Credit-Equity Modeling under a Latent Lévy Firm Process

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September 27, 2013
Plan of My Talk

1. Introduction: Motivation, Literature Review
   - Equity Options, Credit Modeling, Credit-Equity Models
   - CreditGrades, Latent Credit Model

2. Our Model
   - Firm Value, Equity Value
   - Regime-Switching (mid-term spread), Jumps (short-term spread)
   - CDS, Equity Option Pricing

3. Numerical Examples
   - CDS, Equity Option

4. Conclusion, Future Research
Recent credit crisis shows the intimate relationship between the credit and equity markets.

For example, during the credit crisis, both CDS premiums and equity volatilities were at their historical high.

However, until recently, the equity and credit modelings are two separate themes in the finance literature.

The difficulty to construct the credit-equity model stems from the fact that the debt and equity possess different properties.

Hence, new attempts are required to construct the credit-equity modeling in a unified manner.
Literature Review: Equity Options

- Mostly, based on Stochastic Differential Equations (SDEs)
  - Diffusion: Black–Scholes Model, Local Volatility Model
  - Stochastic Volatility (SV): Heston (1993), etc.
  - Jump Diffusion: Merton (1976), Kou (2002), etc.
  - Lévy Process: Ask Professor Vostrikova
  - Regime Switching: Kijima and Yoshida (1993), Buffington and Elliott (2002), etc.

- Strength: Highly liquid markets (equity and equity options)
- Shortcoming: No mention about the issuer’s (firm) credit exposure
Literature Review: Credit Modeling

1 Reduced-form approach
   - Jarrow and Turnbull (1995), Madan and Unal (1998a), Duffie and Singleton (1999), etc.
   - Strength: Analytical tractability and ability of generating a flexible and realistic term structure of credit spreads
   - Shortcoming 1: Exogenous hazard rate process
   - Shortcoming 2: Default mechanism is not related to the firm value

2 Structural approach
   - Strength: Economic appeal—links firm value with debt and equity
   - Shortcoming 1: Difficult to incorporate more realistic features without sacrificing tractability
   - Shortcoming 2: Difficult to price equity or equity options
1 Reduced-form approach
   - Carr and Wu (2010), Mendoza-Arriaga et al. (2010) and references therein

2 Structural approach
   - CreditGrades model by Finger et al. (2002) and its extensions
     - by Sepp (2006) for double-exponential jump-diffusion model
     - by Ozeki et al. (2011) for general spectrally-negative Lévy process
   - Time-changed Brownian motion approach by Hurd and Zhou (2011)
   - Latent model by Kijima et al. (2009)
Ordinary structural approach

- Consider a corporate firm that issues a debt and an equity.
- Let $D$ and $S$ be the debt and equity values per share, respectively.
- Let $V$ be the firm value per share, so that $V = D + S$ by the basic accounting assumption.
- $V$ is modeled by a SDE and the default occurs when $V$ reaches a default threshold.
- $D$ and $S$ are evaluated as contingent claims written on $V$.

CreditGrades model by Finger et al. (2002)

- $D$ is the discounted face value of debt and $S$ is modeled by a GBM.
- $V$ is given by $V = D + S$, and default is the first passage time of $V$.
- Strength: Easy to implement and extend.
- Shortcoming 1: $D$ is irrelevant to the credit structure.
- Shortcoming 2: Credit quality is essentially equal to the equity value.
Latent model (in, e.g., medical science)
- Introduce the notion of the marker process that is observable and correlated to the actual status process (unobservable).

Latent structural model by Kijima et al. (2009)
- The actual firm status is latent.
- Debt value is given in terms of the actual firm status.
- Equity value is obtained as a residual value as in Merton (1974).
- Strength: Economically appealing
- Shortcoming 1: The equity has a maturity as in Merton (1974).
- Shortcoming 1: The pricing of equity options is very complicated.
Our Model: Overview

- Structural approach: treat the firm value as a latent variable
- Extension of Kijima et al. (2009) to include jumps (for short-term credit spread) and regime switch (for mid-term spread)
- Source of information: Equity value
- Objective: Price CDS and Equity Option with default feature under a joint framework.
- Contributions: Our model
  1. Introduces the credit status of the firm into the equity process.
  2. Serves as a theoretical support to the existing empirical analyses on the explanatory power of equity’s historical and option-implied volatilities to the CDS spread variation (work in progress).
  3. Has a flexibility in explaining both the short-term and mid-term behaviors of the credit spread and implied volatility curves.
Our Model: Firm value process

- \( A_t \): Actual firm value at time \( t \) where
  \[ A_t = \exp(X_t), \quad t \geq 0 \]

- \( A_t \) is latent, i.e. unobservable and non-tradable.

