A Market Model of Interest Rate with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies *

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Global Derivatives Tradaing and Risk Management 2011

*This research is supported by CARF (Center for Advanced Research in Finance) and the global COE program "The research and training center for new development in mathematics." All the contents expressed in this research are solely those of the authors and do not represent the views of any institutions. The authors are not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research. M.Fujii is grateful for friends and former colleagues of Morgan Stanley, especially in IDEAS, IR option, and FX Hybrid desks in Tokyo for fruitful and stimulating discussions. The contents of the research do not represent any views or opinions of Morgan Stanley.

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Outlines

OTC Market and Collateralization

- Collateralization
  - The most important credit risk mitigation tool.
    - margin call, settlement and associated procedures.
    - legal specifications are provided by CSA (Credit Support Annex).
  - Dramatic increase in recent years (ISDA [4])
    - 30%(2003) $\longrightarrow$ 70%(2009) in terms of trade volume for all OTC.
    - Coverage goes up to 78% (for all OTC) and 84% (for fixed income) among major financial institutions.
    - More than 80% of collateral is Cash.
    - About half of the cash collateral is USD.
    - Almost all the credit derivatives are collateralized.

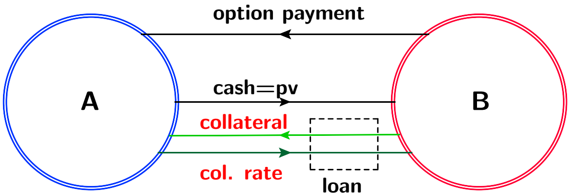
Impact of Collateralization

Impact of collateralization :

- **Reduction of Counter-party Exposure.**
  - **Associated change in CVA has been actively studied.**
- **Change of Funding Cost (topic of this talk)**
  - **Require new term structure model to distinguish discounting and reference rates.**
  - **Cost of collateral is differ from currency to currency.**
  - **"cheapest-to-deliver" option.**
  - **Significant impact on derivative pricing and risk management.**

Source of Funding Cost Difference

- Collateralized (Secured) Contract (current picture)



- No outright cash flow (collateral=PV)
- No external funding is needed.
- Funding is determined by over-night (ON) rate.

$\Rightarrow$ Libor discounting is inappropriate.

Fundamental Market Instruments

Historical behavior of IRS (1Y)-OIS (1Y) spreads (bps)

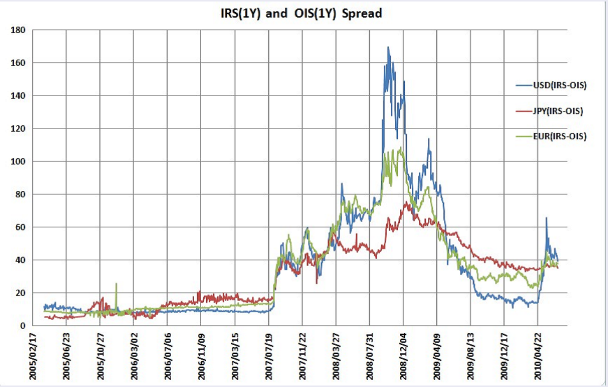


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of JPY TS spreads (bps)

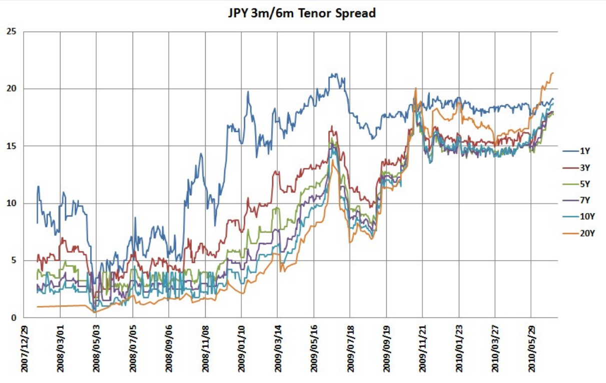


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of USD TS spreads (bps)

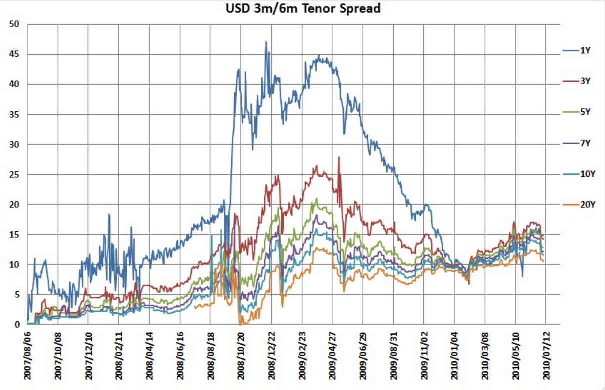


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of EUR TS spreads (bps)

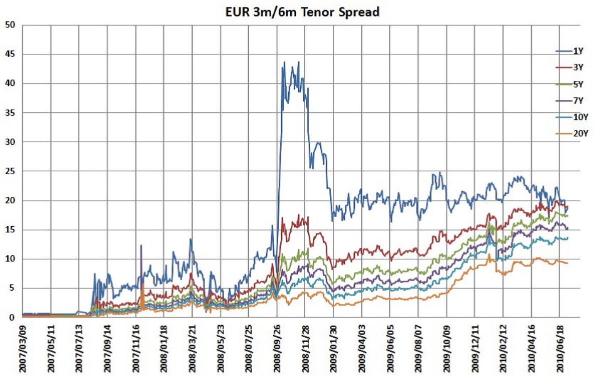


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of USDJPY CCS spreads (bps)

