstanding Issues

- **Wide and Volatile Basis Spreads**:
  - Libor-OIS, Cross Currency basis, 3m/6m basis etc...
- **Pricing consistent with CSA (Credit Support Annex)**:
  - Take into cash flow arising from the collateral account.
  - Collateral Asset, particularly, the choice of collateral currency.
- **New Regulations**
  - CVA/DVA, LCR, mandatory clearing at CCP, stricter collateral control in OTC.
- **Forthcoming ISDA CSA (SCSA)**
  - Daily electric margin call with MTA=Threshold=0.
  - USD collateralization for non-G5
- **New Financial Services**
  - Collateral transformation to allow clients to access CCPs.

All these developments require clear understanding of counter party risk and cost of collateralization.

Market Data: LIBOR-OIS Spread (3m)

Market Data: Cross Currency Basis Spread (1yr)

Market Data: CCS Term Structure

Market Development

What is going on ?

- **Loss of Price-Transparency:**
  - **Counter-party Credit Risk**
  - **Collateralization**
- **New market benchmark price is appearing:**
  - **Collateralized Deals**
    - **Moving to CSA-consistent pricing**
    - **Model change in Swapclear in LCH.Clearnet. (2010)**
  - **Un-collateralized Deals**
    - **Credit Value Adjustment (CVA/DVA)**

Survey by KPMG

Survey by KPMG (collateralized deals)

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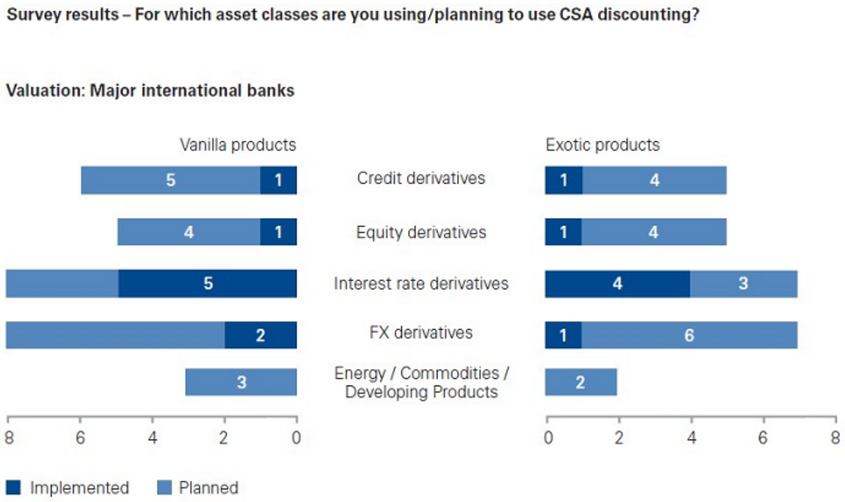
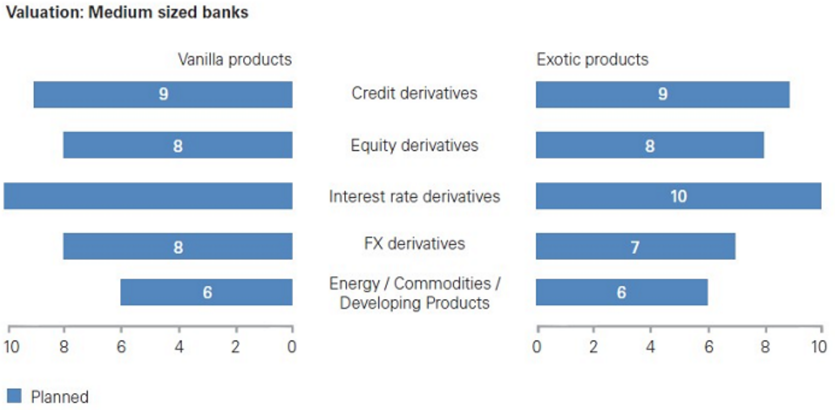


Survey by KPMG (collateralized deals)

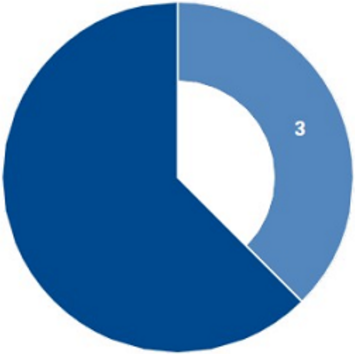


Survey by KPMG (collateralized deals)

