Essays on the Valuation Problems of Contingent Claims

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Abstract

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This dissertation consists of three essays on valuation problems of contingent claims in financial markets. The first essay proposes a new methodology for the valuation problems of financial contingent claims when the underlying asset prices follow the general class of continuous Ito processes. My method can be applicable to a wide range of the valuation problems including the complicated contingent claims associated with a stock, a foreign exchange rate and the term structure of interest rates. I illustrate the method by giving a series of examples both in the Black-Sholes' economy and in the arbitrage-free based forward rate model of the term structure. My method gives some explicit formulae for solutions which are numerically accurate enough for practical purpose in most cases. Moreover, I present a rigorous proof on the validity of the method by utilizing the Malliavin-Watanabe Calculus in stochastic analysis.

The second essay propose a new valuation technique for the valuation problem of the average-rate options under log-normally distributed underlying asset prices. This method transforms the valuation problem into an evaluation of a conditional expectation that is determined by a one-dimensional Markov process as suppose to a two-dimensional Markov process commonly known. This transformation is extremely useful since numerically it is much easier to handle a one-dimensional problem than a two-dimensional problem. Alternatively, I also derive a partial differential equation that the value function must satisfy. I illustrate this technique in the simple Black-Scholes' economy and in the term structure model.

The third essay examines the valuation problems of the securities with default risks. I propose two models in a general equilibrium framework, one of which utilizes a predictable stopping time and the other of which makes use of a totally inaccesible

stopping time to characterize the state of default. In the both models, the state of the default may be related to the other economic variables which are determined in equilibrium inside the models. Moreover, for the both models, I explicitly derive the partial differential equations with boundary conditions which allow us to evaluate any securities subject to default risks in a unified framework.

Chapter 1

An Asymptotic Expansion Approach to Pricing Financial Contingent Claims

1 Introduction

We propose a new approach to the valuation of contingent claims where we extensively develop the unified method of the asymptotic expansion technique for the asset pricing in a continuous time framework. The approach is general enough to be applicable in the broad class of Ito processes of assets and of their functionals, and is powerful in evaluating the complicating payoffs such as average options under the general asset processes. The asymptotic expansion technique is, to our knowledge, firstly applied to financial economics by Kunitomo-Takahashi(1992). They make use of the first order expansion in the pricing of average option under the assumption of the log-normal process of the underlying asset when the volatility is small. That is, they approximate the distribution of the average price by the log-normal distribution which is obtained by the first order stochastic expansion around the volatility parameter being zero. This method gives relatively accurate values, but those are not accurate enough for practical purpose when the volatility parameter is large. Hence, they propose another method where the arithmetic average is replaced by the geometric average which follows the log-normal distribution with its mean and variance adjusted to match the mean and variance of the arithmetic average. Although their second approach gives more accurate values, it is not general enough to be applied to the other processes and the other types of financial claims. Our method is valid in much more general situation and gives the formula for the expansion upto the higher orders which includes their formula as a special case. Moreover, the method gives numerical values which are accurate enough for practical purpose. Following this approach, we can evaluate various types of contingent claims in the unified fashion in a sense that we may apply the same procedure to various types of payoffs under the general class of continuous Ito processes shown below. The processes we consider in the paper can be expressed in the following manner. In the simple Black-Scholes' economy, the underlying asset processes are given by

$$dS^\delta = rS^\delta dt + \delta \sum_{i=1}^{N} \sigma_i(S^\delta, t) d\tilde{w}_{it}$$

where $0 < \delta < 1$, $\tilde{w}_{it}$ is a one-dimensional standard Brownian motion and r is a positive constant. In the term structure model of interest rates, the stochastic processes of instantaneous forward rates are given by

$$f^\varepsilon(t,T) = f(0,T) + \varepsilon^2 \int_0^t b^\varepsilon(v,T)dv + \varepsilon \int_0^t \sum_{i=1}^{N} \sigma_i^\varepsilon(v,T) d\tilde{w}_i(v)$$

where

$$b^\varepsilon(v,T) = \sum_{i=1}^{N} \sigma_i^\varepsilon(v,T) \int_v^T \sigma_i^\varepsilon(v,y)dy,$$

and in particular,

$$r^\varepsilon(t) = f(0,t) + \varepsilon^2 \int_0^t b^\varepsilon(v,t)dv + \varepsilon \int_0^t \sum_{i=1}^{N} \sigma_i(v,t) d\tilde{w}_i(v).$$

Especially, we note that the continuous stochastic processes for spot interest rates and forward rates are not necessarily Markovian or diffusion in the usual sense. The stochastic process is expanded around the deterministic process where the diffusion term (that is, δ or ε) is zero. The expansion is taken along the polynomial order of the volatility coefficients (that is, along δ^k or ε^k where $k = 1, 2, \cdots$). We obtain stochastic differential equations which the coefficient of each term in the expansion must satisfy. By making use of those stochastic differential equations, we can derive the density function (or distribution function) at any date the process follows. We can show that the coefficient of the first order follows normal distribution and the coefficients of the higher orders are considered adjusted terms whose density can be described as the normal density function multiplied by a polynomial function.

Theoretically, any orders can be evaluated, but in practice, the expansion upto the second is enough for most cases. Once the density function of the (discounted) payoff under the equivalent martingale measure is obtained, it is easy to evaluate the expectation of the discounted payoffs. We finally note that this approach is justified in a rigorous fashion by making use of the theory recently developed by Malliavin-Watanabe in the stochastic analysis.

The advantage of this method may be explained as follows. First, this is applicable in the unified manner to the pricing of various types of assets and their functionals in the economy evolved by very general class of continuous Ito processes, which are usually very difficult to evaluate. Second, this method is computationally efficient compare to the other methods such as the partial differential equation(**PDE**) approach and the Monte Carlo method since it is very fast to obtain the answer by, for example spread sheet in ordinary PC. Third, the distributions of the underlying assets and their functionals at any date can be obtained. This is very useful, for example in various kinds of simulation analysis. We also note that the pricing formula obtained by this method can be used as a control variate to improve the efficiency of Monte Carlo simulations. We briefly discuss the other methods. The **PDE** method requires tough task in implementation especially when the underlying assets follows multifactor or complicating processes. This often happens in term structure models. It also requires special consideration for boundary conditions as well as transformation of variables in each case and there is no unified technique. The Monte Carlo method is easy to implement, but is not computaionally efficient. This matters especially when fast response is required such as in forex dealing. Moreover, the Monte Carlo simulation is quite time-consuming in the cases where the process of zero coupon bond can not be solved explicitly in a term structure model and hence need a special treatment.

The organization of the paper is as follows. In the second section, we explain the basic concept of this technique in the simple Black-Scholes' economy where there is a constant risk-free interest rate and the underlying asset following one-factor diffusion process. First, we show the asymptotic expansion of the underlying asset and then, show how to evaluate basket options including "spread" options and average options as well as plain vanilla option by making use of this method.

Next, we present another method, the asymptotic expansion by using a log-normal distribution, which can be thought as a direct extension of the method proposed in Kunitomo-Takahashi(1992). In the last part of this section, We briefly explain how to apply the technique to options with stochastic volatilities.

In the third section, we extend this method to pricing problem of term structure model where we take arbitrage-free based forward rate model as our basics. First, we present a series of the asymptotic expansions of spot interest rates, instantaneous forward rates and zero coupon bonds as well as discount factors. Then, we evaluate the bond options including caps, floors and swaptions, and average options on interest rates by applying this technique.

In the forth section, in order to show that this method is also useful in the Black-Scholes' economy combined with term structure model, we present the pricing formula of average options on foreign exchange rate in the (cross-currency) stochastic interest rates economy.

In the fifth section, we show the several numerical examples. In the Black-Scholes' economy, first we give the numerical values of plain vanilla call options under the square-root process of the underlying asset. Next, we present numerical examples of average options for two types of the underlying asset process, one of which is a square-root process and the other of which is a log-normal process. In particular, we examine the log-normal case in detail which is widely used in practice. We numerically compare the values obtained by the asymptotic log-normal expansion to those obtained by our original method. In a term structure model, we present numerical examples of average options on interest rates for the constant volatility model of instantaneous forward rates. In the final section, we discuss the validity of our method in detail. We shall explain that our method is not an ad-hoc approximation method because it can be rigorously justified by using the Malliavin-Watanabe theory in stochastic analysis and we shall also emphasize that our simple inversion technique gives the exact same formulae as those obtained by the Malliavin calculus. In the appendix, we give the proofs of mathematical formulae freqently appearing in the paper and show the formulae in the multi-dimensional cases.

We finally note that this technique may also be applied to the valuation of various kinds of options in a multicurrency economy combined with term structure models, which are considered, in general, very difficult task.

2 The Asymptotic Expansion in the Black and Scholes' Economy

In this section, we present the asymptotic expansion method in the simple Black-Scholes' economy where the interest rate of the riskless asset is a constant and the risky assets follow some diffusion processes whose volatility functions may depend on the current level of the assets as well as on the current time. First, we derive the asymptotic expansion of the density function of the normalized price of the risky asset and that of the price of the plain vanilla call options in order to explain our method in detail. We next consider the valuation of basket options which is a natural extension of the plain vanilla options. Finally, we show that this technique is also valid in the pricing of average options which is a tough task especially when the underlying asset has a general volatility function.

2.1 The Asymptotic Expansion of Underlying Assets

We consider economy where there is one risky asset and a riskless asset. The volatility function in the risky asset process may depend on the current level of the asset and the current time. That is, the processes of the risky asset and the riskless asset are described as

$$\begin{aligned} dS^{\delta} &= rS^{\delta}dt + \delta\sigma(S^{\delta}, t)d\tilde{w}_t \qquad (1.1) \\ dB &= rBdt \end{aligned}$$

where $0 < \delta < 1$, $\tilde{w}_t$ is a one-dimensional standard Brownian motion and r is a positive constant. Alternatively, the integral form of the risky asset process is expressed as

$$S^{\delta}(t) = S(0) + r\int_0^t S^{\delta}(s)ds + \delta\int_0^t \sigma(S^{\delta}, s)d\tilde{w}(s).$$

Our first objective is to expand $S^{\delta}(t)$ around $\delta = 0$. The deterministic process where $\delta = 0$ is obtained by

$$S^0(t) \equiv \lim_{\delta\to 0} S^{\delta}(t) = S(0) + r\int_0^t S^0 ds.$$

Then, we easily have

$$S^0(t) = e^{rt}S(0). \tag{1.2}$$

Next, we calculate the coefficient of the first order of δ. Let

$$A(t) = \frac{\partial S^\delta(t)}{\partial \delta}|_{\delta=0}.$$

Then, we obtain the stochastic differential equation which $A(t)$ must follow.

$$dA(t) = rA(t)dt + \sigma(S^0, t)d\tilde{w}(t)$$

This stochastic differential equation can be solved as

$$A(t) = \int_0^t e^{r(t-s)}\sigma(S^0, s)d\tilde{w}(s). \tag{1.3}$$

The coefficients of the second and the third orders of δ are obtained in the similar manner. That is, let

$$B(t) = \frac{\partial^2 S^\delta(t)}{\partial \delta^2}|_{\delta=0}.$$

Then, we obtain the stochastic differential equation of $B(t)$.

$$dB(t) = rB(t)dt + 2\partial\sigma(S^0, t)A(t)d\tilde{w}(t)$$

where

$$\partial\sigma(S^0, t) \equiv \frac{\partial\sigma(S^\delta, t)}{\partial S^\delta}|_{S^\delta=S^0}.$$

Hence, $B(t)$ is solved as

$$B(t) = 2\int_0^t e^{r(t-s)}\partial\sigma(S^0, s)A(s)d\tilde{w}(s). \tag{1.4}$$

Similarly, let

$$C(t) = \frac{\partial^3 S^\delta(t)}{\partial \delta^3}|_{\delta=0},$$

and then,

$$dC(t) = rC(t)dt + 3\partial^2\sigma(S^0, t)A(t)^2d\tilde{w}(t) + 3\partial\sigma(S^0, t)B(t)d\tilde{w}(t).$$

Hence,

$$C(t) = 3\int_0^t e^{r(t-s)}\partial^2\sigma(S^0, s)A(s)^2d\tilde{w}(s) + 3\int_0^t e^{r(t-s)}\partial\sigma(S^0, s)B(s)d\tilde{w}(s). \tag{1.5}$$

Finally, we obtain the asymptotic expansion of $S^\delta(t)$. We state the reslut in the following proposition.

Proposition 1.1 *The asymptotic expansion of the price of the risky asset,* $S^\delta(t)$ *at any particular time point,* t *is given by*

$$S^\delta(t) = S^0(t) + \delta A(t) + \delta^2 \frac{B(t)}{2} + \delta^3 \frac{C(t)}{6} + o(\delta^3) \tag{1.6}$$

where $A(t), B(t)$ *and* $C(t)$ *are defined by (1.3), (1.4) and (1.5) respectively.*

Here, we can easily see that $A(t)$ follows a normal distribusion. That is,

$$A(t) \sim N(0, \Sigma_{A_t}) \tag{1.7}$$

where

$$\Sigma_{A_t} = \int_0^t e^{2r(t-s)} \sigma(S^0, s)^2 ds.$$

We next define the new variable for which we explicitly calculate the density function. Let

$$\begin{aligned} X_t^\delta = \{\frac{S^\delta(t) - S^0(t)}{\delta}\} &= A(t) + \delta \frac{B(t)}{2} + \delta^2 \frac{C(t)}{6} + \cdots \\ &\equiv g_1 + \delta g_2 + \delta^2 g_3 + \cdots . \end{aligned} \tag{1.8}$$

We know that

$$g_1 \sim N(0, \Sigma_{A_t}) = N(0, \Sigma_{g_1}). \tag{1.9}$$

Then, intuitively, we see that the density function of $X^\delta(t)$ can be obtained as the normal density function combined with the adjusted terms. To obtain the explicit functional form of the adjusted terms, we use the characteristic function method which is explained in detail below. First, we make the following assumption which is valid in all the subsequent analyses of this paper.

Assumption

$$\Sigma_{g_1} > 0 \tag{1.10}$$

Next, we define the characteristic function of $X^\delta(t)$ as

$$\psi(\xi) = \mathbf{E}[e^{i\xi X_t^\delta}].$$

Then, $\psi(\xi)$ itself can be expanded along the polynomial orders of δ .

$$
\begin{aligned}
\psi(\xi) &= \mathbf{E}[e^{i\xi(g_1+\delta g_2+\delta^2 g_3+\cdots)}] \\
&= \mathbf{E}[e^{i\xi g_1}\{1+\delta(i\xi)g_2+\delta^2(i\xi)g_3+\frac{\delta^2}{2}(i\xi)^2 g_2^2+\cdots\}] \\
&= \mathbf{E}[e^{i\xi g_1}]+\delta(i\xi)\mathbf{E}[e^{i\xi g_1}g_2]+\delta^2(i\xi)\mathbf{E}[e^{i\xi g_1}g_3] \\
&+ \frac{\delta^2}{2}(i\xi)^2\mathbf{E}[e^{i\xi g_1}g_2^2]+\cdots \\
&= e^{\frac{(i\xi)^2\Sigma_{g_1}}{2}}+\delta(i\xi)\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_2|g_1=x]\right]+\delta^2(i\xi)\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_3|g_1=x]\right] \\
&+ \frac{1}{2}\delta^2(i\xi)^2\mathbf{E}\left[e^{i\xi x}\mathbf{E}[g_2^2|g_1=x]\right]+\cdots.
\end{aligned}
$$

Next, we explicitly evaluate the expansion of this characteristic function. First, we will show that $\mathbf{E}[g_2|g_1=x]$, $\mathbf{E}[g_3|g_1=x]$ and $\mathbf{E}[g_2^2|g_1=x]$ are some polynomial functions of x , $h_2(x)$, $h_3(x)$, and $h_{22}(x)$, respectively. We present useful formulae to evaluate those conditional expectations.

Lemma 1.1 *Suppose $\tilde{w}(t)$ is a one-dimensional standard Brownian motion and $q_i(t),\ i=1,2,3,4$ are $R^1\mapsto R^1$ non-stochastic functions. Let x denote a scalar. Then, the following formulae for conditional expectations hold.*

(1)

$$
\begin{aligned}
&\mathbf{E}\left[\int_0^t\left[\int_0^s q_2(u)d\tilde{w}(u)\right]q_3(s)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right] \\
=& -\left[\frac{1}{\Sigma_{g_1}}\int_0^t q_3(s)q_1(s)\int_0^s q_2(u)q_1(u)duds\right] \\
+& \ x^2\left[\frac{1}{\Sigma_{g_1}^2}\int_0^t q_3(s)q_1(s)\int_0^s q_2(u)q_1(u)duds\right]
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\mathbf{E}\left[\left[\int_0^t q_2(u)d\tilde{w}(u)\right]\left[\int_0^t q_3(u)d\tilde{w}(u)\right]|\int_0^T q_1(u)d\tilde{w}(u)=x\right] \\
=& -\frac{1}{\Sigma_{g_1}}\left[\int_0^t q_2(u)q_1(u)du\right]\left[\int_0^t q_3(u)q_1(u)du\right]+\int_0^t q_3(u)q_2(u)du \\
+& \ x^2\frac{1}{\Sigma_{g_1}^2}\left[\int_0^t q_2(u)q_1(u)du\right]\left[\int_0^t q_3(u)q_1(u)du\right]
\end{aligned}
$$

(3)

$$\mathbf{E}\left[\int_0^t \left[\int_0^s q_2(u)d\tilde{w}(u)\right]\left[\int_0^s q_3(u)d\tilde{w}(u)\right] q_4(s)d\tilde{w}(s) | \int_0^T q_1(u)d\tilde{w}(u) = x\right]$$

$$\begin{aligned}
= \quad & x\left[\frac{1}{\Sigma_{g_1}}\int_0^t \left[\int_0^s q_2(u)q_3(u)du\right] q_4(s)q_1(s)ds\right. \\
& \left. -3\frac{1}{\Sigma_{g_1}^2}\int_0^t \left[\int_0^s q_2(u)q_1(u)du\right]\left[\int_0^s q_3(u)q_1(u)du\right] q_4(s)q_1(s)ds\right] \\
+ \quad & x^3\left[\frac{1}{\Sigma_{g_1}^3}\int_0^t \left[\int_0^s q_2(u)q_1(u)du\right]\left[\int_0^s q_3(u)q_1(u)du\right] q_4(s)q_1(s)ds\right]
\end{aligned}$$

(4)

$$\mathbf{E}\left[\int_0^t \int_0^s \left[\int_0^v q_2(u)d\tilde{w}(u)\right] q_3(v)d\tilde{w}(v)q_4(s)d\tilde{w}(s) | \int_0^T q_1(u)d\tilde{w}(u) = x\right]$$

$$\begin{aligned}
= \quad & -3x\left[\frac{1}{\Sigma_{g_1}^2}\int_0^t q_4(s)q_1(s)\int_0^s q_3(v)q_1(v)\int_0^v q_2(u)q_1(u)dudvds\right] \\
+ \quad & x^3\left[\frac{1}{\Sigma_{g_1}^3}\int_0^t q_4(s)q_1(s)\int_0^s q_3(u)q_1(u)\int_0^v q_2(u)q_1(u)dudvds\right]
\end{aligned}$$

(5)

$$\mathbf{E}\left[\left[\int_0^t \left[\int_0^s q_2(u)d\tilde{w}(u)\right] q_3(s)d\tilde{w}(s)\right]^2 | \int_0^T q_1(u)d\tilde{w}(u) = x\right]$$

$$\begin{aligned}
= \quad & \left[\int_0^t Q_{31}(s)\int_0^s Q_{21}(u)duds\right]^2\left[\frac{4}{\Sigma_{g_1}^2}x^4 - \frac{6}{\Sigma_{g_1}^3}x^2 + \frac{3}{\Sigma_{g_1}^2}\right] \\
+ \quad & \left[\int_0^t Q_{31}(s)\int_0^s Q_{32}(v)\int_0^v Q_{21}(u)dudvds\right]\left[\frac{2}{\Sigma_{g_1}^2}x^2 - \frac{2}{\Sigma_{g_1}}\right] \\
+ \quad & \left[\int_0^t Q_{31}(s)^2\int_0^s Q_{22}(u)duds\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2 - \frac{1}{\Sigma_{g_1}}\right] \\
+ \quad & \left[\int_0^t Q_{33}(s)\left[\int_0^s Q_{21}(u)du\right]^2 ds\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2 - \frac{1}{\Sigma_{g_1}}\right] \\
+ \quad & \int_0^t Q_{33}(s)\int_0^s Q_{22}(u)duds
\end{aligned}$$

where

$$Q_{ij}(s) \equiv q_i(s)q_j(s).$$

Proof.

See appendix. ∎

The formulae for multi-dimensional cases are also given by *Lemma 1.1'* in the appendix whose proof is quite similar to that of *Lemma 1.1* and hence it is omitted.

Hence, we can evaluate the conditional expectations by using *Lemma 1.1.* First, by applying $formula(1)$ to $\mathbf{E}[g_2|g_1 = x]$, we have

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{1.11}$$

where

$$c = \frac{1}{\Sigma_{g_1}^2}\int_0^t e^{r(t-s)}\sigma(S^0,s)\partial\sigma(S^0,s)\int_0^s e^{2r(t-v)}\sigma(S^0,v)^2 dvds,$$

and

$$f = -c\Sigma_{g_1}.$$

Second, we obtain $\mathbf{E}[g_3|g_1 = x]$ by using $formula(3)$ and $formula(4)$. First, we note that

$$\begin{aligned} g_3 &= \frac{C(t)}{6} \\ &= \frac{1}{2}e^{rt}\int_0^t \left[e^{rs}\partial^2\sigma(S^0,s)\right]\left[\int_0^v \sigma(S^0,v)d\tilde{w}(v)\right]^2 d\tilde{w}(s) \\ &+ e^{rt}\int_0^t \partial\sigma(S^0,s)\int_0^s \partial\sigma(S^0,v)\int_0^v e^{-ru}\sigma(S^0,u)d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s). \end{aligned}$$

Hence, applying $formula(3)$ and $formula(4)$ to the first and second term of $\mathbf{E}[g_3|g_1 = x]$ respectively, we obtain

$$\mathbf{E}\left[g_3|g_1 = x\right] = \frac{1}{2}e^{rt}[x^3c_{11} + xf_{11}] + e^{rt}[x^3c_{12} + xf_{12}] \tag{1.12}$$

where

$$c_{11} \equiv \frac{1}{\Sigma_{g_1}^2}e^{3rt}\int_0^t \left[\int_0^s e^{-2rv}\sigma(S^0,v)^2dv\right]^2 \partial^2\sigma(S^0,s)\sigma(S^0,s)ds,$$

$$f_{11} \equiv \frac{1}{\Sigma_{g_1}}e^{rt}\int_0^t \left[\int_0^s e^{-2rv}\sigma(S^0,v)^2dv\right]\partial^2\sigma(S^0,s)\sigma(S^0,s)ds - 3\Sigma_{g_1}c_{11},$$

$$\begin{aligned} c_{12} \equiv\ & \frac{1}{\Sigma_{g_1}^3}e^{3rt}\int_0^t \left[e^{-rs}\sigma(S^0,s)\sigma(S^0,s)\right] \\ & \int_0^s \left[e^{-rv}\sigma(S^0,v)\sigma(S^0,v)\right]\int_0^v \left[e^{-2ru}\sigma(S^0,u)^2\right] dudsdt, \end{aligned}$$

and

$$f_{12} \equiv -3\Sigma_{g_1}c_{12}.$$

Then, we can write

$$\mathbf{E}\left[g_3|g_1 = x\right] = c_1x^3 + f_1x \tag{1.13}$$

where

$$c_1 \equiv \frac{1}{2}e^{rt}c_{11} + e^{rt}c_{12},$$

and

$$f_1 \equiv \frac{1}{2}e^{rt}f_{11} + e^{rt}f_{12}.$$

Similarly, by using (5), we can show that for some constants c_2, f_2 and k_2,

$$\mathbf{E}\left[g_2^2|g_1 = x\right] = c_2x^4 + f_2x^2 + k_2. \tag{1.14}$$

Therefore, we obtain the characteristic function as

$$\begin{aligned}\psi(\xi) &= e^{\frac{(i\xi)^2\Sigma_{g_1}}{2}} + \delta(i\xi)\mathbf{E}\left[e^{i\xi x}h_2(x)\right] + \delta^2(i\xi)\mathbf{E}\left[e^{i\xi x}h_3(x)\right] \\ &+ \frac{\delta^2}{2}(i\xi)^2\mathbf{E}\left[e^{i\xi x}h_{22}(x)\right] + \cdots.\end{aligned}$$

We need to evaluate the expectations such as $\mathbf{E}\left[e^{i\xi x}h_2(x)\right]$, $\mathbf{E}\left[e^{i\xi x}h_3(x)\right]$, and $\mathbf{E}\left[e^{i\xi x}h_{22}(x)\right]$ to obtain $\psi(\xi)$.

In the final step to obtain the density function of $X^\delta(t)$, we need to inverse $\psi(\xi)$ (inverse the Fourier transformation). We make use of the following formula which is given by *Fujikoshi etal.(1982)* to summarize both steps of evaluating characteristic function and implementing the inverse-Fourier transformation.

Lemma 1.2 *Suppose that $\vec{x}$ follows N-dimensional normal distribution with mean $\vec{0}$ and variance-covariance matrix $\underline{\Sigma}$. Then, for any polynomial functions $h(\cdot)$ and $g(\cdot)$,*

$$\mathcal{F}^{-1}\left[g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]\right]_{<\vec{\omega}>} = g\left[\frac{\partial}{\partial\vec{\omega}}\right]h(\vec{\omega})n[\vec{\omega};\vec{0},\underline{\Sigma}], \tag{1.15}$$

where

$$\mathcal{F}^{-1}\left[g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]\right]_{<\vec{\omega}>} = (\frac{1}{2\pi})^N\int_{R^N} e^{-i\xi^\top\vec{\omega}}g(-i\vec{\xi})\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]d\vec{\xi},$$

the expectation $\mathbf{E}[\cdot]$ is taken over x, and $\mathcal{F}^{-1}[\cdot]_{<\vec{\omega}>}$ denotes $\mathcal{F}^{-1}[\cdot]$ being evaluated at $\vec{\omega}$.

PROOF. It holds that

$$(\frac{1}{2\pi})^N\int_{R^N} e^{-i\xi^\top\vec{\omega}}\mathbf{E}\left[h(\vec{x})e^{i\xi^\top\vec{x}}\right]d\vec{\xi} = h(\vec{\omega})n[\omega;\vec{0},\underline{\Sigma}]$$

Differenciating both sides with respect to the elements of $\vec{\omega}$, we obtain the result. ▮ Finally, we obtain the asymptotic expansion of the density function of X_t^δ, $f_{X_t^\delta}$. That is,

$$
\begin{aligned}
f_{X_t^\delta} &\sim n[x;0,\Sigma_{g_1}] + \delta\left[-\frac{\partial}{\partial x}\{h_2(x)n[x;0,\Sigma_{g_1}]\}\right] \\
&+ \delta^2\left[-\frac{\partial}{\partial x}\{h_3(x)n[x;0,\Sigma_{g_1}]\}\right] + \frac{1}{2}\delta^2\left[\frac{\partial^2}{\partial x^2}\{h_{22}(x)n[x;0,\Sigma_{g_1}]\}\right] + \cdots .
\end{aligned}
$$

where

$$X_t^\delta = \frac{S^\delta(t) - S^0(t)}{\delta},$$

and

$$n[x;0,\Sigma_{g_1}] = \frac{1}{\sqrt{2\pi\Sigma_{g_1}}}\exp\left[-\frac{x^2}{2\Sigma_{g_1}}\right].$$

Using polynomial functions of $h_2(x)$, $h_3(x)$, and $h_{22}(x)$, we can obtain more explicit form of the density function. We state this result as the following theorem.

Theorem 1.1 *The density function of* $X_t^\delta = \frac{S^\delta(t)-S^0(t)}{\delta}$, $f_{X_t^\delta}$ *as* $\delta \to 0$ *can be expressed as*

$$
\begin{aligned}
f_{X_t^\delta} &= n[x;0,\Sigma_{g_1}] \\
&+ \delta\left[\{\frac{c}{\Sigma_{g_1}}x^3 + (\frac{f}{\Sigma_{g_1}} - 2c)x\}n[x;0,\Sigma_{g_1}]\right] \\
&+ \delta^2\left[\{\frac{c_2}{2\Sigma_{g_1}^2}x^6 + (\frac{f_2}{2\Sigma_{g_1}^2} - \frac{9c_2}{2\Sigma_{g_1}} + \frac{c_1}{\Sigma_{g_1}})x^4\right. \\
&+ \left.(\frac{k_2}{2\Sigma_{g_1}^2} - \frac{5f_2}{2\Sigma_{g_1}} + \frac{f_1}{\Sigma_{g_1}} - 3c_1 + 6c_2)x^2 + (-f_1 - \frac{k_2}{2\Sigma_{g_1}} + f_2)\}n[x;0,\Sigma_{g_1}]\right] \\
&+ o(\delta^2).
\end{aligned}
\tag{1.16}
$$

2.2 Plain Vanilla Options

We next, show how to evaluate plain vanilla options with general volatility functions by using the density function of X_t^δ obtained previously. First, we define the payoffs of plain vanilla options as

$$V(T) = (S(T) - K)^+ \tag{1.17}$$

or

$$V(T) = (K - S(T))^+.$$

Then, from the well-known martingale technique, the value at the initial date is given by

$$V(0) = e^{-rT}\mathbf{E}^*[V(T)] \tag{1.18}$$

where the expectation is taken under the equivalent martingale measure. In the following, we only consider the asymptotic expansion of a call option because that of a put option is obtained in the similar manner. First, we note that by using X_t^δ, the $V(T)$ can be expressed as

$$\begin{aligned} V(T) &= \delta\left[\frac{S^0(T)-K}{\delta} + X_t^\delta\right]^+ = \delta\left[y + X_t^\delta\right]^+ \\ \text{where} & \\ y &\equiv \frac{S^0(T)-K}{\delta}. \end{aligned}$$

Hence, using $\mathbf{E}[g_2|g_1 = x] = cx^2 + f$, $\mathbf{E}[g_3|g_1 = x] = c_1x^3 + f_1x$ and $\mathbf{E}[g_2^2|g_1 = x] = c_2x^4 + f_2x^2 + k_2$ where c, f, c_1, f_1, c_2, f_2 and k_2 are defined in the previous subsection, together with the valuation formula, we can obtain the initial value of the call option. That is,

$$\begin{aligned} V(0) &= e^{-rT}\delta\mathbf{E}^*[(y + X_T^\delta)^+] \\ &= e^{-rT}\delta\left[y\int_{-y}^{\infty} f_{X_T^\delta}(x)dx + \int_{-y}^{\infty} x f_{X_T^\delta}(x)dx\right] \\ &\sim e^{-rT}\delta\Bigg[y\int_{-y}^{\infty} n[x;0,\Sigma_{g_1}]dx + \delta y\int_{-y}^{\infty}\frac{-\partial\{(cx^2+f)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\ &+ \delta^2 y\int_{-y}^{\infty}\frac{-\partial\{(c_1x^3+f_1x)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\ &+ \frac{1}{2}\delta^2 y\int_{-y}^{\infty}\frac{\partial^2\{(c_2x^4+f_2x^2+k_2)n[x;0,\Sigma_{g_1}]\}}{\partial x^2}dx \\ &+ \int_{-y}^{\infty} xn[x;0,\Sigma_{g_1}]dx + \delta\int_{-y}^{\infty} x\frac{-\partial\{(cx^2+f)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\ &+ \delta^2\int_{-y}^{\infty} x\frac{-\partial\{(c_1x^3+f_1x)n[x;0,\Sigma_{g_1}]\}}{\partial x}dx \\ &+ \frac{1}{2}\delta^2\int_{-y}^{\infty} x\frac{\partial^2\{(c_2x^4+f_2x^2+k_2)n[x;0,\Sigma_{g_1}]\}}{\partial x^2}dx\Bigg]. \end{aligned}$$

In the following theorem, we present an more explicit formula which may be convienient to evaluate the value of the call option.

Theorem 1.2 *The asymptotic expansion of the price of a call option with the general volatility function is given by*

$$\begin{aligned} V(0) &= e^{-rT}\Bigg[\delta y N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) + \delta \int_{-y}^{\infty} xn[x;0,\Sigma_{g_1}]dx \\ &+ \delta^2 \int_{-y}^{\infty} (cx^2+f)n[x;0,\Sigma_{g_1}]dx \\ &+ \delta^3 \int_{-y}^{\infty} (c_1x^3+f_1x)n[x;0,\Sigma_{g_1}]dx + \frac{1}{2}\delta^3(c_2y^4+f_2y^2+k_2)n[y:0,\Sigma_{g_1}]\Bigg] \\ &+ o(\delta^3). \end{aligned} \tag{1.19}$$

PROOF. From (1.19), the straightforward calculation shows the result. ∎

In order to evaluate the integrals in (1.19) and those wnich may appear in the coefficients of $\delta^k, k \geq 4$, the following formulae are useful. We omit the proofs because they are easily obtained from integration by parts.

$$\begin{aligned} \int_{-y}^{\infty} xn[x;0,\Sigma_{g_1}]dx &= \Sigma_{g_1} n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^2n[x;0,\Sigma_{g_1}]dx &= \Sigma_{g_1} N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - y\Sigma_{g_1} n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^3n[x;0,\Sigma_{g_1}]dx &= (2\Sigma_{g_1}^2 + \Sigma_{g_1}y^2)n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^4n[x;0,\Sigma_{g_1}]dx &= 3\Sigma_{g_1}^2 N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - (3\Sigma_{g_1}^2 y + \Sigma_{g_1}y^3)n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^5n[x;0,\Sigma_{g_1}]dx &= (8\Sigma_{g_1}^3 + 4\Sigma_{g_1}^2 y^2 + \Sigma_{g_1}y^4)n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^6n[x;0,\Sigma_{g_1}]dx &= 15\Sigma_{g_1}^3 N(\frac{y}{\Sigma_{g_1}^{\frac{1}{2}}}) - (15\Sigma_{g_1}^3 y + 5\Sigma_{g_1}^2 y^3 + \Sigma_{g_1}y^5)n[y;0,\Sigma_{g_1}] \\ \int_{-y}^{\infty} x^7n[x;0,\Sigma_{g_1}]dx &= (48\Sigma_{g_1}^4 + 24\Sigma_{g_1}^3 y^2 + 6\Sigma_{g_1}^2 y^4 + \Sigma_{g_1}y^6)n[y;0,\Sigma_{g_1}] \end{aligned}$$

2.3 Basket Options

Next, we consider the pricing of basket options (including so called "spread" options) which is a natural extension of plain vanilla options by using our method. First, we formally define "basket", $I(t)$ as

$$I(t) = \sum_{j=1}^{N} \alpha_j S_j(t) \tag{1.20}$$

where $S_j(t)$ denotes the the price of the j th risky asset which is a component of the basket. We note that, as a special case, the "spread" is defined by $\alpha_{j1} = 1$, $\alpha_{j2} = -1,\ j2 \neq j1$ and $j = 0$ for $j \neq j1, j2$ in $I(t)$. That is,

$$I(t) = S_{j1}(t) - S_{j2}(t).$$

Then, the payoffs of the basket options are expressed as

$$\begin{aligned} V(T) &= (I(T) - K)^+ \\ &\text{or} \\ V(T) &= (K - I(T))^+. \end{aligned} \tag{1.21}$$

In what follows, we consider call options. That is,

$$V(T) = (I(T) - K)^+.$$

For the pricing, we consider the Black-Sholes' economy where there are risky assets, each of which may depend on N independent Brownian motions.

$$\begin{aligned} dS_j^\delta(t) &= rS_j^\delta(t)dt + \delta \sum_{i=1}^{N} \sigma_i(t, S_j^\delta(t))d\tilde{w}_i(t) \\ dB(t) &= rB(t)dt \end{aligned} \tag{1.22}$$

where r is a positive constant and $0 < \delta < 1$.

Note: δ can differ in j that is, δ , but if we redefine δ such that $\delta = min[\delta_i]_i$, then we have the same expression of the processes as above.

Following the steps in the previous subsection, we can show for each j,

$$\begin{aligned} S_j^0(t) &\equiv \lim_{\delta \to 0} S_j^\delta(t) = e^{rt} S_j(0) \\ A_j(t) &\equiv \frac{\partial S_j^\delta(t)}{\partial \delta}|_{\delta=0} = \int_0^t e^{r(t-s)} \sum_{i=1}^{N} \sigma_{ij}^0(t) d\tilde{w}_i(s) \\ B_j(t) &\equiv \frac{\partial^2 S_j^\delta(t)}{\partial \delta^2}|_{\delta=0} = 2\int_0^t e^{r(t-s)} \sum_{i=1}^{N} \partial\sigma_{ij}^0(s) A_j(s) d\tilde{w}_i(s) \\ &\text{and} \\ C_j(t) &\equiv \frac{\partial^3 S_j^\delta(t)}{\partial \delta^3}|_{\delta=0} = 3\int_0^t e^{r(t-s)} \sum_{i=1}^{N} \partial^2\sigma_{ij}^0(s) A_j(s)^2 d\tilde{w}_i(s) \\ &+3\int_0^t e^{r(t-s)} \sum_{i=1}^{N} \partial\sigma_{ij}^0(s) B_j(s) d\tilde{w}_i(s). \end{aligned}$$

Then, we obtain the asymptotic expansion of each risky asset, $S_j^\delta(t)$, $j = 1, 2, \cdots, N$.

$$S_j^\delta(t) = S_j^0(t) + \delta A_j(t) + \delta^2 \frac{B_j(t)}{2} + \delta^3 \frac{C_j(t)}{6} \cdots .$$

As the "basket", $I_j^\delta(t)$ is a linear combination of risky assets $S_j^\delta(t)$,$j = 1, 2, \cdots, N$, we can easily obtain the asymptotic expansion of a basket.

$$I_j^\delta(t) \sim \sum_{j=1}^N \alpha_j S_j^0(t) + \delta \sum_{j=1}^N \alpha_j A_j(t) + \frac{\delta^2}{2} \sum_{j=1}^N \alpha_j B_j(t) + \frac{\delta^3}{6} \sum_{j=1}^N \alpha_j C_j(t) + \cdots$$

Next, as in the previous subsection, we define $X^\delta(t)$ for which we explicitly obtain the density function. Let

$$X^\delta(t) \equiv \frac{I^\delta(t) - I^0(t)}{\delta} \sim g_1 + \delta g_2 + \delta^2 g_3 + \cdots \tag{1.23}$$

where

$$g_1 = \int_0^t e^{r(t-s)} \sigma_I^0(s)^\top d\tilde{w}(s), \ \sigma_I^0(s) \equiv \left[\sum_{j=1}^N \alpha_j \sigma_{ij}^0(s) \right]_i$$

and

$$g_2 = e^{rt} \sum_{j=1}^N \alpha_j \int_0^t \left[\int_0^s e^{-rv} \sigma_j^0(v)^\top d\tilde{w}(v) \right] \partial \sigma_j^0(s)^\top d\tilde{w}(s).$$

We can easily see g_1 follows the normal distribution,

$$g_1 \sim N(0, \Sigma_{g_1}) \tag{1.24}$$

where

$$\Sigma_{g_1} \equiv \int_0^t e^{2r(t-s)} \sigma_I^0(s)^\top \sigma_I^0(s) ds.$$

For the pricing of the call option, we need to evaluate conditional expectations such as $\mathbf{E}[g_2|g_1 = x]$, $\mathbf{E}[g_3|g_1 = x]$ and $\mathbf{E}[g_2^2|g_1 = x]$. By using formulae appearing in *Lemma 1.1*, we can evaluate those expectations, for example, by applying $formula(1)$,

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{1.25}$$

$$c \equiv e^{rt} \sum_{j=1}^N \alpha_j c_j$$

and

$$f \equiv e^{rt} \sum_{j=1}^N \alpha_j f_j$$

where

$$c_j = \frac{1}{\Sigma_{g_1}^2} e^{2rt} \int_0^t \left[\int_0^s e^{-2rv} \sigma_I^0(v)^\top \sigma_j^0(v) dv \right] e^{-rs} \sigma_I^0(s)^\top \partial \sigma_j^0(s) ds$$

and

$$f_j = -\Sigma_{g_1} c_j.$$

Therefore, applying the pricing formula in *Theorem 1.2* by replacing y by $y \equiv \frac{I^0(T)-K}{\delta}$, we can obtain the initial value of the basket (call) option.