- Nature of default: Default epoch \( \tau \) is defined by
  \[ \tau = \inf\{t \geq 0 : A_t \leq \Gamma\} = \inf\{t \geq 0 : X_t \leq L\} \]
  for some \( \Gamma = e^L \) (default barrier).

- Remark: Easy to extend to include a stochastic boundary.
Our Model: Equity value process

- $S_t$: Equity value of the firm at time $t$
- $S_t$ is observable to investors and tradable.
- Let $Y_t = \log S_t$, and assume that (for each regime)
  \[ Y_t = \rho X_t + Z_t \]
- $Z_t$: Non-firm specific shocks, independent of $X_t$ (given each regime).
- $\rho$: The impact factor of firm's credit exposure on equity

Remark: The model does not satisfy the basic accounting condition.
It can be considered as an extension of CreditGrades by taking $\rho = 1$
and $Z_t = -e^{-rT}F$. 
Introduce the regime-switching for the mid-term spread.

Let \( \{ J_t : t \geq 0 \} \) be a Markov chain on state space \( E \).

\( E \) is finite and contains \( d \) elements, i.e., \( E = \{1, 2, \ldots, d\} \).

Let \( Q \) be the intensity matrix of \( J_t \) with respect to the Lebesgue measure, i.e.,

\[
Q = \{ q_{ij} \}_{i,j \in E}
\]

where

\[
q_{ii} = - \sum_{i \neq j} q_{ij}
\]
Let $X_t = \log A_t$ be defined by

$$X_t = \int_0^t b^X(J_s) \, ds + \int_0^t \sigma^X(J_s) \, dW^X_s$$

$$+ \sum_{j \in E} \int_0^t 1\{J_s = j\} \, dN^X_s(j)$$

where, given $J_t = j \in E$, $b^X(J_t) \equiv b^X_j$ is a drift, $\sigma^X(J_t) \equiv \sigma^X_j$ is a volatility, and $\{N^X_t(j) : t \geq 0\}$ is a compound Poisson process.

$N^X_t(j)$ has arrival rate $\lambda^X_j$ and double-exponential jumps $Y^X_j$ with distribution $\nu^X_j(dy)$, where

$$\nu^X_j(dy) = \lambda^X_j \left[ p^X_j \eta^X_j e^{-\eta^X_j y} 1\{y \geq 0\} 
+ (1 - p^X_j) \eta^X_{j2} e^{\eta^X_{j2} y} 1\{y < 0\} \right] dy$$
Moment Generating Function of $X_t$

The moment generating function (MGF) of $X_t$, $\mathbb{E}[\exp(uX_t)]$, is obtained by Kijima and Siu (2013) as

$$\mathbb{E}[\exp(uX_t)] \equiv \exp(\mathbf{K}^X[u]t)$$

where

$$\mathbf{K}^X[u] \equiv \{\kappa_j^X(u)\}_{\text{diag}} + Q$$

with

$$\kappa_j^X(u) = b_j^X u + \frac{(\sigma_j^X u)^2}{2} + \lambda_j^X \left( \frac{p_j^X \eta_{j1}^X}{\eta_{j1}^X - u} + \frac{(1 - p_j^X) \eta_{j2}^X}{\eta_{j2}^X + u} - 1 \right)$$

for regime-switching, double-exponential jumps.
Recall that $Y_t = \log S_t$ and, for each regime, $Y_t = \rho X_t + Z_t$.

We assume that $Z_t$ has the following canonical representation:

$$Z_t = \int_0^t b^Z(J_s)\,ds + \int_0^t \sigma^Z(J_s)\,dW_s^Z + \int_0^t \int_{\mathbb{R}} y(\mu^Z(J_s) - \nu^Z(J_s))\,(dy)\,ds$$

where, for $J_t = j$, $b^Z(J_t) \equiv b^Z_j$ is a drift, $\sigma^Z(J_t) \equiv \sigma^Z_j$ is a volatility, $\mu^Z(J_t) \equiv \mu^Z_j$ is a random jump measure, and $\nu^Z(J_t) \equiv \nu^Z_j$ is the compensator of $\mu^Z_j$.