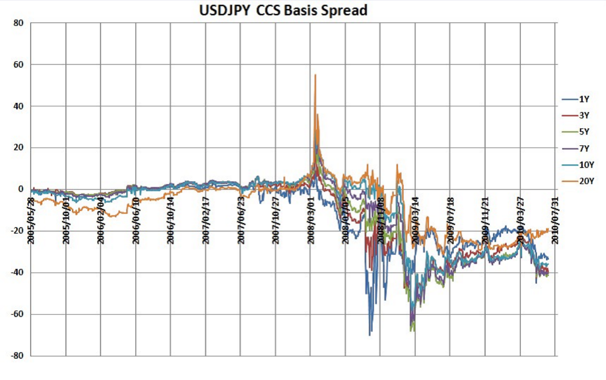


Figure: Source:Bloomberg

Fundamental Market Instruments

Historical behavior of EURUSD CCS spreads (bps)

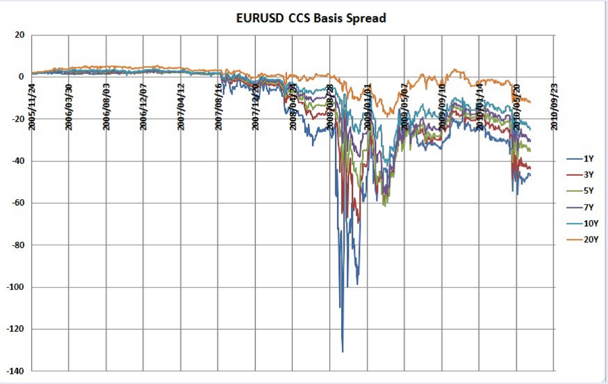


Figure: Source:Bloomberg

Fundamental Market Instruments

Traditional IR model (such as LMM) has become ill-suited for actual derivative business, because...

- Impossible to calibrate fundamental instruments, such as:
  - Tenor Swap (TS) (or IRS with different tenor/frequency)
  - Cross Currency Swap (CCS) $\Rightarrow$ useless for long-dated FX products
  - Overnight Index Swap (OIS)
- Unable to recognize the important delta exposure, such as to Libor-OIS spread.
  - Proper control of risk exposure is impossible.

Criteria for Models Workable in Real Business

Criteria

- **Consistent discounting/forward curve construction**
  - **Price all types of IR swaps correctly:**
    - **OIS, IRS and TS**
  - **Take collateralization into account.**
  - **Maintain consistency in multi-currency environment**
    - **CCS basis spreads need to be recovered.**
    - **Cost of cash collateral and its difference among major currencies should be taken into account.**
- **Stochastic Modeling of Basis spreads**
  - **Allow systematic calibration procedures**
  - **Flexible enough to allow non-trivial term structure of spreads.**

Pricing under the Collateralization

- **Assumption**
  - Continuous adjustment of collateral amount
  - Perfect collateralization by Cash
  - Zero minimum transfer amount

- **Comments**
  - Daily margin call/settlement is becoming popular.
  - By making use of Repo / Reverse-Repo, other collateral assets can be converted into the equivalent amount of cash collateral.
  - General Collateral (GC) repo rate closely tracks overnight rate.

Pricing under the Collateralization

Proposition:

T-maturing European option under the collateralization is given by [a]

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] \\ &= D^{(i)}(t,T)E_t^{\mathcal{T}^c_{(i)}}\left[\left(e^{-\int_t^T y^{(i,j)}(s)ds}\right)h^{(i)}(T)\right] \end{aligned}$$

where,

$$\begin{aligned} y^{(j)}(s) &= r^{(j)}(s) - c^{(j)}(s) \quad , \quad y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \\ D^{(i)}(t,T) &= E_t^{Q_i}\left[e^{-\int_t^T c^{(i)}(s)ds}\right] \end{aligned}$$

- $h^{(i)}(T)$: option payoff at time T in currency i
- collateral is posted in currency j
- $c^{(j)}(s)$: instantaneous collateral rate of currency j at time s
- $r^{(j)}(s)$: instantaneous risk-free rate of currency j at time s
- $E^{\mathcal{T}^c_{(i)}}[\cdot]$: expectation under the fwd measure associated with $D^{(i)}(\cdot,T)$

[a] Fujii,Shimada,Takahashi (2009) [1]

Pricing under the Collateralization

- Collateral amount in currency j at time s is given by $\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}$, which is invested at the rate of $y^{(j)}(s)$:

$$
\begin{aligned}
h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}h^{(i)}(T)\right] \\
&\quad + f_x^{(i,j)}(t)E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du}y^{(j)}(s)\left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right)ds\right] \\
&= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du}y^{(j)}(s)h^{(i)}(s)ds\right].
\end{aligned}
$$

Note that $X(t) = e^{-\int_0^t r^{(i)}(s)ds}h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)h^{(i)}(s)ds$ is a Q_i-martingale. Then, the process of the option value is written by

$$
dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right)h^{(i)}(t)dt + dM(t)
$$

with some Q_i-martingale M. This establishes the proposition.