2.4 Average Options

We next consider more complicating example, the average options commonly known as "Asian Options", in the simple Black-Scholes' economy with a general volatility function. The payoffs of the average options are defined as

$$V(T) = (Z^\delta(T) - K)^+ \tag{1.26}$$

or

$$V(T) = (K - Z^\delta(T))^+$$

where

$$Z^\delta(T) \equiv \frac{1}{T} \int_0^T S^\delta(t) dt.$$

Then, the value at the initial date is expressed as, by using the martingale technique,

$$V(0) = e^{-rT} \mathbf{E}^*[V(T)].$$

In what follows, we evaluate the call option as an example. In this case, we consider the asymptotic expansion of the functional of the risky asset, $Z^\delta(T)$.

$$Z^\delta(T) = Z^0(T) + \frac{1}{T} \int_0^T A(t) dt + \delta \frac{1}{T} \int_0^T \frac{B(t)}{2} dt \int_0^T \frac{C(t)}{6} dt + \cdots \tag{1.27}$$

where

$$Z^0(T) \equiv \lim_{\delta \to 0} Z^\delta(T) = \frac{1}{T} \int_0^T S^0(t) dt.$$

We define X_T^δ for which we obtain the density function.

$$X_T^\delta = \frac{Z^\delta(T) - Z^0(T)}{\delta},$$

Then, the asymptotic expansion of X_T^δ is given by

$$X_T^\delta = g_1 + \delta g_2 + \delta^2 g_3 + \cdots.$$

where

$$g_1 = \int_0^T A(t)dt,$$

$$g_2 = \int_0^T \frac{B(t)}{2}dt$$

and

$$g_3 = \int_0^T \frac{C(t)}{6}dt.$$

We note that

$$g_1 = \int_0^T \frac{1}{T}\left[\frac{e^{r(T-s)}-1}{r}\right]\sigma(S^0,s)d\tilde{w}(s). \tag{1.28}$$

Then, we can see that g_1 follows a normal distribution. In fact,

$$g_1 \sim N(0,\Sigma_{g_1}) \tag{1.29}$$

where

$$\Sigma_{g_1} = \int_0^T \frac{1}{T^2}\left[\frac{e^{r(T-s)}-1}{r}\right]^2 \sigma(S^0,s)^2 ds.$$

Following the same method as in the previous sebsection, we can express the asymptotic expansion of the density function as

$$\begin{aligned} f_{X_T^\delta} \sim\ & n[x;0,\Sigma_{g_1}] \\ +\ & \delta\frac{-\partial\{\mathbf{E}[g_2|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x} \\ +\ & \delta^2\frac{-\partial\{\mathbf{E}[g_3|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x} \\ +\ & \frac{1}{2}\delta^2\frac{\partial^2\{\mathbf{E}[g_2^2|g_1=x]n[x;0,\Sigma_{g_1}]\}}{\partial x^2} + \cdots. \end{aligned}$$

Then, we need to evaluate conditional expectations such as $\mathbf{E}[g_2|g_1=x]$, $\mathbf{E}[g_3|g_1=x]$ and $\mathbf{E}[g_2^2|g_1=x]$. Using the $formula(1)$ in *Lemma 1.1*, we can show

$$\mathbf{E}[g_2|g_1=x] = cx^2+f \tag{1.30}$$

where

$$c = \frac{1}{\Sigma_{g_1}^2}\frac{1}{T^3}\int_0^T\int_0^t e^{r(t-s)}\left[\frac{e^{r(T-s)}-1}{r}\right]\sigma(S^0,s)\partial\sigma(S^0,s)$$

$$\int_0^s e^{r(s-v)}\left[\frac{e^{r(T-v)}-1}{r}\right]\sigma(S^0,v)^2 dvdsdt,$$

and

$$f \;=\; -c\Sigma_{g_1}.$$

Next, using $formula(3)$ and $formula(4)$ in *Lemma 1.1*, we can obtain

$$\begin{aligned}
\mathbf{E}[g_3|g_1=x] &= c_1x^3+f_1x \qquad (1.31)\\
c_1 &= c_{11}+c_{12},\\
f_1 &= f_{11}+f_{12},\\
c_{11} &= \frac{1}{2T^4r^3}\frac{1}{\Sigma_{g_1}^3}\int_0^T e^{rt}\int_0^t\left[\int_0^s e^{-ru}\{e^{r(T-u)}-1\}\sigma(S^0,u)^2du\right]^2\\
&\quad \left[e^{rs}\{e^{r(T-s)}-1\}\partial^2\sigma(S^0,s)\sigma(S^0,s)\right]dsdt,\\
f_{11} &= \frac{1}{2T^2r}\frac{1}{\Sigma_{g_1}}\int_0^T e^{rt}\int_0^t\left[\int_0^s e^{-2rv}\sigma(S^0,v)^2dv\right],\\
&\quad \left[e^{rs}\{e^{r(T-s)}-1\}\partial^2\sigma(S^0,s)\sigma(S^0,s)\right]dsdt-3\Sigma_{g_1}c_{11},\\
c_{12} &= \frac{1}{T^4r^3}\frac{1}{\Sigma_{g_1}^3}\int_0^T e^{rt}\int_0^t\left[\{e^{r(T-s)}-1\}\partial\sigma(S^0,s)\sigma(S^0,s)\right]\\
&\quad \int_0^s\left[\{e^{r(T-v)}-1\}\partial\sigma(S^0,v)\sigma(S^0,v)\right]\\
&\quad \int_0^v\left[e^{-ru}\{e^{r(T-u)}-1\}\sigma(S^0,u)^2\right]dudvdsdt,
\end{aligned}$$

and

$$f_{12} \;=\; -3\Sigma_{g_1}c_{12}.$$

We can also show

$$\mathbf{E}[g_2^2|g_1=x]=c_2x^4+f_2x^2+k_2 \qquad (1.32)$$

where c_2 , f_2 and k_2 are defined by using the following formula.

(5)′

$$\begin{aligned}
&\mathbf{E}\left[\left[\int_0^T\int_0^t\left[\int_0^s q_2(u)d\tilde{w}(u)\right]q_3(s)d\tilde{w}(s)dt\right]^2 \mid \int_0^T q_1(u)d\tilde{w}(u)=x\right]\\
=&\left[\int_0^T\int_0^t Q_{31}(s)\int_0^s Q_{21}(u)dudsdt\right]^2\left[\frac{4}{\Sigma_{g_1}^2}x^4-\frac{6}{\Sigma_{g_1}^3}x^2+\frac{3}{\Sigma_{g_1}^2}\right]\\
+&\left[\int_0^T\int_0^t Q_{31}(s)\int_0^s Q_{32}(v)\int_0^v Q_{21}(u)dudvdsdt\right]\left[\frac{2}{\Sigma_{g_1}^2}x^2-\frac{2}{\Sigma_{g_1}}\right]
\end{aligned}$$

$$
\begin{aligned}
+ & \left[\int_0^T \int_0^t Q_{31}(s)^2 \int_0^s Q_{22}(u)dudsdt\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2 - \frac{1}{\Sigma_{g_1}}\right] \\
+ & \left[\int_0^T \int_0^t Q_{33}(s)\left[\int_0^s Q_{21}(u)du\right]^2 dsdt\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2 - \frac{1}{\Sigma_{g_1}}\right] \\
+ & \int_0^T \int_0^t Q_{33}(s)\int_0^s Q_{22}(u)dudsdt
\end{aligned}
$$

where

$$Q_{ij}(s) \equiv q_i(s)q_j(s).$$

PROOF. The proof is a slight modification of that of $formula(5)$ in *Lemma 1.1.* ▮ Therefore, by using *Theorem 1.2* where we replace y by $y \equiv \frac{Z^0(T)-K}{\delta}$, we can finally obtain the vale of the average (call) option.

2.5 Stochastic Log-normal Expansion of Average Option

In this subsection, we present another approach to pricing average options in the simple Black-Scholes' economy with a log-normally distributed risky asset process. In what follows, we consider an average call option in the economy where the processes of the risky asset and the riskless asset follow respectively

$$dS^\delta(t) = (r-q)S_t dt + \delta S_t d\tilde{w}(t) \tag{1.33}$$

and

$$dB(t) = rB_t dt.$$

The basic idea is that we expand the distribution of

$$\frac{\log Z^\delta(T) - \log Z^0(T)}{\delta} \tag{1.34}$$

by using a normal distribution while we expand

$$\frac{Z^\delta(T) - Z^0(T)}{\delta}$$

by using a normal distribution in the original method. We will formally describe this method. First, we define $Z^\delta(T)$ as

$$Z^\delta(T) = \frac{1}{T}\int_0^T S^\delta(t)dt.$$

Let

$$Z^0(T) = \frac{1}{T}\int_0^T S^0(t)dt = S(0)\frac{e^{\alpha}T-1}{\alpha T}$$

where

$$\alpha = r - q - \sigma^2/2.$$

Next, we define

$$g^\delta = \frac{1}{\sigma}\log\frac{Z(T)}{Z^0(T)} \tag{1.35}$$

for which we obtain the asymptotic expansion.

Then, the asymptotic expansion of g^δ is given by

$$\begin{aligned} g^\delta &\sim g_1 + \delta g_2 + \cdots, \qquad (1.36)\\ g_1 &= \int_0^T \frac{k}{\alpha}(e^{\alpha T} - e^{\alpha u})d\tilde{w}(u) \equiv \int_0^T a(u)d\tilde{w}(u),\\ g_2 &= \frac{1}{2}\int_0^T k_T(t)\tilde{w}(t)^2 dt - \frac{1}{2}\left[\int_0^T k_T(t)\tilde{w}(t)dt\right]^2 \end{aligned}$$

where

$$\begin{aligned} k &= \frac{\alpha}{e^{\alpha T}-1},\\ a(u) &= \frac{k}{\alpha}(e^{\alpha T} - e^{\alpha u}) \end{aligned}$$

and

$$k_T(t) = ke^{\alpha t}.$$

We easily see that g_1 follows a normal distribution. That is,

$$g_1 \sim N(0, \Sigma_{g_1})$$

where

$$\Sigma_{g_1} = (\frac{k}{\alpha})^2\left[Te^{2\alpha T} - 2e^{\alpha T}\frac{e^{\alpha T}-1}{\alpha} + \frac{e^{2\alpha T}-1}{2\alpha}\right].$$

We need to evaluate $\mathbf{E}\left[g_2|g_1 = x\right]$ to obtain the asymptotic expansion. This can be obtained by using the $formula(2)$ in *Lemma 1.1.* Thus,

$$\mathbf{E}\left[g_2|g_1 = x\right] = cx^2 + f \tag{1.37}$$

where

$$c \equiv \frac{1}{2}\left[\frac{1}{\Sigma_{g_1}^2}\int_0^T k_T(t)\left[\int_0^t a(u)du\right]^2 dt - 1\right]$$

and

$$f \equiv -\Sigma_{g_1} c + \frac{1}{2}\int_0^T k_T(t)tdt.$$

Therefore, by using the same technique as in the previous subsection, we have the asymptotic expansion of the density function of g^{δ} as

$$f_{g^{\delta}} \sim n[x;0,\Sigma_{g_1}] + \sigma\left[\frac{c}{\Sigma_{g_1}}x^3 + (\frac{f}{\Sigma_{g_1}} - 2c)x\right] n[x;0,\Sigma_{g_1}] + \cdots.$$

Finally, we need to evaluate

$$C(0) = e^{-rt}\mathbf{E}^*\left[(Z(T)-K)^+\right].$$

This can be written as

$$\begin{aligned} e^{-rt}\mathbf{E}^*\left[(Z(T)-K)^+\right] &= e^{-rt}\mathbf{E}^*\left[(Z^0(T)e^{\sigma x}-K)^+\right] \\ &\sim e^{-rt}Z^0(T)\int_y^{\infty} e^{\sigma x} f_{g_1}(x)dx - e^{-rt}K\int_y^{\infty} f_{g_1}(x)dx \end{aligned}$$

where

$$y \equiv \frac{1}{\sigma}\log\left[\frac{K}{Z^0(T)}\right].$$

The first term is given by

$$\begin{aligned} \int_y^{\infty} e^{\sigma x} f_{g_1}(x)dx &= \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right]\left[1+\sigma(\frac{c}{\Sigma_{g_1}})(\sigma\Sigma_{g_1})^3\right. \\ &+ \left.\sigma(\frac{f}{\Sigma_{g_1}} - 2c)(\sigma\Sigma_{g_1})\right] N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) \\ &+ \sigma\exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right]\left[3\sigma(\frac{c}{\Sigma_{g_1}})(\sigma\Sigma_{g_1})^2 + (\frac{f}{\Sigma_{g_1}} - 2c)\right]\times \\ &\int_{y_1}^{\infty} zn[z;0,\Sigma_{g_1}]dz \\ &+ 3\sigma\exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right](\frac{c}{\Sigma_{g_1}})(\sigma\Sigma_{g_1})\int_{y_1}^{\infty} z^2 n[z;0,\Sigma_{g_1}]dz \\ &+ \sigma\exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right](\frac{c}{\Sigma_{g_1}})\int_{y_1}^{\infty} z^3 n[z;0,\Sigma_{g_1}]dz \end{aligned}$$

where

$$y_1 = y - \sigma\Sigma_{g_1}.$$

This can be simplified to

$$\begin{aligned}
\int_y^\infty e^{\sigma x} f_{g_1}(x)dx &= \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) \\
&+ \sigma \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[(\frac{c}{\Sigma_{g_1}}) \int_{y_1}^\infty z^3 n[z;0,\Sigma_{g_1}]dz \right.\\
&+ \left. (\frac{f}{\Sigma_{g_1}} - 2c) \int_{y_1}^\infty zn[z;0,\Sigma_{g_1}]dz \right] \\
&+ \sigma^2 \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[(f - 2c\Sigma_{g_1})N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) + 3c\int_{y_1}^\infty z^2 n[z;0,\Sigma_{g_1}]dz\right] \\
&+ \sigma^4 \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[c\Sigma_{g_1}^2 N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) + 3c\Sigma_{g_1}\int_{y_1}^\infty zn[z;0,\Sigma_{g_1}]dz\right].
\end{aligned}$$

The second term is given by

$$\int_y^\infty f_{g_1}(x)dx = N(\frac{-y}{\Sigma_{g_1}^{\frac{1}{2}}}) + \sigma(cy^2 + f)n[y;0,\Sigma_{g_1}].$$

Finally, we obtain the initial value of the average call option. We state the result in the following theorem.

Theorem 1.3 *The asymptotic log-normal expansion of the price of the average call option is given by*

$$\begin{aligned}
C(0) \sim\; & e^{-rT}\left[\exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) - KN(\frac{-y}{\Sigma_{g_1}^{\frac{1}{2}}}) \right. \qquad (1.38)\\
+\; & \sigma \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[(\frac{c}{\Sigma_{g_1}}) \int_{y_1}^\infty z^3 n[z;0,\Sigma_{g_1}]dz + (\frac{f}{\Sigma_{g_1}} - 2c) \int_{y_1}^\infty zn[z;0,\Sigma_{g_1}]dz\right] \\
+\; & \sigma^2 \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[(f - 2c\Sigma_{g_1})N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) + 3c\int_{y_1}^\infty z^2 n[z;0,\Sigma_{g_1}]dz\right] \\
+\; & \sigma^4 \exp\left[\frac{\sigma^2\Sigma_{g_1}}{2}\right] \left[c\Sigma_{g_1}^2 N(\frac{-y_1}{\Sigma_{g_1}^{\frac{1}{2}}}) + 3c\Sigma_{g_1}\int_{y_1}^\infty zn[z;0,\Sigma_{g_1}]dz\right] \\
-\; & \left. \sigma(cy^2 + f)n[y;0,\Sigma_{g_1}]\right].
\end{aligned}$$

2.6 Options with a Stochastic Volatility

We may also apply the asymptotic expansion method to the evaluation of the options with stochastic volatilities which He(1992) develop in an equilibrium framework. We assume that there exists a risk-free rate, r which is a positive constant. In general, under the equivalent martingale measure, the processes of the underlying asset and a state variable are defined respectively by

$$\begin{aligned} dS_1^\delta(t) &= \mu_1(S_1^\delta, Y^\delta, t)dt + \delta\vec{\sigma_1}(S_1^\delta, Y^\delta, t)^\top d\vec{\tilde{w}}_t \\ &\text{and} \\ dY^\delta(t) &= \mu_2(S_1^\delta, Y^\delta, t)dt + \delta\vec{\sigma_2}(S_1^\delta, Y^\delta, t)^\top d\vec{\tilde{w}}_t. \end{aligned} \quad (1.39)$$

where $0 < \delta < 1$, $\vec{\tilde{w}}_t$ is the standard two dimensional Brownian motion, and $\vec{\sigma_1}(S_1^\delta, Y^\delta, t)$ and $\vec{\sigma_2}(S_1^\delta, Y^\delta, t)$ denote two dimensional vectors. By using vector form, these can be rewritten as

$$d\vec{S_t^\delta} = \vec{\mu}(S_1^\delta, Y^\delta, t)dt + \delta\underline{\Sigma}(S^\delta, Y^\delta, t)d\vec{\tilde{w}}_t \quad (1.40)$$

where

$$\vec{S_t^\delta} = \begin{bmatrix} S_{1t}^\delta \\ Y_t^\delta \end{bmatrix},$$

$$\vec{\mu}(S_1^\delta, Y^\delta, t) = \begin{bmatrix} \vec{\mu_1}(S_1^\delta, Y^\delta, t) \\ \vec{\mu_2}(S_1^\delta, Y^\delta, t) \end{bmatrix},$$

and

$$\underline{\Sigma}(S^\delta, Y^\delta, t) = \begin{bmatrix} \vec{\sigma_1}(S^\delta, Y^\delta, t)^\top \\ \vec{\sigma_2}(S^\delta, Y^\delta, t)^\top \end{bmatrix}.$$

In fact, we know that

$$\mu_1(S_1^\delta, Y^\delta, t) = rS_{1t}^\delta.$$

Next, we define $\underline{G}_t$ which satisfies a (deterministic) differential equation.

$$d\underline{G}_t = \partial\underline{\mu}^0\underline{G}_t dt \quad (1.41)$$

where

$$\partial\underline{\mu}^\delta = \begin{bmatrix} \partial_1\mu_1^0 & \partial_2\mu_1^0 \\ \partial_1\mu_2^0 & \partial_2\mu_2^0 \end{bmatrix},$$

and

$$\partial_i\mu_j^\delta \equiv \frac{\partial\mu_j^\delta}{\partial S_i^\delta}|_{\delta=0}.$$

We derive the asymptotic expansion of $\vec{S_t^\delta}$. First, $\vec{S_t^0}$ is defined so that this solves the differential equation.

$$d\vec{S_t^0} = \vec{\mu}(S_t^0, Y^0, t)dt \quad (1.42)$$

Next, we define $\vec{A}_t$ as

$$\vec{A}_t = \frac{\partial \vec{S}_t^\delta}{\partial \delta}|_{\delta=0}$$

$\vec{A}_t$ must satisfy the following stochastic differential equation.

$$d\vec{A}_t = \partial \underline{\mu}^0 \vec{A}_t dt + \underline{\Sigma}^0 d\vec{\tilde{w}}_t$$

This can be solved as

$$\vec{A}_t = \underline{G}_t \int_0^t \underline{G}_s^{-1} \underline{\Sigma}^0 d\vec{\tilde{w}}_t. \tag{1.43}$$

Third, we define $\vec{B}_t$ as

$$\vec{B}_t = \frac{\partial^2 \vec{S}_t^\delta}{\partial \delta^2}|_{\delta=0}.$$

We can show that $\vec{B}_t$ must satisfy the following stochastic differential equation.

$$d\vec{B}_t = \left[\sum_{i=1}^2 \sum_{j=1}^2 \partial_i \partial_j \vec{\mu^0} A_{it} A_{jt} + \sum_{i=1}^2 \partial_i \vec{\mu^0} B_{it}\right] dt + 2 \sum_{i=1}^2 \partial_i \underline{\Sigma}^0 A_{it} d\vec{\tilde{w}}_t$$

This can be solved as

$$\vec{B}_t = \int_0^t \underline{G}_t \underline{G}_s^{-1} \left[\sum_{i=1}^2 \sum_{j=1}^2 \partial_i \partial_j \vec{\mu^0} A_{is} A_{js}\right] ds + 2 \int_0^t \underline{G}_t \underline{G}_s^{-1} \left[\sum_{i=1}^2 \partial_i \underline{\Sigma}^0 A_{is}\right] d\vec{\tilde{w}}_s. \tag{1.44}$$

Hence, as in the previous sections, the stochastic expansion of $\vec{S}_t^\delta$ is given by

$$\vec{S}_t^\delta \equiv \vec{S^0} + \delta \vec{g_1} + \delta^2 \vec{g_2} + \cdots \tag{1.45}$$

where

$$\vec{g_1} = \vec{A}_t \text{ and } \vec{g_2} = \frac{1}{2}\vec{B}_t.$$

We observe that

$$\vec{g_1} \sim N(\vec{0}, \underline{\Sigma}_{g_1}) \tag{1.46}$$

where

$$\underline{\Sigma}_{g_1} \equiv \underline{G}_t \int_0^t \left[\underline{G}_s^{-1} \underline{\Sigma}^0 \underline{\Sigma}^{0\top} \underline{G}_s^{-1\top}\right] ds \underline{G}_t^\top.$$

We note that we need only the first element of the asymptotic expansion of $\vec{S}_t^\delta$ in order to obtain the option prices. Once the asymptotic expansion of S_{1t}^δ is obtained explicitly, the same argument holds for evaluating option prices as in the plain vanilla options and so we omit the detail arguments.

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3 The Asymptotic Expansion in the Term Structure Model

In this section, we extend our result to the term structure model of interest rates. The valuation problems in the term structure models are usually considered difficult tasks especially when the processes of state variables have multifactors and state and time dependent volatility functions. We show that the asymptotic expansion method is quite useful even in this situation. In what follows, we consider the general class of arbitrage-free forward rate based model to which we apply asymptotic expansion techinique. The stochastic processes of forward rates considered in the section are fairly general, where the number of factors is a some positive number, N and volatility functions are allowed to depend on the current level of forward rates as well as on the current time and maturities. We note that in this case the continuous stochastic processes for spot rates and interest rates are not not necessarily Markovian in the usual sense. Under this general setting, we shall present the formulae for the values of options on a coupon bond(and swaptions) and options on average interest rates as well as the formulae for the stochastic expansions of (instantaneous) forward rates, spot rates, discount factors and zero coupon bonds.

3.1 The Forward Rates and Spot Rates

In the arbitrage-free forward rate based models, We first consider the processes of an instantaneous forward rates and spot rate. Under the equivalent martingale measure, the process of instantaneous forward rate is described as

$$f^{\varepsilon}(t,T) = f(0,T) + \varepsilon^2 \int_0^t b^{\varepsilon}(v,T)dv + \varepsilon \int_0^t \sum_{i=1}^{N} \sigma_i^{\varepsilon}(v,T)d\tilde{w}_i(v) \tag{1.47}$$

where

$$\sigma_i^{\varepsilon}(v,T) = \sigma_i^{\varepsilon}(v,T,f^{\varepsilon}(t,T)),$$

and

$$b^{\varepsilon}(v,T) = \sum_{i=1}^{N} \sigma_i^{\varepsilon}(v,T) \int_v^T \sigma_i^{\varepsilon}(v,y)dy.$$

In particular, the spot rate process is described as

$$r^{\varepsilon}(t) = f(0,t) + \varepsilon^2 \int_0^t b^{\varepsilon}(v,t)dv + \varepsilon \int_0^t \sum_{i=1}^{N} \sigma_i^{\varepsilon}(v,t)d\tilde{w}_i(v). \tag{1.48}$$

We will derive the asymptotic expansions for those processes. First, we note that

$$\lim_{\varepsilon \to 0} f^{\varepsilon}(v,T) = f(0,T). \tag{1.49}$$

Next, we define $A_t^{(T)}$ and $B_t^{(T)}$ as

$$A_t^{(T)} \equiv \frac{\partial f^{\varepsilon}(t,T)}{\partial \varepsilon}|_{\varepsilon=0},$$

and

$$B_t^{(T)} \equiv \frac{\partial^2 f^{\varepsilon}(t,T)}{\partial \varepsilon^2}|_{\varepsilon=0}.$$

Then, we can show

$$A_t^{(T)} = \int_0^t \sum_{i=1}^{N} \sigma_i^0(v,T) d\tilde{w}_i(v) \tag{1.50}$$

where

$$\sigma_i^0(v,T) \equiv \sigma_i^{\varepsilon}(v,T)|_{\varepsilon=0}.$$

Here, we note that $f^{\varepsilon}(t,T)$ included in the $\sigma_i^{\varepsilon}(v,T)$ is evaluated at $\varepsilon = 0$, that is,

$$f^{\varepsilon}(t,T)|_{\varepsilon=0} = f(0,T).$$

we easily see $A_t^{(T)}$ follows a normal distribution.

$$A_t^{(T)} \sim N(0, \Sigma_{A_t^{(T)}}) \tag{1.51}$$

where

$$\Sigma_{A_t^{(T)}} = \int_0^t \sum_{i=1}^{N} \sigma_i^0(v,T)^2 dv.$$

We can also show that

$$B_t^{(T)} = 2\int_0^t b^0(v,T)dv + 2\int_0^t \sum_{i=0}^{N} A_v^{(T)} \partial\sigma_i^0(v,T) d\tilde{w}_i(v) \tag{1.52}$$

where

$$\partial\sigma_i^0(v,T) \equiv \frac{\partial \sigma_i^{\varepsilon}(v,T)}{\partial f^{\varepsilon}(v,T)}|_{\varepsilon=0}.$$

Therefore, we obtain the asymptotic expansions of the processes of $f^{\varepsilon}(t,T)$ and $r^{\varepsilon}(t)$.

Proposition 1.2 *The asymptotic expansion of an instantaneous forward rate is given by*

$$f^{\varepsilon}(t,T) = f(0,T) + \varepsilon A_t^{(T)} + \varepsilon^2 \frac{1}{2} B_t^{(T)} + o(\varepsilon^2). \tag{1.53}$$

In particular, that of a spot rate is given by

$$r^{\varepsilon}(t) = f(0,t) + \varepsilon A_t^{(t)} + \varepsilon^2 \frac{1}{2} B_t^{(t)} + o(\varepsilon^2). \tag{1.54}$$

3.2 Discount Factor and Zero Coupon Bonds

We will present, in this subsection, the asymptotic expansions of a discount factor and a zero coupon bond which are functionals of spot rates and forward rates respectively, and play an important role in the pricing problems of term structure models. We obtain an asymptotic expansion of the discount factor as

$$
\begin{aligned}
e^{-\int_0^T r(s)ds} &\sim P(0,T)\exp\left[-\varepsilon\int_0^T A_t^{(t)}dt-\varepsilon^2\int_0^T\frac{1}{2}B_t^{(t)}dt\right] \qquad (1.55)\\
&\sim P(0,T)\left[1-\varepsilon\int_0^T A_t^{(t)}dt-\varepsilon^2\int_0^T\frac{1}{2}B_t^{(t)}dt+\varepsilon^2\frac{1}{2}(\int_0^T A_t^{(t)}dt)^2+\cdots\right].
\end{aligned}
$$

First, we note

$$
\int_0^T A_t^{(t)}dt=\int_0^T\sigma_T^0(v)d\tilde{w}(v),\ \ \sigma_T^0(v)\equiv\left[\int_v^T\sigma_i^0(v,t)dt\right]_i \qquad (1.56)
$$

where $\tilde{w}(t)$ denotes N dimentional independent Brownian motion and we note that $\sigma_T(v)$ is a $1\times N$ vector.

We can also show

$$
\begin{aligned}
\int_0^T\frac{1}{2}B_t^{(t)}dt &= k_3(T)+\int_0^T\int_0^t\left[\int_0^s\sigma^0(v,t)d\tilde{w}(v)\right]\partial\sigma^0(s,t)d\tilde{w}(s)dt \qquad (1.57)\\
&\text{where}\\
k_3(T) &= \int_0^T\int_v^T b^0(v,t)dtdv.
\end{aligned}
$$

Similarly, by using the well-known relation,

$$
P(t,T)=\exp\left[-\int_t^T f(t,u)du\right],
$$

we obtain the asymptotic expansion for the value process of zero coupon bond as

$$
\begin{aligned}
P(t,T) &\sim \frac{P(0,T)}{P(0,t)}\exp\left[-\varepsilon\int_t^T A_t^{(u)}du-\varepsilon^2\int_t^T\frac{1}{2}B_t^{(u)}du\right] \qquad (1.58)\\
&\sim \frac{P(0,T)}{P(0,t)}\left[1-\varepsilon\int_t^T A_t^{(u)}du-\varepsilon^2\int_t^T\frac{1}{2}B_t^{(u)}du+\varepsilon^2\frac{1}{2}(\int_t^T A_t^{(u)}du)^2+\cdots\right].
\end{aligned}
$$

We can show

$$
\int_t^T A_t^{(u)}du=\int_0^t\sigma_{tT}^0(v)d\tilde{w}(v),\ \ \sigma_{tT}^0(v)\equiv\left[\int_t^T\sigma_i^0(v,u)du\right]_i \qquad (1.59)
$$

where we note that $\sigma_{tT}(v)$ is a $1 \times N$ vector. We can also show

$$\int_t^T \frac{1}{2} B_t^{(u)} du = k_4(t,T) + \int_t^T \int_0^t \left[\int_0^s \sigma^0(v,u) d\tilde{w}(v)\right] \partial\sigma^0(s,u) d\tilde{w}(s) du \quad (1.60)$$

where

$$k_4(t,T) = \int_0^t \int_t^T b^0(v,u) du dv.$$

Hence, we summarize above results as a proposition.

Proposition 1.3 *The asymtotic expansions of the discount factor and a zero coupon bond are respectively given by*

$$e^{-\int_0^T r(s)ds} = P(0,T)\left[1 - \varepsilon \int_0^T A_t^{(t)} dt - \varepsilon^2 \int_0^T \frac{1}{2} B_t^{(t)} dt + \varepsilon^2 \frac{1}{2} (\int_0^T A_t^{(t)} dt)^2 + \right] + o(\varepsilon^2) \quad (1.61)$$

and

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\left[1 - \varepsilon \int_t^T A_t^{(u)} du - \varepsilon^2 \int_t^T \frac{1}{2} B_t^{(u)} du + \varepsilon^2 \frac{1}{2} (\int_t^T A_t^{(u)} du)^2\right] + o(\varepsilon^2). \quad (1.62)$$

Moreover, we easily obtain the asymptotic expansion of a coupon bond and a swap since thier value processes can be written as linear combinations of those of zero coupon bonds. That is, the value process of a coupon bond or of a swap is expressed in the form of

$$P_{m,\{T_j\},\{c_j\}}(t) = \sum_{j=1}^m c_j P(t,T_j) \quad (1.63)$$

where $t < T_1 < \cdots < T_m$ and c_j are some constants. Hence, the asymptotic expansions of coupon bonds and swaps are just linear combinations of those of zero coupon bonds.

Corollary 1.1 *The asymptotic expansion of a coupon bond or of a swap is a linear combination of those of zero coupon bonds.*

3.3 Options on Coupon Bonds and Swaptions

Based on the results in the previous subsection, we will explicitly derive the formula for options on coupon bonds and swaptions. We formally define the payoffs of options on coupon bonds and swaptions at expiry, T as

$$V(T) = \left[P_{m,\{T_j\},\{c_j\}}(t) - K\right]^+ \tag{1.64}$$

or

$$V(T) = \left[K - P_{m,\{T_j\},\{c_j\}}(t)\right]^+ .$$

Then, we can evaluate the value of the option at $t < T$ by using the martingale technique. That is,

$$V(t) = \mathbf{E}_t^* \left[e^{-\int_t^T r(s)ds} V(T)\right]$$

In what follows, for simplicity, we assume $T < T_1$ and evaluate $V(0)$ for

$$V(T) = \left[P_{m,\{T_j\},\{c_j\}}(T) - K\right]^+ .$$

Based on the *corollary* in the previous subsection, we have the asymptotic expansion of $P_{m,\{T_j\},\{c_j\}}(T)$ as

$$P_{m,\{T_j\},\{c_j\}}(T) \sim \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,T)} \exp\left[-\varepsilon \int_T^{T_j} A_T^{(u)} du - \varepsilon^2 \int_T^{T_j} \frac{1}{2} B_T^{(u)} du\right]. \tag{1.65}$$

Hence, together with the asymptotic expansion of the discount factor appearing in the previous proposition, we can derive the asymptotic expansion of
$e^{-\int_0^T r(s)ds}\left[P_{m,\{T_j\},\{c_j\}}(T) - K\right]$.

$$e^{-\int_0^T r(s)ds}\left[P_{m,\{T_j\},\{c_j\}}(T) - K\right] \sim g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots \tag{1.66}$$

where g_0, g_1, g_2 are defined as follows.

$$g_0 \equiv \sum_{i=1}^{m} c_j P(0,T_j) - KP(0,T). \tag{1.67}$$

$$g_1 \equiv \int_0^T \sigma_{g_1}^*(v) d\tilde{w}(v) \tag{1.68}$$

where

$$\sigma_{g_1}^*(v) = -g_0 \sigma_T^0(v) + \sum_{i=1}^{m} -c_j P(0,T_j) \sigma_{T,T_j}^0(v).$$

We note that g_1 follows a normal distribution.

$$g_1 \sim N(0, \Sigma_{g_1}) \tag{1.69}$$

where

$$\Sigma_{g_1} = \int_0^T \sigma_{g_1}^*(v) \sigma_{g_1}^*(v)^\top dv$$

g_2 is expressed as a rather complicating form.

$$\begin{aligned} g_2 &\equiv \frac{1}{2}g_0\{\int_0^T A_s^{(s)}ds\}^2 + \{\int_0^T A_s^{(s)}ds\}\sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} A_T^{(u)}du\} \\ &+ \frac{1}{2}\sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} A_T^{(u)}du\}^2 - g_0\int_0^T \frac{1}{2}B_s^{(s)}ds \\ &- \sum_{j=1}^m c_j P(0,T_j)\{\int_T^{T_j} \frac{1}{2}B_T^{(u)}du\} \end{aligned} \tag{1.70}$$

We next define X_T^{ε} for which we explicitly obtain the density function.

$$\begin{aligned} X_T^{\varepsilon} &= \frac{e^{-\int_0^T r(s)ds}\left[P_{m,\{T_j\},\{c_j\}}(T) - K\right] - g_0}{\varepsilon} \\ &= g_1 + \varepsilon g_2 + \cdots \end{aligned} \tag{1.71}$$

We already know that g_1 follows a normal distribution. Then, we need to evaluate $\mathbf{E}[g_2|g_1 = x]$ to obtain the density function upto the second order. We can evaluate this conditional expectation by using $formula(1)$ and$formula(2)$ in *Lemma 1.1.*

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{1.72}$$

where

$$\begin{aligned} c &\equiv \frac{1}{2}\frac{g_0}{\Sigma_{g_1}^2}\left[\int_0^T \sigma_T^0(v)\sigma_{g_1}^*(v)^\top dv\right]^2 \\ &+ \frac{1}{\Sigma_{g_1}^2}\left[\int_0^T \sigma_T^0(v)\sigma_{g_1}^*(v)^\top dv\right]\sum_{j=1}^m c_j P(0,T_j)\left[\int_0^T \sigma_{T,T_j}^0(v)\sigma_{g_1}^*(v)^\top dv\right] \\ &+ \frac{1}{2}\frac{1}{\Sigma_{g_1}^2}\sum_{j=1}^m c_j P(0,T_j)\left[\int_0^T \sigma_{T,T_j}^0(v)\sigma_{g_1}^*(v)^\top dv\right]^2 \\ &- \frac{g_0}{\Sigma_{g_1}^2}\left[\int_0^T \int_0^t \sigma_{g_1}^*(s)\partial\sigma^0(s,t)^\top \int_0^s \sigma^0(v,t)\sigma_{g_1}^*(v)^\top dvdsdt\right] \\ &- \frac{1}{\Sigma_{g_1}^2}\sum_{j=1}^m c_j P(0,T_j)\left[\int_T^{T_j}\int_0^T \sigma_{g_1}^*(s)\partial\sigma^0(s,u)^\top \int_0^s \sigma^0(v,u)\sigma_{g_1}^*(v)^\top dvdsdu\right] \end{aligned}$$

and

$$\begin{aligned} f &\equiv -g_0 k_3(T) - \sum_{j=1}^m c_j P(0,T_j)k_4(T,T_j) \\ &- \frac{1}{2}\frac{g_0}{\Sigma_{g_1}}\left[\int_0^T \sigma_T^0(v)\sigma_{g_1}^*(v)^\top dv\right]^2 + \frac{g_0}{2}\left[\int_0^T \sigma_T^0(v)\sigma_T^0(v)^\top dv\right] \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Sigma_{g_1}} \left[\int_0^T \sigma_T^0(v)\sigma_{g_1}^*(v)^\top dv \right] \sum_{j=1}^m c_j P(0,T_j) \left[\int_0^T \sigma_{T,T_j}^0(v)\sigma_{g_1}^*(v)^\top dv \right] \\
&- \frac{1}{2}\frac{1}{\Sigma_{g_1}} \sum_{j=1}^m c_j P(0,T_j) \left[\int_0^T \sigma_{T,T_j}^0(v)\sigma_{g_1}^*(v)^\top dv \right]^2 \\
&+ \sum_{j=1}^m c_j P(0,T_j) \int_0^T \sigma_T^0(v)\sigma_{T,T_j}^0(v)^\top dv \\
&+ \frac{1}{2}\sum_{j=1}^m c_j P(0,T_j) \left[\int_0^T \sigma_{T,T_j}^0(v)\sigma_{T,T_j}^0(v)^\top dv \right] \\
&+ \frac{1}{2}\sum_{j=1}^m c_j P(0,T_j) \left[\int_0^T \sigma_{g_1}^*(v)\sigma_{T,T_j}^0(v)^\top dv \right] \\
&+ \frac{g_0}{\Sigma_{g_1}} \left[\int_0^T \int_0^t \sigma_{g_1}^*(s)\partial\sigma^0(s,t)^\top \int_0^s \sigma^0(v,t)\sigma_{g_1}^*(v)^\top dv ds dt \right] \\
&+ \frac{1}{\Sigma_{g_1}} \sum_{j=1}^m c_j P(0,T_j) \left[\int_T^{T_j} \int_0^T \sigma_{g_1}^*(s)\partial\sigma^0(s,u)^\top \int_0^s \sigma^0(v,u)\sigma_{g_1}^*(v)^\top dv ds du \right].
\end{aligned}
$$

Hence, according to *Theorem 1.2* in the previous section where we replace y by

$$
\begin{aligned}
y &\equiv \frac{1}{\varepsilon} g_0 \\
&= \frac{1}{\varepsilon}\left[\sum_{j=1}^m c_j P(0,T_j) - KP(0,T) \right],
\end{aligned}
$$

we obtain the asymptotic expansion of $V(0)$ which can be used as a valuation formula.