$Z_t$ can be a general Lévy, because it is irrelevant to default.
The MGF $\mathbb{E}[\exp(uZ_t)]$ is given by

$$
\mathbb{E}[\exp(uZ_t)] \equiv \exp(K^Z[u]t)
$$

where

$$
K^Z[u] \equiv \{\kappa^Z_j(u)\}_\text{diag} + Q
$$

with

$$
\kappa^Z_j(u) = b^Z_j u + \frac{1}{2}(\sigma^Z_j u)^2 + \int_{\mathbb{R}} (e^{uy} - 1 - y1_{\{|y| \leq 1\}})\nu^Z_j(dy)
$$
Assume \( J_t = j \). The discounted process \( \tilde{S}_t \equiv e^{-rt} S_t \) is a \( \mathbb{P} \)-martingale with respect to \( \mathcal{F}_t \) if and only if

\[
\rho b^X_j + b^Z_j = r - \frac{1}{2} (\rho \sigma^X_j)^2 - \frac{1}{2} (\sigma^Z_j)^2 - \lambda^X_j \left( \frac{p^X_j \eta_{j1}}{\eta_{j1} - \rho} + \frac{(1 - p^X_j) \eta^X_{j2}}{\eta^X_{j2} + \rho} - 1 \right) \]

\[
- \lambda^Z_j \left( \frac{p^Z_j \eta_{j1}}{\eta_{j1} - 1} + \frac{(1 - p^Z_j) \eta^Z_{j2}}{\eta^Z_{j2} + 1} - 1 \right)
\]

where \( r > 0 \) is the risk-free interest rate.
Standard CDS premium formula: For $J_0 = i \in E$,

\[
e^{(i)}_T = (1 - R) \frac{\int_0^T e^{-rt} dP_i(\tau \leq t)}{\int_0^T e^{-rt} P_i(\tau > t) dt}
\]

\[
= (1 - R) r \frac{\mathbb{E}_i [e^{-r\tau 1_{\{\tau < T\}}}]}{1 - \mathbb{E}_i [e^{-r\tau 1_{\{\tau < T\}}} - e^{-rT} P_i(\tau > T)]}
\]

where $R$ is the recovery rate and $r$ is the risk-free interest rate.

Hence, we need to evaluate $\mathbb{E}_i [e^{-r\tau 1_{\{\tau < T\}}}]$, $i \in E$.

Following a similar discussion to Kijima and Siu (2013), these values are obtained as a solution of a linear equation, when jumps are double-exponential.
Denote $x = -\log\left(\frac{L}{A_0}\right)$ and $J_0 = i$. Then,

$$\lim_{T \downarrow 0} c_{T}^{(i)} = r(1 - R)\nu_{i}^{X}((−\infty, x])$$

where $\nu_{i}^{X}$ denotes the Lévy measure under regime $i$.

**Implication:** Regime-switching Brownian motion alone CANNOT produce non-zero credit spread at $T \downarrow 0$!

$\Rightarrow$ We need jumps for plausible short-term spreads.
Lemma

Assume $\mathbb{P}[\tau < \infty] = 1$ and $J_0 = i$. Then,

$$
\lim_{T \to \infty} c_T^{(i)} = (1 - R)r \frac{\mathbb{E}_\Pi[e^{-r\tau}]}{1 - \mathbb{E}_\Pi[e^{-r\tau}]},
$$

where $\Pi$ denotes the stationary distribution of $J_t$.

**Implication:** Impact of the regime-switching factor appears in the medium part of the CDS term structure!
Corollary

In our model, the CDS premium $c$ is given by

$$c_{T}^{(i)} = (1 - R)r \frac{P_{2}^{RS}}{1 - P_{2}^{RS} - e^{-rT} P_{1}^{RS}}$$

where $J_0 = i$,

$$P_{1}^{RS} = \mathcal{L}_T^{-1} \left( \frac{1}{a} - \frac{1}{a} \mathbb{E}_i \left[ e^{a\tau}; J_\tau \right] \right)$$

and

$$P_{2}^{RS} = \mathcal{L}_T^{-1} \left( \frac{1}{a} \mathbb{E}_i \left[ e^{-(r+a)\tau}; J_\tau \right] \right)$$

Here, $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.
Model Parameters for $X_t$:

### Base Parameters

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### Regime 1

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Regime 1 (Regime 2, resp.) is of high (low) volatility and bigger (smaller) jumps.
Regime-Switching Factor: BM only

Effect of regime-switching on CDS premium: BM case

Regime 1: $\sigma_1 = 0.4$

Regime 2: $\sigma_2 = 0.2$

No Regime: $\sigma_1 = 0.4$

No Regime: $\sigma_2 = 0.2$
Effect of regime-switching on CDS premium, Markov-modulated Levy measure only

Regime 1: Big jumps
Regime 2: Small jumps
Big jumps (no regime)
Small jumps (no regime)

CDS premium (bp)

Time to maturity (year)
Effect of Regime-Switching Intensity: $q_2 = 0.5$
Summary of Numerical Examples

- **Hump** and **inverted-hump** shapes of the CDS curves can be constructed by changing the **regime-switching intensities** of $J_t$.
  - Possible explanation: When buying CDS, investors are concerned with (1) **current state** of the firm, and (2) **persistence** of a firm staying in one particular economic/credit regime.