> $f_x^{(i,j)}(t)$: Foreign exchange rate at time t representing the price of the unit amount of currency "j" in terms of currency "i".

Pricing under the Collateralization

Corollary

- **If payment and collateral currencies are the same, the option value is given by**

$$\begin{aligned} h(t) &= E_t^Q\left[e^{-\int_t^T c(s)ds} h(T)\right] \\ &= D(t,T) E_t^{\mathcal{T}^c}\left[h(T)\right] . \end{aligned}$$

- **The discounting is determined by "collateral rate", which is consistent with the schematic picture seen before.**

Building Blocks for IR Term Structure Model

Building Blocks

$$
\begin{aligned}
c^{(i)}(t,T) &= -\frac{\partial}{\partial T}\ln D^{(i)}(t,T) \\
B^{(i)}(t,T_k;\tau) &= E_t^{\mathcal{T}^c_{k,(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)} - 1\right) \\
y^{(i,k)}(t,T) &= -\frac{\partial}{\partial T}\ln\left(E_t^{Q_i}\left[e^{-\int_t^T y^{(i,k)}(s)ds}\right]\right)
\end{aligned}
$$

- **These building blocks are enough to calibrate all the relevant OIS, IRS, TS and CCS.**

Construction of Term Structure

Term structure construction procedures:[1]

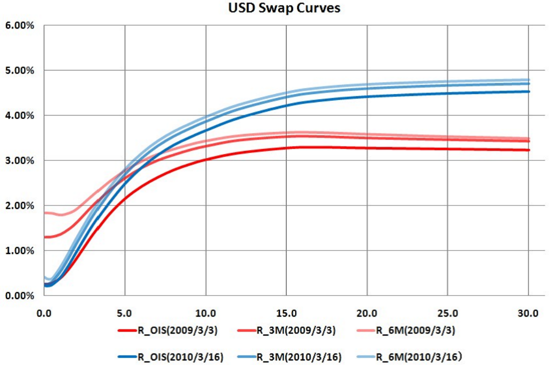
- **(1), OIS** $\Rightarrow c^{(i)}(t,s)$
- **(2), results of (1) + IRS + TS** $\Rightarrow B^{(i)}(t,s;\tau)$
- **(3), results of (1,2) +CCS** $\Rightarrow y^{(i,j)}(t,s)$

- **Assume collateralization in domestic currency for OIS, IRS and TS [2].**
- **Assume collateralization in USD for CCS (USD crosses).**

[1] See, (Fujii, Shimada, Takahashi 2009) [1] for details.

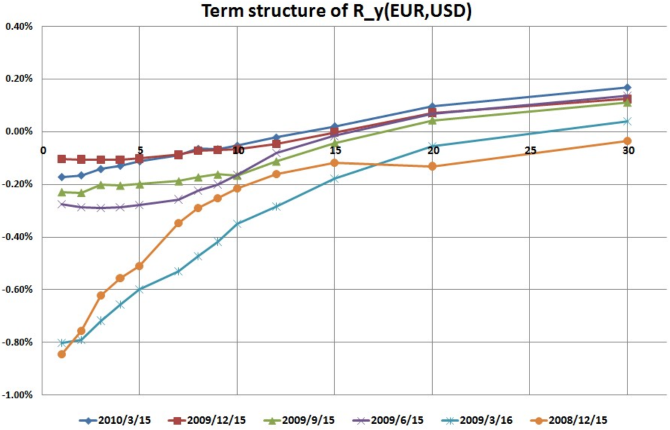
[2] Assumption on collateral currency has only minor impact on the market par quotes.

Construction of Term Structure



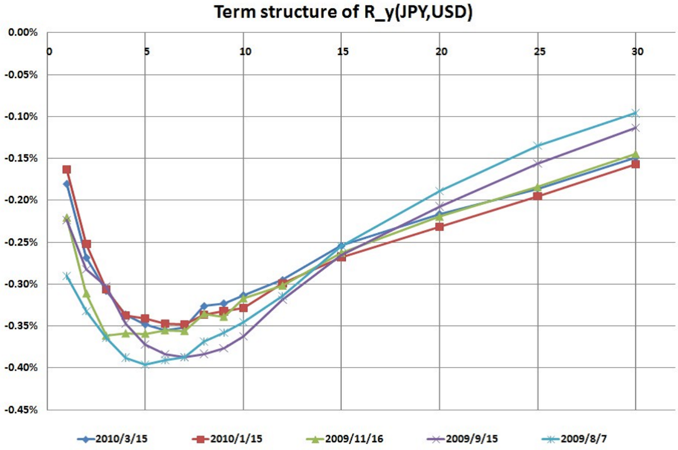
$$
\begin{aligned}
R_{\mathrm{OIS}}(T) &= -\ln(D(0,T))/T \\
E^{\mathcal{T}^c_m}[L(T_{m-1},T_m;\tau)] &= \frac{1}{\delta_m}\left(\frac{e^{-R_\tau(T_{m-1})T_{m-1}}}{e^{-R_\tau(T_m)T_m}}-1\right)
\end{aligned}
$$