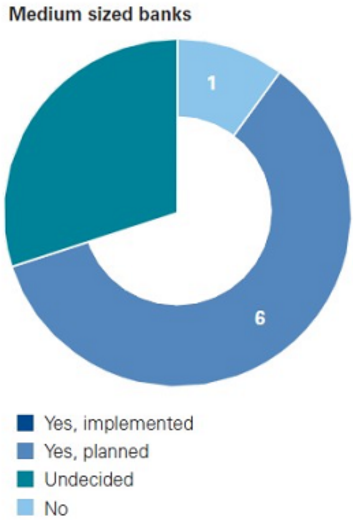
Figure 4

Survey Result – Do you (plan to) consider the currency of the respective collateral posted when building the discount curve?

Major international banks



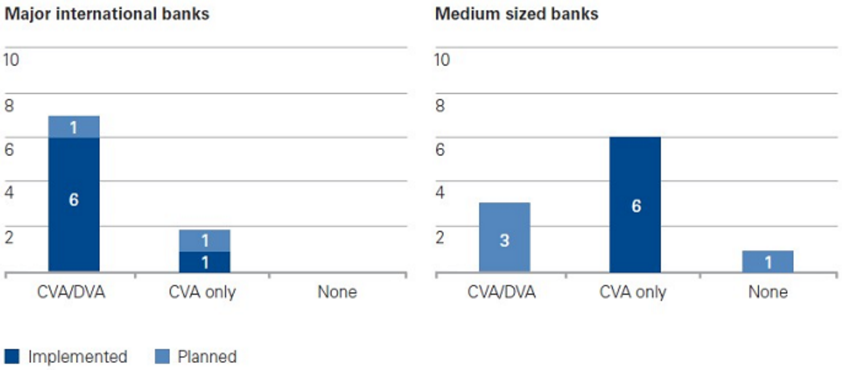
Survey by KPMG (collateralized deals)



Survey by KMPG (un-collateralized deals)

Figure 5

Survey Result – Do you use/plan to use a CVA/DVA approach in pricing uncollateralized derivatives?



Motivation of This Talk

Our Goal

- **Explain the Current Market Issues and Developments (done).**
- **Explain CSA-consistent pricing and CVA**
  - **Pricing under Perfect Collateralization**
  - **Effect of Collateral Currency Choice**
  - **Embedded Cheapest-to-Deliver Option in CSA**
  - **Imperfect Collateralization and Bilateral CVA**

under the unified framework.

Setup

- Probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F}$ contains all the market information including defaults.
- Consider two firms, $i \in \{1, 2\}$, whose default time is $\tau^i \in [0, \infty]$, and $\tau = \tau^1 \wedge \tau^2$.
- τ^i (and hence τ) is assumed to be totally-inaccessible $\mathbb{F}$-stopping time.
- Indicator functions: $H_t^i = 1_{\{\tau^i \le t\}}$, $H_t = 1_{\{\tau \le t\}}$
- Assume the existence of absolutely continuous compensator for H^i:
$$A_t^i = \int_0^t h_s^i 1_{\{\tau^i > s\}} ds, \quad t \ge 0$$
- Assume no simultaneous default, and hence the hazard rate of H is
$$h_t = h_t^1 + h_t^2 \ .$$
- Money market account: $\beta_t = \exp\left(\int_0^t r_u du\right)$

Collateralization

- When party $i \in \{1,2\}$ has negative mark-to-market, it has to post collateral (cash) to party $j(\neq i)$, and it is assumed to be done continuously.
- collateral coverage ratio is $\delta_t^i \in \mathbb{R}_+$, and the value of collateral at time t is given by $\delta_t^i(-V_t^i)$.
  - δ_t^i effectively takes into account under- as well as over-collateralization. Thus $\delta_t^i < 1$, and also $\delta_t^i > 1$ are possible.
- party j has to pay the collateral rate c_t^i on the posted cash continuously.
- c_t^i is determined by the currency posted by party i.
  - In general, $c_t^i \neq r_t^i$, which is the risk-free interest rate of the same currency. This is necessary to explain CCS swap market consistently.