3.4 Options on Average Interest Rates

Our method is general enough to be applied to more complicating pricing problem. As an example, we will derive a pricing formula for options on average interest rates under general forward rate processes. This problem was solved by the **PDE** approach under the constant volatility assumption in Chapter 2 of this dissertation. Our approach allows the problem to be evaluated under more general setting. We first define the payoffs of options on average interest rates. The payoff of the option on an average interest rate is given by

$$
V(T) = (Z(T) - K)^+ \tag{1.73}
$$

or

$$
\begin{aligned}
V(T) &= (K - Z(T))^+ \\
\text{where} & \\
Z(T) &= \frac{1}{T}\int_0^T L^\tau(t)dt,
\end{aligned}
$$

that is, $Z(T)$ denoes the average of interest rates $L^\tau(t)$ from *time* 0 to *time* T and K denotes a strike rate. Specifically, $L^\tau(t)$ represents the yield of a zero coupon bond at time t with the time to maturity of τ years.

$$
L^\tau(t) = \left[\frac{1}{P(t,t+\tau)} - 1\right]\frac{1}{\tau}
$$

In what follows, we evaluate the initial value of a call option whose payoff is given by

$$
\begin{aligned}
V(T) &= (Z(T) - K)^+ \\
&= \frac{1}{T\tau}\left(\int_0^T \frac{1}{P(t,t+\tau)}dt - k\right)^+
\end{aligned}
$$

where

$$
k = (1 + K\tau)T.
$$

Then, by using martingale technique, we express the initial value of a call option on average interest rates as

$$
V(0) = \frac{1}{T\tau}\mathbf{E}^*\left[e^{-\int_0^T r(s)ds}\left(\int_0^T \frac{1}{P(t,t+\tau)}dt - k\right)^+\right]. \tag{1.74}
$$

To evaluate $V(0)$ explicitly, we first expand

$$
e^{-\int_0^T r(s)ds}\int_0^T \frac{1}{P(t,t+\tau)}dt
$$

as

$$
\begin{aligned}
e^{-\int_0^T r(s)ds}\int_0^T \frac{1}{P(t,t+\tau)}dt &\sim P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)} \\
&\times \left[1 + \varepsilon\left\{\int_t^{t+\tau} A_t^{(u)}du - \int_0^T A_s^{(s)}ds\right\}\right. \\
&+ \varepsilon^2\frac{1}{2}\left\{\int_t^{t+\tau} A_t^{(u)}du - \int_0^T A_s^{(s)}ds\right\}^2 \\
&+ \left.\varepsilon^2\left\{\int_t^{t+\tau} \frac{1}{2}B_t^{(u)}du - \int_0^T \frac{1}{2}B_s^{(s)}ds\right\}\right] dt.
\end{aligned}
$$

Hence, together with the asymptotic expansion of the discount function, we obtain

$$e^{-\int_0^T r(s)ds}\left[\int_0^T \frac{1}{P(t,t+\tau)}dt - k\right] \sim g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots. \tag{1.75}$$

g_0, g_1 and g_2 are given by

$$g_0 = P(0,T)\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right], \tag{1.76}$$

$$g_1 = \int_0^T \sigma^0_{T\tau k}(v)d\tilde{w}(v) \tag{1.77}$$

and

$$\begin{aligned} g_2 &= \frac{1}{2}P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \sigma^0_{t\tau}(v)d\tilde{w}(v)\right]^2 dt \qquad (1.78)\\ &- P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \sigma^0_{t\tau}(v)d\tilde{w}(v)\right]\left[\int_0^T \sigma^0_T(v)d\tilde{w}(v)\right]dt \\ &+ \frac{1}{2}g_0\left[\int_0^t \sigma^0_T(v)d\tilde{w}(v)\right]^2 dt \\ &+ P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_t^{t+\tau} \frac{1}{2}B_t^{(u)}du\right]dt - g_0\left[\int_0^T \frac{1}{2}B_t^{(t)}dt\right] \end{aligned}$$

where

$$\sigma^0_{t\tau} = \left[\int_t^{t+\tau} \sigma^0_i(v,u)du\right]_i,$$
$$\sigma^0_T = \left[\int_v^T \sigma^0_i(v,u)du\right]_i,$$

and

$$\sigma^0_{T\tau k}(v) = P(0,T)\int_v^T \frac{P(0,t)}{P(0,t+\tau)}\sigma^0_{t\tau}(v)dt - P(0,T)\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right]\sigma^0_T(v).$$

Next, We define X^ε_T as

$$\begin{aligned} X^\varepsilon(T) &= \frac{e^{-\int_0^T r(s)ds}\left[\int_0^T \frac{1}{P(t,t+\tau)}dt - k\right] - g_0}{\varepsilon} \qquad (1.79)\\ &\sim g_1 + \varepsilon g_2 + \cdots. \end{aligned}$$

We will evaluate $\mathbf{E}[g_2|g_1 = x]$ to obtain the the density function upto the second order. We can evaluate the conditional expectation by applying $formula(1)$ and $formula(2)$ in *Lemma 1.1.*

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{1.80}$$

where

$$\begin{aligned}
c &\equiv \frac{1}{2}\frac{1}{\Sigma_{g_1}^2} P(0,T) \int_0^T \frac{P(0,t)}{P(0,t+\tau)} \left[\int_0^t \sigma_{t\tau}^0(v)\sigma_{T\tau k}^0(v)^\top dv\right]^2 dt \\
&- P(0,T)\frac{1}{\Sigma_{g_1}^2}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)} \int_0^t \sigma_{t\tau}^0(v)\sigma_{T\tau k}^0(v)^\top dvdt\right]\left[\int_0^T \sigma_T^0(v)\sigma_{T\tau k}^0(v)^\top dv\right] \\
&+ \frac{1}{2}\frac{1}{\Sigma_{g_1}^2} g_0 \left[\int_0^t \sigma_T^0(v)\sigma_{T\tau k}^0(v)^\top dv\right]^2 \\
&+ \frac{P(0,T)}{\Sigma_{g_1}^2}\int_0^T \frac{P(0,t)}{P(0,t+\tau)} \times \\
&\quad \int_t^{t+\tau}\int_0^t \sigma_{T\tau k}^0(s)\partial\sigma^0(s,u)^\top \int_0^s \sigma^0(v,u)\sigma_{T\tau k}^0(v)^\top dvdsdudt \\
&- \frac{g_0}{\Sigma_{g_1}^2}\int_0^T\int_0^t \sigma_{T\tau k}^0(s)\partial\sigma^0(s,t)^\top \int_0^s \sigma^0(v,t)\sigma_{T\tau k}^0(v)^\top dvdsdt,
\end{aligned}$$

and

$$\begin{aligned}
f &\equiv -\frac{1}{2}P(0,T)\frac{1}{\Sigma_{g_1}}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \sigma_{t\tau}^0(v)\sigma_{T\tau k}^0(v)^\top dv\right]^2 dt \\
&+ \frac{1}{2}P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\int_0^t \sigma_{t\tau}^0(v)\sigma_{t\tau}^0(v)^\top dvdt \\
&+ P(0,T)\frac{1}{\Sigma_{g_1}}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \sigma_{t\tau}^0(v)\sigma_{T\tau k}^0(v)^\top dvdt\right]\left[\int_0^T \sigma_T^0(v)\sigma_{T\tau k}^0(v)^\top dv\right] \\
&- P(0,T)\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\int_0^t \sigma_{t\tau}^0(v)\sigma_T^0(v)^\top dvdt \\
&- \frac{1}{2}\frac{1}{\Sigma_{g_1}} g_0\left[\int_0^t \sigma_T^0(v)\sigma_{T\tau k}^0(v)^\top dv\right]^2 \\
&+ \frac{1}{2}g_0\left[\int_0^t \sigma_T^0(v)\sigma_T^0(v)^\top dv\right] \\
&+ P(0,T)\left[\int_0^T k_4(t,t+\tau)\frac{P(0,t)}{P(0,t+\tau)}dt\right] - g_0 k_3(T) \\
&- \frac{P(0,T)}{\Sigma_{g_1}}\int_0^T \frac{P(0,t)}{P(0,t+\tau)} \times \\
&\quad \int_t^{t+\tau}\int_0^t \sigma_{T\tau k}^0(s)\partial\sigma^0(s,u)^\top \int_0^s \sigma^0(v,u)\sigma_{T\tau k}^0(v)^\top dvdsdudt \\
&+ \frac{g_0}{\Sigma_{g_1}}\int_0^T\int_0^t \sigma_{T\tau k}^0(s)\partial\sigma^0(s,t)^\top \int_0^s \sigma^0(v,t)\sigma_{T\tau k}^0(v)^\top dvdsdt.
\end{aligned}$$

Finally, according to *Theorem 1.2* where we replace y by

$$\begin{aligned} y &\equiv \frac{1}{\varepsilon} g_0 \\ &= \frac{1}{\varepsilon} P(0,T) \left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)} dt - k \right], \end{aligned}$$

we obtain the asymptotic expansion of $V(0)$ which is used as a pricing formula of a call option on the average interest rate.

4 Average Options in a Stochastic Interest Rate Economy

In this section, we will show that our method may be applied to the pricing problem in a multi-currency economy combined with a term structure model. As an example, we will derive a pricing formula of the call option on average foreign exchange rates in a stochastic interest rate economy. Clearly, we can apply our technique to the other cross-currency or multi-currency contingent claims such as "differential swaps" and options on "differential swaps". In what follows, we consider a cross-currency (J-currency and U-currency) economy which is represented by foreign exchange rate process, J-currency's instantaneous forward rate processes and U-currency's instantaneous forward rate processes. Under the equivalent martingale measure (of J-currency denominated world), those processes are given by

$$S^\varepsilon(t) = S(0) + \int_0^t \{r_J^\varepsilon(s) - r_U^\varepsilon(s)\} S^\varepsilon(s) ds + \varepsilon \int_0^t \sum_1^N \sigma_{iS}^\varepsilon(s) d\tilde{w}_i(s), \tag{1.81}$$

$$f_J^\varepsilon(t,T) = f_J(0,T) + \varepsilon^2 \int_0^t b_J^\varepsilon(v,T) dv + \varepsilon \int_0^t \sum_{i=1}^N \sigma_{iJ}^\varepsilon(v,T) d\tilde{w}_i(v), \tag{1.82}$$

and

$$f_U^\varepsilon(t,T) = f_U(0,T) + \varepsilon^2 \int_0^t b_U^\varepsilon(v,T) dv + \varepsilon \int_0^t \sum_{i=1}^N \sigma_{iU}^\varepsilon(v,T) d\tilde{w}_i(v) \tag{1.83}$$

where

$$\begin{aligned} \sigma_{iS}^\varepsilon(s) &\equiv \sigma_{iS}(s, S^\varepsilon(s)), \\ \sigma_{iJ}^\varepsilon(v,T) &\equiv \sigma_{iJ}(v,T,f_J^\varepsilon(v,T)), \end{aligned}$$

$$
\begin{aligned}
\sigma^{\varepsilon}_{iU}(v,T) &\equiv \sigma_{iU}(v,T,f^{\varepsilon}_U(v,T)),\\
b^{\varepsilon}_J(v,T) &= \sum_{i=1}^N \sigma^{\varepsilon}_{iJ}(v,T)\int_v^T \sigma^{\varepsilon}_{iJ}(v,y)dy\\
&\text{and}\\
b^{\varepsilon}_U(v,T) &= \sum_{i=1}^N \sigma^{\varepsilon}_{iU}(v,T)\{\int_v^T \sigma^{\varepsilon}_{iU}(v,y)dy - \frac{\sigma^{\varepsilon}_{iS}(v)}{S^{\varepsilon}(v)}\}.
\end{aligned}
$$

In particular, both currencies' spot rate processes are given by

$$
r^{\varepsilon}_J(t) = f_J(0,t) + \varepsilon^2 \int_0^t b^{\varepsilon}_J(v,t)dv + \varepsilon \int_0^t \sum_{i=1}^N \sigma^{\varepsilon}_{iJ}(v,t)d\tilde{w}_i(v) \tag{1.84}
$$

and

$$
r^{\varepsilon}_U(t) = f_U(0,t) + \varepsilon^2 \int_0^t b^{\varepsilon}_U(v,t)dv + \varepsilon \int_0^t \sum_{i=1}^N \sigma^{\varepsilon}_{iU}(v,t)d\tilde{w}_i(v). \tag{1.85}
$$

Hence, the foreign exchange rate process is rewritten as

$$
\begin{aligned}
S^{\varepsilon}(t) &= S(0) + \int_0^t \{f_J(0,s) - f_U(0,s)\}S^{\varepsilon}(s)ds\\
&+ \varepsilon^2 \int_0^t \int_0^s \{b^{\varepsilon}_J(v,s) - b^{\varepsilon}_U(v,s)\}dvS^{\varepsilon}(s)ds\\
&+ \varepsilon \int_0^t \sum_{i=1}^N \left[\int_v^t \{\sigma^{\varepsilon}_{iJ}(v,s) - \sigma^{\varepsilon}_{iU}(v,s)\}S^{\varepsilon}(s)ds + \sigma^{\varepsilon}_{iS}(v)\right] d\tilde{w}_i(v).
\end{aligned}
$$

We first need to evaluate explicitly

$$
S^0(t) \equiv \lim_{\varepsilon\to 0} S^{\varepsilon}(t),
$$

$$
A(t) \equiv \frac{\partial S^{\varepsilon}(t)}{\partial \varepsilon}|_{\varepsilon=0}
$$

and

$$
B(t) \equiv \frac{\partial^2 S^{\varepsilon}(t)}{\partial \varepsilon^2}|_{\varepsilon=0}
$$

to obtain the asymptotic expansion of average-rate of foreign exchange rates. We can easily evaluate $S^0(t)$ as

$$
S^0(t) = S(0) + \int_0^t \{f_J(0,s) - f_U(0,s)\}S^0(t)ds.
$$

Thus, we obtain

$$
S^0(t) = S(0)\frac{P_U(0,t)}{P_J(0,t)}. \tag{1.86}
$$

Next, we can show that $A(t)$ satisfies the following stochastic differential equation.

$$dA(t) = \{f_J(0,t) - f_U(0,t)\}A(t)dt + \sum_{i=1}^{N} \sigma_{iS}^0(t)d\tilde{w}_i(t)$$

This can be solved as

$$A(t) = \int_0^t \sum_{i=1}^{N} \frac{F_U(v,t)}{F_J(v,t)} \sigma_{iS}^0(v)d\tilde{w}_i(v) \equiv \int_0^t \sigma_{A_t}(v)d\tilde{w}(v), \tag{1.87}$$

where

$$\sigma_{A_t}(v) \equiv \left[\frac{F_U(v,t)}{F_J(v,t)}\sigma_{iS}^0(v)\right]_i ,$$

$$F_U(v,t) \equiv \frac{P_U(0,t)}{P_U(0,v)}$$

and

$$F_J(v,t) \equiv \frac{P_J(0,t)}{P_J(0,v)}.$$

We can also show that $B(t)$ satisfies a stochastic differencial equation,

$$\begin{aligned} dB(t) &= \{f_J(0,t) - f_U(0,t)\}B(t)dt \\ &+ 2\left[\int_0^t \{b_J^0(v,t) - b_U^0(v,t)\}dvS^0(t)\right]dt + 2\sum_{i=1}^{N} \partial\sigma_{iS}^0(t)A(t)d\tilde{w}_i(t) \end{aligned}$$

where

$$\partial\sigma_{iS}^0(t) \equiv \frac{\partial\sigma_{iS}^\varepsilon}{\partial S^\varepsilon}|_{\varepsilon=0}.$$

This can be solved as

$$\begin{aligned} B(t) &= 2\int_0^t \left[\frac{F_U(s,t)}{F_J(s,t)}S^0(s)\right][\int_0^s \{b_J^0(v,s) - b_U^0(v,s)\}dv]ds \\ &+ 2\int_0^t \sum_{i=1}^{N} \frac{F_U(s,t)}{F_J(s,t)} A(s)\partial\sigma_{iS}^0(s)d\tilde{w}_i(s). \end{aligned} \tag{1.88}$$

Next, we define ths average-rate of foreign exchange rates $Z^\varepsilon(T)$ as

$$Z^\varepsilon(T) = \frac{1}{T}\int_0^T S^\varepsilon(t)dt.$$

Therefore, the asymptotic expansion of the average-rate of forign exchange rates is given by

$$Z^\varepsilon(T) \sim \frac{1}{T}\int_0^T S^0(t)dt + \varepsilon\frac{1}{T}\int_0^T A(t)dt + \varepsilon^2\frac{1}{T}\int_0^T \frac{1}{2}B(t)dt + \cdots. \tag{1.89}$$

We note that the first term of the right hand side is easily evaluated by $S^0(t)$ obtained above. Next, we can evaluate the second term as

$$\int_0^T A(t)dt = \int_0^T \sigma_{A_{s,T}}(s)d\tilde{w}(s), \tag{1.90}$$

where $d\tilde{w}(s)$ is a $1 \times N$ vector and $\sigma_{A_{s,T}}(s)$ is a $1 \times N$ vector

$$\sigma_{A_{s,T}}(s) \equiv \left[\{ \int_s^T \frac{F_U(s,t)}{F_J(s,t)} dt \} \sigma_{iS}^0(s) \right]_i .$$

Third, we can show

$$\int_0^T \frac{B(t)}{2} dt = k_1(T) + \int_0^T \int_0^t \left[\int_0^s \sigma_{A_s}(v) d\tilde{w}(v) \right] \sigma_{B_{s,t}}(s) d\tilde{w}(s) dt, \tag{1.91}$$

where

$$k_1(T) = \int_0^T \int_0^t \frac{F_U(s,t)}{F_J(s,t)} S^0(s) \int_0^s \{ b_J^0(v,s) - b_U^0(v,s) \} dv ds dt$$

and

$$\sigma_{B_{s,t}}(s) \equiv \left[\frac{F_U(s,t)}{F_J(s,t)} \partial \sigma_{iS}^0(s) \right]_i .$$

We already know from the previous section that the asymptotic expansion of the discount factor is given by

$$e^{-\int_0^T r_J(s)ds} \sim P_J(0,T) \left[1 - \varepsilon \int_0^T A_t^{(t)} dt - \varepsilon^2 \int_0^T \frac{1}{2} B_t^{(t)} dt + \varepsilon^2 \frac{1}{2} (\int_0^T A_t^{(t)} dt)^2 \right].$$

We note that

$$\int_0^T A_t^{(t)} dt = \int_0^T \sigma_{JT}^0(v) d\tilde{w}(v) \tag{1.92}$$

where $\sigma_{JT}(v)$ is a $1 \times N$ vector,

$$\sigma_{JT}^0(v) \equiv \left[\int_v^T \sigma_{iJ}^0(v,t) dt \right]_i ,$$

and

$$\int_0^T \frac{1}{2} B_t^{(t)} dt = k_2(T) + \int_0^T \int_0^t \left[\int_0^s \sigma_J^0(v,T) d\tilde{w}(v) \right] \partial \sigma_J^0(s,t) d\tilde{w}(s) dt \tag{1.93}$$

where

$$k_2(T) = \int_0^T \int_v^T b_J^0(v,t) dt dv.$$

Therefore, we combine the asymptotic expansions of the average-rate of foreign exchange rates and that of the discount factor, which leads to

$$e^{-\int_0^T r_J(s)ds}(Z(T)-K) \sim g_0 + \varepsilon g_1 + \varepsilon^2 g_2, \tag{1.94}$$

that is useful to evaluate the average option. g_0, g_1 and g_2 are defined respectively as

$$g_0 \equiv \left[\frac{S(0)}{T}\int_0^T \frac{P_U(0,t)}{P_J(0,t)}dt - K\right] P_J(0,T), \tag{1.95}$$

$$g_1 \equiv \int_0^T \sigma^*_{g_1}(s)d\tilde{w}(s) \tag{1.96}$$

where

$$\sigma^*_{g_1}(s) \equiv \frac{1}{T}P_J(0,T)\sigma_{A_{s,T}}(s) - g_0\sigma^0_{JT}(s).$$

and

$$\begin{aligned} g_2 &= -g_0\int_0^T \frac{1}{2}B^{(t)}_t dt + \frac{P_J(0,T)}{T}\int_0^T \frac{1}{2}B(t)dt \\ &+ g_0\frac{1}{2}\left[\int_0^T A^{(t)}_t dt\right]^2 - \frac{P_J(0,T)}{T}\left[\int_0^T A(t)dt\right]\left[\int_0^T A^{(t)}_t dt\right]. \end{aligned} \tag{1.97}$$

We easily see that g_1 follows a normal distribution.

$$g_1 \sim N(0,\Sigma_{g_1}) \tag{1.98}$$

where

$$\Sigma_{g_1} \equiv \int_0^T \sigma^*_{g_1}(s)\sigma^*_{g_1}(s)^\top ds.$$

As in the previous sections, we define X^ε_T as

$$X^\varepsilon_T = \frac{e^{-\int_0^T r_J(s)ds}(Z(T)-K) - g_0}{\varepsilon} \sim g_1 + \varepsilon g_2 + \cdots \tag{1.99}$$

for which we obtain the density function.

Next, to obtain the density function of X^ε_T , we need to calculate $\mathbf{E}[g_2|g_1 = x]$. Then, we can show ,by using $formula(1)$ and $formula(2)$ in *Lemma 1.1* again, that

$$\mathbf{E}[g_2|g_1 = x] = cx^2 + f \tag{1.100}$$

where

$$
\begin{aligned}
c &\equiv \frac{1}{2}\frac{g_0}{\Sigma_{g_1}^2}\left[\int_0^T \sigma_{JT}^0(v)\sigma_{g_1}^*(v)^\top dv\right]^2 \\
&- \frac{1}{T}\frac{P_J(0,T)}{\Sigma_{g_1}^2}\left[\int_0^T \sigma_{A_{v,T}}(v)\sigma_{g_1}^*(v)^\top dv\right]\left[\int_0^T \sigma_{JT}^0(v)\sigma_{g_1}^*(v)^\top dv\right] \\
&- \frac{g_0}{\Sigma_{g_1}^2}\left[\int_0^T\int_0^t \sigma_{g_1}^*(s)\partial\sigma^0(s,T)^\top \int_0^s \sigma_J^0(v,T)\sigma_{g_1}^*(v)^\top dvdsdt\right] \\
&+ \frac{P_J(0,T)}{T\Sigma_{g_1}^2}\left[\int_0^T\int_0^t \sigma_{g_1}^*(s)\sigma_{B_{s,t}}(s)^\top \int_0^s \sigma_{A_s}(v)\sigma_{g_1}^*(v)^\top dvdsdt\right],
\end{aligned}
$$

and

$$
\begin{aligned}
f &\equiv \frac{1}{T}P_J(0,T)k_1(T) - g_0k_2(T) \\
&- \frac{1}{2}\frac{g_0}{\Sigma_{g_1}}\left[\int_0^T \sigma_{JT}^0(v)\sigma_{g_1}^*(v)^\top dv\right]^2 + \frac{g_0}{2}\left[\int_0^T \sigma_{JT}^0(v)\sigma_{JT}^0(v)^\top dv\right] \\
&+ \frac{1}{T}\frac{P_J(0,T)}{\Sigma_{g_1}}\left[\int_0^T \sigma_{A_{v,T}}(v)\sigma_{g_1}^*(v)^\top dv\right]\left[\int_0^T \sigma_{JT}^0(v)\sigma_{g_1}^*(v)^\top dv\right] \\
&- \frac{P_J(0,T)}{T}\left[\int_0^T \sigma_{A_{v,T}}(v)\sigma_{JT}^0(v)^\top dv\right] \\
&+ \frac{g_0}{\Sigma_{g_1}}\left[\int_0^T\int_0^t \sigma_{g_1}^*(s)\partial\sigma^0(s,T)^\top \int_0^s \sigma_J^0(v,T)\sigma_{g_1}^*(v)^\top dvdsdt\right] \\
&- \frac{P_J(0,T)}{T\Sigma_{g_1}}\left[\int_0^T\int_0^t \sigma_{g_1}^*(s)\sigma_{B_{s,t}}^\top \int_0^s \sigma_{A_s}(v)\sigma_{g_1}^*(v)^\top dvdsdt\right].
\end{aligned}
$$

Clearly, by using the martingale technique, the initial value of the average call option is expressed as

$$
V(0) = \mathbf{E}^*\left[e^{-\int_0^T r(s)ds}(Z(T)-K)^+\right].
$$

By using the the density funcion of X_T^ε and applying *Theorem 1.2* where y is replaced by

$$
\begin{aligned}
y &\equiv \frac{1}{\varepsilon}g_0 \\
&= \frac{1}{\varepsilon}\left[\frac{S(0)}{T}\int_0^T \frac{P_U(0,t)}{P_J(0,t)}dt - K\right]P_J(0,T),
\end{aligned}
$$

we can evaluate the expectation on the right hand side and obtain the pricing formula.

5 Numerical Examples

In this section, we will present several numerical results by applying our method introduced in the previous sections. In the simple Black-Scholes' economy, first we will show the numerical examples of plain vanilla call options for the square-root process of the underlying asset. Next, we will give numerical values of average call options for the square-root process of the underlying asset as well as for the log-normal process of the underlying asset which is commonly used in practice. That is, under the equivalent martingale measure, the processes of the underlying asset are given by

$$dS^{\delta} = (r - q)S^{\delta}dt + \delta(S^{\delta})^{\frac{1}{2}}d\tilde{w}_t$$

or

$$dS^{\delta} = (r - q)S^{\delta}dt + \delta S^{\delta}d\tilde{w}_t$$

where r and q denote the risk-free interest rate and a dividend yield respectively, both of which are assumed to be positive constants in the simple Black-Scholes' economy ,and $\tilde{w}_t$ denotes the one dimensional Brownian motion.

In the term structure model, we show the numerical results of call options on average interest rates under the constant volatility assumption for the instantaneous forward rates, which is examined in Chapter 2 of this dissertation by the **PDE** appoach. That is, the process of instantaneous forward rates under the equivalent martingale measure is given by

$$df^{\varepsilon}(t;T) = \varepsilon^2(T - t)dt + \varepsilon d\tilde{w}_t.$$

We easily see that the process of zero coupon bond follows a log-normal process in this case.

Table 1.1-1.3 show the numerical values of plain vanilla call options for square-root processes of the underlying asset which represents an equity index with no dividend. The values obtained by the stochastic expansions upto the first and second order are given respectively. For comparative purpose, the values by the Monte Carlo simulations are also given, where 500,000 trials are implemented in each case. We note that all the "difference" or "difference rate" appearing in Tables 1.1-1.17 are those from, or those relative to, the corresponding values by the Monte

Carlo simulations. The spot prices and the risk-free interest rate are assumed to be 40.00 and 5 % respectively, and the term to expiry is assumed to be one year. The volatilities(δ) are set so that the instantaneous variances at time 0 are equivalent to those of the log-normal process whose volatilities are 10 % in Table 1.1, 20 % in Table 1.2, and 30 % in Table 1.3. The values of out-of-the money (strike price K= 45), at-the-money (K=40), and in-the-money(K=35) are given. We observe that the values obtained by the stochastic expansions upto the higher order are improved.

Tables 4-10 show the numerical values of average call options when the underlying assets follow square-root processes, where the underlying asset is an equity index with no dividend (that is, $q = 0$) in Tables 1.4-1.6 and in Table 1.7-1.10, that is the foreign exchange rate of Japanese yen and US dollar (that is, q is a US Interest rate). The results given by the stochastic expansion are those from the computation upto the second order. For comparative purpose, the values by the Monte Carlo simulations are also shown, where 500,000 trials are implemented in each case.

In Tables 1.4-1.6, the spot prices and the risk-free interest rate are assumed to be 40.00 and 5 % respectively, and the volatilities(δ) are set so that the instantaneous variances at time 0 are equivalent to those of log-normal process where the volatilities are 30 %. The vales of out-of-the money (strike price K= 45), at-the-money (K=40), and in-the-money(K=35) are shown for each of the time to maturities, three months, six months and one year.

In Tables 1.7-1.10, the spot prices, the risk-free Japanese interest rate and the US interest rate are assumed to be 100.00, 3 %, and 5 % respectively. The volatilities(δ) are set so that the instantaneous variances at time 0 are equivalent to those of the log-normal process where the volatilities are 10 % in Tables 1.7-1.9, and 30 % in Table 1.10. The values of out-of-the money (K= 105 for Tables 1.7-1.9 and K=110 for Table 1.10), at-the-money (K=100 for Tables 1.7-1.10), and in-the-money(K=95 for Tables 1.7-1.9 and K=90 for Table 1.10) are shown for each of the time to maturities, three months, six months and one year.

Tables 1.11-1.14 show the numerical values of average call options when the underlying assets follow log-normal processes, where the underlying asset is the foreign exchange rate of Japanese yen and US dollar. The assumptions for the spot prices, the risk-free Japanese and US interest rates are same as in Tables 1.7-1.10. The volatilities are assumed to be 10 % in Tables 1.11-1.13, and 30 % in Table 1.14. The vales of out-of-the money (K= 105 for Tables 1.11-1.13 and K=110 for Table 1.14), at-the-money (K=100 for Tables 1.11-1.14), and in-the-money(K=95 for Tables 1.11-1.13 and K=90 for Table 1.14) are shown for each of the time to maturities, three months, six months and one year. The results given by the asymptotic expansion are those from the computation upto the second order as well as from computation upto the first order. The results from the asymptotic log-normal expansion upto the first order and the second orderare also shown. We observe that the values from the asymptotic expansion upto the second order are much more improved than those upto the first order for both our original method and asymptotic log-normal expansion. Figure 1.1 and Figure 1.2 show the difference of the distributions of the $\frac{X_T^\delta}{\Sigma_{g_1}^{0.5}}$ and $\frac{g^\delta}{\Sigma_{g_1}^{0.5}}$ respectively, obtained by the asymptotic expansions from those obtained by the Monte Carlo simulations. The asymptotic expansion in Figure 1.1 is given by our original method and that in Figure 1.2 is given by the asymptotic log-normal expansion. We can observe that the difference is significantly smaller in the asymptotic expansions upto the second order than those upto the first order, which leads to the much improved values of the option prices. We also note that, in the first order expansion, the log-normal expansion gives the closer distribution (and hence gives better values of the options prices) than the original method does while the second order expansion in our original method adjusts most of the difference.

For comparative purpose, the values by the Monte Carlo simulations are shown, where 500,000 trials are implemented in each case, and moreover, the values obtained by the **PDE** method developed in Chapter 2 are given.

Finally, Tables 1.15-1.17 show the numerical values of call options on average interest rates in the constant volatility model of the instantaneous forward rates where the time to maturity of the underlying interest rates is one year. That is, the average is taken over interest rates whose maturities are one year. For simplicity, the term structure is assumed to be flat of 5 % per year. Moreover, the volatilities of instantaneous forward rates are assumed to be 150 basis point per year (that is, $\varepsilon = 0.015$) which is a reasonable level in practice. The vales of out-of-the money

(K= 5.5 % for Table 1.15 and K=6 % for Tables 1.16-1.17), at-the-money (K=5 % for Tables 1.15-1.17), and in-the-money(K=4.5 % for Table 1.15 and K=4 % for Tables 1.16-1.17) are given for each of the time to expiries, three months, six months and one year. Again, for comparative purpose, the values by the Monte Carlo simulations are shown, where 500,000 trials are implemented in each case, and the values from the **PDE** method in Chapter 2 are also shown.

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6 The Validity of the Asymptotic Expansion Approach

The validity of the asymptotic expansion approach in this paper could be obtained along the line based on the remarkable work by Watanabe(1987) on the Malliavin calculus in stochastic analysis. Yoshida(1992) has also shown some useful results on the validity of the asymptotic expansions of some functionals on continuous time homogenous diffusion processes. In this section, We shall rigorously prove the validity of the asymptotic expansions both in the Black-Scholes' economy and in the term structure model and show that our simple inversion technique for the characteristic functions of random variables is justified. The validity of our method will be given by applying the results and method originally developed by Watanabe(1987) and Yoshida(1992) to the some stochastic differential equations. In fact, it is obtained by the similar arguments used by Chapter V of Ikeda and Watanabe(1989) and Yoshida(1992). However, we should mention that those existing asymptotic expansion methods in stochastic analysis and statistics have been developed for the case of continuous time homogenous diffusion processes. Hence, we need to substantially modify and extend their results mainly because the processes we encounter in the Black-Scholes' economy include the continuous diffusion processes which are not necessarily time homogenous, and in the term structure model, the continuous stochastic processes for spot interest rates and instantaneous forward rates are not necessarily Markovian in the usual sense.

In the first subsection, we shall prepare the fundamental results and notations in the Malliavin calculus and the definitions of the asymtotic expansions of the Wiener functionals. Based on the results and the theorems, we shall give in the subsequent subsections the proofs of the vailidty of the asymptotic expansions both in the Black-Scholes' economy and in the term structure model. We shall only discuss the validity of the asymptotic expansion approach based on the one-dimensional Wiener space without loss of generality. We only need more complicated notations in the general case.

6.1 Preliminary

We shall first prepare the fundamental results including *Theorem 2.2* of Yoshida(1992), the truncated version of *Theorem 2.3* of Watanabe(1987) which is the key result to show the vailidity of the asymptotic expansions in this paper. For this purpose, we shall freely use the notations by Ikeda and Watanabe (1989) as a standard text-book. The results in this subsection are given without any proofs. The interested readers may see Watanabe(1984), Watanabe(1987), Ikeda and Watanabe(1989) and Yoshida(1992) for details.

The Fundamental Results of Malliavin Calculus

- First, we formally define Wiener space (W, H, μ).
  - Let W a Banach space be the totality of continuous function

    $$w : [0, T] \to R; w(0) = 0$$

    with the topology induced by countable system of norms

    $$\| w \|_n = max_{0 \le t \le n} |w(t)|; n = 1, 2, \cdots .$$

  - Let μ Wiener measure.
  - H denotes the Cameron-Martin subspace of W, a Hilbert space. That is, $h(t) \in H$ is in W and is absolutely continuous on $[0, T]$ with square integrable derivative $\dot{h}(t)$ endowed with an inner product defined by

    $$< h_1, h_2 >_H = \int_0^T \dot{h}_1(s) \dot{h}_2(s) ds.$$

- A function $f : W \mapsto R$ is called a polynomial functional if there exist an $n \in N$, $h_1, h_2, \cdots, h_n \in H$ and a real polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that

  $$f(w) = p([h_1](w), [h_2](w), \cdots, [h_n](w))$$

  where $h_i \in H$ and

  $$[h_i](w) = \int_0^T \dot{h}_s d\tilde{w}_s.$$

- $\mathcal{P}$ denotes the totality of all polynomial functions.
  - We note $\mathcal{P}$ is dense in L_2.
- Wiener-Chaos decomposition of L_2 is defined by

$$L_2 = C_0 \oplus C_1 \oplus C_2 \cdots .$$

  - Note that $\mathcal{P}_n$ is dense in C_n where n denotes the order of polynomial functions.
- Let F Wiener functional $W \to R$.
- We define $J_n(F)$ as a projection of F to C_n, then

$$F = \sum_{n=0}^{\infty} J_n(F).$$

- The Ornstein-Uhlenbeck semigroup $\{T_t\}_t$ is defined by

$$T_t(F) = \sum_{n=0}^{\infty} e^{-tn} J_n(F).$$

- The Ornstein-Uhlenbeck operator L is defined by

$$L(F) = \sum_{n=0}^{\infty} (-n) J_n(F)$$

  and we can show

$$L(F) = \frac{d}{dt}|_{t=0} T_t(L).$$

- The norm $\| \cdot \|_{p,s}, s \in R, p \in (1, \infty)$ is defined by

$$\| F \|_{p,s} = \| (I - L)^{\frac{s}{2}} F \|_p .$$

- H derivative of F in the direction of $h \in H$ is denoted by

$$D_h F(w) = \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} F(w + \varepsilon h)$$

  Then, H derivative of F is defined by

$$D_h F(w) = < DF, h >_H .$$

- Hence we can show

$$L(F(w)) = D^2F(W) - [DF](w).$$

- *Meyer(1983)* shows the equivalence between the norm $\| \cdot \|_{p,s}$ and L_p norm of H derivative. That is, for $p \in (1, \infty)$ and $s \in \{1, 2, \cdots\}$, there exists constants $c_{p,s}$ and $C_{p,s}$ such that

$$c_{p,s} \parallel D^s F \parallel_p \leq \parallel F \parallel_{p,s} \leq C_{p,s} \sum_{l=0}^{s} \parallel D^l F \parallel_p .$$

Due to this result, we can identify the $\| \cdot \|_{p,s}$ finiteness of a Wiener functional.

- The Sobolev space of Wiener functionals is defined by making use of the norm $\| \cdot \|_{p,s}$ as follows.

$\boldsymbol{D}_p^s$: the completion of $\mathcal{P}$ with respect to $\| \cdot \|_{p,s}$ which is a Banach space

$\boldsymbol{D}_q^{-s}$: the dual of $\boldsymbol{D}_p^s$ where $s \in R$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$\boldsymbol{D}^{\infty} = \cap_{s>0} \cap_{1<p<\infty} \boldsymbol{D}_p^s$
(the set of Wiener test functionals)

$\tilde{\boldsymbol{D}}^{-\infty} = \cup_{s>0} \cap_{1<p<\infty} \boldsymbol{D}_p^{-s}$
(the set of generalized Wiener functionals)

- The relation among those spaces are given. For $1 \leq p < p1$ and $0 < s < s1$,

$$\begin{aligned} \boldsymbol{D}_p^{s1} \subset \boldsymbol{D}_p^s \subset \boldsymbol{D}_p^0 &= L_p \subset \boldsymbol{D}_p^{-s} \subset \boldsymbol{D}_p^{-s1} \\ \boldsymbol{D}_{p1}^{s1} \subset \boldsymbol{D}_{p1}^s \subset \boldsymbol{D}_{p1}^0 &= L_{p1} \subset \boldsymbol{D}_{p1}^{-s} \subset \boldsymbol{D}_{p1}^{-s1}. \end{aligned}$$

- The Malliavin-covariance of is defined by F $\sigma(F) =< DF, DF >_H$.

The Asymptotic Expansion

Next, we rigorously define the asymptotic expansion of Wiener functional $X^{\delta}(w), \delta \in (0, 1)$.

- For $k > 0$, $X^\delta(w) = O(\delta^k)$ in $\boldsymbol{D}_p^s$ as $\delta \downarrow 0$ means that

$$limsup_{\delta\downarrow 0}\frac{\| X^\delta \|_{p,s}}{\delta^k} < \infty.$$

- If, for all $p > 1, s > 0$ and every $k = 1, 2, \cdots$,

$$X^\delta(w) - (g_1 + \delta g_2 + \cdots + \delta^{k-1} g_k) = O(\delta^k)$$

  in $\boldsymbol{D}_p^s$ as $\delta \downarrow 0$,
  then we can say that $X^\delta(w)$ has the asymptotic expansion :

$$X^\delta(w) \sim g_1 + \delta g_2 + \cdots$$

  in $\boldsymbol{D}^\infty$ as $\delta \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^\infty$.

- If, for every $k = 1, 2, \cdots$, there exists $s > 0$
  such that, for all $p > 1$, $X^\delta(w), g_1, g_2, \cdots \in \tilde{\boldsymbol{D}}_p^{-s}$ and

$$X^\delta(w) - (g_1 + \delta g_2 + \cdots + \delta^{k-1} g_k) = O(\delta^k)$$

  in $\boldsymbol{D}_p^{-s}$ as $\delta \downarrow 0$,
  then we can say that $X^\delta(w) \in \tilde{\boldsymbol{D}}^{-\infty}$ has the asymptotic expansion :

$$X^\delta(w) \sim g_1 + \delta g_2 + \cdots$$

  in $\boldsymbol{D}^{-\infty}$ as $\delta \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^{-\infty}$.

Finally, we state a simple version of *Theorem 2.2* of Yoshida(1992) which is a truncated version of *Theorem 2.3* of Watanabe(1987). The validity of the asymptotic expansion in this paper is obtained by showing that the conditions of this theorem are met.

Theorem 1.4 *Yoshida(1992)*
Let $\psi(x)$ *be a smooth function such that* $0 \le \psi(x) \le 1$ *for* $x \in R$,$\psi(x) = 1$ *for* $|x| \le 1/2$ *and* $\psi = 0$ *for* $|x| \ge 1$. *Suppose a set of conditions given below is satisfied.*

Conditions

1. $X^{\delta}(w) \in \boldsymbol{D}^{\infty}$

2. $X^{\delta}(w)$ *has the asymptotic expansion:*

$$X^{\delta}(w) \sim g_1 + \delta g_2 + \cdots$$

in $\boldsymbol{D}^{\infty}$ *as* $\delta \downarrow 0$ *with* $g_1, g_2, \cdots \in \boldsymbol{D}^{\infty}$.

3. $\{\eta_c^{\delta}(w); \varepsilon \in (0,1]\}$ *is* $O(1)$ *in* $\boldsymbol{D}^{\infty}$ *as* $\delta \downarrow 0$ *where* $c > 0$.

4. *There exists* $c_0 > 0$ *such that for* $c > c_0$ *and any* $p > 1$,

$$sup_{\delta \in (0,1)} \mathbf{E}[1_{\{|\eta_c^{\delta}| \leq 1\}} \left(det\ \sigma(X^{\delta})\right)^{-p}] < \infty.$$

That is, the Malliavin covariance of $X^{\delta}(w)$ *is uniformly non-degenerate.*

5. *For any* $n \geq 1$,

$$\lim_{\delta \to 0} \delta^{-n} P\{|\eta_c^{\delta}| > \frac{1}{2}\} = 0 \ .$$

6. $\phi^{\delta}(x)$ *is a smooth function in* (x, δ) *on* $R \times [0,1]$ *with all derivatives of polynomial growth order in* x *uniformly in* δ.

Then, $\psi(\eta_c^{\delta})\phi^{\delta}(X^{\delta})I_{\mathcal{B}}(X^{\delta})$ *has the asymptotic expansion:*

$$\psi(\eta_c^{\delta})\phi^{\delta}(X^{\delta})I_{\mathcal{B}}(X^{\delta}) \sim \Phi_0 + \delta\Phi_1 + \cdots$$

in $\tilde{\boldsymbol{D}}^{-\infty}$ *as* $\delta \downarrow 0$ *where* $\mathcal{B}$ *is a Borel set and* $\Phi_0, \Phi_1, \cdots$ *are determined by the formal Taylor expansion.*

6.2 The Validity in the Black-Scholes' Economy

Now we give the proof of validity of our method. For ease of exposition, we consider a one dimensional stochastic differentail equation. The validity of the multidimentional case could be obtained by almost the same argument although it is more tedious. We begin with the definition of a stochastic differential equation. That is, for the fixed $T < 0$ and $\delta \in (0,1)$,

$$S_T^\delta = S_0 + \int_0^T \mu(S_s^\delta, s)ds + \int_0^T \delta\sigma(S_s^\delta, s)d\tilde{w}_s \tag{1.101}$$

where $\mu(S_s^\delta, s)$ and $\sigma(S_s^\delta, s)$ are $R \times [0,T] \to R$ and Borel measurable in (S^δ, s). Moreover, they are $\boldsymbol{C}^\infty(R \to R)$ for $s \in [0,T]$ with bounded derivatives of any orders in the first arguments. That is, for the first arguments there exists $M > 0$ such that

$$sup_{0\leq s\leq T}|\frac{\partial^k \mu(S_s, s)}{\partial S^k}| < M \tag{1.102}$$

and

$$sup_{0\leq s\leq T}|\frac{\partial^k \sigma(S_s, s)}{\partial S^k}| < M$$

for any $k = 1,2,3,\cdots$. These conditions imply that there exists some positve K such that for all $s \in [0,T]$,

$$\begin{aligned} |\mu(S^\delta, s)| &< K(1 + |S_s^\delta|) \\ |\sigma(S^\delta, s)| &< K(1 + |S_s^\delta|) \end{aligned} \tag{1.103}$$

or

$$\begin{aligned} |\mu(S1^\delta, s) - \mu(S2^\delta, s)| &< K|S1^\delta - S2^\delta| \\ |\sigma(S1^\delta, s) - \sigma(S2^\delta, s)| &< K|S1^\delta - S2^\delta|. \end{aligned} \tag{1.104}$$

Hence the standard argument(e.g. Ikeda and Watanabe(1989)) shows the existence of the unique strong solution which has continuous sample paths and is in L_p for any $1 \leq p < \infty$. In the remaining of the section, we will discuss the validity of the asymptotic expansion of $\phi(X_T^\delta)I_{\mathcal{B}}(X_T^\delta)$ where X_T^δ is defined by $X_T^\delta = \frac{S_T^\delta - S_T^0}{\delta}$ and $\mathcal{B}$ is a Borel set. We will also discuss the validity of the asymptotic expansion of $\phi(Z^\delta)I_{\mathcal{B}}(Z^\delta)$. Z^δ is defined by

$$Z^\delta = \frac{1}{\delta}\int_0^T f(S_s^\delta)ds \tag{1.105}$$

where $f(x)$ is a $\boldsymbol{C}^\infty(R \to R)$ function. First, we show the S_T^δ is a smooth Wiener functional in the sense of Malliavin.