Construction of Term Structure



$$R_{y(i,j)}(T) = -\frac{1}{T}\ln\left(E^{Q_i}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right]\right) = \frac{1}{T}\int_0^T y^{(i,j)}(0,s)ds$$

Construction of Term Structures



HJM-framework under the collateralization

SDEs for HJM-framework

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW_t^{Q_i}$$

$$dy^{(i,k)}(t,s) = \sigma_y^{(i,k)}(t,s)\cdot\left(\int_t^s \sigma_y^{(i,k)}(t,u)du\right)dt + \sigma_y^{(i,k)}(t,s)\cdot dW_t^{Q_i}$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW_t^{Q_i}$$

$$\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW_t^{Q_i}$$

- **For construction of swap curves, the independence of y is useful assumption.**
- **See Fujii, Shimada, Takahashi (2009,2010) [2, 3].**

Choice of Collateral Currency

Role of $y^{(i,j)}$

- **Payment currency i with Collateral currency j**

$$D^{(i)}(t,T) \Rightarrow E_t^{Q_i}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right] D^{(i)}(t,T)$$

after neglecting small corrections from possible non-zero correlations.

- **To choose "strong" currency, such as USD, is expensive (for the collateral payer).**

Choice of Collateral Currency

Role of $y^{(i,j)}$

Optimal behavior of collateral payer can significantly change the derivative value.

- **Payment currency i with multiple currencies as eligible collateral choice $\mathcal{C}$**

$$D^{(i)}(t,T) \Rightarrow E_t^{Q_i}\left[e^{-\int_t^T \max_{j\in\mathcal{C}}\{y^{(i,j)}(s)\}ds}\right] D^{(i)}(t,T)$$

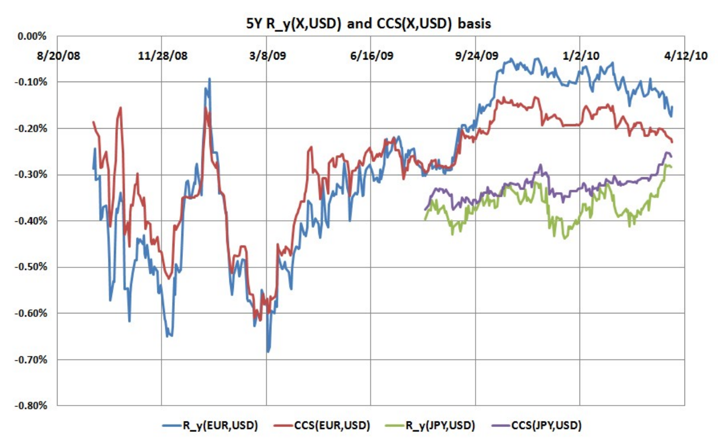
- **Payment currency and USD as eligible collateral is relatively common.**

$$D^{(i)}(t,T) \Rightarrow E_t^{Q_i}\left[e^{-\int_t^T \max\{y^{(i,USD)}(s),0\}ds}\right] D^{(i)}(t,T)$$

- **Volatility of $y^{(i,j)}$ is an important determinant.**

Choice of Collateral Currency

Close relationship to CCS basis spread



Choice of Collateral Currency

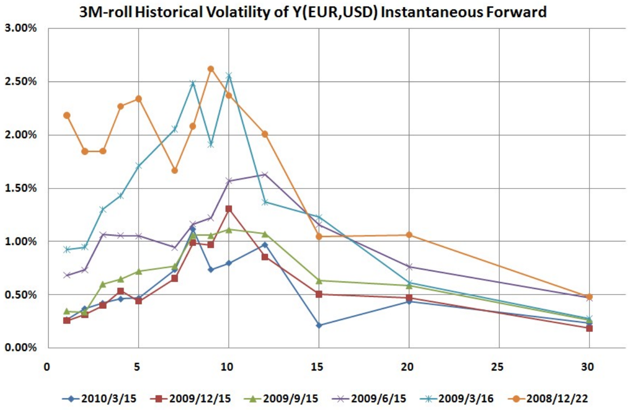


Figure: 3M-Roll historical volatility of $y^{(EUR,USD)}$ instantaneous forward. Annualized in absolute terms.

Choice of Collateral Currency

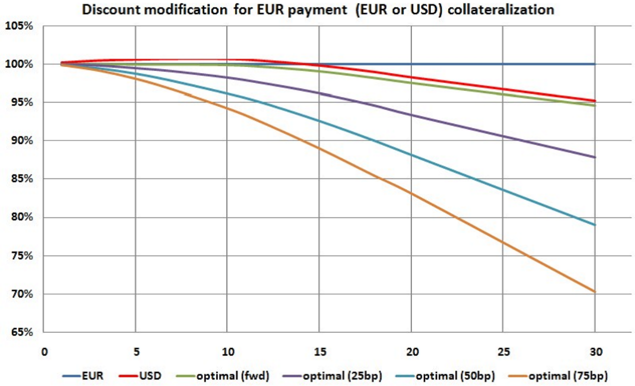


Figure: Modification of EUR discounting factors based on HW model for $\boldsymbol{y}^{(EUR,USD)}$ as of 2010/3/16. The mean-reversion parameter is $\mathbf{1.5\%}$, and the volatility is given at each label.