- Short-term spreads become more realistic by the jump effects.

- Introduction of **regime-switching, jump-diffusion** results in more flexible CDS term structures!
Recall that $S_t = \exp(Y_t)$ with $Y_t = \rho X_t + Z_t$.

The call option price written on $S$ under $\{\tau > T\}$ is given by

$$C(S, K, T) = \mathbb{E}[e^{-rT}(S_T - K)^+1\{\tau > T\}]$$

$$= \mathbb{E}[e^{-rT}(S_T - K)^+]$$

$$- \mathbb{E}[e^{-rT}(S_T - K)^+1\{\tau \leq T\}]$$

Hence, equivalently,

Defaultable call = Non-defaultable call — Down-and-in call
Theorem

The double Laplace transform of \( \mathbb{E}[e^{-rT}(S_T - K)^+1_{\{\tau \leq T\}}] \) with respect to \( k = \log K \) and \( T \) is obtained as

\[
\mathcal{L}_{\xi,\beta}(\mathbb{E}[e^{-rT}(S_T - K)^+1_{\{\tau \leq T\}}]) = \frac{S_0^{\xi+1}}{\xi(\xi + 1)} \sum_j \tilde{E}_i \left[ e^{-(\beta + r) - \kappa_j^Z(\xi+1))\tau + (\xi+1)\rho X \tau} 1_{\{J_\tau = j\}} \right] \\
\times \sum_n \left( (r + \beta)I - \left\{ \kappa_j^Z(\xi + 1) + \kappa_j^X(\rho(\xi + 1)) \right\}_{\text{diag}} + Q \right)_{jn}^{-1}
\]

where \( \tilde{E}_i \) is the expectation under which \( Z_t \) is taken as the numeraire.
Numerical Results

Model Parameters for $X_t$:

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<th>Base Parameters</th>
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Model Parameters for $Z_t$ (double-exponential for simplicity):

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<td>0.1</td>
<td>60</td>
<td>60</td>
<td>0.4</td>
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Impact of Jump Factor

- Regime-switching BM produces symmetric smiles.
- Negative skewness is a common feature found in equity markets.
- The negative skewness is more pronounced as the probability of upper jumps $p_i^X$ decreases, since the probability of default is decreased.
- Regime 1 (Regime 2, resp.) is of high (low) volatility and bigger (smaller) jumps.
Effect of $\rho$ without Regime-Switch

- The curvature of IV curve decreases with increasing correlation $\rho$.
- That is, the increase in $\rho$ augments the negative skewness of IV.
- The negative skewness reflects the credit nature on the equity.
Volatility curves flatten with increasing maturity $T$.

In Regime 1, the IV curve moves downward as it flattens, whereas it elevates as its curvature decreases in Regime 2.

Of course, they converge to coincide as $T \to \infty$. 

![Effect of time to maturity on implied volatility, $\rho=1, J_0=1$](image1)

![Effect of time to maturity on implied volatility, $\rho=1, J_0=2$](image2)
Regime-switching and jump factors play a significant role in the equity option with default feature.

In particular, the IV curve of the high regime decreases, while it increases under the low regime, as the switching-intensity or the maturity lengthens.

The default probability contributes to the negative skewness of IV.

However, the degree of negative skewness is limited, in comparison with the reduced-form credit-equity model in Carr and Wu (2010).

The assumption of independent and stationary increments of Lévy processes makes it inflexible in capturing the IV observed in the market (see Konikov and Madan, 2002).
Conclusion

- Increasing evidence of the linkage between the equity and credit aspects of a corporate firm demands a unified equity-credit model.
- We propose one approach to the problem: Latent structural model.
- Extend Kijima et al. (2009) to include jumps and regime-switch.
- Application: Price CDS and equity option under one framework.
- Strength: Separate jumps and regime-switch effects.
- Strength: More flexible CDS term structures and IV surfaces.
- Strength: Clarify the role of impact factor $\rho$ to the skewness of volatility smiles.
- Numerical scheme: Inverse Laplace transform is very easy and stable.
Future Research

- Need to develop a *calibration* scheme.
- Want to *extract credit quality* (e.g., distance to default) under the physical measure from the marker process (i.e., equity value process).
- These can lead to more empirical works.
- Extend the model to *include the Heston-type SV* (to increase the negative skewness).
- The pricing of *equity default swap*, which has both the equity and credit components of a firm.
Thank You for Your Attention