Counterparty Exposure and Recovery Scheme

- **Counterparty exposure to party j at time t (from the view point of party i)**

$$\max(1-\delta_t^j,0)\max(V_t^i,0)+\max(\delta_t^i-1,0)\max(-V_t^i,0)$$

- **Assume party-j's recovery rate at time t as $R_t^j \in [0,1]$**
- **Recovery value at the time of j's default:**

$$R_t^j\left([1-\delta_t^j]^+[V_t^i]^+ + [\delta_t^i-1]^+[-V_t^i]^+\right)$$

Pricing Formula

- Pricing from the view point of party 1.

$$S_t = \beta_t E^Q \left[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \left\{ dD_u + (y_u^1 \delta_u^1 1_{\{S_u < 0\}} + y_u^2 \delta_u^2 1_{\{S_u \geq 0\}}) S_u du \right\} \right.$$
$$\left. + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \left(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right) \middle| \mathcal{F}_t \right]$$

- D: cumulative dividend to party 1.
- $y_t^i = r_t^i - c_t^i, \ (i \in \{1,2\})$ denotes the instantaneous return at time t from the cash collateral posted by party i, or stochastic dividend yield.
- Default payoff:

$$Z^1(t,v) = \left(1 - (1 - R_t^1)(1 - \delta_t^1)^+\right) v 1_{\{v<0\}} + \left(1 + (1 - R_t^1)(\delta_t^2 - 1)^+\right) v 1_{\{v \geq 0\}}$$
$$Z^2(t,v) = \left(1 - (1 - R_t^2)(1 - \delta_t^2)^+\right) v 1_{\{v \geq 0\}} + \left(1 + (1 - R_t^2)(\delta_t^1 - 1)^+\right) v 1_{\{v<0\}}$$

Pricing Formula

Pre-default value of the contract $V_t 1_{\{\tau>t\}} = S_t$ is given by

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u,V_u))\right) dD_s \middle| \mathcal{F}_t\right], \quad t \le T$$

where

$$\begin{aligned}\mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\ge 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-R_t^i)(1-\delta_t^i)^+ h_t^i + (1-R_t^j)(\delta_t^i - 1)^+ h_t^j\end{aligned}$$

- **Duffie & Huang (1996) derives under some technical conditions.**
- **More precisely, the measure is different from the usual money-market measure, but rather "survival measure" introduced by Schönbucher (2000) and Collin-Dufresne et.al. (2004).**
- **It has an important meaning when dealing with the credit derivatives, such as CDS (Fujii & Takahashi, 2011).**
- **In this talk, we focus on the usual fixed income derivatives, and do not go into details about this.**

Symmetric Case

Effective discount factor is non-linear

$$r_t - \mu(t,v) = r_t - \left(\tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}}\right)$$

which makes the portfolio value non-additive.
If $\tilde{y}_t^1 = \tilde{y}_t^2 = \tilde{y}_t$, then we have

$$\mu(t,v) = \tilde{y}_t \ .$$

If $\tilde{y}$ is not explicitly dependent on V, we can recover the linearity.

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

Portfolio valuation can be decomposed into that of each payment.

$\Downarrow$

A good characteristic for market benchmark price.

Generic Situations

- If $\tilde{y}^1 \neq \tilde{y}^2$,

$$\begin{aligned} V_t &= E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s \left(r_u - \mu(u, V_u)\right)\right) dD_s \middle| \mathcal{F}_t\right] \\ \mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-R_t^i)(1-\delta_t^i)^+ h_t^i + (1-R_t^j)(\delta_t^i - 1)^+ h_t^j \end{aligned}$$

V and the other processes form a system of non-linear FBSDE.

Marginal Impact of asymmetry:

- Make use of Gateaux derivative as the first-order Approximation: Duffie & Skiadas (1994), Duffie & Huang (1996)

$$\lim_{\epsilon\downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta};\eta) - \frac{V_t(\bar{\eta}+\epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0$$

η and $\bar{\eta}$ are bounded and predictable

Generic Situations

Marginal Impact of Asymmetry

- We want to expand the price around symmetric benchmark price.

$$\begin{aligned}\mu(t,v) &= \tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}} \\ &= y_t + \Delta\tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \Delta\tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}}\end{aligned}$$

$$\Delta\tilde{y}_t^i = \tilde{y}_t^i - y_t$$

- Calculate GD at symmetric $\mu = y$ point.

$$V_t(\mu) \simeq V_t(y) + \nabla V_t(y, \mu - y)$$

Generic Situations

- Applying Gateaux Derivative at $\mu = y$ point:

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T$$

is decomposed as $V_t \simeq \overline{V}_t + \nabla V_t$, where

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\nabla V_t = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\overline{V}_s \left(\Delta\tilde{y}_s^1 1_{\{\overline{V}_s < 0\}} + \Delta\tilde{y}_s^2 1_{\{\overline{V}_s \geq 0\}}\right) ds \middle| \mathcal{F}_t\right]$$

If y is chosen in such a way that it reflects the funding cost of the standard collateral agreements, $\overline{V}$ turns out to be the market benchmark price, and ∇V represents the corrections (CVA, non-standard CSA, etc). For analytical approximation including higher order corrections for generic non-linear FBSDEs, see Fujii & Takahashi (2011).