Lemma 1.3 S_T^δ *is in* $\boldsymbol{D}^\infty$ *and has the asymptotic expansion*

$$S_T^\delta \sim S_T^0 + \delta g_{1T} + \delta^2 g_{2T} + \cdots \tag{1.106}$$

is in $\boldsymbol{D}^\infty$ *as* $\delta \downarrow 0$ *with* $g_{1T}, g_{2T}, \cdots \in \boldsymbol{D}^\infty$.

Proof. First, we prove S_T^δ in $\boldsymbol{D}^\infty$. We define Y^δ by

$$dY^\delta = \partial\mu(S^\delta, t)Y^\delta dt + \delta\partial\sigma(S^\delta, t)Y^\delta dw_t, Y_0^\delta = 1$$

where $\partial\mu$ and $\partial\sigma$ denotes the $\frac{\partial\mu}{\partial S^\delta}$ and $\frac{\partial\sigma}{\partial S^\delta}$ respectively. We easily see Y^δ has the unique strong solution and $Y^\delta \in L_p$. Let $W_t^\delta = Y_t^{(\delta)-1}$. Then, W_t^δ satisfies the stochastic differential equation

$$dW^\delta = -\{\partial\mu(S^\delta, t) - \delta^2\partial\sigma(S^\delta, t)^2\}W^\delta dt - \delta\partial\sigma(S^\delta, t)W^\delta dw_t, W_0^\delta = 1.$$

W_t^δ has also the unique strong solution and $Y^{(\delta)-1} \in L_p$.

In the first step, we calculate the first order H-derivative of S_T^δ. For any $h \in H$, we note that $D_h S_T^\delta$ satisfies

$$D_h S_T^\delta = \int_0^T \delta\partial\sigma(S^\delta, s)D_h S_s^\delta dw(s) + \int_0^T \partial\mu(S^\delta, s)D_h S_s^\delta ds + \int_0^T \delta\sigma(S^\delta, s)\dot{h}_s ds.$$

Thus for $h \in H$,

$$D_h S_T^\delta = \int_0^T Y_T^\delta Y_s^{(\delta)-1}\delta\sigma(S^\delta, s)\dot{h}_s ds.$$

We note

$$|D_h S_T^\delta| \le \delta|Y_T^\delta| \left[\int_0^T |Y_s^{(\delta)-1}|^2 K^2(1 + |S_s^\delta|)^2 ds\right]^{\frac{1}{2}} \left[\int_0^T |\dot{h}_s|^2 ds\right]^{\frac{1}{2}}.$$

Then,

$$\mathbf{E}\left[|D_h S_T^\delta|^2\right] \le \delta^2 \mathbf{E}\left[|Y_T^\delta|^2\{\int_0^T |Y_s^{(\delta)-1}|^2 K^2(1 + |S_s^\delta|)^2 ds\}\right] M_h^2$$

where

$$M_h = \left[\int_0^T |\dot{h}_s|^2 ds\right]^{\frac{1}{2}} < \infty.$$

Likewise for any $2 < p < \infty$, we can show

$$\mathbf{E}\left[|D_h S_T^\delta|^p\right] \le (\delta K M_h)^p T^{(\frac{p-2}{p})}\mathbf{E}\left[|Y_T^\delta|^p\{\int_0^T |Y^{\delta-1}|^p(1 + |S_s^\delta|)^p ds\}\right].$$

By using Hölder inequality for expectations:

$$\mathbf{E}[|x_s y_s|] \leq \mathbf{E}[|x_s|^p]^{\frac{1}{p}}\mathbf{E}[|y_s|^q]^{\frac{1}{q}}$$

where $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, the inequality,

$$(|x| + |y|)^p \leq 2^{(p-1)}(|x|^p + |y|^p)$$

for $p \geq 1$ and Fubini's theorem, we can evaluate the right hand side of the last equation as

$$\begin{aligned}
& \mathbf{E}\left[|Y_T^\delta|^p\{\int_0^T |Y^{(\delta)-1}|^p(1+|S_s^\delta|)^p ds\}\right] \\
\leq\ & \mathbf{E}\left[|Y_T^\delta|^{2p}\right]^{\frac{1}{2}}\mathbf{E}\left[(\int_0^T \{|Y^{(\delta)-1}|(1+|S_s^\delta|)\}^p ds)^2\}\right]^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^\delta|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}}\mathbf{E}\left[\int_0^T |Y^{(\delta)-1}|^{2p}(1+|S_s^\delta|)^{2p} ds\right]^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^\delta|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}}\{\int_0^T \mathbf{E}\left[|Y^{(\delta)-1}|^{4p}\right]^{\frac{1}{2}}\mathbf{E}\left[(1+|S_s^\delta|)^{4p}\right]^{\frac{1}{2}} ds\}^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^\delta|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}}\{\int_0^T \mathbf{E}\left[\{|Y^{(\delta)-1}|^{4p}\right]^{\frac{1}{2}}\mathbf{E}\left[2^{(4p-1)}(1+|S_s^\delta|^{4p})\right]^{\frac{1}{2}}\}^{\frac{1}{2}}.
\end{aligned}$$

Hence, by $S_s^\delta, Y_s^\delta, Y_s^{(\delta)-1} \in L_p$ for $s \in [0,T]$ and any $1 < p < \infty$, we have $\mathbf{E}\left[|D_h S_T^\delta|^p\right] < \infty$ for any $p > 1$. Therefore, we conclude $S_T^\delta \in \cap_{1<p<\infty}\boldsymbol{D}_p^1$.

As for the second order H-derivative, $D^2_{h_1,h_2}S_T^\delta$ satisfies a stochastic integral equation.

$$\begin{aligned}
D^2_{h_1,h_2}S_T^\delta =\ & \int_0^T \delta\partial\sigma(S^\delta,s)D^2_{h_1,h_2}S_s^\delta dw_s + \int_0^T \partial\mu(S^\delta,s)D^2_{h_1,h_2}S_s^\delta ds \\
+\ & \left[\int_0^T \delta\partial^2\sigma(S^\delta,s)D_{h_1}S_s^\delta D_{h_2}S_s^\delta dw_s \right.\\
+\ & \left.\int_0^T \partial^2\mu(S^\delta,s)D_{h_1}S_s^\delta D_{h_2}S_s^\delta ds + \int_0^T \delta\partial\sigma(S^\delta,s)D_{h_1}S_s^\delta h_{2s} ds\right].
\end{aligned}$$

Then, we can obtain the second order H-derivative by

$$\begin{aligned}
D^2_{h_1,h_2}S_T^\delta =\ & \int_0^T Y_T^\delta Y_s^{\delta-1}[\partial^2\mu(S^\delta,s)D_{h1}S_s^\delta D_{h_2}S_s^\delta ds \\
+\ & \delta\partial^2\sigma(S^\delta,s)D_{h_1}S_s^\delta D_{h_2}S_s^\delta dw_s + \delta\partial\sigma(S^\delta,s)D_{h_1}S_s^\delta \dot{h}_{2s} ds].
\end{aligned}$$

To show the L_p-boundedness of the second order H-derivative, we first note for any $p > 1$,

$$\begin{aligned} |D^2_{h_1,h_2} S^\delta_T|^p &\leq 3^{p-1} | \int_0^T Y^\delta_T Y^{\delta-1}_s \partial^2 \mu(S^\delta, s) D_{h1} S^\delta_s D_{h_2} S^\delta_s ds|^p \\ &+ | \int_0^T Y^\delta_T Y^{\delta-1}_s |\delta \partial^2 \sigma(S^\delta, s) D_{h_1} S^\delta_s D_{h_2} S^\delta_s dw_s|^p \\ &+ | \int_0^T Y^\delta_T Y^{\delta-1}_s \delta \partial \sigma(S^\delta, s) D_{h_1} S^\delta_s \dot{h}_{2s} ds]|^p. \end{aligned}$$

By the boundedness of $\partial^2 \mu(S^\delta, s)$ and L_p-boundedness of $Y^\delta_T, Y^{\delta-1}_s, D_{h1} S^\delta_s$, and $D_{h_2} S^\delta_s$, the similar argument as before shows the L_p-boundedness of the first term in the last equation. It is also easily seen

$$| \int_0^T Y^\delta_T Y^{\delta-1}_s \delta \partial \sigma(S^\delta, s) D_{h_1} S^\delta_s \dot{h}_{2s} ds]|^p \leq \left[\int_0^T |Y^\delta_T Y^{\delta-1}_s \delta \partial \sigma(S^\delta, s) D_{h_1} S^\delta_s|^2 ds \right]^{\frac{p}{2}} M^p_{h_2}$$

where

$$M_{h_2} = \left[\int_0^T |\dot{h}_{2s}|^2 ds \right]^{\frac{1}{2}}.$$

Then, again we can show L_p-boundedness of the last term in the last equation by the boundedness of $\partial \sigma(S^\delta, s)$ and L_p-boundedness of $Y^\delta_T, Y^{(\delta)-1}_s$, and $D_{h1} S^\delta_s$.

As for the second term, we first note

$$\begin{aligned} &\mathbf{E} \left[| \int_0^T Y^\delta_T Y^{\delta-1}_s \delta \partial^2 \sigma(S^\delta, s) D_{h_1} S^\delta_s D_{h_2} S^\delta_s dw_s|^p \right] \\ \leq\ &\mathbf{E} \left[|Y^\delta_T|^{2p} \right]^{\frac{1}{2}} \mathbf{E} \left[| \int_0^T Y^{\delta-1}_s \delta \partial^2 \sigma(S^\delta, s) D_{h_1} S^\delta_s D_{h_2} S^\delta_s dw_s|^{2p} \right]^{\frac{1}{2}}. \end{aligned}$$

Clearly, the first part is bounded for any $p > 1$. As for the second part, we first state the following well-known (local) martingale inequality(e.g. *Theorem III-3.1* of Ikeda and Watanabe (1989)). There exists a positive constant $c_p, 0 < p < \infty$ such that for every square integrable continuous local martingale M_s,

$$\mathbf{E}[(max_{0 \leq s \leq t} |M_s|)^{2p}] \leq c_p \mathbf{E}[< M, M >^p_t].$$

Then, by using this inequality, we evaluate the second part as

$$\mathbf{E} \left[| \int_0^T Y^{(\delta)-1}_s \delta \partial^2 \sigma(S^\delta, s) D_{h_1} S^\delta_s D_{h_2} S^\delta_s dw_s|^{2p} \right]^{\frac{1}{2}}$$

$$\leq \ (c_p)^{\frac{1}{2}}\mathbf{E}\left[\{\int_0^T \left[Y_s^{(\delta)-1}\delta\partial^2\sigma(S^\delta,s)D_{h_1}S_s^\delta D_{h_2}S_s^\delta\right]^2 ds\}^p\right]^{\frac{1}{2}}$$

$$\leq \ (c_p)^{\frac{1}{2}}T^{\frac{p-2}{2}}\mathbf{E}\left[\int_0^T \left[Y_s^{(\delta)-1}\delta\partial^2\sigma(S^\delta,s)D_{h_1}S_s^\delta D_{h_2}S_s^\delta\right]^{2p} ds\right]^{\frac{1}{2}}.$$

Hence, we can see this part is also bounded for any $p > 1$ by the boundedness of $\partial\sigma(S^\delta, s)$ and L_p-boundedness of $Y_s^{(\delta)-1}$,$D_{h_1}S_s^\delta$ and $D_{h_2}S_s^\delta$. Therefore, we have $S_T^\delta \in \cap_{1<p<\infty}\boldsymbol{D}_p^2$.
Repeating this argument, we can show the boundedness of higher order H-derivatives with L_p estimates of S_T^δ. Finally, we conclude $S^\delta \in \boldsymbol{D}^\infty$.

Next, we prove the second part of this lemma. The coefficients of the asymptotic expansion of S_T^δ is given by the Taylor formula. For instance,

$$g_{1T} = \int_0^T Y_T Y_s^{-1}\sigma(S^0,s)ds$$

$$g_{2T} = \int_0^T \frac{1}{2}Y_T Y_s^{-1}\{\partial^2\mu(S^0,s)g_{1s}^2 ds + \sigma(S^0,s)g_{1s}dw_s\}$$

and

$$g_{3T} = \int_0^T Y_T Y_s^{-1}\{\partial^2\mu(S^0,s)g_{1s}g_{2s}ds + \frac{1}{6}\partial^3\mu(S^0,s)g_{1s}^3 dw_s + \partial\sigma(S^0,s)g_{2s}dw_s\}$$

where $Y_t = Y_t^0$ is the solution of the differential equation

$$dY = \partial\mu(S^0,t)Ydt, Y_0 = 1.$$

That is, $Y_t = exp(\int_0^t \mu(S^0,s)ds)$. By the boundedness of $Y_T, Y_s^{-1}, \sigma(S^0,s)$ on $[0,T]$, it is easily seen $\mathbf{E}[|g_{1s}|^p] < \infty, s \in [0,T]$ for any $1 < p < \infty$. Given $g_{1s} \in L_p$, we can easily see by the (local) martingale inequality $\mathbf{E}[|g_{2s}|^p] < \infty$ for any $1 < p < \infty$. Likewise, $g_{ks} \in L_p$ is obtained recursively given $g_{js} \in L_p, j = 1,2,\cdots k-1$. Hence $g_{1T}, g_{2T},\cdots \in \cap_{1<p<\infty}\boldsymbol{D}_p^0$. Next, we note $D_h g_{1T} = Y_T\int_0^T Y_s^{-1}\sigma(S^0,s)\dot{h}_s ds$ and $D_{h_1,\cdots,h_k}^k g_1 = 0$ for $k = 2,3,\cdots$. Thus, clearly, $g_{1T} \in \boldsymbol{D}^\infty$. We also have

$$D_h g_{2T} = Y_T\int_0^T Y_s^{-1}\partial^2\mu(S^0,s)g_{1s}D_h g_{1s}ds + \int_0^T \partial\sigma(S_s^0,s)D_h g_{1s}dw_s + \int_0^T \partial\sigma(S^0,s)\dot{h}_s ds,$$

$$D_{h_1,h_2}^2 g_{2T} = \int_0^T Y_T Y_s^{-1}\partial^2\mu(S^0,s)D_{h_1}g_{1s}D_{h_2}g_{1s}ds + \int_0^T \partial\sigma(S^0,s)D_{h_1}g_{1s}\dot{h}_{2s}ds.$$

and $D_{h_1,\cdots,h_k}^k g_{2T} = 0 fork = 3,4,\cdots$. Then, given $g_{1s} \in \boldsymbol{D}^\infty$ for any $s \in [0,T]$, we can easily conclude $g_{2T} \in \boldsymbol{D}^\infty$.

Recursively we can show the L_p-boundedness of any order H-derivatives of $g_{kT}, k = 3, 4, \cdots$. Therefore, we have proven the second part. ▮

Next, we define X_T^δ by $X_T^\delta = \frac{S_T^\delta - S_T^0}{\delta}$. By the above lemma, we easily see X_T^δ is in $\boldsymbol{D}^\infty$ and has the asymptotic expansion

$$X_T^\delta \sim g_{1T} + \delta g_{2T} + \cdots$$

is in $\boldsymbol{D}^\infty$ as $\delta \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^\infty$. We also have the first order H-derivative as

$$D_h X_T = \int_0^T Y_T^\delta Y_s^{\delta-1} \sigma(S^\delta, s) \dot{h}_s ds.$$

Then, the Malliavin covariance $\sigma(X_T^\delta) =< DX_T^\delta, DX_T^\delta >_H$ is given by

$$\int_0^T \{Y_T^\delta Y_s^\delta \sigma(S^\delta, s)\}^2 ds. \tag{1.107}$$

Note

$$\sigma(X_T^\delta) \to \Sigma_{g_1} = \int_0^T \{Y_T Y_s \sigma(S^0, s)\}^2 ds \tag{1.108}$$

as $\delta \downarrow 0$ where Σ_{g_1} denotes the variance of g_1. Next, we consider the uniform non-degeneracy of Malliavin covarince, which is the important step of the application of *Theorem 2.2* of Yoshida(1992). First, we make the following assumption.

Assumption 1

$$\Sigma_{g_1} = \int_0^T \{Y_T Y_s \sigma(S^0, s)\}^2 ds > 0 \tag{1.109}$$

Next, we define η_c^δ by for any $c > 0$,

$$\eta_c^\delta = c \int_0^T |Y_T^\delta (Y_s^\delta)^{-1} \sigma(S_s^\delta) - Y_T Y_s^{-1} \sigma(S_s^0)|^2 ds.$$

Hence, we have the following lemma.

Lemma 1.4 *Under Assumption 1, Malliavin covariance $\sigma(X_T^\delta)$ is uniformly non-degenerate. That is, there exists $c_0 > 0$ such that for $c > c_0$ and any $p > 1$,*

$$sup_{\delta \in (0,1]} \mathbf{E}\left[1_{\{\eta_c^\delta \leq 1\}} \{det\ \sigma(X_T^\delta)\}^{-p}\right] < \infty. \tag{1.110}$$

PROOF. Let $\xi^{\delta}_{s,t} = Y^{\delta}_t (Y^{\delta}_s)^{-1} \sigma(S^{\delta}_s)$ and $\xi_{s,t} = Y_t Y_s^{-1} \sigma(S^0_s)$.
Then, $|\eta^{\delta}_c| \leq 1$ implies $\int_0^T |\xi^{\delta}_{s,T} - \xi_{s,T}|^2 ds \leq \frac{2}{c}$.
Note

$$\begin{aligned} |\sigma(X^{\delta}_T) - \Sigma_{g_1}| &= |\int_0^T (\xi^{\delta}_{s,T})^2 - (\xi_{s,T})^2 ds| \\ &\leq \int_0^T |\xi^{\delta}_{s,T} - \xi_{s,T}|^2 ds + 2\int_0^T |\xi_{s,T}||\xi^{\delta}_{s,T} - \xi_{s,T}| ds \\ &\leq \frac{2}{c} + 2\Sigma_{g_1}^{\frac{1}{2}} (\frac{2}{c})^{\frac{1}{2}}. \end{aligned}$$

Hence we can take $c_0 > 0$ such that for $c > c_0$,

$$0 < \Sigma_{g_1} - |\sigma(X^{\delta}_T) - \Sigma_{g_1}| < \sigma(X^{\delta}_T)$$

holds uniformly for $\delta \in (0, 1]$. Thus, (1.110) is concluded. ∎

æ Next, we present two inequalities which are useful to show the truncation by $\psi(\eta^{\delta}_c)$ is negligible in the asymptotic expansion.

Lemma 1.5 • *(A)There exist positive constants $a_i; i = 1, 2$ independent of δ such that*

$$P(sup_{0\leq s\leq T}|S^{\delta}_s - S^0_s| > a_0) \leq \frac{a_1}{a_0}(a_0 + C)exp(-\frac{a_2 a_0^2}{(a_0 + C)^2}\delta^{-2}) \qquad (1.111)$$

for all $a_0 > 0$.

- *(B)There exist positive constants $a_i; i = 1, 2$ independent of δ such that*

$$P(sup_{0\leq s\leq T}|Y^{\delta}_s - Y_s| > a_0) \leq \frac{a_1}{a_0}(a_0 + C)exp(-\frac{a_2 a_0^2}{(a_0 + C)^2}\delta^{-2}) \qquad (1.112)$$

for all $a_0 > 0$.

PROOF.

- (A)

$$S^{\delta}_T = S_0 + \int_0^T \mu(S^{\delta}, s) ds + \int_0^T \sigma(S^{\delta}, s) dw_s$$

$$S^0_T = S_0 + \int_0^T \mu(S^0, s) ds$$

Using the Lipschitz continuity of $\mu(S^{\delta}, t)$ in the first argument,

$$|S^{\delta}_t - S^0_t| \leq K\int_0^t |S^{\delta}_s - S^0_s| ds + sup_{0\leq s\leq t}|\int_0^s \delta\sigma(S^{\delta}, u) dw_u|.$$

Let us recall the useful Gronwall's inequality(e.g. Elliott(1982) pp.192): Suppose $\alpha(t)$ is a Lebesgue integrable function on $[a, b]$, and that B and H are constants such that

$$\alpha(t) \leq B + H \int_a^t \alpha(s) ds$$

for all $t \in [a, b]$. Then, $\alpha(t) \leq Be^{H(t-a)}$.

Then, by the Gronwall's inequality,

$$sup_{0 \leq s \leq T} |S_s^\delta - S_s^0| \leq sup_{0 \leq s \leq T} | \int_0^s \delta\sigma(S^\delta, u) dw_u | e^{KT}.$$

Note that by the method of time change(e.g. Ikeda and Watanabe(1989) pp.197) there exists a Brownian motion $B(t)$ such that $B(A_t) = \int_0^t \delta\sigma(S^\delta, s) dw_s$ where $A_t = \int_0^t \delta^2 \sigma(S^\delta, s)^2 ds$. Then,

$$sup_{0 \leq s \leq T} |S_s^\delta - S_s^0| \leq sup_{0 \leq s \leq T} |B(A_s)| e^{KT}.$$

Let $\tau = inf\{s; |S_s^\delta - S_s^0| > a_0\}$.
Then, noting $\tau < T$ implies $\{sup_{0 \leq s \leq \tau < T} |B(A_s)| e^{KT} > a_0\}$, we have

$$P(\{sup_{0 \leq s \leq T} |S_s^\delta - S_s^0| > a_0\}) = P(\{\tau < T, sup_{0 \leq s \leq \tau} |B(A_s)| e^{KT} > a_0\}).$$

We also note for $s \leq \tau$,

$$|S_s^\delta| - |S_s^0| \leq |S_s^\delta - S_s^0| \leq a_0.$$

Then, we have

$$|S_s^\delta| \leq a_0 + |S_s^0| \leq a_0 + sup_{0 \leq s \leq T} |S_s^0|.$$

Hence,

$$|\sigma(S^\delta, s)| \leq (1 + |S_s^\delta|) \leq (a_0 + C)$$

where $C \equiv 1 + sup_{0 \leq s \leq T} |S_s^0|$. Thus, A_s for $s \in [0, \tau]$ is evaulated as

$$A_s = \int_0^s \delta^2 \sigma(S^\delta, u)^2 du \leq \delta^2 T (a_0 + C)^2.$$

Therefore,

$$\begin{aligned} & P(\{\tau < T, sup_{0 \leq s \leq \tau} |B(A_s)| e^{KT} > a_0\}) \\ \leq \ & P(\{sup_{0 \leq u \leq \delta^2 K^2 T (a_0 + C)^2} |B(u)| e^{KT} > a_0\}). \end{aligned}$$

Using the reflection principle and the inequality

$$\frac{1}{\sqrt{2\pi}}\int_{a_0}^{\infty} e^{\frac{-x^2}{2}}dx < \frac{1}{\sqrt{2\pi}}a_0^{-1}e^{\frac{-a_0^2}{2}},$$

we obtain

$$\begin{aligned} P(\{sup_{0\le s\le T}|S_s^\delta - S_s^0| > a_0\}) &\le 2P(\{MaxB(\delta^2K^2T(a_0+C)^2) > a_0e^{-KT}\}) \\ &= 4P(\{B(\delta^2K^2T(a_0+C)^2) > a_0e^{-KT}\}) \\ &\le \frac{4e^{KT}\delta K\sqrt{T}}{\sqrt{2\pi}}\frac{(a_0+C)}{a_0}\times \\ &\quad exp(-\frac{e^{-2KT}a_0^2}{2K^2T(a_0+C)^2}\delta^{-2}). \end{aligned}$$

Finally, defining appropriately a_1 and a_2, we conclude the result. That is,

$$a_1 = \frac{4e^{KT}\delta K\sqrt{T}}{\sqrt{2\pi}}$$

and

$$a_2 = \frac{e^{-2KT}}{2K^2T}.$$

- (B)

$$dY^\delta = \partial\mu(S^\delta,t)Y^\delta dt + \partial\sigma(S^\delta,t)Y^\delta dw_t$$

$$dY = \partial\mu(S^0,t)Y dt$$

By the smoothness and the boundedness of the derivatives of $\mu(S^\delta,s)$ in the first argument and the boundedness of Y_s on $[0,T]$, there exist positive M and M_1 such that

$$\begin{aligned} |\partial\mu(S^\delta,s)Y_s^\delta - \partial\mu(S^0,s)Y| &\le ||\partial\mu(S^\delta,s)||Y_s^\delta - Y_s| + \\ |Y_s||\partial\mu(S^\delta,s) - \partial\mu(S^0,s)| &\le M|Y_s^\delta - Y_s| + M_1|S_s^\delta - S_s^0|. \end{aligned}$$

Thus, we have

$$\begin{aligned} |Y_s^\delta - Y_s| &\le \left[M_1Tsup_{0\le s\le T}|S_s^\delta - S_s^0| + sup_{0\le s\le T}|\int_0^s \delta\partial\sigma(S^\delta,u)Y_u^\delta dw_u|\right] \\ &+ M\int_0^s |Y_u^\delta - Y_u|du. \end{aligned}$$

Then, by the Gronwall's inequality, there exists a positive M_2 such that

$$sup_{0\le s\le T}|Y_s^\delta - Y_s| \le M_2\left[sup_{0\le s\le T}|S_s^\delta - S_s^0| + sup_{0\le s\le T}|\int_0^s \delta\partial\sigma(S^\delta,u)Y_u^\delta dw_u|\right].$$

Let $\varepsilon = \frac{a_0}{2M_2}$ and $\tau = inf\{s; |S_s^\delta - S_s^0| > \varepsilon$ or $|Y_s^\delta - Y_s| > a_0\}$. Then, we have

$$
\begin{aligned}
& P(\{sup_{0\le s\le T}|Y_s^\delta - Y_s| > a_0\}) \\
= \ & P(\{sup_{0\le s\le T}|Y_s^\delta - Y_s| > a_0,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon\}) \\
+ \ & P(\{sup_{0\le s\le T}|Y_s^\delta - Y_s| > a_0,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| > \varepsilon\}) \\
\le \ & P(\{sup_{0\le s\le T}|Y_s^\delta - Y_s| > a_0,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon\}) \\
+ \ & P(\{sup_{0\le s\le T}|S_s^\delta - S_s^0| > \varepsilon\}) \\
\le \ & P(\{\tau < T,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon, \\
& (a_0 - M_2\varepsilon) \le M_2 sup_{0\le s\le \tau}|\int_0^s \partial\sigma(S^\delta,u)Y_u^\delta dw_u|\}) \\
+ \ & P(\{sup_{0\le s\le T}|S_s^\delta - S_s^0| > \varepsilon\}) \\
= \ & P(\{\tau < T,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon, \\
& \frac{a_0}{2} \le M_2 sup_{0\le s\le \tau}|B(\delta^2\int_0^s \partial\sigma(S^\delta,u)^2 Y_u^{\delta 2}du)|\}) \\
+ \ & P(\{sup_{0\le s\le T}|S_s^\delta - S_s^0| > \varepsilon\})
\end{aligned}
$$

The second term in the last equation is equivalent to (A). Hence, what we have to do is to evaluate the first term. We first note $0 \le s \le \tau$ and $sup_{0\le s\le T}|S_s^\delta - S_s^0|$ implies $|Y_s^\delta - Y_s| \le a_0$ for $s \in [0,\tau]$. Then, for $s \in [0,\tau]$, we have

$$|Y_s^\delta| \le a_0 + C$$

where $C \equiv sup_{0\le s\le T}Y_s$. Together with $|\partial\sigma(S^\delta,u)| \le M$, we can show

$$\delta^2\int_0^s \partial\sigma(S^\delta,u)^2 Y_u^{\delta 2}du \le \delta^2 TM^2(a_0+C)^2.$$

Thus,

$$
\begin{aligned}
& P(\{\tau < T,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon, \\
& \frac{a_0}{2} \le M_2 sup_{0\le s\le \tau}|B(\delta^2\int_0^s \partial\sigma(S^\delta,u)^2 Y_u^{\delta 2}du)|\}) \\
\le \ & P(\{\frac{a_0}{2M_2} \le sup_{[0\le u\le \delta^2 TM^2(a_0+C)^2]}|B(u)|\}).
\end{aligned}
$$

Therefore, repeating the same argument as in (A), we can conclude there exists positive constants $a_{i1}, i = 1,2$ independent of δ such that

$$P(\{\tau < T,\ sup_{0\le s\le T}|S_s^\delta - S_s^0| \le \varepsilon,$$

$$\frac{a_0}{2} \leq M_2 sup_{0\leq s\leq \tau}|B(\delta^2 \int_0^s \partial\sigma(S^\delta,u)^2 Y_u^{\delta 2} du)|\})$$
$$\leq \frac{a_{11}}{a_0}(a_0+C)exp(-\frac{a_{21}a_0^2}{(a_0+C)^2}\delta^{-2}).$$

∎

We now can show the truncation is negligible by utilizing the above large deviation inequalities.

Lemma 1.6 *For* $c>0$, η_c^δ *is* $O(1)$ *in* $\boldsymbol{D}^\infty$ *and for* $c_0>0$, *there exist some constants* $c_i,\ i=1,2,3,$ *such that*

$$P(\{|\eta_c^\delta|>c_0\}) \leq c_1 exp(-c_2\delta^{-c_3}) \tag{1.113}$$

PROOF. the result follows from the inequalities (A) and (B) in the previous lemma. First, we note

$$|\eta_c^\delta| = c\int_0^T |Y_T^\delta Y_s^{(\delta)-1}\sigma(S_s^\delta) - Y_T Y_s^{-1}\sigma(S_s^0)|^2 ds$$
$$\leq cT sup_{0\leq s\leq T}\left[Y_T^\delta Y_s^{(\delta)-1}\sigma(S_s^\delta) - Y_T Y_s^{-1}\sigma(S_s^0)\right]^2.$$

Then, $|\eta_c^\delta|>c_0$ implies

$$\{|\eta_c^\delta|>c_0\} \subset \{sup_{0\leq s\leq T}\left[Y_T^\delta Y_s^{(\delta)-1}\sigma(S_s^\delta) - Y_T Y_s^{-1}\sigma(S_s^0)\right] > (\frac{c_0}{cT})^{\frac{1}{2}}\}$$

Let $c_4 = \frac{c_0}{cT}^{\frac{1}{2}}$.

$$\{|\eta_c^\delta|>c_0\} \subset \{sup_{0\leq s\leq T}|Y_T Y_s^{-1}||\sigma(S^\delta,s)-\sigma(S^0,s)|>\frac{c_4}{3}\}$$
$$\cup \{sup_{0\leq s\leq T}|Y_s^{\delta-1}||\sigma(S_s^\delta,s)||Y_T^\delta - Y_T|>\frac{c_4}{3}\}$$
$$\cup \{sup_{0\leq s\leq T}|Y_T||\sigma(S_s^\delta,s)||Y_s^{\delta-1}-Y_s^{-1}|>\frac{c_4}{3}\}$$

By the boundedness of $|Y_T Y_s^{-1}|$, the Lipschitz continuity of $\sigma(S^\delta,s)$ in the first argument and *Lemma 1.5*, the first term of the right hand side implies that there exist c_{11}, c_{21}, and c_{31} such that

$$P(\{sup_{0\leq s\leq T}|Y_T Y_s^{-1}||\sigma(S^\delta,s)-\sigma(S^0,s)|>\frac{c_4}{3}\})$$
$$\leq P(\{sup_{0\leq s\leq T}|S_s^\delta - S_s^0|>c_5\}) \leq c_{11}exp(-c_{21}\delta^{-c_{31}})$$

where $c_5 = \frac{c_4}{3Ksup_{0\le s\le T}|Y_T Y_s^{-1}|}$. We note that for any $c_{03} > 0$, $|Y_s^{(\delta)-1} - Y_s^{-1}| > c_{03}$ is implied by $|Y^{(\delta)-1}| > max(|c_{03} + Y_s^{-1}|, |c_{03} - Y_s^{-1}|)$. Then, by $Y_s^{-1} > 0$ for $0 \le s \le T$,

$$\{|Y_s^{\delta-1}| > m_1\} \subset \{|Y_s^{\delta-1} - Y_s^{-1}| > c_{03}\}$$

where $m_1 = c_{03} + sup_{0\le s\le T} Y_s^{-1}$. We also note that for any $c_{01} > 0$, $|S_s^\delta - S_s^0| > c_{01}$ is implied by $|S_s^\delta| > max(|c_{01} + S_s^0|, |c_{01} - S_s^0|)$, which is also implied by $|S_s^\delta| > c_{01} + sup_{0\le s\le T}|S_s^0|$. Hence, by $|\sigma(S^\delta, s)| \le M(1 + |S_s^\delta|)$,

$$\{|\sigma(S^\delta, s)| > m_2\} \subset \{|S_s^\delta - S_s^0| > c_{01}\}.$$

where $m_2 = K(1 + c_{01} + sup_{0\le s\le T}|S_s^0|)$. Therefore, as for the second term,

$$\begin{aligned}
& \{sup_{0\le s\le T}|Y_s^{\delta-1}||\sigma(S_s^\delta, s)||Y_T^\delta - Y_T| > \frac{c_4}{3}\} \\
\subset\ & \{|Y_t^\delta - Y_t| > \frac{c_4}{3m_1m_2}\}, sup_{0\le s\le T}|Y^{(\delta)-1}| \le m_1, sup_{0\le s\le T}|\sigma(S^\delta, s)| \le m_2\} \\
\cup\ & \{sup_{0\le s\le T}|Y_s^{\delta-1}||\sigma(S_s^\delta, s)||Y_T^\delta - Y_T| > \frac{c_4}{3}, sup_{0\le s\le T}|Y^{(\delta)-1}| > m_1\} \\
\cup\ & \{sup_{0\le s\le T}|Y_s^{\delta-1}||\sigma(S_s^\delta, s)||Y_T^\delta - Y_T| > \frac{c_4}{3}, sup_{0\le s\le T}|\sigma(S_s^\delta, s)| > m_2\} \\
\subset\ & \{|Y_T^\delta - Y_T| > \frac{c_4}{3m_1m_2}\} \cup \{sup_{0\le s\le T}|Y_s^{\delta-1} - Y_s^{-1}| > c_{03}\} \\
\cup\ & \{sup_{0\le s\le T}|S_s^\delta - S_s^0| > c_{01}\}.
\end{aligned}$$

Then, by *Lemma 1.5* , there exist $c_{12}, c_{22}, c_{32} > 0$ such that

$$P(\{sup_{0\le s\le t}|Y_s^{\delta-1}||\sigma(S_s^\delta, s)||Y_t^\delta - Y_t| > \frac{c_4}{3}\}) < c_{12}exp(-c_{22}\delta^{-c_32}).$$

The similar argemnt holds for the third term. Thus, the result is concluded. ∎

Therefore, the conditions of *Theorem 2.2* of Yoshida(1992) are satisfied, which leads to the desired result.

Proposition 1.4 *For a smooth function $\phi^\delta(x)$ with all derivatives of polynomial growth orders,*
$\psi(\eta_c^\delta)\phi^\delta(X_T^\delta)I_{\mathcal{B}}(X_T^\delta)$ has the asymptotic expansion:

$$\psi(\eta_c^\delta)\phi^\delta(X_T^\delta)I_{\mathcal{B}}(X_T^\delta) \sim \Phi_0 + \delta\Phi_1 + \cdots \tag{1.114}$$

in $\tilde{\boldsymbol{D}}^{-\infty}$ as $\delta \downarrow 0$ where $\mathcal{B}$ is a Borel set, $\psi(x)$ is a smooth function such that $0 \le \psi(x) \le 1$ for $x \in R$, $\psi(x) = 1$ for $|x| \le 1/2$ and $\psi = 0$ for $|x| \ge 1$, and $\Phi_0, \Phi_1, \cdots$ are determined by the formal Taylor expansion.

Finally, we obtain the asymptotic expansion of the expectation of $\phi^\delta(X_T^\delta)I_{\mathcal{B}}(X_T^\delta)$.

Theorem 1.5 *The asymptotic expansion of* $\mathbf{E}[\phi^\delta(X^\delta)I_{\mathcal{B}}(X^\delta)]$ *is given by*

$$\begin{aligned}\mathbf{E}[\phi^\delta(X^\delta)I_{\mathcal{B}}(X^\delta)] &\sim \mathbf{E}[\psi(\eta_c^\delta)\phi^\delta(X^\delta)I_{\mathcal{B}}(X^\delta)] \\ &\sim \mathbf{E}[\Phi_0]+\delta\mathbf{E}[\Phi_1]+\cdots\end{aligned} \tag{1.115}$$

as $\delta\downarrow 0$.

Proof. The first assertion follows from uniform integrability of $\{|\phi^\delta(x)|^p;\delta\in(0,1]\},p\geq 1$ and the *Lemma 1.6*, and the second assertion is obtained by *Proposition 1.4*. ∎

We next consider the validity of the asymptotic expansion of

$$Z^\delta \equiv \frac{Z_T^\delta}{\delta}=\frac{1}{\delta}\int_0^T f(X_s^\delta)ds$$

where $f(x)$ is a smooth function that is, $\boldsymbol{C}^\infty(R\to R)$. Then, the expansion of Z_T^δ is formally given by

$$\begin{aligned}Z_T^\delta &\sim \int_0^T f(S_s^0)ds+\delta\int_0^T \partial f(S_s^0)g_{1s}ds \\ &+ \delta^2\int_0^T\{\frac{1}{2}\partial^2 f(S_s^0)g_{1s}^2+\partial f(S_s^0)g_{2s}\}ds \\ &+ \delta^3\int_0^T\{\frac{1}{6}\partial^3 f(S_s^0)g_{1s}^3+\partial^2 f(S_s^0)g_{1s}g_{2s}+\partial f(S_s^0)g_{3s}\}ds+\cdots \\ &\equiv Z_T^0+\delta g_{1T}^Z+\delta^2 g_{2T}^Z+\delta^3 g_{3T}^Z+\cdots.\end{aligned}$$

By the smoothness of $f(x)$, $X_T^\delta\in\boldsymbol{D}^\infty$ and $g_1,g_2,g_3,\cdots\in\boldsymbol{D}^\infty$, we can easily see $Z_T\in\boldsymbol{D}^\infty$ and Z_T^δ has the asymptotic expansion is in $\boldsymbol{D}^\infty$ as $\delta\downarrow 0$ with $g_{kT}^Z,k=1,2,\cdots$. æ Next, the Malliavin covariance of Z^δ,$\sigma(Z^\delta)$ is given by

$$\sigma(Z^\delta)=\int_0^T\left[\{\int_u^T\partial f(S_s^\delta)Y_s^\delta ds\}Y_u^{\delta-1}\sigma(S_u^\delta,u)\right]^2du.$$

We note $\sigma(Z^\delta)\to\Sigma_{g_{1T}^Z}$ as $\delta\downarrow 0$ where

$$\Sigma_{g_{1T}^Z}=\int_0^T\left[\{\int_u^T\partial f(S_s^0)Y_s ds\}Y_u^{-1}\sigma(S_u^0,u)\right]^2du.$$

If we define $\eta_c^{\delta Z}$ as before by

$$\eta_c^{\delta Z}=c\int_0^T\left[\{\int_u^T\partial f(S_s^\delta)Y_s^\delta ds\}Y_u^{\delta-1}\sigma(S_u^\delta,u)-\{\int_u^T\partial f(S_s^0)Y_s ds\}Y_u^{-1}\sigma(S_u^0,u)\right]^2du,$$

then we can have *Lemma 1.4* and *Lemma 1.6* for $\eta_c^{\delta Z}$ instead of η_c^{δ} in a similar way by making use of *Lemma 1.5* and smoothness of $f(x)$. Consequently, we can apply *Theorem 2.2* of Yoshida(1992) to $\psi(\eta_c^{\delta Z})\phi(Z^\delta)I_{\mathcal{B}}$, and the same results as in *Proposition 1.4* and *Theorem 1.5* hold for Z^δ as follows.