Choice of Collateral Currency

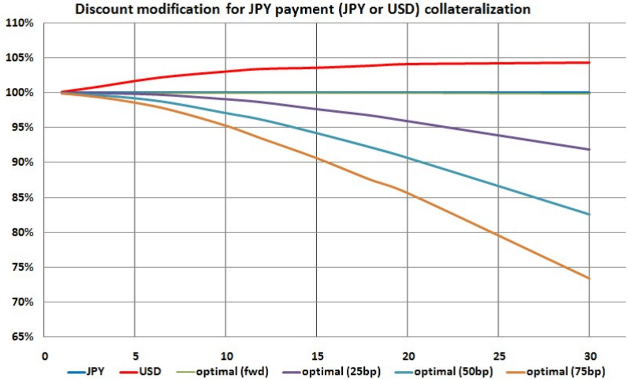


Figure: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

Conclusions

Conclusions

- We proposed a term structure model under the collateralization, which directly relates the cost of cash-collateral to cross currency basis spreads.
- We pointed out
  - importance of choice of collateral currency.
  - potential impact of the embedded cheapest-to-deliver option.

Comments:

- Including collateral cost for modeling will be particularly important for CCP-driven derivatives markets.

New Issues

Questions to ask

- **Impacts of Asymmetric Collateralization:**
  - **Asymmetric CSA, such as one-way collateralization with Sovereigns, Central Banks, etc.**

    Called Funding Time-Bomb in the article of Risk (2011, Feb).
  - **Symmetric CSA but significantly different level of sophistication of collateral management between the two firms.**
- **Impacts of Imperfect Collateralization:**
  - **Counterparty credit risk.**
  - **Collateral cost adjustment.**
  - **Interplay between funding cost and credit risk, etc.**
- **Implications for Trading Behavior**

Setup

- Probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F}$ contains all the market information including defaults.
- Consider two firms, $i \in \{1, 2\}$, whose default time is $\tau^i \in [0, \infty]$, and $\tau = \tau^1 \wedge \tau^2$.
- τ^i (and hence τ) is assumed to be totally-inaccessible $\mathbb{F}$-stopping time.
- Indicator functions: $H_t^i = 1_{\{\tau^i \leq t\}}$, $H_t = 1_{\{\tau \leq t\}}$
- Assume the existence of absolutely continuous compensator for H^i:

$$A_t^i = \int_0^t h_s^i 1_{\{\tau^i > s\}} ds, \quad t \geq 0$$

- Assume no simultaneous defaults, and hence the hazard rate of H is

$$h_t = h_t^1 + h_t^2 \ .$$

- Money market account: $\beta_t = \exp\left(\int_0^t r_u du\right)$

Collateralization

- When party $i \in \{1, 2\}$ has negative mark-to-market, it has to post collateral (cash) to party $j (\neq i)$, and it is assumed to be done continuously.
- collateral coverage ratio is $\delta_t^i \in \mathbb{R}_+$, and the value of collateral at time t is given by $\delta_t^i(-V_t^i)$.
  - δ_t^i effectively takes into account under- as well as over-collateralization. Thus $\delta_t^i < 1$, and also $\delta_t^i > 1$ are possible.
- party j has to pay the collateral rate c_t^i on the posted cash continuously.
- c_t^i is determined by the currency posted by party i.
  - market convention is to use overnight (O/N) rate at time t of corresponding currency.
    $\Rightarrow$ Traded through OIS (overnight index swap), which is also collateralized.
  - In general, $c_t^i \neq r_t^i$, which is the risk-free interest rate of the same currency. This is necessary to explain CCS swap market consistently. See, Fujii&Takahashi (2010, 2011).

Counterparty Exposure and Recovery Scheme

- **Counterparty exposure to party j at time t (from the view point of party i)**

$$\max(1-\delta_t^j,0)\max(V_t^i,0)+\max(\delta_t^i-1,0)\max(-V_t^i,0)$$

- **Assume party-j's recovery rate at time t as $R_t^j \in [0,1]$**
- **Recovery value at the time of j's default:**

$$R_t^j\left([1-\delta_t^j]^+[V_t^i]^+ + [\delta_t^i-1]^+[-V_t^i]^+\right)$$

Pricing Formula

- Pricing from the view point of party 1.

$$S_t = \beta_t E^Q \Bigg[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \Big\{ dD_u + \big(y_u^1 \delta_u^1 1_{\{S_u < 0\}} + y_u^2 \delta_u^2 1_{\{S_u \geq 0\}} \big) S_u du \Big\}$$
$$+ \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \Big(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \Big) \Bigg| \mathcal{F}_t \Bigg]$$

- D: cumulative dividend to party 1.
- $y_t^i = r_t^i - c_t^i$, $(i \in \{1,2\})$ denotes the instantaneous return at time t from the cash collateral posted by party i.
- Default payoff:

$$Z^1(t,v) = \Big(1 - (1 - R_t^1)(1 - \delta_t^1)^+\Big) v 1_{\{v<0\}} + \Big(1 + (1 - R_t^1)(\delta_t^2 - 1)^+\Big) v 1_{\{v \geq 0\}}$$
$$Z^2(t,v) = \Big(1 - (1 - R_t^2)(1 - \delta_t^2)^+\Big) v 1_{\{v \geq 0\}} + \Big(1 + (1 - R_t^2)(\delta_t^1 - 1)^+\Big) v 1_{\{v<0\}}$$

Pricing Formula

Following the method in Duffie&Huang (1996), pre-default value of the contract $V_t 1_{\{\tau>t\}} = S_t$ is given by

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T$$

where

$$\begin{aligned}\mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-R_t^i)(1-\delta_t^i)^+ h_t^i + (1-R_t^j)(\delta_t^i - 1)^+ h_t^j\end{aligned}$$

if $\Delta V_\tau = 0$ a.s. and if appropriate regularity conditions are satisfied.