Perfect Collateralization with a Domestic Currency

Special Cases

Case 1 : Benchmark for single currency product

- **bilateral perfect collateralization** $(\delta^1 = \delta^2 = 1)$
- **both parties use the same currency** (i) **as collateral, which is also the payment (evaluation) currency.**
- $\mu(t, v) = y^{(i)} = r^{(i)} - c^{(i)}$

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du \right) dD_s \middle| \mathcal{F}_t \right]$$

The valuation method for single currency swap adopted by LCH Swapclear (2010) is the same with this formula.

Perfect Collateralization with a Foreign Currency

Special Cases

Case 2 : Collateral posted by a Foreign Currency

Particularly relevant for non-G5 currencies allocated in USD-silo in SCSA.

- bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)
- both parties use the same currency (k) as collateral, which is different from the payment (evaluation) currency (i)
- $r^{(i)} - \mu(t, v) = r^{(i)} - y^{(k)}$

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

Funding Spread between the two currencies

$$y^{(i,k)} = y^{(i)} - y^{(k)} = \left(r^{(i)} - c^{(i)}\right) - \left(r^{(k)} - c^{(k)}\right)$$

Cheapest-to-Deliver Option

Special Cases

Case 3 : Multiple Eligible Collaterals

- bilateral perfect collateralization $(\delta^1 = \delta^2 = 1)$
- both parties choose the optimal currency from the eligible collateral set $\mathcal{C}$. Currency (i) is used as the evaluation currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s \left(c_u^{(i)} + \max_{k\in\mathcal{C}}[y_u^{(i,k)}]\right)du\right) dD_s \middle| \mathcal{F}_t\right]$$

- The party who posts collateral has the optionality of collateral choice.
- The cheapest collateral currency should be chosen based on CCS information.

Term Structure Modeling under Collateralization

Modeling of Collateral Rate : $c(t,s)$

- **OIS collateralized with Domestic Currency:**
  - **Both of the reference and discounting rate is equal to collateral rate (O/N-rate).**
  - **Market OIS rates determine collateral rate term structure.**

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) dt + \sigma_c(t,s) \cdot dW_t^Q$$

Term Structure Modeling under Collateralization

Modeling of LIBOR-OIS Spread : $B(t;T,\tau)$

- **IRS and TS (tenor swap, such as 3m/6m basis swap) collateralized with Domestic Currency:**
  - **Discounting is given by the collateral rate c.**
  - **Market IRS and TS rates determine the term structures of LIBOR-OIS spreads.**

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t;T,\tau)\cdot\left(\int_t^T \sigma_c(t,s)ds\right)dt + \sigma_B(t;T,\tau)\cdot dW_t^Q$$

Term Structure Modeling under Collateralization

Currency Funding Spread $y^{(i,j)}$

- Cross Currency Swap (USD vs JPY, EUR, GBP, etc)
  - Collateralized by USD (to be standardized by SCSA).
  - USD leg : all the ingredients are already fixed by USD domestic IR market.
  - X leg : c(X), LIBOR-OIS(X) are already fixed by X domestic IR market. Only freedom to fix is $y^{\{X,USD\}}$.
  - Term structure of $y^{\{X,USD\}}$ is fixed by market quotes of CCS basis.

$$dy^{(i,j)}(t,s) = \sigma_y^{(i,j)}(t,s) \cdot \left(\int_t^s \sigma_y^{(i,j)}(t,u)du \right) dt + \sigma_y^{(i,j)}(t,s) \cdot dW_t^{Q^{(i)}}$$

Term Structure Modeling under Collateralization

Generic IR-FX Framework under the Perfect Collateralization

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s) \cdot \left(\int_t^s \sigma_c^{(i)}(t,u)du \right) dt + \sigma_c^{(i)}(t,s) \cdot dW_t^{Q^{(i)}}$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t;T,\tau) \cdot \left(\int_t^T \sigma_c^{(i)}(t,s)ds \right) dt + \sigma_B^{(i)}(t;T,\tau) \cdot dW_t^{Q^{(i)}}$$

$$dy^{(i,j)}(t,s) = \sigma_y^{(i,j)}(t,s) \cdot \left(\int_t^s \sigma_y^{(i,j)}(t,u)du \right) dt + \sigma_y^{(i,j)}(t,s) \cdot dW_t^{Q^{(i)}}$$

$$\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t)\right)dt + \sigma_x^{(i,j)}(t) \cdot dW_t^{Q^{(i)}}$$