- *Proposition 1.4'*
  For a smooth function $\phi^\delta(x)$ with all derivatives of polynomial growth orders, $\psi(\eta_c^{\delta Z})\phi^\delta(Z^\delta)I_{\mathcal{B}}(Z^\delta)$ has the asymptotic expansion:

$$\psi(\eta_c^{\delta Z})\phi^\delta(Z^\delta)I_{\mathcal{B}}(Z^\delta) \sim \Phi_0 + \delta\Phi_1 + \cdots \tag{1.116}$$

  in $\tilde{D}^{-\infty}$ as $\delta \downarrow 0$ where $\mathcal{B}$ is a Borel set, $\psi(x)$ is a smooth function such that $0 \leq \psi(x) \leq 1$ for $x \in R$,$\psi(x) = 1$ for $|x| \leq 1/2$ and $\psi = 0$ for $|x| \geq 1$, and $\Phi_0, \Phi_1, \cdots$ are determined by the formal Taylor expansion.

- *Theorem 1.5'*
  The asymptotic expansion of $\mathbf{E}[\phi^\delta(Z^\delta)I_{\mathcal{B}}(Z^\delta)]$ is given by

$$\begin{aligned}\mathbf{E}[\phi^\delta(Z^\delta)I_{\mathcal{B}}(Z^\delta)] &\sim \mathbf{E}[\psi(\eta_c^\delta)\phi^\delta(Z^\delta)I_{\mathcal{B}}(Z^\delta)] \\ &\sim \mathbf{E}[\Phi_0] + \delta\mathbf{E}[\Phi_1] + \cdots\end{aligned} \tag{1.117}$$

  as $\delta \downarrow 0$.

Our next objective is to show that the resulting formulae is equivalent to those from our method which is based on the simple inversion technique for the characteristic function. To do so, we only discuss the case of X_T^δ because the same argument holds for the asymptotic expansion of Z^δ. In particular, we explicitly derive the formula of asymptotic distribution function or density function and that of the expectation of X_T^δ in a certain range and show they are equivalent to those by our simple method. We start with the explicit evaluation of the expectation in *Theorem 1.5*. We first observe that in *Proposition 1.4*, Φ_0, Φ_1 *and* Φ_2 are given by

$$\begin{aligned}\Phi_0 &= \phi^0(g_1)I_{\mathcal{B}}(g_1) \\ \Phi_1 &= \left[\frac{\partial\phi^\delta}{\partial\delta}|_{\delta=0}(g_1) + \partial\phi(g_1)g_2\right]I_{\mathcal{B}}(g_1) + \phi^0(g_1)\partial I_{\mathcal{B}}(g_1)g_2\end{aligned}$$

$$
\begin{aligned}
\Phi_2 &= \left[\frac{\partial \phi^\delta}{\partial \delta}|_{\delta=0}(g_1) + \partial\phi^\delta(g_1)g_2\right]\partial I_{\mathcal{B}}(g_1)g_2 \\
&+ \left[\frac{1}{2}\frac{\partial^2\phi^\delta}{\partial\delta^2}|_{\delta=0}(g_1) + \{\frac{\partial^2\phi^\delta(x)}{\partial x\partial\delta}|_{\delta=0,x=g_1}\}g_2 + \partial\phi^0(g_1)g_3 + \frac{1}{2}\partial^2\phi^0(g_1)g_2^2\right] I_{\mathcal{B}}(g_1) \\
&+ \phi^0(g_1)\{\frac{1}{2}\partial^2 I_{\mathcal{B}}(g_1)g_2^2 + \partial I_{\mathcal{B}}g_3\}.
\end{aligned}
$$

Hence we have the follwing proposition.

Proposition 1.5 $\mathbf{E}[\Phi_i], i=0,1,2$ *are given respectively by*

$$
\begin{aligned}
\mathbf{E}[\Phi_0] &= \int_{\mathcal{B}}\phi^0(x)n[x,0,\Sigma_{g_1}]dx, \qquad (1.118)\\
\mathbf{E}[\Phi_1] &= \int_{\mathcal{B}}\{\frac{\partial\phi}{\partial\delta}|_{\delta=0}(x)n[x:0,\Sigma_{g_1}] \\
&+ \phi^0(x)\left[\frac{-\partial\mathbf{E}[g_2|g_1=x]n[x:0,\Sigma_{g_1}]}{\partial x}\right]\}dx
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}[\Phi_2] &= \int_{\mathcal{B}} -\frac{\partial\phi^\delta}{\partial\delta}|_{\delta=0}(x)\frac{\partial}{\partial x}\{\mathbf{E}[g_2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \frac{1}{2}\frac{\partial^2\phi^\delta}{\partial\delta^2}|_{\delta=0}(x)n[x:0,\Sigma_{g_1}] \\
&+ \frac{1}{2}\phi^0(x)\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \phi^0(x)\frac{\partial}{\partial x}\{-\mathbf{E}[g_3|g_1=x]n[x:0,\Sigma_{g_1}]\}.
\end{aligned}
$$

PROOF. It is easily seen the formula for $\mathbf{E}[\Phi_0]$.

The expectation of the first term of Φ_1 is given by

$$
\begin{aligned}
&\mathbf{E}\left[\{\frac{\partial\phi}{\partial\delta}|_{\delta=0}(g_1) + \partial\phi(g_1)g_2\}I_{\mathcal{B}}(g_1)\right] \\
&= \int_{\mathcal{B}}\{\frac{\partial\phi}{\partial\delta}|_{\delta=0}(x) + \partial\phi(x)\mathbf{E}[g_2|g_1=x]\}n[x,0,\Sigma_{g_1}]dx.
\end{aligned}
$$

As for the expectation of $\phi^0(g_1)\partial I_{\mathcal{B}}(g_1)g_2$, by noting $\phi^0(g_1)g_2 \in \boldsymbol{D}^\infty$ and using the integration by parts formula for Wiener functional(see Ikeda and Watanabe(1989) for the detail.), we have

$$
\begin{aligned}
\mathbf{E}\left[\phi^0(g_1)\partial I_{\mathcal{B}}(g_1)g_2\right] &= \mathbf{E}\left[\phi^0(g_1)g_2\partial I_{\mathcal{B}}(g_1)\right] \\
&= \mathbf{E}[G(w)I_{\mathcal{B}}(g_1)] \\
&= \mathbf{E}[\mathbf{E}[G(w)|g_1=x]I_{\mathcal{B}}(g_1)]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathcal{B}} \mathbf{E}[G(w)|g_1 = x] n[x:0,\Sigma_{g_1}]dx \\
&\equiv \int_{\mathcal{B}} p_1(x)dx
\end{aligned}
$$

for some smooth Wiener functional $G(w)$. To obtain $p_1(x)$, taking $\mathcal{B}_x = (-\infty, x]$, we can see

$$
\begin{aligned}
\mathbf{E}\left[\phi^0(g_1)\partial I_{\mathcal{B}_x}(g_1)g_2\right] &= \int_{-\infty}^{\infty} \phi^0(y)E[g_2|g_1 = y]\partial I_{\mathcal{B}_x}(y)n[y:0,\Sigma_{g_1}]dy \\
&= -\int_{-\infty}^{\infty} \phi^0(y)E[g_2|g_1 = y]\delta_x(y)n[y:0,\Sigma_{g_1}]dy \\
&= -\phi^0(x)E[g_2|g_1 = x]n[x:0,\Sigma_{g_1}]
\end{aligned}
$$

where $\delta_x(y)$ denotes the delta function of y at x. Then,

$$
p_1(x) = \frac{\partial}{\partial x}\left[-\phi^0(g_1)\mathbf{E}[g_2|g_1 = x]n[x:0,\Sigma_{g_1}]\right].
$$

Hence, we have the explicit formula for $\mathbf{E}\left[\Phi_1\right]$ as

$$
\mathbf{E}\left[\Phi_1\right] = \int_{\mathcal{B}} \{\frac{\partial \phi}{\partial \delta}|_{\delta=0}(x)n[x:0,\Sigma_{g_1}] + \phi^0(x)\left[\frac{-\partial \mathbf{E}[g_2|g_1 = x]n[x:0,\Sigma_{g_1}]}{\partial x}\right]\}dx.
$$

Similarly, we can write $\mathbf{E}\left[\Phi_2\right]$ as

$$
\begin{aligned}
\mathbf{E}\left[\Phi_2\right] &= \int_{\mathcal{B}} p_2(x)dx \\
&= \int_{\mathcal{B}} p_{21}(x)dx + \int_{\mathcal{B}} p_{22}(x)dx + \int_{\mathcal{B}} p_{23}(x)dx.
\end{aligned}
$$

Then, $p_{21}(x)$ is given in the similar way as in $\mathbf{E}\left[\Phi_1\right]$ by

$$
p_{21}(x) = \frac{\partial}{\partial x}\left[-\{\frac{\partial \phi^\delta}{\partial \delta}|_{\delta=0}(x)\mathbf{E}[g_2|g_1 = x] + \partial\phi(x)\mathbf{E}[g_2^2|g_1 = x]\}n[x:0,\Sigma_{g_1}]\right].
$$

$p_{22}(x)$ is also easily obtained by

$$
\begin{aligned}
p_{22}(x) &= \left[\frac{1}{2}\frac{\partial^2 \phi^\delta}{\partial \delta^2}|_{\delta=0}(x) + \{\frac{\partial^2 \phi^\delta(x)}{\partial x \partial \delta}|_{\delta=0}\}\mathbf{E}[g_2|g_1 = x] + \partial\phi^0(x)\mathbf{E}[g_3|g_1 = x]\right. \\
&+ \left.\frac{1}{2}\partial^2\phi^0(x)\mathbf{E}[g_2^2|g_1 = x]\right] n[x:0,\Sigma_{g_1}].
\end{aligned}
$$

To obtain $p_{23}(x)$, as for the term $\mathbf{E}\left[\frac{1}{2}\phi^0(g_1)\partial^2 I_{\mathcal{B}}(g_1)g_2^2\right]$, taking $\mathcal{B} = \mathcal{B}_x = (-\infty, x]$, we have

$$
\mathbf{E}\left[\frac{1}{2}\phi^0(g_1)\partial^2 I_{\mathcal{B}_x}(g_1)g_2^2\right]
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \partial^2 I_{\mathcal{B}_x}(y)\{\frac{1}{2}\phi^0(y)\mathbf{E}[g_2^2|g_1=y]n[y:0,\Sigma_{g_1}]\}dy \\
&= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \delta_x(y)\{\frac{1}{2}\phi^0(y)\mathbf{E}[g_2^2|g_1=y]n[y:0,\Sigma_{g_1}]\}dy \\
&= \frac{\partial}{\partial x}\{\frac{1}{2}\phi^0(x)\mathbf{E}[g_2^2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&= \int_{-\infty}^{x} \frac{\partial^2}{\partial y^2}\{\frac{1}{2}\phi^0(y)\mathbf{E}[g_2^2|g_1=y]n[y:0,\Sigma_{g_1}]\}dy.
\end{aligned}
$$

For the second term, we obtain as before

$$
\mathbf{E}\left[\phi^0(g_1)\partial I_{\mathcal{B}_x} g_3\right] = \int_{-\infty}^{x} \frac{\partial}{\partial y}\{-\phi^0(y)\mathbf{E}[g_3|g_1=y]n[y:0,\Sigma_{g_1}]\}dy.
$$

Hence, $p_{23}(x)$ is given by

$$
\begin{aligned}
p_{23}(x) &= \frac{\partial^2}{\partial x^2}\{\frac{1}{2}\phi^0(x)\mathbf{E}[g_2^2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \frac{\partial}{\partial x}\{-\phi^0(x)\mathbf{E}[g_3|g_1=x]n[x:0,\Sigma_{g_1}]\}.
\end{aligned}
$$

Finally collecting and rearranging each term of $p_{21}(x), p_{22}(x)$ and $p_{23}(x)$, we conclude

$$
\begin{aligned}
p_2(x) &= -\frac{\partial\phi^\delta}{\partial\delta}|_{\delta=0}(x)\frac{\partial}{\partial x}\{\mathbf{E}[g_2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \frac{1}{2}\frac{\partial^2\phi^\delta}{\partial\delta^2}|_{\delta=0}(x)n[x:0,\Sigma_{g_1}] \\
&+ \frac{1}{2}\phi^0(x)\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \phi^0(x)\frac{\partial}{\partial x}\{-\mathbf{E}[g_3|g_1=x]n[x:0,\Sigma_{g_1}]\}.
\end{aligned}
$$

∎

In particular, if we take $\phi^\delta(x) \equiv 1$ and $\mathcal{B} = (-\infty, x]$, then

$$
\begin{aligned}
P(\{X_T^\delta \le x\}) &\sim \int_{-\infty}^{x} n[y:0,\Sigma_{g_1}]dy + \delta\int_{-\infty}^{x} \frac{-\partial\mathbf{E}[g_2|g_1=y]n[y:0,\Sigma_{g_1}]}{\partial y}dy \\
&+ \delta^2\left[\int_{-\infty}^{x} \frac{1}{2}\frac{\partial^2}{\partial y^2}\{\mathbf{E}[g_2^2|g_1=y]n[y:0,\Sigma_{g_1}]\}\right. \\
&+ \left.\frac{\partial}{\partial y}\{-\mathbf{E}[g_3|g_1=y]n[y:0,\Sigma_{g_1}]\}\right]dy + \cdots \\
&= N\left[\frac{x}{\Sigma_{g_1}^{\frac{1}{2}}}\right] + \delta\{-\mathbf{E}[g_2|g_1=x]n[x:0,\Sigma_{g_1}]\} \\
&+ \delta^2\left[\frac{1}{2}\frac{\partial}{\partial x}\mathbf{E}[g_2^2|g_1=x]n[x:0,\Sigma_{g_1}] - \mathbf{E}[g_3|g_1=x]n[x:0,\Sigma_{g_1}]\right] + \cdots.
\end{aligned}
$$

Moreover, if we take $\phi^\delta(x) = x + y$ for a constant y and $\mathcal{B} = [-y, \infty)$, then

$$\begin{aligned}
\mathbf{E}\left[(x+y)^+\right] &\sim \int_{-y}^{\infty} x\ n[x:0,\Sigma_{g_1}]dx \\
&+ \delta \int_{-y}^{\infty} x\ \frac{-\partial \mathbf{E}[g_2|g_1 = x]n[x:0,\Sigma_{g_1}]}{\partial x} dx \\
&+ \delta^2 \int_{-y}^{\infty} x \left[\frac{\partial}{\partial x} \{-\mathbf{E}[g_3|g_1 = x]n[x:0,\Sigma_{g_1}]\} \right. \\
&+ \left. \frac{1}{2}\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1 = x]n[x:0,\Sigma_{g_1}]\} \right] dx + \cdots .
\end{aligned}$$

These are exactly equivalent to the formulae by our simple inversion technique for the characteristic function. Hence we have confirmed the assertion. æ

æ

6.3 The Validity in the Term Structure Model

Next, we shall show the validity of our method in an arbitrage-free pricing model based on a family of the instantaneous forward rates processes whch obeys the stochastic integral equation.

$$\begin{aligned}
f^{(\varepsilon)}(s,t) = f(0,t) &+ \varepsilon^2 \int_0^s \left[\sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy \right] dv \\
&+ \varepsilon \int_0^s \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t)d\tilde{w}_i(v), \qquad (1.119)
\end{aligned}$$

where $0 < \varepsilon \leq 1$ and $0 \leq s \leq t \leq T \leq \bar{T}$. The volatility function $\sigma_i(f^{(\varepsilon)}(s,t),s,t)$ depends not only on s and t, but also on $f^{(\varepsilon)}(s,t)$ in the general case.

First, we make the follwing two assumptions.

Assumption I : The volatility functions $\{\sigma_i(f^{(\varepsilon)}(s,t),s,t)\}$ are non-negative, bounded, Lipschitz continuous, and smooth in its first argument, and all derivatives are bounded uniformly in ε, where $f^{(\varepsilon)}(s,t)$ are properly defined in $(\varepsilon, s, t, f^{(\varepsilon)}(s,t))$ $\in (0,1] \times \{0 < s \leq t \leq T\} \times R^1$. The initial forward rates $f(0,t)$ are also Lipschitz continuous with respect to t.

Assumption II : For any $0 < t \leq T$,

$$\Sigma_t = \int_0^t \sum_{i=1}^n \sigma_i^{(0)}(v,t)^2 dv > 0, \qquad (1.120)$$

where

$$\sigma_i^{(0)}(v,t) = \sigma_i(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0}.$$

The conditions we have made in Assumption I can exclude the possiblity of explosions for the solution of (1.119) [1] . They are quite strong and could be relaxed considerably, which may be interesting from the view of stochastic analysis. For practical purposes, however, we can often use the truncation arguments as an example given by Heath, Jarrow, and Morton (1992). Assumption II ensures the key condition of non-degeneracy of the Malliavin-covariance in our problem, which is essential for the validity of the asymptotic expansion approach as we shall see in the following derivations. Under these assumptions we can get the stochastic expansions of the forward rates and spot interest rates processes. We show this in the follwing two steps.

[Step 1] : We set $n = 1$ and $\varepsilon = 1$ in (1.119) in Step 1. The starting point of our discussion is the result by Morton (1989) on the existence and uniqueness of the solution of the stochastic integral equation (1.119) for forward rate processes.

Theorem 1.6 *: Under Assumption I, there exists a jointly continuous process $\{f^{(\varepsilon)}(s,t), 0 \leq s \leq t \leq T\}$ satisfying (1.119) with $\varepsilon = 1$. There is at most one solution of (1.119) with $\varepsilon = 1$.*

We shall consider the $H-$derivatives of the forward rate processes $\{f^{(1)}(s,t)\}$. For any $h \in H$, we successively define a sequence of random variables $\{\xi^{(n)}(s,t)\}$ by the integral equation:

$$\begin{aligned}
\xi^{(n+1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\xi^{(n)}(v,t)\right]dv \\
&+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi^{(n)}(v,y)dy\right]dv \\
&+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)\xi^{(n)}(v,t)d\tilde{w}(v) \\
&+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv \qquad (1.121)
\end{aligned}$$

[1] For example, Morton (1989) has shown that there does not exist any meaningful solution when the volatility function is proportional to the forward rate process.

where the initial condition is given by $\xi^{(0)}(s,t) = 0$. Then we have the next result by using the standard method in stochastic analysis.

Lemma 1.7 *: For any $p > 1$ and $0 \leq s \leq t \leq T$,*

$$\mathbf{E}[|\xi^{(n)}(s,t)|^p] < \infty \ , \tag{1.122}$$

and as $n \to \infty$

$$\mathbf{E}[\sup_{0 \leq s \leq t \leq T} |\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2] \to 0 \ . \tag{1.123}$$

PROOF. **[i]** We use the induction argument for n. We have (1.122) when $n = 1$ because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is a square-integrable function in (1.121). Suppose (1.122) hold for $n = m$. Then there exist positive constants $M_i (i = 1, \cdots, 4)$ such that

$$\begin{aligned} |\xi^{(m+1)}(s,t)|^p \ &\leq \ M_1 \int_0^s |\xi^{(m)}(v,t)|^p dv + M_2 [\sup_{0 \leq u \leq s} | \int_0^u \xi^{(m)}(v,t) d\tilde{w}(v)|]^p \\ &\quad + M_3 \int_0^s \int_v^t |\xi^{(m)}(v,y)|^p dy dv + M_4 [\int_0^s |\dot{h}_v|^2 dv]^{p/2} \ . \end{aligned} \tag{1.124}$$

By a (local) martingale inequality (e.g. *Theorem III-3.1* of Ikeda and Watanabe (1989)), the expectation of the second term on the right hand side of (1.124) is less than

$$M_3^{'} \mathbf{E}[\int_0^s |\xi^{(m)}(v,t)|^2 dv]^{p/2} \leq M_3^{''} \int_0^s \mathbf{E}[|\xi^{(m)}(v,t)|]^p dv \ ,$$

where $M_3^{'}$ and $M_3^{''}$ are positive constants. Because $\dot{h}_v$ is square-integrable, we have (1.122) when $n = m + 1$.

[ii] From (1.121), there exist positive constants $M_i (i = 5, 6, 7)$ such that for $0 \leq s \leq t$,

$$\begin{aligned} |\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2 \ &\leq \ M_5 [\int_0^s |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)| dv]^2 \\ &+ \ M_6 [\int_0^s \int_v^t |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)| dy dv]^2 \\ &+ \ M_7 [\int_0^s \partial\sigma(f^{(1)}(v,t),v,t) |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)| d\tilde{w}(v)]^2 \\ &\equiv \ \sum_{i=1}^3 I_i^{(n)}(s,t) \ , \end{aligned} \tag{1.125}$$

where we have defined $I_i^{(n)}(s,t)$ by the last equality. By using the Cauchy-Schwartz inequality,

$$\mathbf{E}[\sup_{0\le u\le s} I_1^{(n)}(u,t)] \le M_5 s \int_0^s \mathbf{E}[|\xi^{(n)}(v,t)-\xi^{(n-1)}(v,t)|^2]dv \ .$$

By repeating the above argument to the second term of (1.125), we have

$$\begin{aligned} I_2^{(n)}(u,t) &\le M_6 u \int_0^u [\int_v^t |\xi^{(n)}(v,y)-\xi^{(n-1)}(v,y)|dy]^2 dv \\ &\le M_6 ut \int_0^u \int_v^t |\xi^{(n)}(v,y)-\xi^{(n-1)}(v,y)|^2 dydv \ . \end{aligned} \tag{1.126}$$

Then

$$\mathbf{E}[\sup_{0\le u\le s} I_2^{(n)}(u,t)] \le M_6 st \int_0^s \int_v^t \mathbf{E}[|\xi^{(n)}(v,y)-\xi^{(n-1)}(v,y)|^2]dydv \ . \tag{1.127}$$

For the third therm of (1.125), we have

$$\mathbf{E}[\sup_{0\le u\le s} I_3^{(n)}(u,t)] \le M_7' \int_0^s \mathbf{E}[|\xi^{(n)}(v,t)-\xi^{(n-1)}(v,t)|^2]dv \tag{1.128}$$

because of the boundedness of $\partial\sigma(\cdot)$, where M_7' is a positive constant. By using (1.126), (1.127), and (1.128), we have

$$\begin{aligned} \mathbf{E}[\sup_{0\le u\le s}|\xi^{(n+1)}(u,t)-\xi^{(n)}(u,t)|^2] \ &\le \ M_8 \Bigg(\int_0^s \mathbf{E}[\sup_{0\le v\le u}|\xi^{(n)}(v,t)-\xi^{(n-1)}(v,t)|^2]du \\ &+ \ \int_0^s \int_u^t \mathbf{E}[\sup_{0\le v\le u}|\xi^{(n)}(v,y)-\xi^{(n-1)}(v,y)|^2]dydu \Bigg) \end{aligned}$$

where M_8 is a positive constant. By defining a sequence of $\{u^{(n)}(s,t)\}$ by

$$u^{(n+1)}(s,t) = \mathbf{E}[\sup_{0\le u\le s}|\xi^{(n+1)}(u,t)-\xi^{(n)}(u,t)|^2] \ ,$$

we have the relation

$$u^{(n+1)}(s,t) \le M_9 \int_0^s [\int_u^t u^{(n)}(u,y)dy + u^{(n)}(u,t)]du \ ,$$

where M_9 is a positive constant. If we have an inequality

$$u^{(n+1)}(s,t) \le \frac{1}{(n+1)!}[M_9(t+1)s]^{n+1} \ , \tag{1.129}$$

we can show (1.123) as $n \to +\infty$. We use the induction argument for $n \geq 1$. When $n = 1$, there exists a positive constant M_9 such that

$$\begin{aligned} u^{(1)}(s,t) &= \mathbf{E}[\sup_{0\leq u\leq s} |\xi^{(1)}(u,t) - \xi^{(0)}(u,t)|^2] \\ &= \mathbf{E}[\sup_{0\leq u\leq s} |\int_0^s \sigma(f^{(1)}(u,t),u,t)\dot{h}_v du|^2] \\ &\leq M_9(1+t)s \end{aligned}$$

because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is square- integrable. Suppose (1.129) hold for $n = m$. Then

$$\begin{aligned} u^{(m+1)}(s,t) &\leq M_9 \int_0^s [\int_u^t u^{(m)}(u,y)dy + u^{(m)}(u,t)]du \\ &\leq M_9 \int_0^s [\int_u^t M_9^m (t+1)^m \frac{s^m}{m!} dy + M_9^m (t+1)^m \frac{s^m}{m!}]du \\ &\leq M_9^{m+1}(t+1)^{m+1} \frac{s^{m+1}}{(m+1)!} \ . \end{aligned}$$

▮

Because of (1.123), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P\{\sup_{0\leq s\leq t\leq T} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)| > \frac{1}{2^n}\} \\ \leq &\sum_{n=1}^{\infty} \frac{1}{n!}[4M_9(T+1)s]^n < +\infty \ . \end{aligned}$$

Then by the Borel-Cantelli lemma, the sequence of random variables $\{\xi^{(n)}(s,t)\}$ converges uniformly on $0 \leq s \leq t \leq T$. Hence we can establish the existence of the $H-$derivative of $f^{(1)}(s,t)$, which is given by the solution of the stochastic integral equation:

$$\begin{aligned} D_h f^{(1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy D_h f^{(1)}(v,t)\right]dv \\ &+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)D_h f^{(1)}(v,y)dy\right]dv \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)D_h f^{(1)}(v,t)d\tilde{w}(v) \\
&+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv \ . \qquad (1.130)
\end{aligned}
$$

Next, we examine the existence of higer order moments of $D_h f^{(1)}(s,t)$ satisfying (1.130). To do this, we prepare the following inequality.

Lemma 1.8 *: Suppose for* $k_0 > 0, k_1 > 0, A_N > 0$ *and* $0 < s \le t \le T$*, a function* $u_N(s,t)$ *satisfies (i)* $0 < u_N(s,t) \le A_N$ *and (ii)*

$$
u_N(s,t) \le k_0 + k_1 \left[\int_0^s u_N(v,t)dv + \int_0^s \int_v^t u_N(v,y)dydv\right] \ . \qquad (1.131)
$$

Then,

$$
u_N(s,t) \le k_0 e^{k_1(1+t)s} \ . \qquad (1.132)
$$

PROOF. By substituting (i) into the right hand side of (1.131), we have

$$
\begin{aligned}
u_N(s,t) &\le k_0 + A_N k_1 \left[\int_0^s ds + \int_0^s \int_v^t dydv\right] \qquad (1.133) \\
&\le k_0 + A_N k_1 (1+t)s \ .
\end{aligned}
$$

By repeating the substitution of (1.133) into the right hand side of (1.131), we have

$$
u_N(s,t) \le k_0 \sum_{k=0}^n \frac{1}{k!}[k_1(1+t)]^k + \frac{1}{(n+1)!}A_N[k_1(1+t)s]^{n+1} \ .
$$

Then we have (1.132) by taking $n \to +\infty$. ∎

In order to use *Lemma 1.8*, we consider the truncated random variable

$$
\zeta_N(s,t) = \left[D_h f^{(1)}(s,t)\right] I_N(s,t) \ , \qquad (1.134)
$$

where $I_N(s,t) = 1$ if

$$
\sup_{0\le v\le s, v\le y\le t}|D_h f^{(1)}(v,y)| \le N
$$

and $I_N(s,t) = 0$ otherwise. By using the boundedness conditions in Assumption I and $\dot{h}_s$ being square-integrable, we can show that there exist positive constants $M_i (i = 10, \cdots, 13)$ such that

$$
\begin{aligned}
|\zeta_N(s,t)|^p &\leq M_{10}\int_0^s |\zeta_N(v,t)|^p dv + M_{11}|\int_0^s \zeta_N(v,t)d\tilde{w}(v)|^p \\
&+ M_{12}\int_0^s\int_v^t |\zeta_N(v,y)|^p dydv + M_{13}|\int_0^s \sigma(v,t)\dot{h}_v dv|^p \ . \\
&\equiv \sum_{i=1}^4 J_i^N(s,t) \ ,
\end{aligned} \tag{1.135}
$$

where we have defined $J_i^N(s,t)(i = 1,\cdots,4)$ by the last equality. By using a (local) martingale inequality (e.g. *Theorem III-3.1* of Ikeda and Watanabe (1989)), we have

$$
\begin{aligned}
\mathbf{E}[J_2^N(s,t)] &\leq M_{11}^{'}\mathbf{E}[\int_0^s |\zeta_N(v,t)|^2 dv]^{p/2} \\
&\leq M_{11}^{''}\mathbf{E}[\int_0^s |\zeta_N(v,t)|^p dv] \ ,
\end{aligned} \tag{1.136}
$$

where $M_{11}^{'}$ and $M_{11}^{''}$ are positive constants. Also by the Cauchy-Schwartz inequality, we have

$$
J_4^N(s,t) \leq M_{13}[\int_0^s \sigma(f^{(1)}(v,t),v,t)^2 dv \int_0^s |\dot{h}_v|^2 dv]^{p/2} \ , \tag{1.137}
$$

which is bounded because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is square-integrable. If we set $u_N(s,t) = \mathbf{E}[|\zeta_N(s,t)|^p]$, then we can directly apply Lemma1.8. By taking the limit of the expectation function $u_N(s,t)$ as $N \to \infty$, we have the following result.

Lemma 1.9 *: For any $p > 1$,*

$$
\mathbf{E}[|D_h f^{(1)}(s,t)|^p] < +\infty \ , \tag{1.138}
$$

By this lemma and the equivalence of two norms stated in Step 1, we can establish that

$$
f^{(1)}(s,t) \in \cap_{1<p<+\infty}\boldsymbol{D}_p^1 \ .
$$

Then by repeating the above procedure, we can derive the higher order $H-$derivatives of $f^{(1)}(s,t)$. Hence we can obtain the following result.

Theorem 1.7 *: Suppose Assumption I hold for the forward rate processes. Then for $0 < s \leq t \leq T$*

$$
f^{(1)}(s,t) \in \boldsymbol{D}^{\infty} \ . \tag{1.139}
$$

[Step 2] : Let a stochastic process $\{Y^{(\varepsilon)}(s,t), 0 \le s \le t \le T\}$ be the solution of the stochastic integral equation:

$$Y^{(\epsilon)}(s,t) = 1 + \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Y^{(\varepsilon)}(v,t)dv + \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y^{(\varepsilon)}(v,t)d\tilde{w}(v) . \tag{1.140}$$

Since the coefficients of $Y^{(\varepsilon)}(s,t)$ on the right hand side of (1.140) are bounded by Assumption I, we can obtain the next result.

Lemma 1.10 *: For any* $1 < p < +\infty, 0 < \varepsilon \le 1,$ *and* $0 < s \le t \le T$,

$$\mathbf{E}[|Y^{(\varepsilon)}(s,t)|^p] + \mathbf{E}[|Y^{(\varepsilon)-1}(s,t)|^p] < +\infty . \tag{1.141}$$

PROOF. : We define a sequence of random variables $\{Y_n^{(\varepsilon)}(s,t)\}$ by

$$Y_{n+1}^{(\epsilon)}(s,t) = 1 + \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Y_n^{(\varepsilon)}(v,t)dv + \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y_n^{(\varepsilon)}(v,t)d\tilde{w}(v) ,$$

where the initial condition is given by $Y_0^{(\varepsilon)}(s,t) = 1$. Then by the same argument as in the proof of *Lemma 1.7*, we have

$$\mathbf{E}[|Y_n^{(\varepsilon)}(s,t)|^p] < \infty ,$$

and as $n \to \infty$

$$\mathbf{E}[\sup_{0\le s\le t\le T} |Y_{n+1}^{(\varepsilon)}(s,t) - Y_n^{(\varepsilon)}(s,t)|^2] \to 0 .$$

Hence we can establish the existence of the random variables $\{Y^{(\varepsilon)}(s,t)\}$ satisfying (1.140). Then by the same argument as (1.134)-(1.137), we have

$$\mathbf{E}[|Y^{(\varepsilon)}(s,t)|^p] < \infty$$

for any $p > 1$. Let $Z^{(\varepsilon)}(s,t) = Y^{(\varepsilon)-1}(s,t)$. Then we can show that

$$d[Z^{(\varepsilon)}(s,t)Y^{(\varepsilon)}(s,t)] = 0$$

and

$$
\begin{aligned}
Z^{(\epsilon)}(s,t) = 1 \quad &- \quad \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Z^{(\varepsilon)}(v,t)dv \\
&- \quad \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Z^{(\varepsilon)}(v,t)d\tilde{w}(v)
\end{aligned}
$$

by using Itô's formula and $Z^{(\varepsilon)}(0,t) = 1$. Hence by the similar argument as on $Y^{(\varepsilon)}(s,t)$, we can establish

$$
\mathbf{E}[|Z^{(\varepsilon)}(s,t)|^p] < \infty
$$

for any $p > 1$. ∎

Now we consider the asymptotic behavior of a functional

$$
F^{(\varepsilon)}(s,t) = \frac{1}{\varepsilon}[f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)] \tag{1.142}
$$

as $\varepsilon \to 0$. By using the stochastic process $\{Y^{(\varepsilon)}(s,t)\}$, the $H-$derivative of $F^{(\varepsilon)}(s,t)$ can be represented as

$$
D_h F^{(\varepsilon)}(s,t) = \int_0^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)C^{(\varepsilon)}(v,t)dv \ ,
$$

where

$$
\begin{aligned}
C^{(\epsilon)}(v,t) \quad &= \quad \sigma(f^{(\varepsilon)}(v,t),v,t)\dot{h}_v + \varepsilon\sigma(f^{(\varepsilon)}(v,t),v,t) \\
&\quad \times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy \ .
\end{aligned}
$$

Let

$$
a_v^{(\varepsilon)}(s,t) = Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)C^{(\varepsilon)}(v,t) \ ,
$$

and

$$
\begin{aligned}
\eta_c^{(\varepsilon)}(s,t) \quad &= \quad c\int_0^s |\varepsilon Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) \\
&\quad \times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy|^2 dv \\
&+ \quad c\int_0^s |Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) - \sigma(f^{(0)}(v,t),v,t)|^2 dv \ .
\end{aligned} \tag{1.143}
$$

Then the condition in Assumption II is equivalent to the non-degeneracy condition:

$$
\Sigma_t = \int_0^s a_v^{(0)}(v,t)^2 \, dv > 0
$$

because $Y^{(0)}(v,t) = 1$ for $0 \le v \le s \le t$. The next lemma shows that the truncation by $\eta_c^{(\varepsilon)}(s,t)$ is negligible.

Lemma 1.11 *: For $0 < s \le t \le T$ and any $n \ge 1$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-n} P\{|\eta_c^{(\varepsilon)}(s,t)| > \frac{1}{2}\} = 0 \ . \tag{1.144}$$

PROOF. : We re-write (1.143) as $\eta_c^{(\varepsilon)}(s,t) = \eta_1^{(\varepsilon)} + \eta_2^{(\varepsilon)}$. By using Assumption I, *Lemma1.9*, and the Markov inequality, it is straightforward to show that for any $p > 1$ and $c_1 > 0$ there exists a positive constant c_2 such that

$$P\{|\eta_1^{(\varepsilon)}| > c_1\} \le c_2 \varepsilon^{2p} \ . \tag{1.145}$$

By the Lipschitz continuity of the volatility function $\sigma(\cdot)$, there exist positive constants M_{14} and M_{15} such that

$$|\eta_2^{(\varepsilon)}| \le M_{14}|f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| + M_{15}|Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t) - 1| \ . \tag{1.146}$$

Then by *Lemma 10.5* of Ikeda and Watanabe (1989), for a positive c_3 and sufficiently small $\varepsilon > 0$, there exist positive constants c_4 and c_5 such that

$$P\{\sup_{0\le s\le t\le T} |f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| > c_3\} \le c_4 exp(-c_5\varepsilon^{-2}) \ . \tag{1.147}$$

For the second term of the right hand side of (1.146) for η_2^{ε}, we re-write

$$\eta_{22}^{(\varepsilon)} = M_{15} Y^{(\varepsilon)}(v,t)^{-1} |Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t)| \ ,$$

where

$$\begin{aligned} Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t) \quad &= \quad \varepsilon^2 \int_v^s \left[\partial\sigma(f^{(\varepsilon)}(u,t),u,t) \int_u^t \sigma(f^{(\varepsilon)}(u,y),u,y)dy \right] \times \\ & \quad Y^{(\varepsilon)}(u,t)du + \varepsilon \int_v^s \partial\sigma(f^{(\varepsilon)}(u,t),u,t)Y^{(\varepsilon)}(u,t)d\tilde{w}(u) \ . \end{aligned}$$

Then by *Lemma1.10*, for any $p \ge 1$ and $c_6 > 0$ there exists a positive constant c_7 such that

$$P\{|\eta_{22}^{(\varepsilon)}| > c_6\} \le c_7 \varepsilon^{2p} \ . \tag{1.148}$$

By using (1.145), (1.147), and (1.148), we have (1.144).

$\blacksquare$

Finally, by the similar argument as in the Black-Sholes' economy, we shall obtain a truncated version of the non-degeneracy condition of the Malliavin-covariance for the spot interest rates and forward rates processes, which is the key step to show the validity.

Lemma 1.12 : *Under Assumptions I and II, the Malliavin-covariance* $\sigma(F^{(\varepsilon)})$ *of* $F^{(\varepsilon)}$ *is uniformly non-degenerate in the sense that there exists* $c_0 > 0$ *such that for any* $c > c_0$ *and any* $p > 1$

$$\sup_{\varepsilon} \mathbf{E}[I(|\eta_c^{(\varepsilon)}| \leq 1)\sigma(F^{(\varepsilon)})^{-p}] < +\infty \ , \tag{1.149}$$

where $I(\cdot)$ *is the indicator function.*

Hence the validity of the asymptotic expansions of the distribution function or the density function of a spot rate and an instantaneous forward rate is obtained under the assumptions **I** and **II** because we have proved that a set of conditions in *Theorem 2.2* of Yoshida(1992) is satisfied. That is, let $\psi : R \rightarrow R$ be a smooth function such that $0 \leq \psi(x) \leq 1, \psi(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\psi(x) = 0$ for $|x| \geq 1$ as before. Then the composite functional $\psi(\eta^{(\varepsilon)})I_A(F^{(\varepsilon)})$ is well-defined for any $A \in \boldsymbol{B}$ in the sense that it is in $\tilde{\boldsymbol{D}}^{-\infty}$, where $\boldsymbol{B}$ is the Borel $\sigma-$field in R and $I_A(\cdot)$ is the indicator function. Hence by using *Theorem 2.2* of Yoshida(1992), it has a proper asymptotic expansion as $\varepsilon \rightarrow 0$ uniformly in $\tilde{\boldsymbol{D}}^{-\infty}$. Then we have a proper asymptotic expansion for the density function of our interest by taking the expectation operations.

Also it is straightforward to obtain the similar non-degeneracy conditions as $\Sigma_{g_1} > 0$ for the discounted coupon bond price process and the average interest rate process given before.

Finally, we mention that the same argument as in the Black-Sholes economy holds to show the equivalence between the formulae by the Schwartz's type distributiion theory for the generalized Wiener functionals and the formulae by the simple inversion technique for the characteristic functions of random variables.