Symmetric Case

If $\tilde{y}_t^1 = \tilde{y}_t^2 = \tilde{y}_t$, then we have

$$\mu(t, v) = \tilde{y}_t \ .$$

If $\tilde{y}$ is not explicitly dependent on V, we can recover the linearity.

$$V_t = E^Q \left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u) du \right) dD_s \middle| \mathcal{F}_t \right]$$

Portfolio valuation can be decomposed into that of each payment.

$\Downarrow$

A good characteristic for market benchmark price.

Symmetric Perfect Collateralization

Special Cases

Case 1

- **bilateral perfect collateralization** $(\delta^1 = \delta^2 = 1)$
- **both parties use the same currency** (i) **as collateral, which is also the payment (evaluation) currency.**

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

The valuation method for single currency swap adopted by LCH Swapclear (2010) is the same with this formula.

Symmetric Perfect Collateralization

Special Cases

Case 2 : bilateral perfect collateralization

- both parties use the same currency (k) as collateral, which is different from the payment (evaluation) currency (i)

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

- both parties choose the optimal currency from the eligible collateral set $\mathcal{C}$. Currency (i) is used as the evaluation currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}}[y_u^{(i,k)}])du\right) dD_s \middle| \mathcal{F}_t\right]$$

- $y^{(i,k)} = y^{(i)} - y^{(k)} = \left(r^{(i)} - c^{(i)}\right) - \left(r^{(k)} - c^{(k)}\right)$

Examples : Fundamental Instruments

Overnight Index Swap (OIS)

- exchange fixed rate with compounded Overnight rate periodically.
- collateralized by domestic currency.

$$dD_s = \sum_{n=1}^{N} \delta_{T_n}(s) \left[\Delta_n S - \left\{ \exp\left(\int_{T_{n-1}}^{T_n} c_u^{(i)} du \right) - 1 \right\} \right]$$

- time t value of T_0 ($> t$)-start T_N-maturing OIS of currency (i);

$$V_t = \sum_{n=1}^{N} E^{Q^{(i)}} \left[e^{-\int_t^{T_n} c_u^{(i)} du} \left(\Delta_n S + 1 - e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right) \middle| \mathcal{F}_t \right]$$

$$= \sum_{n=1}^{N} D^{(i)}(t, T_n) \Delta_n S - \left(D^{(i)}(t, T_0) - D^{(i)}(t, T_N) \right)$$

- $D^{(i)}(t,T) = E^{Q^{(i)}} \left[e^{-\int_t^T c_u^{(i)} du} \middle| \mathcal{F}_t \right]$ is a value of domestically collateralized zero-coupon bond.

Examples : Fundamental Instruments

(i,j) Mark-to-Market Cross Currency OIS

- compounded O/N rate of currency (i) is exchanged by that of currency (j) with additional spread periodically.
- notional of currency (j) is kept constant while that of currency (i) is refreshed every reset time with the spot FX rate. (currency (i) is usually USD.)
- collateralized by currency (i) .
- payoff seen from the spread receiver:

$$dD_s = dD_s^{(j)} + f_x^{(j,i)}(s)dD_s^{(i)}$$

where

$$dD_s^{(j)} = -\delta_{T_0}(s) + \delta_{T_N}(s) + \sum_{n=1}^{N} \delta_{T_n}(s)\left[\left(e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - 1\right) + \delta_n B\right]$$

$$dD_s^{(i)} = \sum_{n=1}^{N} f_x^{(j,i)}(T_{n-1})\left[\delta_{T_{n-1}}(s) - \delta_{T_n}(s)e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du}\right]$$

Fundamental Instruments

(i,j) Mark-to-Market Cross Currency OIS

- time t value of spread receiver of (i,j)-MtMCCOIS:

$$V_t = \sum_{n=1}^{N} E^{Q^{(j)}} \Bigg[e^{-\int_t^{T_n} (c_u^{(j)} + y_u^{(j,i)}) du} \left\{ e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} + \delta_n B - \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right\} \Bigg| \mathcal{F}_t \Bigg]$$