- **SDEs for the other currencies are just given by the measure change.**
- **The above framework allows to take into account LIBOR-OIS basis, Tenor basis, and Cross Currency basis spreads.**

Imperfect Collateralization

CVA as the Deviation from the Perfect Collateralization

- **Assume the both parties use the same currency for simplicity, and hence $y^1 = y^2 = y$.**

$$\mu(t,v) = y_t - \Big\{\Big((1-\delta_t^1)y_t + (1-R_t^1)(1-\delta_t^1)^+ h_t^1 - (1-R_t^2)(\delta_t^1-1)^+ h_t^2\Big) 1_{\{v<0\}}$$
$$+ \Big((1-\delta_t^2)y_t + (1-R_t^2)(1-\delta_t^2)^+ h_t^2 - (1-R_t^1)(\delta_t^2-1)^+ h_t^1\Big) 1_{\{v\geq 0\}}\Big\}$$

- **GD around $\mu = y$ decomposes the price into three parts:**
  - **Symmetric perfectly collateralized benchmark price**
  - **$(1-\delta^i)y1_{\{v \lesseqgtr 0\}} \Rightarrow$ Collateral Cost Adjustment (CCA)**
  - **Remaining h dependent terms $\Rightarrow$ Credit Value Adjustment (CVA)**

$$\begin{aligned} V_t &\simeq \overline{V}_t + \nabla V_t \\ &= \overline{V}_t + \mathrm{CCA} + \mathrm{CVA} \end{aligned}$$

Imperfect Collateralization

Price adjustment of imperfectly collateralized contract

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du} y_s \left((1-\delta_s^1)[-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$
$$E^Q\Bigg[\int_t^T e^{-\int_t^s (r_u - y_u)du}(1-R_s^1)h_s^1\left[(1-\delta_s^1)^+[-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right] ds$$
$$- \int_t^T e^{-\int_t^s (r_u - y_u)du}(1-R_s^2)h_s^2\left[(1-\delta_s^2)^+[\overline{V}_s]^+ + (\delta_s^1 - 1)^+[-\overline{V}_s]^+\right] ds \Bigg| \mathcal{F}_t\Bigg]$$

- $V_t \simeq \overline{V}_t + \text{CCA} + \text{CVA}$

Imperfect Collateralization

A simple case of Imperfect Collateralization.

- Both parties use currency (j) as collateral.
- Evaluation (payment) currency is (i).
- Assume common collateral coverage ratio $\delta < 1$.
- Assume constant recovery ratio R^1 and R^2, respectively.

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = -E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du} y_s^{(j)}(1-\delta_s)\overline{V}_s ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} = (1-R^1)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du}(1-\delta_s)h_s^1[-\overline{V}_s]^+ ds \middle| \mathcal{F}_t\right]$$
$$-(1-R^2)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du}(1-\delta_s)h_s^2[\overline{V}_s]^+ ds \middle| \mathcal{F}_t\right]$$

Dependence among y, δ and other factors, such as $\overline{V}, h^i$ is particularly important. $\Rightarrow$ New type of Wrong (Right)-way Risk.

Devaluation of Collateral

- How to handle the situation where there exists significant risk of the devaluation of collateral at the time of counter party default?
  - EUR posted by European banks as collateral
  - JGB posted by Japanese mega-banks as collateral
  - Domestic currency or government bond posted by the sovereign as collateral

Assume perfect collateralization for simplicity.

- CCA $\Rightarrow$ 0.
- CVA $\Rightarrow$ interpret δ (< 1) as the fraction of devaluation.
- More general situations, one needs to introduce multiple δ processes.

We have explained:

- **Current Market Issues**
- **CSA-consistent Benchmark Pricing**
- **Adjustments for remaining costs: CCA, CVA/DVA**

Challenge for Japanese Financial Firms and Authorities

- **Adopt the new standards as quickly as possible in order not to be arbitraged away by foreign firms and protect the interests of depositors.**
- **Improve the status of JPY as collateral.**
- **Deepen the discussion on CCPs and related topics, such as the accessibility from funds, local banks, and corporates and associated risk-management issues.**

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