æ

7 Appendix

7.1 The Proof of Lemma 1.1

In this subsection, we present the proof of the formulae appearing in *Lemma 1.1.*

We use the following notation in the proof.

$$\delta(u,s) = \begin{cases} 1 & \text{if } u = s \\ 0 & \text{otherwise} \end{cases}$$

(1)

$$\mathbf{E}\left[\int_0^t \left[\int_0^s q_2(u)d\tilde{w}(u)\right] q_3(s)d\tilde{w}(s) | \int_0^T q_1(u)d\tilde{w}(u) = x\right]$$

$$\begin{aligned}
& \mathbf{E}\left[\int_0^t \left[\int_0^s q_2(u)d\tilde{w}(u)\right] q_3(s)d\tilde{w}(s) | \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= & \int_0^t \int_0^s q_2(u)q_3(s)\mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(s)| \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= & \int_0^t \int_0^s q_2(u)q_3(s)\left[\{\delta(u,s)ds - dsq_1(s)\Sigma_{g_1}^{-1}q_1(u)du\}\right. \\
+ & \left. q_1(s)ds\Sigma_{g_1}^{-1}x^2\Sigma_{g_1}^{-1}q_1(u)du\right] \\
= & -\frac{1}{\Sigma_{g_1}}\int_0^t q_3(s)q_1(s)\int_0^s q_2(u)q_1(u)duds \\
+ & x^2\frac{1}{\Sigma_{g_1}^2}\int_0^t q_3(s)q_1(s)\int_0^s q_2(u)q_1(u)duds
\end{aligned}$$

(2)

$$\mathbf{E}\left[\left[\int_0^t q_2(u)d\tilde{w}(u)\right]\left[\int_0^t q_3(u)d\tilde{w}(u)\right] | \int_0^T q_1(u)d\tilde{w}(u) = x\right]$$

$$
\begin{aligned}
& \mathbf{E}\left[\left[\int_0^t q_2(u)d\tilde{w}(u)\right]\left[\int_0^t q_3(u)d\tilde{w}(u)\right] \mid \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= \; & \int_0^t \int_0^t q_2(u)q_3(s)\mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(s)| \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= \; & \int_0^t \int_0^t q_2(u)q_3(s)\left\{\delta(u,s)du + q_1(u)du\Sigma_{g_1}^{-1}(x^2 - \Sigma_{g_1})\Sigma_{g_1}^{-1}q_1(s)ds\right\} \\
= \; & -\frac{1}{\Sigma_{g_1}}\left[\int_0^t q_2(u)q_1(u)du\right]\left[\int_0^t q_3(u)q_1(u)du\right] + \int_0^t q_3(u)q_2(u)du \\
+ \; & x^2\frac{1}{\Sigma_{g_1}^2}\left[\int_0^t q_2(u)q_1(u)du\right]\left[\int_0^t q_3(u)q_1(u)du\right]
\end{aligned}
$$

Note: We use the following relation in (1) and (2).

$$
\begin{aligned}
& \mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(s)| \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= \; & Cov.\left[d\tilde{w}(u), d\tilde{w}(s)|x\right] + \mathbf{E}\left[d\tilde{w}(u)|x\right]\mathbf{E}\left[d\tilde{w}(s)|x\right] \\
= \; & \left[\delta(u,s)du - q_1(u)du\Sigma_{g_1}^{-1}q_1(s)ds\right] + \left[\Sigma_{g_1}^{-1}xq_1(u)du\right]\left[\Sigma_{g_1}^{-1}xq_1(s)ds\right]
\end{aligned}
$$

(3)

$$
\mathbf{E}\left[\int_0^t \left[\int_0^s q_2(u)d\tilde{w}(u)\right]\left[\int_0^s q_3(u)d\tilde{w}(u)\right] q_4(s)d\tilde{w}(s)| \int_0^T q_1(u)d\tilde{w}(u) = x\right]
$$

We first note that, for $s > u$ and $s > v$,

$$
\begin{aligned}
& \mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s)| \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
= \; & Cov.\left[d\tilde{w}(u), d\tilde{w}(v)|x\right]\mathbf{E}\left[d\tilde{w}(s)|x\right] + Cov.\left[d\tilde{w}(u), d\tilde{w}(s)|x\right]\mathbf{E}\left[d\tilde{w}(v)|x\right] \\
+ \; & Cov.\left[d\tilde{w}(v), d\tilde{w}(s)|x\right]\mathbf{E}\left[d\tilde{w}(u)|x\right] + \mathbf{E}\left[d\tilde{w}(u)|x\right]\mathbf{E}\left[d\tilde{w}(v)|x\right]\mathbf{E}\left[d\tilde{w}(s)|x\right] \\
= \; & \left[\delta(u,v)du - q_1(u)du\Sigma_{g_1}^{-1}q_1(v)dv\right]\left[\Sigma_{g_1}^{-1}xq_1(s)ds\right] \\
+ \; & \left[\delta(u,s)ds - q_1(u)du\Sigma_{g_1}^{-1}q_1(s)ds\right]\left[\Sigma_{g_1}^{-1}xq_1(v)dv\right]
\end{aligned}
$$

$$
\begin{aligned}
+ & \left[\delta(v,s)ds - q_1(v)dv\Sigma_{g_1}^{-1}q_1(s)ds\right]\left[\Sigma_{g_1}^{-1}xq_1(u)du\right] \\
+ & \left[\Sigma_{g_1}^{-1}xq_1(u)du\right]\left[\Sigma_{g_1}^{-1}xq_1(v)dv\right]\left[\Sigma_{g_1}^{-1}xq_1(s)ds\right] \\
= & \ x^3\left[\Sigma_{g_1}^{-3}q_1(u)q_1(v)q_1(s)dudvds\right] + x\left[\delta(u,v)\Sigma_{g_1}^{-1}q_1(s)duds\right] \\
- & \ 3x\left[\Sigma_{g_1}^{-2}q_1(u)q_1(v)q_1(s)dudvds\right].
\end{aligned}
$$

Hence, we can conclude

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^t\left[\int_0^s q_2(u)d\tilde{w}(u)\right]\left[\int_0^s q_3(u)d\tilde{w}(u)\right]q_4(s)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right] \\
= & \int_0^t\int_0^s\int_0^s q_2(u)q_3(v)q_4(s)\mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right] \\
= & \ x\left[\frac{1}{\Sigma_{g_1}}\int_0^t\left[\int_0^s q_2(u)q_3(u)du\right]q_4(s)q_1(s)ds\right. \\
& \left.-3\frac{1}{\Sigma_{g_1}^2}\int_0^t\left[\int_0^s q_2(u)q_1(u)du\right]\left[\int_0^s q_3(u)q_1(u)du\right]q_4(s)q_1(s)ds\right] \\
+ & \ x^3\left[\frac{1}{\Sigma_{g_1}^3}\int_0^t\left[\int_0^s q_2(u)q_1(u)du\right]\left[\int_0^s q_3(u)q_1(u)du\right]q_4(s)q_1(s)ds\right].
\end{aligned}
$$

(4)

$$
\mathbf{E}\left[\int_0^t\int_0^s\left[\int_0^v q_2(u)d\tilde{w}(u)\right]q_3(v)d\tilde{w}(v)q_4(s)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right]
$$

By using the formula of $\mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right]$ derived in (3), we obtain

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^t\int_0^s\left[\int_0^v q_2(u)d\tilde{w}(u)\right]q_3(v)d\tilde{w}(v)q_4(s)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right] \\
= & \int_0^t\int_0^s\int_0^v q_2(u)q_3(v)q_4(s)\mathbf{E}\left[d\tilde{w}(u)d\tilde{w}(v)d\tilde{w}(s)|\int_0^T q_1(u)d\tilde{w}(u)=x\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \int_0^s \int_0^v q_2(u)q_3(v)q_4(s)\left[x^3\Sigma_{g_1}^{-3}q_1(u)q_1(v)q_1(s)dudvds - 3x\Sigma_{g_1}^{-2}q_1(u)q_1(v)q_1(s)dudvds\right] \\
&= -3x\left[\frac{1}{\Sigma_{g_1}^2}\int_0^t q_4(s)q_1(s)\int_0^s q_3(v)q_1(v)\int_0^v q_2(u)q_1(u)dudvds\right] \\
&+ x^3\left[\frac{1}{\Sigma_{g_1}^3}\int_0^t q_4(s)q_1(s)\int_0^s q_3(u)q_1(u)\int_0^v q_2(u)q_1(u)dudvds\right].
\end{aligned}
$$

(5)

$$
\mathbf{E}\left[\left[\int_0^t\left[\int_0^s q_2(u)d\tilde{w}(u)\right]q_3(s)d\tilde{w}(s)\right]^2 \mid \int_0^T q_1(u)d\tilde{w}(u) = x\right]
$$

We first note that, for $s > v$ and $s^{'} > u$,

$$
\begin{aligned}
&\mathbf{E}\left[d\tilde{w}(s)d\tilde{w}(s^{'})d\tilde{w}(v)d\tilde{w}(u)| \int_0^T q_1(u)d\tilde{w}(u) = x\right] \\
&= Cov.\left[d\tilde{w}(s), d\tilde{w}(s^{'})|x\right] Cov.\left[d\tilde{w}(v), d\tilde{w}(u)|x\right] \\
&+ Cov.\left[d\tilde{w}(s), d\tilde{w}(v)|x\right] Cov.\left[d\tilde{w}(s^{'}), d\tilde{w}(u)|x\right] \\
&+ Cov.\left[d\tilde{w}(s), d\tilde{w}(u)|x\right] Cov.\left[d\tilde{w}(s^{'}), d\tilde{w}(v)|x\right] \\
&+ Cov.\left[d\tilde{w}(s), d\tilde{w}(s^{'})|x\right] \mathbf{E}\left[d\tilde{w}(u)|x\right] \mathbf{E}\left[d\tilde{w}(v)|x\right] \\
&+ Cov.\left[d\tilde{w}(s), d\tilde{w}(v)|x\right] \mathbf{E}\left[d\tilde{w}(s^{'})|x\right] \mathbf{E}\left[d\tilde{w}(u)|x\right] \\
&+ Cov.\left[d\tilde{w}(s), d\tilde{w}(u)|x\right] \mathbf{E}\left[d\tilde{w}(s^{'})|x\right] \mathbf{E}\left[d\tilde{w}(v)|x\right] \\
&+ Cov.\left[d\tilde{w}(s^{'}), d\tilde{w}(v)|x\right] \mathbf{E}\left[d\tilde{w}(s)|x\right] \mathbf{E}\left[d\tilde{w}(u)|x\right] \\
&+ Cov.\left[d\tilde{w}(s^{'}), d\tilde{w}(u)|x\right] \mathbf{E}\left[d\tilde{w}(s)|x\right] \mathbf{E}\left[d\tilde{w}(v)|x\right] \\
&+ Cov.\left[d\tilde{w}(v), d\tilde{w}(u)|x\right] \mathbf{E}\left[d\tilde{w}(s)|x\right] \mathbf{E}\left[d\tilde{w}(s^{'})|x\right] \\
&+ \mathbf{E}\left[d\tilde{w}(s)|x\right] \mathbf{E}\left[d\tilde{w}(s^{'})|x\right] \mathbf{E}\left[d\tilde{w}(v)|x\right] \mathbf{E}\left[d\tilde{w}(u)|x\right] \\
&= \sum_{3terms}\left[\delta(i,j)di - q_1(i)di\Sigma_{g_1}^{-1}q_1(j)dj\right]\left[\delta(k,l)dk - q_1(k)dk\Sigma_{g_1}^{-1}q_1(l)dl\right] \\
&+ \sum_{6terms}\left[\delta(i,j)di - q_1(i)di\Sigma_{g_1}^{-1}q_1(j)dj\right]\left[\Sigma_{g_1}^{-1}xq_1(k)dk\right]\left[\Sigma_{g_1}^{-1}xq_1(l)dl\right] \\
&+ \left[\Sigma_{g_1}^{-1}xq_1(s)ds\right]\left[\Sigma_{g_1}^{-1}xq_1(s^{'})ds^{'}\right]\left[\Sigma_{g_1}^{-1}xq_1(v)dv\right]\left[\Sigma_{g_1}^{-1}xq_1(u)du\right] \\
&= q_1(u)q_1(v)q_1(s^{'})q_1(s)dudvds^{'}ds\left[\Sigma_{g_1}^{-4}x^4 - 6\Sigma_{g_1}^{-3}x^2 + 3\Sigma_{g_1}^{-2}\right]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \delta(v,s^{'})q_1(u)q_1(s)duds\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]+\delta(v,u)q_1(s^{'})q_1(s)ds^{'}ds\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \delta(u,s)q_1(v)q_1(s^{'})dvds^{'}\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]+\delta(s,s^{'})q_1(u)q_1(v)dudv\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \delta(v,s^{'})dv\delta(u,s)du\\
+ \quad & \delta(u,v)du\delta(s,s^{'})ds.
\end{aligned}
$$

where

$$i,j,k,l\in\{s,s^{'},v,u\}$$

and

$$i\neq j\neq k\neq l.$$

Here, we use $\delta(v,s)=0$ and $\delta(u,s^{'})=0$ under our assumption, $s>v$ and $s^{'}>u$. We also note that $\delta(v,s^{'})\delta(u,s)=0$ under our assumption, $s>v$ and $s^{'}>u$.

Hence,

$$
\begin{aligned}
& \mathbf{E}\left[\left[\int_0^t\left[\int_0^s q_2(u)d\tilde{w}(u)\right]q_3(s)d\tilde{w}(s)\right]^2 \mid \int_0^T q_1(u)d\tilde{w}(u)=x\right]\\
= \quad & \int_0^t\int_0^t\int_0^s\int_0^{s^{'}} q_3(s)q_2(v)q_3(s^{'})q_2(u)\times\\
& \mathbf{E}\left[d\tilde{w}(s)d\tilde{w}(s^{'})d\tilde{w}(v)d\tilde{w}(u)\mid\int_0^T q_1(u)d\tilde{w}(u)=x\right]\\
= \quad & \int_0^t\int_0^t\int_0^s\int_0^{s^{'}}\left[q_1(s)q_3(s)\right]\left[q_1(s^{'})q_3(s^{'})\right]\left[q_1(v)q_2(v)\right]\left[q_1(u)q_2(u)\right]dudvds^{'}ds\\
& \left[\Sigma_{g_1}^{-4}x^4-6\Sigma_{g_1}^{-3}x^2+3\Sigma_{g_1}^{-2}\right]\\
+ \quad & \int_0^t\int_0^s\int_0^v\left[q_1(s)q_3(s)\right]\left[q_2(v)q_3(v)\right]\left[q_1(u)q_2(u)\right]dudvds\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \int_0^t\int_0^s\left[q_1(s)q_3(s)\right]^2q_2(u)^2duds\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \int_0^t\int_0^s\int_0^s\left[q_1(s)q_3(s)\right]\left[q_1(v)q_2(v)\right]\left[q_1(u)q_2(u)\right]dudvds\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \int_0^t\int_0^{s^{'}}\int_0^u\left[q_1(s^{'})q_3(s^{'})\right]\left[q_2(u)q_3(u)\right]\left[q_1(v)q_2(v)\right]dvduds^{'}\left[\Sigma_{g_1}^{-2}x^2-\Sigma_{g_1}^{-1}\right]\\
+ \quad & \int_0^t\int_0^s q_3(s)^2q_2(u)^2duds\\
= \quad & \left[\int_0^t Q_{31}(s)\int_0^s Q_{21}(u)duds\right]^2\left[\frac{4}{\Sigma_{g_1}^2}x^4-\frac{6}{\Sigma_{g_1}^3}x^2+\frac{3}{\Sigma_{g_1}^2}\right]
\end{aligned}
$$

$$+ \quad \left[\int_0^t Q_{31}(s)\int_0^s Q_{32}(v)\int_0^v Q_{21}(u)dudvds\right]\left[\frac{2}{\Sigma_{g_1}^2}x^2-\frac{2}{\Sigma_{g_1}}\right]$$
$$+ \quad \left[\int_0^t Q_{31}(s)^2\int_0^s Q_{22}(u)duds\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2-\frac{1}{\Sigma_{g_1}}\right]$$
$$+ \quad \left[\int_0^t Q_{33}(s)\left[\int_0^s Q_{21}(u)du\right]^2 ds\right]\left[\frac{1}{\Sigma_{g_1}^2}x^2-\frac{1}{\Sigma_{g_1}}\right]$$
$$+ \quad \int_0^t Q_{33}(s)\int_0^s Q_{22}(u)duds$$

where

$$Q_{ij}(s) \equiv q_i(s)q_j(s).$$

7.2 Lemma 1.1'

In this subsection, we show the multi-dimensional version of the formulae appearing in *Lemma 1.1.*

Lemma 1.1' (1) Let $\vec{\tilde{w}}_t$ be N dimensional Brownian motion. Let $\vec{x}$ be k dimensional vector. Suppose $\underline{q_1}(t)$ be $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\underline{q_2}(t)$ and $\underline{q_3}(t)$ $R^1 \mapsto R^{m\times N}$ non-stochastic functions.
Then,

$$\mathbf{E}\left[\int_0^t \left[\int_0^s \underline{q_2}(u)d\vec{\tilde{w}}_u\right]^\top \underline{q_3}(s)d\vec{\tilde{w}}_s | \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right]$$
$$= \quad trace\int_0^t\int_0^s \underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\underline{q_3}(s)^\top\underline{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top-\underline{\Sigma}_{g_1}\right]duds.$$

(2) Let $\vec{\tilde{w}}_t$ be N dimensional Brownian motion. Let $\vec{x}$ be k dimensional vector. Suppose $\underline{q_1}(t)$ be $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\vec{q_2}(t)$ and $\vec{q_3}(t)$ $R^1 \mapsto R^N$ non-stochastic functions.
Then,

$$\mathbf{E}\left[\left[\int_0^t \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\left[\int_0^t \vec{q_3}(s)d\vec{\tilde{w}}_s\right] | \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right]$$
$$= \quad \int_0^t \vec{q_2}(u)\vec{q_3}(u)^\top du + \left[\int_0^t \vec{q_2}(u)\underline{q_1}(u)^\top du\right]\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top-\underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\left[\int_0^t \underline{q_1}(s)\vec{q_3}(s)^\top ds\right].$$

(3) Let $\vec{\tilde{w}}_t$ be N dimensional Brownian motion. Let $\vec{x}$ be k dimensional vector. Suppose $\underline{q_1}(t)$ be $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\vec{q_2}(t)$, $\vec{q_3}(t)$ and $\vec{q_4}(t)$ be $R^1 \mapsto R^N$ non-stochastic functions.
Then,

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t \left[\int_0^s \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\left[\int_0^s \vec{q_3}(u)d\vec{\tilde{w}}_u\right]\vec{q_4}(s)d\vec{\tilde{w}}_s | \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= \;& \int_0^t\int_0^s\int_0^s \Big[\vec{q_2}(v)\vec{q_1}(v)^\top \underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_3}(u)^\top\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x} \\
- \;& \vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\left[\vec{q_3}(v)^\top\vec{q_2}(u) + \vec{q_2}(v)^\top\vec{q_3}(u)\right]\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\Big]dudvds \\
+ \;& \int_0^t\int_0^s \vec{q_4}(s)\underline{q_1}(s)^\top\vec{q_2}(u)\vec{q_3}(u)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}duds
\end{aligned}$$

(4) Let $\vec{\tilde{w}}_t$ be N dimensional Brownian motion. Let $\vec{x}$ be k dimensional vector. Suppose $\underline{q_1}(t)$ be $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\vec{q_2}(t)$, $\vec{q_3}(t)$ and $\vec{q_4}(t)$ be $R^1 \mapsto R^N$ non-stochastic functions.
Then,

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t\int_0^s \left[\int_0^v \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\vec{q_3}(v)d\vec{\tilde{w}}_v\vec{q_4}(s)d\vec{\tilde{w}}_s | \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= \;& \int_0^t\int_0^s\int_0^v \left[\vec{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right]\left[\vec{x}^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_3}(v)^\top\right]\left[\vec{q_4}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\vec{x}\right]dudvds \\
- \;& \int_0^t\int_0^s\int_0^v \left[\vec{q_2}(u)\underline{q_1}(u)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_3}(v)^\top\right]\left[\vec{q_4}(s)\underline{q_1}(s)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds \\
- \;& \int_0^t\int_0^s\int_0^v \left[\vec{q_3}(v)\underline{q_1}(v)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_4}(s)^\top\right]\left[\vec{q_2}(u)\underline{q_1}(u)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds \\
- \;& \int_0^t\int_0^s\int_0^v \left[\vec{q_4}(s)\underline{q_1}(s)^\top\right]\left[\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_2}(u)^\top\right]\left[\vec{q_3}(v)\underline{q_1}(v)^\top\right]\underline{\Sigma}_{g_1}^{-1}\vec{x}dudvds
\end{aligned}$$

(5) Let $\vec{\tilde{w}}_t$ be N dimensional Brownian motion. Let $\vec{x}$ be k dimensional vector. Suppose $\underline{q_1}(t)$ be $R^1 \mapsto R^{k\times N}$ non-stochastic function. Suppose also $\vec{q_2}(t)$ and $\vec{q_3}(t)$ be $R^1 \mapsto R^N$ non-stochastic functions.

Then,

$$
\begin{aligned}
& \mathbf{E}\left[\left[\int_0^t \left[\int_0^s \vec{q_2}(u)d\vec{\tilde{w}}_u\right]\vec{q_3}(s)d\vec{\tilde{w}}_s\right]^2 \mid \int_0^T \underline{q_1}(u)d\vec{\tilde{w}}_u = \vec{x}\right] \\
= & \int_0^t\int_0^t\int_0^s\int_0^v \left[\vec{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_3}(s)^\top\times\right. \\
& \vec{q_2}(u^{'})\underline{q_1}(u^{'})^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(v)\vec{q_3}(v)^\top \\
- & \vec{q_2}(u)\underline{q_1}(u)^\top\underline{\Sigma}_{g_1}^{-1}\left[\underline{q_1}(v)\vec{q_3}(v)^\top\vec{q_2}(u^{'})\underline{q_1}(u^{'}) + \underline{q_1}(v)\vec{q_2}(v)^\top\vec{q_3}(u^{'})\underline{q_1}(u^{'})^\top\right]\times \\
& \underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_3}(s)^\top \\
- & \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\underline{q_1}(v)\vec{q_3}(v)^\top\vec{q_2}(u^{'})\underline{q_1}(u^{'}) + \underline{q_1}(u^{'})\vec{q_2}(u^{'})^\top\vec{q_3}(v)\underline{q_1}(v)^\top\right]\times \\
& \left.\underline{\Sigma}_{g_1}^{-1}\vec{x}\vec{x}^\top\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u)\vec{q_2}(u)^\top\right]du^{'}dudvds \\
+ & 2\int_0^t\int_0^s\int_0^u \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(u^{'})\vec{q_2}(u^{'})^\top\vec{q_3}(u)\vec{q_2}(u)^\top du^{'}duds \\
+ & \int_0^t\int_0^s \vec{q_3}(s)\underline{q_1}(s)^\top\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\underline{q_1}(s)\vec{q_3}(s)^\top\vec{q_2}(u)\vec{q_2}(u)^\top du^{'}duds \\
+ & \int_0^t\left[\int_0^s \vec{q_2}(u)\underline{q_1}(u)^\top du\right]\underline{\Sigma}_{g_1}^{-1}\left[\vec{x}\vec{x}^\top - \underline{\Sigma}_{g_1}\right]\underline{\Sigma}_{g_1}^{-1}\left[\int_0^s \underline{q_1}(u)\vec{q_2}(u)^\top du\right]\vec{q_3}(s)\vec{q_3}(s)^\top ds \\
+ & \int_0^t\int_0^s \vec{q_3}(s)\vec{q_3}(s)^\top\vec{q_2}(u)\vec{q_2}(u)^\top duds
\end{aligned}
$$

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Figure 1.1: Errors in the Expansion around the Normal distribution

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Table 1.1: Plain Vanilla Call Options-Square root process -(vol. = 10%)

Strike price	45	40	35
(1)Monte Carlo	0.5771	2.7226	6.7676
(2)Stochastic Expansion(second)	0.5763	2.7228	6.7640
Diff. Rate%	-0.144	0.007	-0.053
(3)Stochastic Expansion(first)	0.5548	2.7398	6.7796
Diff. Rate%	-3.865	0.632	0.178

Table 1.2: Plain Vanilla Call Options-Square root process -(vol. = 20%)

Strike price	45	40	35
(1)Monte Carlo	2.0005	4.1841	7.4802
(2)Stochastic Expansion(second)	1.9979	4.1858	7.4855
Diff. Rate%	-0.130	0.041	0.071
(3)Stochastic Expansion(first)	1.9460	4.2231	7.5776
Diff. Rate%	-2.724	0.932	1.303

Table 1.3: Plain Vanilla Call Options-Square root process -(vol. = 30%)

Strike price	45	40	35
(1)Monte Carlo	3.5347	5.7069	8.6453
(2)Stochastic Expansion(second)	3.5379	5.7105	8.6502
Diff. Rate%	0.091	0.064	0.057
(3)Stochastic Expansion(first)	3.4573	5.7674	8.8191
Diff. Rate%	-2.189	1.067	2.010

Table 1.4: Average Call Options on Equity -Square root process -(T=0.25y)

Strike price	45	40	35
(1)Monte Carlo	0.1559	1.4985	5.2659
(2)Stochastic Expansion	0.1562	1.4983	5.2679
Difference	0.00029	-0.00020	0.00210
Diff. Rate%	0.18	-0.01	0.04

Table 1.5: Average Call Options on Equity-Square root process-(T=0.50y)

Strike price	45	40	35
(1)Monte Carlo	0.5221	2.1758	5.6468
(2)Stochastic Expansion	0.5228	2.1788	5.6516
Difference	0.00078	0.00301	0.00482
Diff. Rate %	0.15	0.14	0.09

Table 1.6: Average Call Options on Equity-Square root process-(T=1.0y)

Strike price	45	40	35
(1)Monte Carlo	1.2802	3.1848	6.3845
(2)Stochastic Expansion	1.2813	3.1873	6.3881
Difference	0.00112	0.00255	0.00362
Diff. Rate %	0.09	0.08	0.06

Figure 1.2: Errors in the Expansion around the Log-normal distribution

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Table 1.7: Average Call Options on FX-Square root process- (T=0.25y)

Strike price	105	100	95
(1)Monte Carlo	0.0416	1.0217	4.7672
(2)Stochastic Expansion	0.0419	1.0215	4.7698
Difference	0.00031	-0.00025	0.00254
Diff. Rate %	0.75	-0.02	0.05

Table 1.8: Average Call Options on FX-Square root process-(T=0.50y)

Strike price	105	100	95
(1)Monte Carlo	0.1721	1.3625	4.6858
(2)Stochastic Expansion	0.1730	1.3654	4.6931
Difference	0.00090	0.00286	0.00730
Diff. Rate %	0.52	0.21	0.16

Table 1.9: Average Call Options on FX-Square root process-(T=1.0y,Vol.=10%)

Strike price	105	100	95
(1)Monte Carlo	0.4443	1.7700	4.6525
(2)Stochastic Expansion	0.4426	1.7709	4.6585
Difference	-0.00166	0.00090	0.00600
Diff. Rate %	-0.37	0.05	0.13

Table 1.10: Average Call Options on FX-Square root process(T=1.0y,Vol.=30%)

Strike price	110	100	90
(1)Monte Carlo	2.7995	6.18088	11.7334
(2)Stochastic Expansion	2.8045	6.1881	11.7464
Difference	0.00502	0.007221	0.00130
Diff. Rate %	0.18	0.12	0.11

Table 1.11: Average Options on FX -Log-normal process- (T=0.25y)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.0384	1.0199	4.7738
Diff. Rate %	-15.97	-0.19	0.16
(2)Stochastic Expansion(normal)(2nd)	0.0452	1.0220	4.7650
Diff. Rate %	-1.09	-0.02	-0.02
(3)Stochastic Expansion(log-normal)(1st)	0.0434	1.0114	4.7472
Diff. Rate %	-5.29	-1.03	-0.40
(4)Stochastic Expansion(log-normal)(2nd)	0.0454	1.0215	4.7657
Diff. Rate %	-0.65	-0.03	-0.01
(5)Finite difference(Crank-Nicholson method)	0.0457	1.0216	4.7659
Diff. Rate %	0.01	-0.02	-0.00
(6)Monte Carlo simulation method	0.0457	1.0218	4.7660

Table 1.12: Average Options on FX -Log-normal process- (T=0.50y)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.1620	1.3610	4.7040
Diff. Rate %	-11.96	-0.53	0.53
(2)Stochastic Expansion(normal)(2nd)	0.1830	1.3660	4.6800
Diff. Rate %	-0.54	-0.16	0.01
(3)Stochastic Expansion(log-normal)(1st)	0.1753	1.3457	4.6483
Diff. Rate %	-4.96	-1.67	-0.67
(4)Stochastic Expansion(log-normal)(2nd)	0.1830	1.3655	4.6804
Diff. Rate %	-0.54	-0.20	-0.02
(5)Finite difference(Crank-Nicholson method)	0.1831	1.3656	4.6788
Diff. Rate %	-0.49	-0.19	-0.01
(6)Monte Carlo simulation method	0.1840	1.3682	4.6793

Table 1.13: Average Options on FX -Log-normal process- (T=1.00y,Vol.=10%)

Strike price	105	100	95
(1)Stochastic Expansion(normal)(1st)	0.4180	1.7590	4.6750
Diff. Rate %	-10.30	-0.61	0.81
(2)Stochastic Expansion(normal)(2nd)	0.4640	1.7720	4.6410
Diff. Rate %	-0.43	-0.12	0.08
(3)Stochastic Expansion(log-normal)(1st)	0.4428	1.7329	4.5773
Diff. Rate %	-4.98	-2.13	-1.31
(4)Stochastic Expansion(log-normal)(2nd)	0.4640	1.7713	4.6328
Diff. Rate %	-0.43	-0.08	-0.10
(5)Finite difference(Crank-Nicholson method)	0.4640	1.7715	4.6315
Diff. Rate %	-0.43	-0.09	-0.13
(6)Monte Carlo simulation method	0.4660	1.7699	4.6375

Table 1.14: Average Options on FX -Log-normal process- (T=1.00y,Vol.30%)

Strike price	110	100	90
(1)Stochastic Expansion(normal)(1st)	2.6107	6.1516	11.8900
Diff. Rate %	-12.22	-0.76	2.61
(2)Stochastic Expansion(normal)(2nd)	2.9699	6.1910	11.5751
Diff. Rate %	-0.14	-0.12	-0.11
(3)Stochastic Expansion(log-normal)(1st)	2.6563	5.7746	11.0569
Diff. Rate %	-10.68	-6.84	-4.58
(4)Stochastic Expansion(log-normal)(2nd)	2.9505	6.1727	11.5571
Diff. Rate %	-0.79	-0.42	-0.26
(6)Monte Carlo simulation method	2.9740	6.1985	11.5874

Table 1.15: Average Options on 1-Year Interest Rate(T=0.25y)

Strike rate %	5.50	5.00	4.50
(1)Stochastic Expansion	5.36	25.12	63.98
Difference (bp)	-0.034	0.002	0.080
Diff. rate %	-0.56	0.01	0.13
(2)Finite difference	5.36	24.99	63.81
Difference (bp)	0.026	-0.13	-0.09
Diff. rate %	-0.48	-0.52	-0.14
(3)Monte Carlo	5.39	25.12	63.90
(4)European call	16.30	38.05	71.76
(3)/(4) %	33	66	89

Table 1.16: Average Options on 1-Year Interest Rate(T=0.50y)

Strike rate %	6.00	5.00	4.00
(1)Stochastic expansion	2.69	32.10	111.54
Difference (bp)	0.005	0.121	-0.010
Diff. rate %	0.20	0.38	-0.09
(2)Finite difference	2.68	31.98	111.34
Difference (bp)	-0.0066	0.002	-0.21
Diff. rate %	-0.25	-0.16	0.23
(3)Monte Carlo	2.69	31.98	111.55
(4)European call	13.86	50.47	119.64
(3)/(4) %	19	63	93

Table 1.17: Average Options on 1-Year Interest Rate(T=1.00y)

Strike rate %	6.00	5.00	4.00
(1)Stochastic expansion	8.13	41.37	112.30
Difference (bp)	0.040	-0.030	-0.010
Diff. rate %	0.49	0.07	-0.01
(2)Finite difference	8.06	41.32	112.25
Difference (bp)	-0.03	-0.017	-0.060
Diff. rate %	-0.37	-0.04	-0.05
(3)Monte Carlo	8.09	41.34	112.31
(4)European call	28.14	67.26	129.60
(3)/(4) %	29	62	87

Chapter 2

A Variable Reduction Technique for Pricing Average-Rate Options

1 Introduction

"Average-rate options", commonly known as Asian options, are contingent claims whose payoffs depend on the arithmetic average of some underlying index (e.g., stock prices, exchange rates or interest rates) over a fixed time horizon. While no average-rate options are traded as standardized option contracts in any organized options or futures exchange in the world, these options, especially those with the underlying being exchange rates or interest rates, are extremely popular in the over-the-counter market among institutional investors.

There are many economic reasons why average-rate options are so popular. For example, if a corporation expects to receive or pay foreign currency claims on a regular basis, then a foreign currency option based on an average of exchange rates represents one way to reduce its average foreign currency exposure. Similar argument can be made for an interest rate option based on an average of short term LIBOR rates or an average of constant maturity yields (CMS). Since the average of the underlying tends to be much less volatile than the underlying itself, average-rate options are priced more cheaply than the standard (plain vanilla) options. This reduces significantly the hedging costs for corporations in need of average-rate options. In addition, by its very design, the payoff of the average-rate options is less dependent on the closing price of the underlying near the expiration date. Thus, it reduces the significance of market impact or price manipulation at the maturity of the option.

The pricing and hedging of average-rate options raise some interesting issues. First, these options are path-dependent, i.e., the value of an average-rate option at any point in time depends upon the value of the underlying at that time as well as the history of the underlying up to that time. More specifically, if the underlying follows a Markov-diffusion process, then the value of an average-rate option depends on the current underlying as well as the average of the underlying at that time. Thus, when applying standard option pricing techniques (such as the binomial method or the partial differential equation method), a second state variable (in addition to the underlying itself) is often necessary. This makes the pricing problem much more complicated. Second, the arithmetic average is not lognormally distributed when the underlying follows a standard lognormal process. In fact, it is impossible to find analytically the probability distribution of the arithmetic average when the underlying is lognormally distributed. Due to the above reasons, it is well known that no analytical solution exists for the price of European calls or puts written on the arithmetic average when the underlying index follows a lognormal process.[1] Consequently, numerical techniques must be relied upon in order to determine the value of average-rate options.

There are several types of numerical techniques that have become popular for valuing average-rate options. The first one, which perhaps is also the most simple and commonly used one, is the Monte Carlo simulations method as discussed in Kemma and Vorst (1990) for the case when the underlying is a lognormal process. The Monte Carlo simulations method is convenient and flexible. In particular, it is applicable as long as the underlying follows a Markov-diffusion process. For example, it can be applied to a square root process for interest rates as well. However, in terms of the computing time required, this method is not very efficient.

The second type of numerical techniques for average-rate options explores the idea that an arithmetic average can be reasonably approximated by a geometric average with an appropriately adjusted mean and variance. This technique includes i) the modified-strike method (Vorst, 1990), which replaces the arithmetic average by

[1] Geman and Yor (1993) have developed a semi-analytical valuation method using the Laplace transformation technique.

a geometric average with an adjustment in the strike price to correct the mean bias; ii) the modified-geometric method (Kunitomo and Takahashi, 1992), which replaces the arithmetic average by a geometric average with its mean and variance adjusted to match the mean and variance of the arithmetic average; iii) the geometric conditioning method, which replaces the arithmetic average by its conditional expectation conditioning on the geometric average (Curran, 1992); and iv) the Edgeworth series expansion method (Turnbull and Wakeman, 1991), which applies an expansion of the distribution of the arithmetic average around the distribution of the geometric average (which is lognormal). The above mentioned methods have been shown to be reliable whenever the volatility of the underlying is not too large (e.g., less than 30%). However, numerical errors can become significant when the volatility is high.

The third type of numerical techniques for average-rate options addresses the arithmetic average in a more direct way. Specifically, these techniques put forward various discrete time models (e.g., binomial trees or grids) to approximate the continuous time value of the average-rate options. Hull and White (1993) have developed an extended Binomial method in which they construct a binomial tree with a vector of average rates stacking at each node. Conditioning on the current value of the average rate, they apply the standard recursive valuation method for each level of average rates chosen in the vector of average rates. A similar idea has been carried out by Dewynne and Wilmott (1993) in solving numerically the partial differential equation for average-rate options. Carverhill and Clewlon (1990) have developed another approach using the Fourier transformation technique. Their approach involves calculating the distribution function of the arithmetic average through the Fast Fourier Transform technique. All of the three methods mentioned above require intensive computing time, as they have to handle more or less a two-dimensional valuation problem.

In this paper we propose a new valuation technique, called the variable reduction technique, for average-rate options. This method has many advantages over the various techniques described above. The main idea of our method is quite simple. Basically, this method transforms the valuation problem of an average-rate option into an evaluation of a conditional expectation that is determined by a one-dimensional Markov process (as suppose to a two-dimensional Markov process commonly known).[2] This transformation is extremely useful since numerically it is much easier to handle a one-dimensional valuation problem than a two-dimensional problem. Alternatively, we can also derive a partial differential equation that the value function of an average-rate option must satisfy. In this case, the PDE is a second order parabolic one with one state variable and one time variable. Standard numerical techniques can be applied to evaluate the conditional expectation or to solve numerically the partial differential equation that determines the value of an average-rate option.

Compared to the geometric approximation technique, our variable reduction technique works directly with the arithmetic average, and therefore will not encounter the cited approximation errors when the volatility of the underlying is relatively large. Furthermore, compared to the methods proposed by Hull and White (1993) and Dewynne and Wilmott (1993), our technique has reduced the dimensionality by one, which certainly will make our pricing more efficient in terms of computing time. Finally, there is no doubt that this technique is more favorable than the Monte Carlo method when the underlying is a lognormal process. Unfortunately, when the underlying is not lognormal, the variable reduction technique is no longer applicable for average-rate options.

The rest of the paper is organized as follows. In the next section, we illustrate the variable reduction technique in the simple Black-Scholes' economy in which there is one risky asset and one riskless bond. While much of the analysis in the paper assumes continuous averaging or continuous fixing, we will briefly discuss in this section the implementation of discrete averaging. In Section 3, we apply the variable reduction technique to average-rate options where the underlying index is an interest rate (for example, LIBOR rates with a constant maturity). Numerical comparisons of different methods are presented in Section 4. We conclude the paper in Section 5.

[2] Ingersoll (1987) has shown that an average-strike option, i.e., an option whose strike price is the average of the underlying over a fixed horizon, can be handled by factoring out the average-strike and thereby reducing the two-state variables problem into a one-state variable problem. Wilmott, Dewynne and Howison (1993) have provided a similar variable reduction technique to the partial differential equation that the value of an average-strike option must be satisfied. However, Wilmott, Dewynne and Howison (1993) claim that the same technique doesn't work for average-rate options.

2 Average-Rate Options in the Black and Scholes' Economy

To illustrate the idea of the variable reduction technique, we first consider the Black and Scholes' economy (Black and Scholes, 1973) in which there is one riskless bond and one risky asset, and the prices of the riskless bond and risky asset are determined as follows,

$$\begin{aligned} B_t &= e^{rt} \\ S_t &= S_0 e^{(\mu - q - \frac{1}{2}\sigma^2)t + \sigma w_t}, \qquad t \geq 0 \end{aligned}$$

and w is a standard Brownian motion defined on a probability space, coefficients r, μ, q and σ are constants, and q is the implicit payout rate. The risky asset here could be a stock, foreign currency or commodity. The payout rate therefore would be the dividend yield, the foreign riskless rate, and the convenience yield, respectively. We assume that there exists a risk neutral probability or equivalent martingale measure Q under which the price of the risky asset is determined by

$$S_t = S_0 e^{(r - q - \frac{1}{2}\sigma^2)t + \sigma \tilde{w}_t}$$

where $\tilde{w}$ is a standard Brownian motion under Q.

An average-rate European call option is defined to be an option that gives the holder the right (but not the obligation) to receive at the expiration date the arithmetic average of the price of the underlying asset over a finite time horizon for a fixed strike price. A similar definition can be made for an average-rate European put option. In this paper we will not consider average-strike options whose strike price is a fixed percentage of the average of the price of the risky asset over a fixed time horizon. However, the variable reduction technique works for average-strike options as well as shown in Ingersoll (1987) and Wilmott, Dewynne and Howison (1993). However, none of these two works were able to extend their approaches to average-rate options. In fact, Wilmott, Dewynne and Howison claimed that their variable reduction technique doesn't work for average-rate options.

Following Cox and Ross (1975) and Harrison and Kreps (1979), the price at time t of an average-rate call option, $C(t)$, with a maturity date T and a strike price K

can be evaluated by

$$C(t) = e^{-r(T-t)}\,\mathbf{E}_t^*\Big[(\frac{1}{T}A_T - K)^+\Big] \tag{2.1}$$

where $\mathbf{E}_t^*$ denotes the conditional expectation under the risk neutral probability distribution, or equivalently, the equivalent martingale measure Q, conditional on the information set at time t, and

$$A_t = \int_0^t S_u du$$

A similar definition can be made for an average-rate put option.

We note that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\tilde{w}_t \\ dA_t &= S_t dt \end{aligned}$$

under the risk neutral probability distribution Q, and therefore S and A together form a two dimensional Markov process under Q. Thus, the value of an average-rate call option at t $(< T)$ must be a function of S_t, A_t and t, i.e., $C = C(S, A, t)$. Moreover, since $e^{-rt}C$ must be a martingale under Q, the drift of $e^{-rt}C$ under Q must be zero. This leads to the partial differential equation for C,

$$\frac{\sigma^2}{2}S^2 C_{SS} + (r-q)SC_S + SC_A + C_t - rC = 0 \tag{2.2}$$

This is a second-order partial differential equation (PDE) with two space variables and one time variable. Moveover, the second order partial derivative with respect to A is degenerate. Numerical solutions of this partial differential equation is possible but cumbersome as well as time-consuming.