- if $c^{(i)}$ and $y^{(j,i)}$ are independent, then

$$V_t = \sum_{n=1}^{N} \left[\delta_n B D^{(j,i)}(t,T_n) - D^{(j,i)}(t,T_{n-1}) \left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u) du} \right) \right]$$

where

$$y^{(j,i)}(t,s) = -\frac{1}{s} \ln E^{Q^{(j)}} \left[e^{-\int_t^s y_u^{(j,i)} du} \middle| \mathcal{F}_t \right]$$

$$D^{(j,i)}(t,T) = D^{(j)}(t,T) e^{-\int_t^T y^{(j,i)}(t,s) ds}$$

Symmetric Perfect Collateralization

- **Symmetric perfectly collateralized price is becoming the market Benchmark, at least for standardized products.**
- **No-arbitrage dynamics of c,y and Libor-OIS spreads in HJM-framework is given in Fujii,Shimada&Takahashi(2009), and Fujii&Takahashi(2010).**
  - **OIS $\rightarrow$ c**
  - **IRS&TS $\rightarrow$ Libor-OIS for each tenor (1m, 3m, 6m, ...)**
  - **CCS of currency pair (i,j) $\rightarrow$ $y^{(i,j)}$**
- **CCS spread level and volatility are key elements to determine derivative price.**

Asymmetric Collateralization

Marginal Impact of Asymmetry

- Make use of Gateaux derivative as the first-order Approximation: Duffie&Skiadas (1994), Duffie&Huang (1996)

$$\lim_{\epsilon \downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta}; \eta) - \frac{V_t(\bar{\eta} + \epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0$$

  η and $\bar{\eta}$ are bounded and predictable

- We want to expand the price around symmetric benchmark price.

$$\begin{aligned} \mu(t, v) &= \tilde{y}^1_t 1_{\{v<0\}} + \tilde{y}^2_t 1_{\{v\geq 0\}} \\ &= y_t + \Delta\tilde{y}^1_t 1_{\{v<0\}} + \Delta\tilde{y}^2_t 1_{\{v\geq 0\}} \\ \Delta\tilde{y}^i_t &= \tilde{y}^i_t - y_t \end{aligned}$$

- Calculate GD at symmetric $\mu = y$ point.

Asymmetric Collateralization

- Applying Gateaux Derivative at $\mu = y$ point:

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T$$

is decomposed as $V_t = \overline{V}_t + \nabla V_t$, where

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\nabla V_t = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\overline{V}_s \left(\Delta\tilde{y}_s^1 1_{\{\overline{V}_s < 0\}} + \Delta\tilde{y}_s^2 1_{\{\overline{V}_s \geq 0\}}\right) ds \middle| \mathcal{F}_t\right]$$

If y is chosen in such a way that it reflects the funding cost of the standard collateral agreements, $\overline{V}$ denotes the market benchmark price, and ∇V denotes the correction for it.

Asymmetric Collateralization

An example of asymmetric perfect collateralization

- party 1 choose optimal currency from the eligible collateral set $\mathcal{C}$, but the party 2 can only use currency (i) as collateral, either due to the asymmetric CSA or lack of easy access to foreign currency pool. The evaluation (payment) currency is (i).
- very plausible situation for the trades between domestic medium-to-small size financial firms and major global broker-dealers.

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

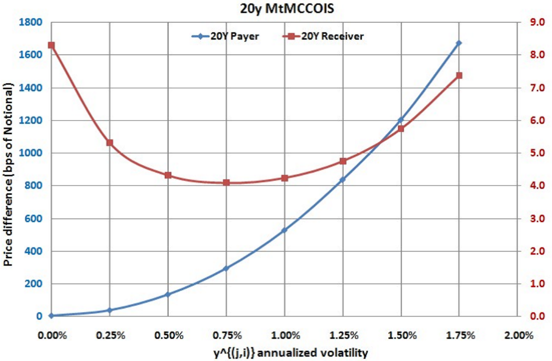
$$\nabla V_t = E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s c_u^{(i)} du\right) [-\overline{V}_s]^+ \max_{k\in\mathcal{C}}[y_s^{(i,k)}] \middle| \mathcal{F}_t\right]$$

$$V_t \simeq \overline{V}_t + \nabla V_t$$

$\Rightarrow$ Expansion around the symmetric collateralization with currency (i).

Asymmetric Collateralization

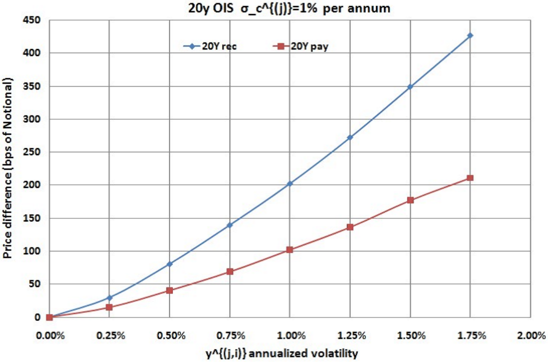
- Numerical Example of ∇V for USD/JPY-MtMCCOIS (one type of CCS)
- Eligible collateral are USD and JPY for party-1 but only USD for party-2. (Right axis for Receiver)



- CCS spread is set to make $\overline{V} = 0$.
- Dependence between y and V creates big difference between CCS spread-payer and spread-receiver.

Asymmetric Collateralization

- Numerical Example of ∇V for JPY-OIS.
- Eligible collateral are USD and JPY for party-1 but only JPY for party-2.



- OIS rate is set to make $\overline{V} = 0$.
- Difference between Receiver and Payer comes from up-ward sloping term structure. (receiver has the optionality in the long end of the contract.)