We now introduce the variable reduction method which transforms (2.2) into a PDE with only one state variable and one time variable. To motivate our transformation, let us re-write the valuation equation (2.1) as follows,

$$\begin{aligned} C(t) &= e^{-r(T-t)}\mathbf{E}_t^*\Big[\frac{1}{T}A_t - K + \frac{1}{T}\int_t^T S_u du\Big]^+ \\ &= \frac{S_t}{T}e^{-r(T-t)}\,\mathbf{E}_t^*\Big[x_t + \int_t^T \frac{S_u}{S_t}du\Big]^+ \end{aligned}$$

where we have introduced a new state variable x determined by

$$x_t = \frac{1}{S}(A_t - TK)$$

Since S_u/S_t $(u > t)$ is independent of the history of S up to t, the conditional expectation in the above equation must be a function of x_t. Thus, C can be written as a function of x_t and t multiplied by S_t, i.e.,

$$C(S_t, A_t, t) = S_t f(x_t, t)$$

for some function f of x and t only:

$$f(x,t) = \frac{e^{-r(T-t)}}{T}\mathbf{E}_t^*\Big[x + \int_t^T \frac{S_u}{S_t}du\Big]^+$$

A simple calculation shows

$$\begin{aligned} C_S &= f - xf_x \\ C_{SS} &= \frac{1}{S}x^2 f_{xx} \\ C_A &= f_x \\ C_t &= Sf_t \end{aligned}$$

Substituting these relations into the above PDE for C, we get

$$S \times \left[\frac{\sigma^2}{2}x^2 f_{xx} + (1-\alpha x)f_x + f_t - qf\right] = 0$$

where $\alpha = r - q$. Since $S_t > 0$ for all t, we obtain the partial differential equation for f,

$$\frac{1}{2}\sigma^2 x^2 f_{xx} + (1-\alpha x)f_x + f_t - qf = 0 \tag{2.3}$$

The boundary condition is given by

$$f(x,T) = \frac{1}{T}\max[x, 0] \tag{2.4}$$

The value of the call option at time t is then given by $S_t f(x_t, t)$. In other words, the variable reduction technique has helped us to factor out S from the call price. We summarize our results in the following proposition.

Proposition 2.1 *The value of an average-rate call option is determined by $S_t f(x_t, t)$, where f satisfies the PDE (2.3) and the boundary condition (2.4), and where $x_t = (A_t - TK)/S_t$.*

It is useful to note that the stochastic process x is a diffusion process by itself, i.e.,

$$dx_t = (1 - \alpha x_t + \sigma^2 x_t)dt - \sigma x_t d\tilde{w}_t$$

This explains why we are able to factor out S from the PDE (2.2) to get the PDE (2.3). Moreover, if we introduce a pseudo probability measure Q' in such a way that

$$dx_t = (1 - \alpha x_t)dt - \sigma x_t dw'_t$$

where w' is a standard Brownian motion under Q', then (2.3) is equivalent to the statement that under Q', the discounted value (f) is a martingale, while the discount rate is the implicit payout rate q, i.e.,

$$f(x_t, t) = \mathbf{E}'_t \left[\frac{e^{-q(T-t)}}{T} \max[x_T, 0]] \right] \tag{2.5}$$

where the expectation is taken under Q'. The above formula is also called the Feynman-Kac representation of the partial differential equation (2.3), see Karatzas and Shreve (1988).

For readers who are familiar with the Harrison and Kreps' argument, it can be shown that $e^{rt}/S_t e^{qt}$ is a martingale under Q', i.e., the riskless asset price discounted by the risky asset price (after adjusted by the payout rate) is a martingale under Q'. Thus, Q' is the equivalent martingale measure when the risky asset is chosen as a numeraire. Moreover, the martingale argument allow us to claim that

$$\frac{C_t}{S_t e^{qt}} = \mathbf{E}'_t \left[\frac{\frac{A_T}{T} - K}{S_T e^{qT}} \right]^+$$

Introducing x_t as we did above and realizing that x is a Markov process by itself under Q', we can immediately conclude that the right-hand side of the above equation must be a function of x_t and t. In other words, S can be factored out using the martingale argument as well.

We also note that when $x_t \geq 0$ $(t < T)$, one can easily obtain an explicit formula for f,

$$f(x, t) = \frac{1}{T} e^{-r(T-t)} x + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r - q)} \tag{2.6}$$

However, for $x_t < 0$, we must utilize standard numerical techniques such as the finite difference method or the Monte carlo simulations method to evaluate f from the PDE (2.3) or from the conditional expectation that defines f in (2.5). Note that if we are to apply the finite difference method, then we need to use the boundary conditions (2.6), $\lim_{x\to-\infty} f(x,t) = 0$ for small values of x and (2.4) for large value of x.

Our variable reduction technique can also be applied to average-rate options where the averaging is taken at a discrete set of time points, i.e., discrete averaging or discrete fixing. For illustrations, we assume that averaging takes place at points $0 = t_1 < t_2 < t \cdots < t_n = T$. For simplicity, let us consider evaluating the option exactly on the points where averaging takes place.[3] Define

$$\begin{aligned} A_{t_k} &\equiv \sum_{i=1}^{k} S_{t_i}, \\ x_{t_k} &\equiv \frac{A_{t_k} - nK}{S_{t_k}}, \quad k = 1, 2, \cdots, n \end{aligned}$$

Applying the same variable reduction technique, we can show that

$$\begin{aligned} C_{t_k} &= e^{-r(t_n - t_k)} \frac{S_{t_k}}{n} \mathbf{E}^*_{t_k} \left[x_{t_k} + \sum_{i=k+1}^{n} \frac{S_{t_i}}{S_{t_k}} \right]^+ \\ &\equiv S_{t_k} f(x_{t_k}, t_k) \end{aligned}$$

where

$$\begin{aligned} f(x_{t_k}, t_k) &= \frac{e^{-r(t_n - t_k)}}{n} \mathbf{E}^*_{t_k} \left[x_{t_k} + \sum_{i=k+1}^{n} \frac{S_{t_i}}{S_{t_k}} \right]^+, \quad k = 1, 2, \cdots, n-1 \\ f(x_{t_n}, t_n) &= \frac{1}{n} x^+_{t_n} \end{aligned}$$

It then follows that

$$\begin{aligned} f(x_{t_k}, t_k) &= e^{-r(t_{k+1} - t_k)} \mathbf{E}^*_{t_k} \left[\frac{S_{t_{k+1}}}{S_{t_k}} f(x_{t_{k+1}}, t_{k+1}) \right] \\ x_{t_{k+1}} &= \frac{S_{t_k}}{S_{t_{k+1}}} x_{t_k} + 1 \end{aligned} \tag{2.7}$$

We note that

$$\begin{aligned} \frac{S_{t_{k+1}}}{S_{t_k}} &= e^{(r - q - \frac{\sigma^2}{2})\Delta t_k + \sigma\sqrt{\Delta}\epsilon} \\ \frac{S_{t_k}}{S_{t_{k+1}}} &= e^{-(r - q - \frac{\sigma^2}{2})\Delta t_k - \sigma\sqrt{\Delta}\epsilon} \end{aligned}$$

[3] A similar approach can be used to value the option at times other than those averaging points.

where ϵ is a random variable distributed as $N(0,1)$, and $\Delta t_k = t_{k+1} - t_k$. Thus, (2.7) can be solved recursively by numerical integrations.[4]

Alternatively, we can also determine f by

$$f(x_{t_k}, t_k) = \frac{e^{-q(t_n - t_k)}}{n} \mathbf{E}_t' \Big[\max[x_{t_n}, 0] \Big]$$

This formula is useful if we would like to value f by Monte-Carlo simulation. Note that in this case we will be simulating the process x under the probability measure Q'.

It is important to note that as in the case of continuous averaging, the valuation problem here is also a one-dimensional problem. We have avoided the complexity of the two dimensional problem encountered by Hull and White (1993) and Dewynne and Wilmott (1993).

Before leaving this section, we point out that all of our analyses so far are equally applicable when σ and r are functions of t. This suggests that we can value average-rate options when we have a deterministic term structure of volatilities and interest rates.

3 Average-Rate Options on Interest Rates

We now apply the variable reduction technique to value average-rate options on interest rates related derivative instruments. Such instruments are commonly traded in the over-the-counter (OTC) markets, and have played an important role in satisfying various needs of institutional investors or borrowers. Among those interest rates related derivative instruments, options on the average of CMT or CMS rates (constant maturity treasury yields or constant maturity swap rates) have been somewhat popular. Those option contracts can also be imbedded in a swap transaction to serve as speculative or hedging purposes for the investors or the issuers.

In this section we present the variable reduction technique for pricing the average-rate options on CMS rates. First, let us define an option on the average of CMS rates (with a fixed time to maturity). Let $L^\tau(t)$ denote the yield at time t for a zero

[4] Specifically, we can fix a set of grid points for x, and evaluate f over these points recursively. For those points that are not on the grids, a second order interpolation can be used to find the value of f on these points.

coupon bond with a time to maturity of τ years.[5] Then, the average of $L^\tau(t)$ in a prespecified time period $[0, T]$ is given by

$$Z(T) = \frac{1}{T}\int_0^T L^\tau(t)dt$$

where τ is a fixed real number, e.g., 0.25, 0.5, or 1. Let $P(t, T)$ denote the price at time t of a zero coupon bond maturing at time T. Then, we can re-express $L^\tau(t)$ as

$$L^\tau(t) = \left(\frac{1}{P(t, t+\tau)} - 1\right)\frac{1}{\tau}$$

The payoff of a European call option on the average rates at the expiration date T with a strike price K is given by

$$C(T) = \max[Z(T) - K, 0]$$

while the payoff for a European put option is given by

$$C(T) = \max[K - Z(T), 0]$$

In the rest of this section, we determine the arbitrage-free value of such European call or put options by using the similar technique introduced in the previous section. Note that once the call price is obtained, the value of a put option can be easily derived through the "put-call parity", which will be shown later in this section.

3.1 Arbitrage-Free Forward Rate Processes

To evaluate an average-rate option on interest rates with a constant maturity, we employ the Heath-Jarrow-Morton's model as our basic model for term structure of interest rates. This model is based on an explicit specification of the instantaneous forward rates and a restriction of no arbitrage, see Heath, Jarrow, and Morton (1992) for details. In this setting, the instantaneous forward rate process under the equivalent martingale measure is described as

$$f(t, T) = f(0, T) + \sum_{i=1}^{N}\int_0^t \left(\sigma_i(s, T)\int_s^T \sigma_i(s, u)du\right)ds + \sum_{i=1}^{N}\int_0^t \sigma_i(s, T)d\tilde{w}_i(s)$$

[5] For simplicity, we will not consider pricing average rate options based on par yields.

where w is an N-dimensional standard Brownian motions. In the above specification, the diffusion term or the volatility process σ can be chosen by the user (subject to a regularity condition) while the drift is completely determined by the choice of σ, due to the no arbitrage condition. In particular, the spot rate process, $r(t) = f(t,t)$, is given by

$$r(t) = f(0,t) + \sum_{i=1}^{N} \int_0^t \left(\sigma_i(s,t) \int_s^t \sigma_i(s,u)du \right) ds + \sum_{i=1}^{N} \int_0^t \sigma_i(s,t) d\tilde{w}_i(s)$$

Given the spot rate, any interest rate contingent claim can be priced through the well-known property that the value process relative to the money market account is a martingale under the equivalent martingale measure or the risk neutral probability:

$$\frac{V(t)}{B(t)} = \mathbf{E}_t \left[\frac{V(T)}{B(T)} \right]$$

where $B(t) = \exp[\int_0^t r(s)ds]$, and $B(t)$ denotes the value process of the money market account.

3.2 A Constant Volatility Model

For simplicity, we shall specify a one-factor model of forward rates (with a constant volatility) in order to evaluate the average-rate options under consideration. That is, we set $N = 1$ in the forward rate process described above. The volatility function in the forward rate process is given by $\sigma_1(s,t) = \sigma$, where σ is a positive constant. This model is known to be a continuous time version of the Ho and Lee (1986)'s model. Specifically, the forward rate process can be described as

$$f(t,T) = f(0,T) + \sigma^2 (Tt - \frac{t^2}{2}) + \sigma \tilde{w}(t)$$

and the spot rate process is given by

$$r(t) = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma \tilde{w}(t)$$

A straightforward calculation shows that the price at time t of a zero coupon bond maturing at time T is given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{\sigma^2}{2} Tt(T-t) - \sigma(T-t)\tilde{w}(t) \right] \tag{2.8}$$

3.3 Pricing Average-Rate Options on Interest Rates

We will evaluate the average-rate options on interest rates under a term structure model with a constant volatility. As in the average rate options under the Black-Scholes' economy, we will show that the valuation problem can be simplified to a partial differential equation with a single state variable and a time variable, after a simple transformation of variables. It is much easier to solve this equation numerically, for example, by the finite difference method. Moreover, in the spacial case where the option is deep-in-the money, an explicit valuation formula is obtained. As mentioned before, we will consider only the call option case, as the put option can be priced through a "put-call parity".

First, we will rewrite our valuation problem in terms of zero coupon bonds. Then, the final payoff of the call option is re-expressed as

$$\begin{aligned} C(T) &= \max[Z(T) - K, 0] \\ &= \max[\frac{1}{T}\int_0^T \{\frac{1}{P(t,t+\tau)} - 1\}\frac{1}{\tau}dt - K, 0\,] \\ &= \frac{1}{T\tau}\max[\int_0^T \frac{1}{P(t,t+\tau)}dt - k, 0\,] \end{aligned}$$

where τ is a positive constant and $k = (1 + K\tau)T$.

By using the expression (2.8), the reciprocal of the price of a zero coupon bond with τ years to maturity, $\frac{1}{P(t,t+\tau)}$, is described as

$$\frac{1}{P(t,t+\tau)} = \frac{P(0,t)}{P(0,t+\tau)}\exp\left[\frac{\sigma^2}{2}(t+\tau)t\tau + \sigma\tau\tilde{w}(t)\right]$$

Hence, the price of the average-rate call option at time t (before the maturity date T) is given by

$$\begin{aligned} C(t) &= \mathbf{E}_t^*\left[\exp\left(-\int_t^T r(u)du\right)C(T)\right] \\ &= \frac{1}{T\tau}\frac{P(0,T)}{P(0,t)}e^{-\frac{\sigma^2(T^3-t^3)}{6}}\mathbf{E}_t^*\left[e^{-\sigma\int_t^T \tilde{w}(u)du} \times C(T)\right] \end{aligned} \qquad (2.9)$$

where we use the relation, $\exp(-\int_t^T f(0,u)du) = P(0,T)/P(0,t)$, and where

$$C(T) = \max\left[\int_0^T \frac{P(0,u)}{P(0,u+\tau)}\exp[\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau\tilde{w}(u)]du - k, 0\right]$$

Our main objective is to evaluate the conditional expectation in (2.9). Although it is a fairly tough task to evaluate this expectation directly, if the new variable defined below is introduced, the problem becomes much easier to handle. We will give this transformation in the next lemma.

Lemma 2.1 *Define a stochastic process X as*

$$X(t) = \frac{\int_0^t \frac{P(0,u)}{P(0,u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau\tilde{w}(u)} du - k}{e^{\sigma\tau\tilde{w}(t)}}$$

Then, X satisfies the following stochastic differential equation,

$$dX_t = \left(\frac{P(0,t)}{P(0,t+\tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2} X_t \right) dt - \sigma\tau\ X_t d\tilde{w}(t) \tag{2.10}$$

where $X(0) = -k$, and the value at time t of the average-rate call can be expressed as

$$\begin{aligned} C(t) \quad = \quad & \frac{1}{T\tau} e^{-\int_t^T f(0,u)du} e^{-\frac{\sigma^2(T^3-t^3)}{6}} e^{\sigma(t-T+\tau)\tilde{w}(t)} \times \\ & \mathbf{E}_t^* \Bigg[e^{-\sigma \int_t^T (\tilde{w}(u)-\tilde{w}(t))du} \times \\ & \max[X_t + \int_t^T \frac{P(0,u)}{P(0,u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u)-\tilde{w}(t)]} du, 0] \Bigg] \end{aligned} \tag{2.11}$$

The expectation on the right-hand side is a function of X_t.

PROOF. Equation (2.10) can be shown by Ito's lemma while (2.11) can be verified easily. The last statement follows from the fact that $\tilde{w}(u) - \tilde{w}(t)$ is independent of $w(t)$. ∎

Here, we note that $\tilde{w}(t)$ can be expressed in terms of r_t. Therefore, $C(t)$ can be separated into the product of a function of (r,t) and a function of (X,t).

$$C(r,X,t) = g(r,t)h(X,t) \tag{2.12}$$

where

$$\begin{aligned} g(r,t) \quad &= \quad \frac{1}{T\tau} \frac{P(0,T)}{P(0,t)} e^{-\frac{\sigma^2(T^3-t^3)}{6}} \times e^{(t-T+\tau)[r_t - f(0,t) - \frac{(\sigma t)^2}{2}]} \\ h(X,t) \quad &= \quad \mathbf{E}_t^* \left[e^{-\sigma \int_t^T (\tilde{w}(u)-\tilde{w}(t))du} \times \max[X_t + P_u^{u+\tau} \int_t^T e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u)-\tilde{w}(t)]} du, 0] \right]. \end{aligned}$$

where $P_u^{u+\tau} = \frac{P(0,u)}{P(0,u+\tau)}$. Clearly, $g(r,t)$ is calculated using the information available at time t while $h(X,t)$ turns out to satisfy a partial differential equation as shown in the following proposition.

Proposition 2.2 *h satisfies the partial differential equation*

$$\begin{aligned}\frac{(\sigma\tau)^2}{2}X^2 h_{XX} + \left[\frac{P(0,t)}{P(0,t+\tau)}e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \{\sigma^2\tau(T-t) - \frac{(\sigma\tau)^2}{2}\}X_t\right] h_X + \\ h_t \ + \ \frac{\sigma^2(t+\tau-T)^2}{2}h = 0 \qquad (2.13)\end{aligned}$$

with the boundary condition

$$h(X,T) = \max[X_T, 0]. \qquad (2.14)$$

Proof. Since $C(t)\exp(-\int_0^t r(s)ds)$ is a martingale under the equivalent martingale measure, the drift of $C(t)\exp(-\int_0^t r(s)ds)$ must be 0. Recall that

$$\begin{aligned} dr &= \{f_t(0,t) + \sigma^2 t\}dt + \sigma d\tilde{w}(t) \\ dX &= \Big(\frac{P(0,t)}{P(0,t+\tau)}e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2}X\Big)dt - \sigma\tau\ X d\tilde{w}(t). \end{aligned}$$

It is easily seen by the Itô's lemma that

$$\begin{aligned}&\frac{\sigma^2}{2}C_{rr} + \frac{(\sigma\tau X)^2}{2}C_{XX} - \sigma^2\tau X C_{rX} + [f_t(0,t) + \sigma^2 t]C_r \\ &+ \left[\frac{P(0,t)}{P(0,t+\tau)}e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2}X_t\right] C_X + C_t - rC = 0 \qquad (2.15)\end{aligned}$$

Since $C(r,X,t) = g(r,t)h(X,t)$, simple calculations show

$$\begin{aligned} C_r &= g_r h = (t+\tau-T)gh \\ C_{rr} &= g_{rr}h = (t+\tau-T)^2 gh \\ C_{rX} &= g_r h_X = (t+\tau-T)gh_X \\ C_X &= gh_X \\ C_{XX} &= gh_{XX} \\ C_t &= g_t h + gh_t = r\ gh - (t+\tau-T)[f_t(0,t)+\sigma^2 t]\ gh + gh_t. \end{aligned}$$

Substituting the above relations into (2.15), we obtain

$$\begin{aligned} g \times \Bigg[\ &\frac{(\sigma\tau X)^2}{2}h_{XX} + \Big(\frac{P(0,t)}{P(0,t+\tau)}e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \{\sigma^2\tau(T-t) - \frac{(\sigma\tau)^2}{2}\}X_t\Big)h_X \\ &+h_t + \frac{\sigma^2(t+\tau-T)^2}{2}h\Bigg] = 0. \qquad (2.16)\end{aligned}$$

Next, noting that $g(r,t) > 0$ for all t and r, we obtain the desired partial differential equation for h. Finally, it is easily seen that the terminal boundary condition is given by

$$h(X,T) = \max[X_T, 0]$$

∎

When $X_t \geq 0$, i.e., when the option is very deep-in-the money, we can show that h can be calculated explicitly. We present this result in the following proposition.

Proposition 2.3 *When $X_t \geq 0$, the price of an average-rate call option on interest rates is given by*

$$C(r,x,t) = g(r,t)h(X,t)$$

where $g(r,t)$ is defined as above and $h(X,t)$ is given by

$$\begin{aligned} h(X,t) &= X_t e^{\frac{\sigma^2(T-t)^3}{6}} + \exp\left(\frac{\sigma^2}{2}(t+\tau)t\tau + \frac{\sigma^2(T-t)^3}{6}\right) \\ &\quad \times \int_0^{T-t} \frac{P(0,s+t)}{P(0,s+t+\tau)} \exp\left[\sigma^2 \tau s^2 + \sigma^2(\tau^2 + 2t\tau - T\tau)s\right] ds \end{aligned} \quad (2.17)$$

Proof. Note that when $X_t \geq 0$,

$$\begin{aligned} &\max[X_t + \int_t^T \frac{P(0,u)}{P(0,u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau \; + \; \sigma\tau[\tilde{w}(u)-\tilde{w}(t)]} du, \; 0\,] \\ &= X_t + \int_t^T \frac{P(0,u)}{P(0,u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau \; + \; \sigma\tau[\tilde{w}(u)-\tilde{w}(t)]} du. \end{aligned}$$

Hence

$$\begin{aligned} h(X,t) &= X_t \mathbf{E}_t^* \left[e^{-\sigma \int_t^T [\tilde{w}(u)-\tilde{w}(t)]} du \right] \\ &\quad + \mathbf{E}_t^* \left[e^{-\sigma \int_t^T [\tilde{w}(u)-\tilde{w}(t)] du} \int_t^T \frac{P(0,u)}{P(0,u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u)-\tilde{w}(t)]} du \right]. \end{aligned}$$

By the strong Markov property of the Brownian motion,

$$\begin{aligned} h(X,t) &= X_t \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \right] + \\ &\quad \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \int_0^{T-t} \frac{P(0,s+t)}{P(0,s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau + \sigma\tau\tilde{w}(s) ds} \right]. \end{aligned}$$

The first term can be calculated using the fact that

$$\sigma \int_0^{T-t} \tilde{w}(s) ds = \sigma \int_0^{T-t} (T-t-u) d\tilde{w}(u)$$

is normally distributed with

$$\mathbf{E}^* \left[\sigma \int_0^{T-t} \tilde{w}(s) ds \right] = 0$$

$$\mathbf{Var}^* \left[\sigma \int_0^{T-t} \tilde{w}(s) ds \right] = \frac{\sigma^2 (T-t)^3}{3}$$

We obtain

$$X_t \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \right] = X_t e^{\frac{\sigma^2 (T-t)^3}{6}}$$

For the second term, we apply the Fubini's theorem to claim that

$$\mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau + \sigma\tau\tilde{w}(s)} ds \right]$$

$$= \int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau} \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s)} \right] ds.$$

Note that

$$-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) = -\sigma \int_0^{T-t} (T-t-u) d\tilde{w}(u) + \sigma\tau \int_0^s d\tilde{w}(u). \quad (2.18)$$

is normally distributed with

$$\mathbf{E}^* \left[-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) \right] = 0$$

$$\mathbf{Var}^* \left[-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) \right] = \sigma^2 \left[\tau s^2 + \{\tau^2 - 2\tau(T-t)\} s + \frac{(T-t)^3}{3} \right]$$

Therefore, the second term is

$$\exp \left[\frac{\sigma^2}{2}(t+\tau)t\tau + \frac{\sigma^2 (T-t)^3}{6} \right] \times$$

$$\int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} \, exp \left[\sigma^2 \tau s^2 + \sigma^2 (\tau^2 + 2t\tau - T\tau) s \right] ds$$

This leads to the desired expression for h. ∎

When $X(t) < 0$, the price of the average rate call must be solved numerically, e.g., applying the finite difference method to find h at time t. The boundary condition at time T for h is

$$h(X, T) = \max[X, 0]$$

When X is large, the formula in the above proposition can be used as a boundary condition. And, when X is small, we note that

$$lim_{X \to -\infty} h(X, t) = 0$$

Numerical examples using the finite difference method are shown in the next section.

As mentioned before, once the prices of European call options are obtained, the prices of European put options are easily derived through the put-call parity shown in the following proposition.

Proposition 2.4 *The put-call parity for the average rate options is given by*

$$P(t) = C(t) + P(t,T)K - \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} Z_T \right] \tag{2.19}$$

Proof. The above relation is clearly true at the expiration date T. Discounting both sides by the money market rates and taking the conditional expectation, we can easily derive the above relation. ▮

Note that

$$\begin{aligned} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} Z_T \right] &= \frac{1}{T\tau} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} \int_0^T \frac{1}{P(t,t+\tau)} dt \right] \\ & \quad - \frac{1}{\tau} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} \right] \\ &= g(r,t)h(X,t) - \frac{1}{\tau} P(t,T) \end{aligned}$$

where g is defined by (2.12) and h is given by the explicit formula for $X_t \geq 0$ in proposition 2 with $k = 0$.

4 Numerical Examples

We now present two numerical examples which illustrate how our variable reduction technique can be efficiently used to value average-rate options. Our first example involves average-rate options written on foreign exchange rates, while our second example focuses on average-rate options on one-year CMS rates.

Tables 2.1-2.3 show the prices of average-rate options on dollar-yen exchange rates with three different expiration dates (i.e., three months, six months and one year). For each expiration date, the prices of out-of-the money, at-the money, and in-the-the money options are shown separately. The spot price and the volatility are assumed to be 100 yen and 10 percent per year, respectively, while the risk-free interest rates for yen and dollar are assumed to be 3 percent and 5 percent, respectively. Option prices are quoted in terms of yen. In the first row of each table, we obtain the option prices by solving the PDE (2.3) using a Crank-Nicholson finite difference scheme, while in the second row of each table, we calculate the option prices by evaluating the expectation of (2.5), i.e., the Feynman-Kac representation of the solution of the PDE (2.3), using the standard Monte Carlo simulation applying to a discretized counterpart for X. Specifically, we discretize X using a standard first order finite difference scheme:

$$x_{n+1} = x_n + (1 - \alpha x_n)\Delta - \sigma x_n \sqrt{\Delta}\tilde{\epsilon}_n$$

A total of 100,000 trials are implemented in each simulation. For purposes of comparisons, we report in the last row of the table the option prices calculated using the more conventional Monte Carlo simulation method, i.e., simulate a sample path of the exchange rate process (i.e., S) and compute the average exchange rates (i.e., A) along each of the sample paths generated. A total of 500,000 paths have been sampled to arrive at the numbers reported. It is clear from these tables that in terms of accuracy, our variable reduction method fares well with Monte-Carlo simulations methods. Moreover, we note that among the three methods, the computational time involved in simulations is much longer than that of the finite difference method which generates the numbers in the first row (in order to achieve the same level of accuracy).[6]

Tables 2.4-2.6 show similar results for average-rate options written on 1-year constant maturity yields (CMS) with three different times to maturity, 3 months, 6 months and 1 year. For each maturity, the prices of three different strikes are shown as in the case for the foreign exchange rate options. For simplicity, the term structure of interest rates is assumed to be flat at 5 percent in all cases, and the volatility of instantaneous forward rates is assumed to be 150 basis point per year. The option prices are expressed in terms of basis point per year. As in the case for foreign exchange rate options, the PDE for $h(X, t)$ is solved numerically by the Crank-Nicholson finite difference scheme. Option prices are also evaluated through Monte-Carlo simulations based on Feynman-Kac representation for h, where 100,000

[6] The computing time required for implementing the finite difference scheme is well under one minute on a SunSparc 20 machine.

trials are implemented for each case, and the more conventional simulations method, where 500,000 trials are implemented for each case.

5 Concluding Remarks

We have presented in this paper a variable reduction technique which values average-rate options by reducing the two-dimensional valuation problem to a one-dimensional problem. In doing so, it reduces significantly the computing time required for average-rate options. While we have shown that this technique is useful when the underlying state variable is lognormally distributed, this technique is potentially applicable for a larger class of asset price dynamics as long as the returns of the underlying are independent of their past histories. An example of such case is the 2-factor stochastic volatility model in which the volatility of the underlying is governed by another one dimensional Markov process, e.g., Hull and White (1987).

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Table 2.1: Average-Rate Options on FX (T=0.25y)

Strike price	105	100	95
Finite Difference	0.046	1.022	4.766
Monte Carlo (I)	0.046	1.023	4.765
Monte Carlo (II)	0.046	1.022	4.766

Table 2.2: Average-Rate Options on FX (T=0.50y)

Strike price	105	100	95
Finite Difference	0.183	1.366	4.679
Monte Carlo (I)	0.183	1.364	4.680
Monte Carlo (II)	0.184	1.368	4.679

Table 2.3: Average-Rate Options on FX(T=1.00y)

Strike price	105	100	95
Finite Difference	0.464	1.772	4.632
Monte Carlo (I)	0.465	1.765	4.616
Monte Carlo (II)	0.466	1.770	4.638

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Table 2.4: Average-Rate Options on 1-Year CMS(T=0.25y)

Strike rate %	5.50	5.00	4.50
Finite Difference	5.36	24.99	63.81
Monte Carlo (I)	5.34	25.11	63.59
Monte Carlo (II)	5.39	25.12	63.90

Table 2.5: Average-Rate Options on 1-Year CMS(T=0.50y)

Strike rate %	6.00	5.00	4.00
Finite Difference	2.68	31.98	111.34
Monte Carlo (I)	2.67	32.05	111.38
Monte Carlo (II)	2.69	31.98	111.55

Table 2.6: Average-Rate Options on 1-Year CMS(T=1.00y)

Strike rate %	6.00	5.00	4.00
Finite Difference	8.06	41.32	112.25
Monte Carlo (I)	8.08	41.36	112.42
Monte Carlo (II)	8.09	41.34	112.31

Chapter 3

Pricing of Securities with Default Risks

1 Introduction

We present a new model for pricing the securities with default risks in a general equilibrium framework. Our model is general enough in a sense that the state of the default can be related to any other economic variables such as macro economic indicators which are determined in equilibrium in the model as well as to the firm's specific factor such as its asset value. In other words, the state of default can be determined endogenously in the model or at least, the determinants of the spreads between the corporate yields and the treasury yields which are the important factor in pricing the securities may be consistently related to the main economic variables. Moreover, any types of the securities with default risks such as defaultable bonds, swaps, caps, options on bonds may be evaluated in a unified framework of the model.

Our starting point is the general equilibrium model of asset prices in Cox, Ingersoll, and Ross(1985a) where they derive the fundamental pricing equation, the partial differential equation for any contingent claims. In this paper, we apply and extend the framework to evaluate contingent claims with default risks.

We characterize the state of default by a set of stopping times. We propose two types of the models, one of which utilizes a predictable stopping time, and the other of which makes use of a totally inaccessible stopping time represented by a jump process to characterize the state of default. For each model, we explicitly derive a partial differential equation with a set of boundary conditions to price any securities subject to default risks. The reason we present two mutually exclusive stopping times to describe the state of the default is that the argument on this matter seems to be unsettled empirically. That is, one of the consequences from the assumption of modelling a default time as a predictable stopping time is that the spreads between the treasury yields and the corporate yields should be close to zero near the maturities while they must be substantially large whenever the related variables approach the default boundaries, which does not seem to be valid.

As for the first model, the state of default is determined by the stopping time when some functions of the wealth and the factors which govern the economy in the model hit the prespecified boundaries. We note the functions may denote some macro economic variables such as the spot interest rate since in equilibrium, those variables can be expressed by the functions of the wealth and the factors. The prespecified boundaries as well as those functions may vary among the firms, which represents the variations of the default states for each company. In this way, it is possible to handle various types of the specific pricing model of the securities subject to default risks in our general framework.

As for the second model, we explicitly introduce a default indicator function for each company following a jump process in the general equilibrium framework. The intensity, that is the possibility of the default in the next instant may depend on the other economic variables such as a set of macro economic indicators determined inside the model while the default itself occurs suddenly due to the jump martingale part of the process. The intuition behind this modelling is that while we can infer the possibility of the default by the other observable indicators, we can not predict the occurence of the default itself since the information and the indicators inside the company which drive the default directly are usually unobservable.

There are mainly three approaches so far taken to model the default risks. The first approach is initiated by Merton(1974) where he applies the contingent claim analysis to pricing a firm's debts. He takes a firm's asset value as a state variable and the default occurs whenever the firm's asset value is less than the firm's liabilities at the coupon payment dates or the maturities. Many researches along this approach has been followed. Recently, Cooper and Mello(1991) apply this technique to the interest rate and currency swap valuations. Although this is a remarkable approach in a sense that the state of the default is endogenously determined in the model, and is closely related to the decision of a firm on its capital structure, it has not become popular in practice. The reason for this is that first of all, the firm's asset is unlikely to be observable or tradable, and moreover, it is extremely complicated to determine the seniority of the firm's liabilities which is the key element to pricing of the corporte debts.

The second approach is taken by Longstaff and Schwartz(1995) where they overcome the problem of the "seniority" in pricing by introducing a predictable stopping time. That is, the default occurs whenever the firm's asset value hits the prespecified boundary and the payment in case of default for each security is made according to its seniority determined by the contract and the related laws. In addition, they take the spot interest rate as the second state variable and emphasize the correlation between the firm's asset and the spot interest rate in determination of the spreads of defaultable bonds. On the other hand, they assume the firm's asset is observable and tradable as the first approach does and the process of the spot interest rate is exogenously given under the equivalent martingale measure.

The third approach is recently initiated by Jarrow and Turnbull (1995), Duffie and Huang(1994), Madan and Unal(1994) and others where they model the time of default as a pure jump process, and evaluate directly the tradable and observable securities with default risks under the arbitrage-free condition. In other words, they start with the securities with default risks traded in a financial market and construct an equivalent martingale measure based on the securities or just assume the existence of an equivalent martingale measure based on the general theory. Hence they assume explicitly or implicitly the liquidity of the securities in the market. For instance, Jarrow and Turnbull(1995) is based on the existence of corporate bonds for each credit class enough to construct their term structures. This assumption, however, is questionable in practice and the level of those corporate bonds itself should be determined by some economic reasoning. In fact, one of the main feature in this approach is that the state of the default is exogenously given, which is contrast to the previous approaches while the model basically depends only on the observable factors given the liquidity of the corporate bonds and hence it may be appropriate for pricing the derivatives of the corporate bonds and the vulnerable options. We should note, however, that the numerical computation is tough task without so called the independent assumption which is usually violated in the practical world.

There are several advantages of our models. In the predictable stopping time model, we take the observable economic variables such as the spot interest rate to endogenize the state of the default and they are internally consistent in the sense that those economic variables themselves are determined in equilibrium inside the model. Clearly, by construction, we can relate the spreads of the corporate bonds to those variables. Moreover, we can diviate from the assumptions that the firm's asset is tradable and that the process of the spot interest rate under an equivalent martingale measure is exogenously given since the risk premium of the firm's asset as well as that of the spot interest rate can be explicitly obtained. In the model where we utilize the jump process to express the time of default, we can freely relate the economic indicators determined in equilibrium to the possibility of the default and the risk premium of the default of a company can be explicitly and consistently obtained inside the model. Finally, the resulting PDE for the pre-default value of the securities does not include any jump part, which reduces the computational burden required in the models including the jump components.

The remainder in the paper is organized as follows. The next section presents the first model where we model the default risks by a predictable stopping time, and also show one-factor and two-factor models as simple examples. The section 3 proposes the second model where we utilize a jump process to model the state of the default and show a numerical example. The section 4 makes concluding remarks.

2 Model I: The Securities with the default risks in the CIR Economy

In this section, we present a framework and simple examples of how to evaluate the securities subject to the default risks. Our model is based on the general equilibrium asset pricing model presented in the Cox, Ingersoll, and Ross(1985a). We forcus on how to characterize the event of default in the model by utilizing a predictable stopping time. In fact, we shall derive the PDE with appropriate boundary conditions at the stopping time which the prices of the securities with default risks must satisfy by making use of the fundamental pricing equation of CIR(1985a) .

2.1 A General Model

We start with the fundamental pricing equation for contingent claims which is a partial differential equation of parabolic type. The fundamental partial differential equation(PDE) is given by Theorem 3 in CIR(1985a) as

$$\frac{1}{2}Var.(dW)F_{WW} + Cov.(dW, d\vec{Y})^{\top} F_{W,\vec{Y}} + \frac{1}{2}tr(F_{\vec{Y}\vec{Y}}\underline{SS}^{\top}) +$$
$$(rW - c)F_W + F_{\vec{Y}}^{\top}(\vec{\mu} - \vec{\phi}_{\vec{Y}}) + F_t - rF + \delta = 0$$

with $F(W, \vec{Y}, T) = \Psi(W, \vec{Y}, T)$ where $c = c(W, \vec{Y}, t)$ and $r = r(W, \vec{Y}, t)$ are the consumption process and the riskless interest rate process in equilibrium. $\delta(W, \vec{Y}, t)$ and $\Psi(W, \vec{Y}, T)$ are specified by payoffs of a contract without default. The default of a company is characterized by a stopping time and the price of a security asscoiated with the defaultable company is given by the fundamental PDE with a set of boundary conditions at the stopping time as well as the boundary information obtained from the payoffs without default. We formally state this below. Suppose the security with the maturity T is to be evaulated. Let $\vec{f_j}(t, W, \vec{Y})$ be a vector of smooth functions such that $[0, T] \times (0, \infty) \times R^k \mapsto R^{n_j}$ Then, the default stopping time for company **j** is defined by

$$\tau_j = \inf\{t; \left(t, \vec{f_j}(t, W(t), \vec{Y}(t))\right) \notin \Omega_j\}$$

where Ω_j is an open set on $(0, T) \times R^{n_j}$, $\partial\Omega_j$ is its boundary, and $\hat{\partial}\Omega_j$ is the closed subset such that $\left(\tau_j, \vec{f_j}(\tau_j, W, \vec{Y})\right) \in \hat{\partial}\Omega_j$ for every choice of an initial point $\left(t, \vec{f_j}(t, W, \vec{Y})\right) \in \Omega_j$. That is, a default time for **j** is defined as the exit time for $\vec{f_j}(t, W, \vec{Y})$ from Ω_j. Then, the fundamental PDE holds on

$$\{(t, W, \vec{Y}); \left(t, \vec{f_j}(t, W, \vec{Y})\right) \in \Omega_j\},$$

and the boundary conditions are given by

$$F(W(\tau_j), \vec{Y}(\tau_j), \tau_j) = \Theta_j(W(\tau_j), \vec{Y}(\tau_j), \tau_j),$$

and $F(W, \vec{Y}, T) = \Psi(W, \vec{Y}, T)$. We note that the functon Θ_j depends on **j** since the payment in case of default may vary among firms. Hence we obtain the following theorem.

Theorem 3.1 *The fundamental pricing equation for any securities subject to the default risk of the company* **j** *is given by the PDE with a set of boundary conditions.*

$$\frac{1}{2}Var.(dW)F_{WW} + Cov.(dW, d\vec{Y})^\top F_{W,\vec{Y}} + \frac{1}{2}tr(F_{\vec{Y}\vec{Y}}\underline{SS}^\top) + $$
$$(rW - c)F_W + F_{\vec{Y}}^\top(\vec{\mu} - \vec{\phi}_{\vec{Y}}) + F_t - rF + \delta = 0 \tag{3.1}$$

with

$$F(W(\tau_j), \vec{Y}(\tau_j), \tau_j) \quad = \quad \Theta_j(W(\tau_j), \vec{Y}(\tau_j), \tau_j) \tag{3.2}$$

and

$$F(W, \vec{Y}, T) \quad = \quad \Psi(W, \vec{Y}, T) \tag{3.3}$$

where

$$\tau_j = \inf\{t; \left(t, \vec{f}_j(t, W, \vec{Y})\right) \notin \Omega_j\}. \tag{3.4}$$

Note $\vec{f}_j(t, W, \vec{Y})$ *be a vector of smooth functions such that* $[0, T] \times (0, \infty) \times R^k \mapsto R^{n_j}$ *and* Ω_j *is an open set on* $(0, T) \times R^{n_j}$.

There exist defaultable securities such as swaps which are subjected to two or more companies' default risks. This approach is still vaild to price those securities if we define properly the default stopping times. That is, if the security is subjected to the default risks of the companies $1, \cdots, n$, the default stopping time is defined by

$$\tau_1^n = \tau_1 \wedge \cdots \wedge \tau_n.$$

Then, the fundamental PDE holds on

$$\{(t, W, \vec{Y})\} = \cap_{j=1}^n \{(t, W, \vec{Y}); \left(t, \vec{f}_j(t, W, \vec{Y})\right) \in \Omega_j\}.$$

The boundary conditions are given by

$$F(W(\tau_1^n), \vec{Y}(\tau_1^n), \tau_1^n) = \Theta(W(\tau_1^n), \vec{Y}(\tau_1^n), \tau_1^n),$$

and $F(W, \vec{Y}, T) = \Psi(W, \vec{Y}, T)$.