Implications for Netting

Proposition

Assume perfect collateralization. Suppose that, for each party i, y_t^i is bounded and does not depend on the contract value directly. Let V^a, V^b, and V^{ab} be, respectively, the value process (from the view point of party 1) of contracts with cumulative dividend process D^a, D^b, and $D^a + D^b$. If $y^1 \geq y^2$, then $V^{ab} \geq V^a + V^b$, and if $y^1 \leq y^2$, $V^{ab} \leq V^a + V^b$.

- Proof can be done in the same way as Duffie&Huang(1996) using stochastic Gronwall-Bellman inequality.
- V^{ab} represents the value under netting agreement.
- Sophisticated financial firm which can achieve lower funding cost y tries to avoid netting.
- May lead to disperse but highly interconnected network of trades rather than the one highly concentrated in major CCPs.

Imperfect Collateralization

CVA as the Deviation from the Perfect Collateralization

- Assume the both parties use the same currency for simplicity, and hence $y^1 = y^2 = y$.

$$\mu(t,v) = y_t -$$
$$\Big\{\Big((1-\delta^1_t)y_t + (1-R^1_t)(1-\delta^1_t)^+ h^1_t - (1-R^2_t)(\delta^1_t - 1)^+ h^2_t\Big) 1_{\{v<0\}}$$
$$+ \Big((1-\delta^2_t)y_t + (1-R^2_t)(1-\delta^2_t)^+ h^2_t - (1-R^1_t)(\delta^2_t - 1)^+ h^1_t\Big) 1_{\{v\geq 0\}}\Big\}$$

- GD around $\mu = y$ decomposes the price into three parts:
  - Symmetric perfect collateralized benchmark price
  - $(1-\delta^i)y1_{\{v \lesseqgtr 0\}}$ ⇒ Collateral Cost Adjustment (CCA)
  - Remaining h dependent terms ⇒ Credit Value Adjustment (CVA)

$$\begin{aligned} V_t &\simeq \overline{V}_t + \nabla V_t \\ &= \overline{V}_t + \mathrm{CCA} + \mathrm{CVA} \end{aligned}$$

Imperfect Collateralization

Price adjustment of imperfectly collateralized contract

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\mathrm{CCA} = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du} y_s \left((1-\delta_s^1)[-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\mathrm{CVA} =$$

$$E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}(1-R_s^1)h_s^1 \left[(1-\delta_s^1)^+[-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right] ds\right.$$

$$\left. - \int_t^T e^{-\int_t^s (r_u - y_u)du}(1-R_s^2)h_s^2 \left[(1-\delta_s^2)^+[\overline{V}_s]^+ + (\delta_s^1 - 1)^+[-\overline{V}_s]^+\right] ds \middle| \mathcal{F}_t\right]$$

- $V_t = \overline{V}_t + \mathrm{CCA} + \mathrm{CVA}$

Dependence among y, δ^i and other factors, such as $\overline{V}, h^i$ is particularly important. $\Rightarrow$ New type of Wrong (Right)-way Risk.

Imperfect Collateralization

A simple case of Imperfect Collateralization.

- Both parties use currency (j) as collateral.
- Evaluation (payment) currency is (i).
- Assume common constant collateral coverage ratio $\delta < 1$.
- Assume constant recovery ratio R^1 and R^2, respectively.

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = -(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du} y_s^{(j)} \overline{V}_s ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} = (1-R^1)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du} h_s^1 [-\overline{V}_s]^+ ds \middle| \mathcal{F}_t\right]$$
$$-(1-R^2)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(i)}+y_u^{(i,j)}\right)du} h_s^2 [\overline{V}_s]^+ ds \middle| \mathcal{F}_t\right]$$

Unilateral Collateralization

One-way CSA

- An interesting example of Asymmetric & Imperfect collateralization.
- Market standard for trades with sovereigns, central banks, and government sponsored agencies.
- These special entities do not post but receive collateral.
- The counterpart financial firms are required to have two-way CSA when they enter hedge positions in financial market.

- Cash-flow mismatch clearly exists.
- What about the mark-to-market risk ?

Unilateral Collateralization

- party 1 is a sovereign entity and does not post collateral.
- party 2 is required to post currency (i) as collateral.

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\mathrm{CCA} = E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du} y_s^{(i)} \left([-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\mathrm{CVA} = E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du}(1-R_s^1)h_s^1 \left([-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right) \middle| \mathcal{F}_t\right]$$
$$-E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du}(1-R_s^2)(1-\delta_s^2)^+ h_s^2 [\overline{V}_s]^+ ds \middle| \mathcal{F}_t\right]$$

- Stringent collateral requirement $\delta^2 \simeq 1$, and loose monetary policy (kept "c" lower) in an improving economy (r goes higher), one-way CSA can lead to significant funding benefit to the party 1 (sovereigns), or in other words, Big Loss to the counterparty.

Summary

- **Extended our previous works on collateralized derivative pricing to Asymmetric and Imperfect situations.**
  - **relevance of sophisticated collateral management**
  - **incentives for advanced financial firms to avoid netting to exploit funding benefit**
  - **deviation from the fully collateralized benchmark price involves CCA and CVA**
  - **dependence of collateral cost and other factors, such as hazard rates, is important. May contain new type of wrong-way risk.**
  - **existence of the significant mark-to-market risk of one-way CSA.**

Main References

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