Corollary 3.1 *The fundamental pricing equation for any securities subject to the default risk of the companies* $1, \cdots, n$ *is given by the PDE with a set of boundary conditions stated in the Theorem 3.1 except that the default stopping time is redefined by*

$$\tau_1^n = \tau_1 \wedge \cdots \wedge \tau_n. \tag{3.5}$$

We shall present two specifications in the model.

- **[i]** $\vec{f}_j(W,\vec{Y},t)$ is taken to be a vector of macro economic variables, which implies $\vec{f}_j(W,\vec{Y},t)$ are common among **j**, that is, $\vec{f}(W,\vec{Y},t)$ while Ω_j is the region of the macro economic variables specific to each company **j**. In other words, the default occurs if a set of macro economic indicators hit some levels specific to the company. For instance, let $f^i(W,\vec{Y},t)$,$i=1,2$ be some macro economic indicators defined on $[0,\infty)$ and let f^i_{jL} and f^i_{jH} denote the level of the lower boundary and that of the higher boundary respectively of $f^i(W,\vec{Y},t)$,$i=1,2$ for the company **j**. Then, we may define Ω_j by

$$\Omega_j = (0,T)\times \Pi^2_{i=1}\{f^i_{jL} < f^i(W,\vec{Y},t) < f^i_{jH}\} \tag{3.6}$$

or

$$\begin{aligned}\Omega_j &= (0,T)\times\Big(\{f^1_{jL} < f^1(W,\vec{Y},t) < f^1_{jH}\}\times[0,\infty) \\ &\quad \cup [0,\infty)\times\{f^2_{jL} < f^2(W,\vec{Y},t) < f^2_{jH}\}\Big).\end{aligned} \tag{3.7}$$

To define the default stopping time, we first introduce the a set of stopping times.

$$\tau^i_{jL} = \inf\{t; f^i(W,\vec{Y},t) \le f^i_{jL}\} \quad \text{i=1,2} \tag{3.8}$$

$$\tau^i_{jH} = \inf\{t; f^i(W,\vec{Y},t) \ge f^i_{jH}\} \quad \text{i=1,2} \tag{3.9}$$

Then, the default stopping time may be defined by

$$\tau_j = \tau^1_{jL}\wedge\tau^1_{jH}\wedge\tau^2_{jL}\wedge\tau^2_{jH} \tag{3.10}$$

or

$$\tau_j = \inf\{\tau^1_j : \tau^1_j = \tau^2_j\} \tag{3.11}$$

where $\tau^1_j = \tau^1_{jL}\wedge\tau^1_{jH}$ and $\tau^2_j = \tau^2_{jL}\wedge\tau^2_{jH}$.

More specifically, $f^i(W,\vec{Y},t)$,$i=1,2$ can be taken to be the aggregate wealth W and the spot interest rate $r(W,\vec{Y},t)$. That is, the PDE holds on

$$\{(t,W,\vec{Y}); W_L < W \text{and } r_L < r(W,\vec{Y},t) < r_H\},$$

or

$$\{(t, W, \vec{Y}); W_L < W \text{or } r_L < r(W, \vec{Y}, t) < r_H\},$$

and the boundary conditions are given by $F(W, \vec{Y}(\tau_j), \tau_j) = \Theta_j(W, \vec{Y}(\tau_j), \tau_j)$, and $F(W, \vec{Y}, T) = \Psi(W, \vec{Y}, T)$.

- **[ii]** Another way to capture default is to introduce a state variable Y_j specific to the company **j**. That is, some components of $\vec{Y}$ represent the firm **j**'s specific state variables $Y_{j_1}, Y_{j_2}, \cdots Y_{j_k}$. Y_{js} for some $s \in \{1, 2, \cdots, k\}$ may be taken as the firm's asset value V_j. Then, the default of the company **j** is characterized by the stopping time when $Y_{js}, s \in \{1, 2, \cdots, k\}$ hit some boundaries. Hence the PDE still holds with the appropriate boundaries conditions. For example, the default for the company **j** occurs whenever Y_j hits the lower boundary $Y_{jL}(t)$. That is, the PDE holds on

$$\{(t, W(t), \vec{Y}(t)); Y_{jL}(t) < Y_j(t)\}, \tag{3.12}$$

and the boundary conditions are given by

$$F(W, Y_1, \cdots, Y_{jL}, \cdots, Y_k, \tau_j) = \Theta_j(W, Y_1, \cdots, Y_{jL}, \cdots, Y_k, \tau_j),$$

and $F(W, \vec{Y}, T) = \Psi(W, \vec{Y}, T)$.

More specifically, suppose a defaultable security is considered as a function of $r(W, \vec{Y^*}, t)$ and V, that is $F(r(W, \vec{Y^*}, t), V, t)$ where the last(k th) component of $\vec{Y}$ is V, the firm's asset value and $\vec{Y^*}$ consists of the other factors. Then, if the default boundary is taken as V_L, the PDE may be expressed as

$$\begin{aligned}&\frac{1}{2}\vec{\sigma}_r^\top \vec{\sigma}_r F_{rr} + \frac{1}{2}\vec{\sigma}_V^\top \vec{\sigma}_V F_{VV} + \vec{\sigma}_r^\top \vec{\sigma}_V F_{rV} + \\ &\mu_r F_r + (\mu_V - \phi_V) F_V + F_t - rF = 0\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\mu_r &= (rW - c) r_W + (\vec{\mu}_{\vec{Y}^*} - \vec{\phi}_{\vec{Y}^*})^\top \vec{r}_{\vec{Y}^*} \\ &+ \frac{1}{2} Var.(dW) r_{WW} + Cov.(dW, d\vec{Y}^*)^\top \vec{r}_{W\vec{Y}^*} + \frac{1}{2} tr.(\underline{r}_{\vec{Y}^*\vec{Y}^*} \underline{S}^* \underline{S}^{*\top}) + \frac{\partial r}{\partial t}\end{aligned} \tag{3.14}$$

and

$$\vec{\sigma}_r = r_W \vec{a}^\top \underline{G} + \vec{r}_{\vec{Y}^*} \underline{S}^* \tag{3.15}$$

for $\{(W, \vec{Y^*}, V, t); t < T, V_L < V\}$. ϕ_V and $\vec{\phi}_{\vec{Y}^*}$ denote the risk premia of V and $\vec{Y}^*$ respectively, and r_W, $r_{\vec{Y}^*}$, r_{WW}, $\vec{r}_{W\vec{Y}^*}$ and $\underline{r}_{\vec{Y}^*\vec{Y}^*}$ denote the partial derivatives of r with respect to W or/and $\vec{Y}^*$. $Var.(dW)$ and $Cov.(dW, d\vec{Y}^*)^\top$ are determined by $Var(dW) = \vec{a}^\top \underline{GG}^\top \vec{a}$ and $Cov.(dw, d\vec{Y}^*)^\top = \vec{a}^\top \underline{GS}^{*\top}$ where $\vec{a}$ denotes the vector of the propotion of wealth invested in real assets with $\vec{a}^\top \vec{1} = 1$.

The boundary condition may be given by $F(T) = X(r(W, \vec{Y}, T), V, T)$ where X denotes a terminal payoff of a contract and $F(\tau) = \Theta(r(W, \vec{Y}, \tau), V, \tau)$ where $\tau = \inf.\{t; V_t \leq V_L\}$.

We next show that the model of Longstaff and Schwartz(1995) can be considered an example in this case. In fact, we first assume the utility function of the representative agent to be a logarithmic one, $u(c_t, t) = e^{-\delta t} \log c_t$, $\delta > 0$. Next, we suppose that there is only one real asset, η and that there exist two factors, Y_i, $i = 1, 2$. The stochastic processes of η and Y_i, $i = 1, 2$ are assumed to be

$$
\begin{aligned}
d\eta &= (\alpha_1 + \alpha_2 Y_1)\eta dt + \sigma_\eta \eta(\rho_\eta dw_{1t} + \sqrt{1 - \rho_\eta^2} dw_{2t}) \\
dY_1 &= (\mu_{11} - \mu_{12} Y_1)dt + \sigma_1 dw_{1t} \\
dY_2 &= \mu_2 Y_2 dt + \sigma_2 Y_2(\rho dw_{1t} + \sqrt{1 - \rho^2} dw_{3t})
\end{aligned}
$$

where $\alpha_i, i = 1, 2$, σ_η, $\mu_{1i}, i = 1, 2$, μ_2 and $\sigma_i, i = 1, 2$ are some constants, ρ_η and ρ are constants with $|\rho_\eta| \leq 1$ and $|\rho| \leq 1$, and $w_i, i = 1, 2, 3$ are mutually independent Brownian motions. That is, we assume that the two factors follow an Ornstein-Uhlenbeck process and a lognormal process respectively, and that the process of the return of the real asset depends only on the first factor. Then, we can easily obtain the process of r and the risk premia of Y, $i = 1, 2$ since as in CIR(1985b), the indirect utility function is given by $J(W, Y_1, t) = f(t) \log W + g(Y_1, t)$, where $f(t) = \frac{e^{-\delta t} - e^{-\delta T}}{\delta}$. That is, the equilibrium spot interest rate is obtained by

$$
r = \alpha_1 - \sigma_\eta^2 + \alpha_2 Y_1,
$$

and the factor risk premia are obtained by $\phi_{Y_1} = \rho_\eta \sigma_\eta \sigma_1$ and $\phi_{Y_2} = \rho_\eta \rho \sigma_\eta \sigma_2 Y_2$ respectively. Consequently, if we interpret the factor Y_2 as the firm's asset V

and further assume that V is tradable, which implies $(\mu_V - \phi_V)$ is replaced by rV, then the PDE (3.13) becomes the exactly same one as they derive.

Next we briefly examine the boundary conditions for several defaultable securities. We suppose the price of a default-free security such as a zero coupon bond, an optioin or a swap is obtained by the fundamental PDE with the appropriate boundary conditions specified by the contract's payoff.

- **Zero coupon bonds**
  $\psi(W,\vec{Y},T) = 1$ and $\Theta_j(W,\vec{Y},\tau_j) = \varphi_j P(\tau_j,T)$ where $P(\tau_j,T)$ denotes the price of the default-free zero coupon bond at the time τ_j with the maturity T, and φ_j denotes the recovery rate for the company **j** which may be a well-behaved function of $(W,\vec{Y},t)$.

- **Vulnerable call options on a zero coupon bond**
  $\psi(W,\vec{Y},T) = (P(T,T^*) - K)^+$ and $\Theta_j(W,\vec{Y},\tau_j) = \varphi_j C(P(\tau_j,T^*),K,\tau_j)$ where $P(T,T^*)$ denotes the price of the default-free zero coupon bond at the time T with the maturity T^*, and $C(P(\tau_j,T^*),K,\tau_j)$ is the price of default-free call option at τ_j. φ_j denotes also the recovery rate of the company **j** which sells the option.

- **Call options on a defaultable zero coupon bond**
  $\psi(W,\vec{Y},T) = (P_j(T,T^*) - K)^+$ and $\Theta_j(W,\vec{Y},\tau_j) = P(\tau_j,T)(\varphi_j P(\tau_j,T^*) - K)^+$ where $P_j(T,T^*)$ is the price of the defaultable zero coupon bond of the company **j** at T with the maturity T^*.

- **(One Period) Swaps**
  $\psi(W,\vec{Y},T) = \bar{F} - \tilde{L}$ where $\bar{F}$ denotes a fixed rate which is received by the company 1 and $\tilde{L}$ denotes the LIBOR which is received by the company 2.

  $$\Theta(W,\vec{Y},\tau_1^2) = 1_{\{V_f(\tau_1^2)>0\}} V_f(\tau_1^2) + 1_{\{V_f(\tau_1^2)\leq 0\}} \varphi_1 V_f(\tau_1^2)$$

  if $\tau_1^2 = \tau_1$ and

  $$\Theta(W,\vec{Y},\tau_1^2) = 1_{\{V_f(\tau_1^2)>0\}} \varphi_2 V_f(\tau_1^2) + 1_{\{V_f(\tau_1^2)\leq 0\}} V_f(\tau_1^2)$$

  if $\tau_1^2 = \tau_2$ where $V_f(\tau_1^2)$ denotes the price of the default-free swap at the time τ_1^2 with the same terminal payoff.

2.2 An Example: A One Factor Model

We present the model with one factor of the square-root process as an example of the specification **[i]** in the previous subsection. The state of the default is described by the stopping time when the spot interest rate or/and the level of the wealth which is equivalent to the price of the market portfolio in this model hit some prespecified boundaries. Those levels may be specific to each company. We start with the set of assumptions in the model.

First, we fix the period of the economy in this model as $[0, T^*]$, where T^* is a positive finite number. Then, the representative agent is described by a logarithmic utility function which is explicitly given by

$$\mathbf{E}\left[\int_0^{T^*} e^{-\rho t} \log c_t \; dt\right].$$

where $\rho \in (0, \infty)$ denotes the time preference of the agent.

The State variable Y which governs the economy follows a square-root process as in CIR(1985b).

$$dY = (\xi_1 Y + \xi_2)dt + \sigma\sqrt{Y}dw_t$$

where ξ_1 and ξ_2 are some constants with $\xi_2 \geq 0$ and w_t is a one-dimensional Brownian motion. There is a single capital stock or real asset denoted by η in the economy. The process of the logarithm of η is completely determined by the state variable Y and also follows a square-root process.

$$d\eta = \alpha Y \eta dt + G\sqrt{Y}\eta dw_t$$

where α and G are some constants with $\alpha - G^2 > 0$.

Finally, there are four financial assets in the economy as follows.

- The riskless money market account B_t with riskless rate r
- The equity (market portfolio) S_t which claim the capital stock at T^*, that is η_{T^*}.
- The default-free zero-coupon bond with the maturity $T(< T^*)$, $P(t, T)$ with the payoff $P(T, T) = 1$.

- The defaultable zero-coupon bond for the company **j** with the maturity T, $P_j(t,T)$ with the payoff $P_j(T,T) = 1$ without default. The payoffs in case of the default are discussed later.

Next, we characterize the states of the default. In particular, we consider the two cases for the states which drive the **j**'s default.

(i) The default of **j** occurs if the price of the market portfolio hits the lower boundary S_{jL} for the company **j**, or the spot interest rate hits the lower boundary r_{jL} or the higher boundary r_{jH} for the company **j**. That is, the default occurs whenever the market portfolio or the spot interest rate hits its boundary.

(ii) The default of **j** occurs if the price of the market portfolio hits the lower boundary S_{jL}, and the short term interest rate hits the lower boundary r_{jL} or the higher boundary r_{jH}. That is, the default occurs whenever both the market portfolio and the spot interest rate hit their boundaries.

We next define three stopping times associated with the default of the company **j**.

$$\begin{aligned}\tau_{11} &= \inf\{t; r_t \le r_{jL}\}\\ \tau_{12} &= \inf\{t; r_t \ge r_{jH}\}\\ \tau_{2} &= \inf\{t; S_t \le S_{jL}\}\end{aligned}$$

In equilibrium, simple calculation shows as in CIR(1985b) that the stochastic processes of the consumption, the spot interest rate, the stock price, and the price of the default-free zero coupon bond are given respectively by

$$\begin{aligned}c_t &= \frac{\rho}{1-e^{-\rho(T^*-t)}}W\\ r_t &= (\alpha - G^2)Y_t\\ S_t &= W_t\\ &\text{and}\\ P(t,T) &= A(T-t)e^{B(T-t)r_t}.\end{aligned}$$

Now we turn to our main objective, pricing the defaultable bond. In order to evaluate the price of the defaultable zero coupon bond, we first note that the fundamental PDE may be rewritten in terms of S and r. First, we note that the process of the wealth is expressed by using $c_t = \frac{\rho}{1-e^{-\rho(T^*-t)}}W$ and $Y = \frac{1}{(\alpha-G^2)}r$ as

$$\begin{aligned} dW &= (\alpha YW - c)dt + G\sqrt{Y}W dw_t \\ &= \left[\alpha Y - \left(\frac{\rho}{1-e^{-\rho(T^*-t)}}\right)\right] W + G\sqrt{Y}W dw_t \\ &= \left[\left(\frac{\alpha}{\alpha - G^2}\right) r - \left(\frac{\rho}{1-e^{-\rho(T^*-t)}}\right)\right] W + G\sqrt{\left(\frac{r}{\alpha - G^2}\right)} W dw_t. \end{aligned}$$

Next, we note

$$\begin{aligned} P_{jY} &= P_{jr}(\alpha - G^2) \\ P_{jYY} &= P_{jrr}(\alpha - G^2)^2 \\ P_{jWY} &= P_{jWr}(\alpha - G^2). \end{aligned}$$

Finally, replacing W and Y by S and $\frac{1}{\alpha-G^2}r$ respectively, we can re-express the fundamental PDE in terms of W and r.

$$\frac{1}{2}\left(\frac{G^2}{\alpha - G^2}\right) rS^2 P_{jSS} + \frac{1}{2}\sigma^2(\alpha - G^2) r P_{jrr} + G\sigma r S P_{jSr} + \left(r - \frac{\rho}{\left[1-e^{-\rho(T^*-t)}\right]}\right) SP_{jS} + \left[(\xi_1 - G\sigma)r + (\alpha - G^2)\xi_2\right] P_{jr} + P_{jt} - rP_j = 0$$

for the case (i) $S_L < S$ and $r_L < r < r_H$ and for the case (ii) $S_L < S$ or $r_L < r < r_H$.

The boundary conditions are given by

$$P_j(T,T) = 1$$

and for $t < T$,

$$\begin{aligned} \text{(i) } P_j(S_L, r, \tau_2, T) &= \varphi_j P(\tau_2, T) \\ P_j(S, r_L, \tau_{11}, T) &= \varphi_j P(\tau_{11}, T) \\ P_j(S, r_H, \tau_{12}, T) &= \varphi_j P(\tau_{12}, T) \end{aligned}$$

in the case (i), and

$$\begin{aligned} \text{(ii) } P_j(S_L, r_L, \tau, T) &= \varphi_j P(\tau, T) \ \text{ if } \tau_1 = \tau_{11} \\ P_j(S_L, r_H, \tau, T) &= \varphi_j P(\tau, T) \ \text{ if } \tau_1 = \tau_{12} \end{aligned}$$

in the case (ii) where $\tau_1 = \tau_{11} \wedge \tau_{12}$ and $\tau = \inf\{\tau_2; \tau_2 = \tau_1\}$, and $\varphi_j \in [0, 1)$ is the recovery rate.

Finally, we note that the solution of the PDE with a set of the boundary conditions may be represented by the conditional expectation. That is,

$$\begin{aligned} P_j(t,T) &= \mathbf{E}_t^* \left[e^{-\int_t^{\tau} r_u du} 1_{\{t \le \tau < T\}} \varphi_j P_j(\tau, T) \right] \\ &+ \mathbf{E}_t^* \left[e^{-\int_t^{T} r_u du} 1_{\{\tau \ge T\}} \right] \end{aligned}$$

where $\tau = \tau_{11} \wedge \tau_{12} \wedge \tau_2$ in (i) and $\tau = \inf\{\tau_2; \tau_2 = \tau_1\}$ in (ii). The conditional expectation is taken under, for $t \in [0, T^*)$

$$dS = \left(r - \frac{\rho}{[1 - e^{-\rho(T^* - t)}]} \right) S dt + \frac{G}{\sqrt{\alpha - G^2}} \sqrt{r} S dw_t^*$$

and

$$dr = \left[(\xi_1 - G\sigma) r + (\alpha - G^2) \xi_2 \right] dt + \sigma \sqrt{\alpha - G^2} \sqrt{r} dw_t^*.$$

A Numerical Example

We compute a defaultable zero coupon bond as a numerical example. We suppose that the default occurs if both the spot rate r and the market portfolio S hit their lower boundaries for the company **j**, r_{jL} and S_{jL} respectively. We next define a set of the stopping times.

$$\begin{aligned} \tau_1 &= \inf\{t; r_t \le r_L\} \\ \tau_2 &= \inf\{t; S_t \le S_L\} \\ \tau &= \inf\{\tau_1; \tau_1 = \tau_2\} \end{aligned}$$

Then, the payoffs of the defaultable zero coupon bond is given by

$$\begin{aligned} P_j(T,T) &= 1 \ \text{ if } \tau \ge T \\ P_j(\tau,T) &= \varphi_j P(\tau, T) \ \text{ if } \tau < T \end{aligned}$$

where φ_j is a constant $\in [0, 1)$.

Finally the parameters are specified numerically as follows.

$$
\begin{aligned}
\text{The maturity of the bond}(T) &= \text{5year} \\
\text{The correlation}(\rho) &= 0.07 \\
\text{The recovery rate}(\varphi) &= 0.7 \\
\text{The current spot rate}(r_0) &= 7\% \\
\text{The expectation of r in 5 y} &= 7.0\% \\
\text{The standard deviation of r in 5y} &= 5.38\% \\
\text{The current market portfolio}(S_0) &= 100.00 \\
\text{The expectation of S in 5 y} &= 103.31 \\
\text{The standard deviation of S in 5y} &= 36.13
\end{aligned}
$$

We compute $P_j(t,T)$ by the Monte Carlo simulation and the result is given in Table 3.1. We list the spot yields of the five-year zero coupon bonds for different default boundaries of the spot rate and the market portfolio. We also show the spread between the yield of each defaultable bond and that of the default-free bond with the same maturity. We note that the yield of the default-free bond is 6.73 % for all the cases. In the first three rows, we fix the default boundary of the spot rate as 0.5 % and compute the spot yileds for three different default boundaries of the market portfolio,60, 50 and 40. As we expect, the lower is the boudary of the market portfolio, the larger is the spread. In fact, the spread for the boundary of 60 is 114 basis point where the spread for the boundary of 40 is 27 basis point. In the last three rows, we compute the spot yields for two different default boundaries of the spot rate, 0.25 % and 1.00 % while we fix the default boundary of the market portfolio as 50. Again, the spread is larger when the boundary is higher; the spread for the boundary of 1.00 % is 111 basis point and that for the boundary of 0.25 % is 63 basis point. All in all, the spread is largest when the boundary of the spot rate is 0.5 % and that of the market portfolio is 60 and is smallest when the boundary of the spot rate is 0.5 % and that of the market portfolio is 40.

2.3 An Example: A Two Factor Model

We introduce another state variable Y_j which is specific to the company **j** in the previous one-factor model. Y_j may be regarded as the asset value of the company **j** while Y represent the factor dominating macro economy. Y_j also follows square-root process where the Brownian motion may be correlated with the one in the first factor. The correlation is denoted by $\rho_j \in [0,1]$. Then, the state variables in the model are given by

$$\begin{aligned} dY_j &= (\xi_{1j}Y_j + \xi_{2j})dt + \sigma_j\sqrt{Y_j}(\rho_j dw_t + \sqrt{1-\rho_j^2}dw_{jt}) \\ dY &= (\xi_1 Y + \xi_2)dt + \sigma\sqrt{Y}dw_t. \end{aligned}$$

The other specification is same as in the previous example. That is, the representative agent's preference is a logarithmic utility function and the real asset process is given by

$$d\eta = \alpha Y\eta dt + G\sqrt{Y}\eta dw_t.$$

Note that the capital stock's movement is dominated by the factor Y only. Hence as in the previous one-factor model, the interest rate r is given by $r = (\alpha - G^2)Y$ in equilibrium and the factor premiums are obtained by a simple calculation.

$$\begin{aligned} \phi_Y &= \sigma G Y \\ \phi_{Y_j} &= \rho_j\sigma_j G\sqrt{Y}\sqrt{Y_j}. \end{aligned}$$

Hence the fundamental PDE for any defaultable securities of the company **j** becomes

$$\begin{aligned} &\frac{1}{2}\sigma^2 r P_{jrr} + \frac{1}{2}\sigma_j^2 Y_j P_{jY_jY_j} + \rho_j\sigma\sigma_j\sqrt{\alpha - G^2}\sqrt{r}\sqrt{Y_j}P_{jrY_j} \\ &+ \left[(\xi_1 - \sigma G)r + (\alpha - G^2)\xi_2\right]P_{jr} + \left(\xi_{1j}Y_j + \xi_{2j} - \rho_j\sigma_j G\frac{1}{\sqrt{\alpha - G^2}}\sqrt{r}\sqrt{Y_j}\right)P_{jY_j} \\ &+ P_{jt} - rP_j = 0. \end{aligned}$$

In particular, when the Y_j itself is a traded asset, the coeficient of P_{jrY_j} is replaced by rY_j.

$$\left(\xi_{1j}Y_j + \xi_{2j} - \rho_j\sigma_j G\frac{1}{\sqrt{\alpha - G^2}}\sqrt{r}\sqrt{Y_j}\right) = rY_j.$$

In this model, the default of the company **j** is characterized by the stopping time τ_j.

$$\tau_j = \inf\{t; Y_{jt} \leq Y_{jL}\}$$

Hence the default of the company **j** occurs whenever the factor Y_j hits the lower boundary Y_{jL}. Consequently, the price of each defaultable security of the company **j** is obtained by solving the PDE with a set of appropriate boundary conditions of the payoffs at the maturity as well as at the default stopping time τ_j . For instance, the boundary conditions of the defaultable zero coupon bond with the maturity T is given by

$$\begin{aligned} P_j(T,T) &= 1 \ \text{ if } \tau_j \geq T \\ P_j(\tau_j,T) &= \varphi_j P(\tau_j,T) \ \text{ if } \tau_j < T \end{aligned}$$

where φ_j denotes the recovery rate and $P(\tau_j, T)$ denotes the price of the default-free zero coupon bond with the maturity T at τ_j. Although this example is obtained in the framework of CIR equilibrium model, it is similar to the model of Longstaff and schwartz(1995) except the assumption of the stochastic processes which Y and Y_j follow if the Y_j is interpreted as the firm's asset value and the asset is assumed to be tradable.

æ

3 Model II: The Securities with the default risks in the CIR Economy(II)

A default risk is modelled by utilizing a predictable stopping time in the previous section since the time of a default is an exit time of some diffusion processes from a certain region. One may argue that as the default occurs surprisingly, the stopping time of the default is not predictable, but totally inaccessible. For instance, Unal and Madan(1995) reports the some empirical evidence shows that the spreads of corporate bonds may be substantially large even just prior to their maturities while those are limited for the companies which operate for long periods in state of so called technical default. If this is the case, a predictable stopping time is not suitable for modelling the time of default. Rather, a totally inaccessible stopping time is appropriate, which implies a jump process may represent the event of a default.

We in this section present a new model for the case by introducing a set of new state variables which follows a certain jump process and extend the original CIR general equilibrium model. By using the state variable, we model the empirical observation that the default itself is driven suddenly by the specific factors in the firm which are usually unobservable or hidden just prior to the default while the possibility of the default in the next instant given the company's current solvency may be infered by the other economic variables such as macro economic indicators. For example, the observation is made by Longstaff and Schwatz(1995) and Duffee(1995) where they report the change in the level of interest rate represented by the treasury yields are very important to the variation in credit spreads of the corporate bonds and they are negatively correlated.

3.1 A General Model

We start with the definition of the default indicator function. First, let τ_j denote the time of default of the company **j** which is a totally inaccessible stopping time and

$$H_j(t) = 1_{\{t \geq \tau_j\}} \tag{3.16}$$

is called the default indicator function. Next, a counting process is defined by

$$N_j(t) = \sum_{k=1} 1_{\{t \geq \tau_{jk}\}}$$

where $\tau_j = \tau_{j1} < \tau_{j2} < \cdots$ is an increasing sequence of stopping times. By using Doob-Meyer decomposition, $N_j(t)$ is decomposed into the finite variation part and the martingale part. We know that the finite variation part is continuous if τ_j is a totally inaccessible stopping time. Furthermore, we assume absolutely continuous. Then,

$$N_j(t) = \int_0^t \lambda_j(s)ds + M_j(t)$$

where $\lambda_j(t)$ is an intensity function which may be random that is, in general, $\lambda_j(t)$ may depend on times and the realization of the other variables as well as on the value of the contract itself, and $M_j(t)$ ia a martingale. In particular, we will make $\lambda_j(t)$ related to the wealth W and the factors $\vec{Y}$. Under this setting, $H_j(t)$ is expressed by the stopped process of $N_j(t)$ at $t = \tau_j$.

$$H_j(t) = N_j(t \wedge \tau_j) = \int_0^t \lambda_j(s) 1_{\{s \leq \tau_j\}} ds + m_j(t) \tag{3.17}$$

where $m_j(t) = M_j(t \wedge \tau_j)$ is a martingale and $1_{\{t \leq \tau_j\}} = 1 - H_j(t-)$. We will extend the CIR(1985a) model by utilizing the new factors.

For ease of exposition, we introduce two types of the default indicator functions in the model and assume that the simultaneous defaults do not occur. We first fix the period of the economy as $[0, T^*], T^* < \infty$. We define the factors governing the economy of the model.

$$d\vec{Y} = \vec{\mu}(\vec{Y}, t)dt + \underline{S}(\vec{Y}, t)d\vec{w}_t \tag{3.18}$$

$$dH_j = \lambda_j(\vec{Y}, W, t)(1 - H_{jt-})dt + dm_j \quad \text{j=1,2} \tag{3.19}$$

where $\vec{Y}$ and $\underline{S}$ denote the $k \times 1$ vector and $k \times (n + k)$ matrix respectively, and $\vec{w}_t$ is the $(n + k) \times 1$ Brownian motion.

We assume that there exist n types of real assets or capital stocks following the stochastic processes which are exogenously specified. The uncertainity of the real assets in the next instant is generated by the Brownian motions while the coefficients of the processes may depend on the factors $H_j, j = 1, 2$ as well as $\vec{Y}$.

$$d\vec{\eta} = \underline{I}_\eta \vec{\alpha}(\vec{Y}, H_1, H_2, t)dt + \underline{I}_\eta \underline{G}(\vec{Y}, H_1, H_2, t)d\vec{w}_t \tag{3.20}$$

where $\vec{\eta}$ and $\underline{G}$ denotes the $n \times 1$ vector and $n \times (n+k)$ respectively.

We also suppose that there exist the financial assets of the riskless bond and $(k+2)$ types of defaultable securities. We note that the processes of the securities with default risks include not only the continuous components generated by Brownian motions, but also the jump components due to their default risks.

$$
\begin{aligned}
dB_t &= rB_t dt && (3.21)\\
dP_i &= \beta_i P_i dt + \vec{\sigma}_i^\top P_i d\vec{w}_t && (3.22)\\
&+ (\Theta_{i1} P_i - \varphi_{i1} P_i) dH_{1t} + (\Theta_{i2} P_i - \varphi_{i2} P_i) dH_{2t};\ i = 1, 2, \cdots, (k+2)
\end{aligned}
$$

where $\vec{\sigma}_i,\ k = 1, 2, \cdots, (k+2)$ denote $(n+k) \times 1$ vectors. r, $\beta_i, i = 1, 2, \cdots, (k+2)$, $\vec{\sigma}_i, i = 1, 2, \cdots, (k+2)$, $\Theta_{ij}, i = 1, 2, \cdots, (k+2), j = 1, 2$ are endogenously determined in equilibrium while $\vec{\alpha}$ and $\underline{G}$ are exogenously given.

We assume that the contracts of the defaultable securities are terminated once a default occurs. The term $-\varphi_{ij} P_i dH_{jt}, j = 1, 2$ for each $i = 1, 2, \cdots, (k+2)$ means that once the default of the company **j**, **j**=1,2 occurs, the dividend $\varphi_{ij} P_i, j = 1, 2$ respectively is paid at that default time where $\varphi_{ij}, j = 1, 2$ for each $i = 1, 2, \cdots, (k+2)$ are exogenously given recovery rates. In other words, we assume that the contracts are terminated once a default occurs and the payoffs in case of the default are paid.

Then, the wealth process for the representative agent is expressed as

$$
\begin{aligned}
dW_t &= [(rW - c) + \vec{a}^\top (\vec{\alpha} - r\vec{1})W + \sum_{i=1}^{(k+2)} b_i (\beta_i - r) W] dt && (3.23)\\
&+ W \vec{a}^\top \underline{G} d\vec{w}_t + +W \sum_{i=1}^{(k+2)} b_i \vec{\sigma}_i^\top d\vec{w}_t + W \sum_{i=1}^{(k+2)} b_i \sum_{j=1}^{2} \Theta_{ij} dH_{jt}.
\end{aligned}
$$

Hence the "dynamic programming problem" in this economy for the representative agent is defined by

$$
max_{c \geq 0, \vec{a} \geq 0, b_1, b_2, \cdots, b_{k+2}} \mathbf{E}_0 \left[\int_0^{T^*} u(c, \vec{Y}, H_1, H_2, t) dt \right] \qquad (3.24)
$$

subject to the wealth process (3.23).

Then, an equilibrium in the economy is defined as in the original CIR(1985a). That is, an equilibrium is defined by a set of the equilibrium stochastic processes

$(r, P_1, P_2, \cdots, P_{k+2})$ with $\vec{a} \geq \vec{0}$ and $\vec{a}^\top \vec{1} = 1$ so that given $(r, P_1, P_2, \cdots, P_{k+2})$, $(c, \vec{a}, b_1, b_2, \cdots, b_{k+2})$ where $c \geq 0$, $\vec{a} \geq \vec{0}$, $\vec{a}^\top \vec{1} = 1$, and $b_i = 0, i = 1, 2, \cdots, (k+2)$ solves the "dynamic programming problem" for the representative agent. We solve this problem in the following three steps, [A], [B] and [C]. We denote the indirect utility function in equilibrium by $J(W, \vec{Y}, H_1, H_2, t)$.

[A] We first solve the optimization problem in the economy where there are only real assets.

$$max_{c\geq 0, \vec{a}\geq 0, \vec{a}=1} \mathbf{E}_0 \left[\int_0^{T^*} u(c, \vec{Y}, H_1, H_2, t) dt \right]$$

subject to

$$dW_t = (\vec{a}^\top \vec{\alpha} W - c)dt + W \vec{a}^\top \underline{G} d\vec{w}_t.$$

The Bellman equation is obtained by

$$\begin{aligned} 0 &= J_t + max_{c\geq 0, \vec{a}\geq 0} \Big[u(c_t) + (\vec{a}^\top \vec{\alpha} W - c) J_W \\ &+ \frac{1}{2} \vec{a}^\top \underline{GG}^\top \vec{a} W^2 J_{WW} + \vec{a}^\top \underline{GS}^\top W \vec{J}_{W\vec{Y}} + \vec{\mu}^\top \vec{J}_{\vec{Y}} + \frac{1}{2} tr(\underline{SS}^\top \vec{J}_{\vec{Y}\vec{Y}}) \\ &+ \lambda_1 (1 - H_{1t-}) \{ J(W, \vec{Y}, 1, H_2) - J(W, \vec{Y}, H_1, H_2) \} \\ &+ \lambda_2 (1 - H_{2t-}) \{ J(W, \vec{Y}, H_1, 1) - J(W, \vec{Y}, H_1, H_2) \} + \gamma (1 - \vec{a}^\top \vec{1}) \Big] . \end{aligned}$$

where γ denotes the Lagrangian multiplier associated with $\vec{a}^\top \vec{1} = 1$. The first order condition of the Bellman equation with respect to $\vec{a}$ is given by

$$(W J_W) \vec{\alpha} + \underline{GG}^\top \vec{a} W^2 J_{WW} + \underline{GS}^\top W \vec{J}_{W\vec{Y}} - \gamma \vec{1} \leq 0 \quad (3.25)$$

where the equality holds for $\vec{a} > 0$.

[B] Next, we consider the optimization problem in the economy where the riskless asset exists in addition to real assets.

$$max_{c\geq 0, \vec{a}\geq 0} \mathbf{E}_0 \left[\int_0^{T^*} u(c, \vec{Y}, H_1, H_2, t) dt \right]$$

subject to

$$dW_t = [\vec{a}^\top (\vec{\alpha} - r\vec{1}) W + (rW - c)] dt + W \vec{a}^\top \underline{G} d\vec{w}_t.$$

The Bellman equation is obtained by

$$0 = J_t + max_{c\geq 0, \vec{a}\geq 0} \Big[u(c_t) + \{ (rW - c) + \vec{a}^\top (\vec{\alpha} - r\vec{1}) W \} J_W$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\vec{a}^\top \underline{GG}^\top \vec{a} W^2 J_{WW} + \vec{a}^\top \underline{GS}^\top W \vec{J}_{W\vec{Y}} + \vec{\mu}^\top \vec{J}_{\vec{Y}} + \frac{1}{2} tr(\underline{SS}^\top \vec{J}_{\vec{Y}\vec{Y}}) \\
&+ \quad \lambda_1(1 - H_{1t-})\{J(W, \vec{Y}, 1, H_2) - J(W, \vec{Y}, H_1, H_2)\} \\
&+ \quad \lambda_2(1 - H_{2t-})\{J(W, \vec{Y}, H_1, 1) - J(W, \vec{Y}, H_1, H_2)\}\Big].
\end{aligned}
$$

The first order condition with respect to $\vec{a}$ in this Bellman equaition is given by

$$
(W J_W)(\vec{\alpha} - r\vec{1}) + \underline{GG}^\top \vec{a} W^2 J_{WW} + \underline{GS}^\top W \vec{J}_{W\vec{Y}} \leq 0 \tag{3.26}
$$

where the equality holds for $\vec{a} > 0$. Comparing the first order condition (3.25) and (3.26), we obtain

$$
r = \frac{\gamma}{W J_W}.
$$

We also note that

$$
\vec{a}^\top \vec{\alpha} W J_W + \vec{a}^\top \underline{GG}^\top \vec{a} W^2 J_{WW} + \vec{a}^\top \underline{GS}^\top W \vec{J}_{W\vec{Y}} = \gamma.
$$

Hence we obtain

$$
r = \vec{a}^\top \vec{\alpha} + (\vec{a}^\top \underline{GG}^\top \vec{a}) \left(\frac{W J_{WW}}{J_W} \right) + (\vec{a}^\top \underline{GS}^\top) \left(\frac{\vec{J}_{W\vec{Y}}}{J_W} \right). \tag{3.27}
$$

Note if we define the factor premium for the wealth W, ϕ_W by

$$
\phi_W = (\vec{a}^\top \underline{GG}^\top \vec{a}) \left(\frac{-W^2 J_{WW}}{J_W} \right) + (\vec{a}^\top \underline{GS}^\top) \left(-W \frac{\vec{J}_{W\vec{Y}}}{J_W} \right), \tag{3.28}
$$

the equilibrium r may be written by

$$
r = \vec{a}^\top \vec{\alpha} - \frac{1}{W} \phi_W.
$$

[C] Finally, we turn to the main objective of deriving the pricing equation for the pre-default values of the defaultable securities. First, we show the Bellman equation for the dynamic programming problem where all the contingent claims are included in addition to real assets and the riskless asset.

Lemma 3.1 *The Bellman equation of the dynamic programming problem for the representative agent is given by*

$$
0 \quad = \quad J_t + max_{c \geq 0, \vec{a} \geq 0, b_1, b_2, \cdots, b_{k+2}} \Big[u(c_t) + \{(rw - c) + \vec{a}^\top (\vec{\alpha} - r\vec{1}) W \tag{3.29}
$$

$$
\begin{aligned}
&+ \sum_{i=1}^{(k+2)} b_i(\beta_i - r)W\} J_W \\
&+ \frac{1}{2}\vec{a}^\top \underline{GG}^\top \vec{a} W^2 J_{WW} + \frac{1}{2}\vec{b}^\top \underline{\Sigma\Sigma}^\top \vec{b} W^2 J_{WW} + \vec{a}^\top \underline{G\Sigma}^\top \vec{b} W^2 J_{WW} \\
&+ \vec{\mu}^\top \vec{J}_{\vec{Y}} + \frac{1}{2} tr(\underline{SS}^\top \vec{J}_{\vec{Y}\vec{Y}}) + \vec{a}^\top \underline{GS}^\top W \vec{J}_{W\vec{Y}} + \vec{b}^\top \underline{\Sigma S}^\top W \vec{J}_{W\vec{Y}} \\
&+ \lambda_1(1 - H_{1t-})\{J(W[1 + \sum_{i=1}^{(k+2)} b_i \Theta_{i1}], \vec{Y}, 1, H_2, t) - J(W, \vec{Y}, H_1, H_2, t)\} \\
&+ \lambda_2(1 - H_{2t-})\{J(W[1 + \sum_{i=1}^{(k+2)} b_i \Theta_{i2}], \vec{Y}, H_1, 1, t) - J(W, \vec{Y}, H_1, H_2, t)\} \Big]
\end{aligned}
$$

where $\vec{b} = [b_1, b_2, \cdots, b_{k+2}]^\top$ *and* $\Sigma = [\vec{\sigma_1}, \vec{\sigma_2}, \cdots, \vec{\sigma_{k+2}}]^\top$.

The first order conditions with respect to b_i are given by

$$
\begin{aligned}
0 &= (\beta_i - r)WJ_W + \left[b_i \vec{\sigma_i}^\top \vec{\sigma_i} + \sum_{i' \neq i} b_{i'} \vec{\sigma_i}^\top \vec{\sigma_{i'}} + \vec{a}^\top \underline{G} \vec{\sigma_i} \right] W^2 J_{WW} \\
&+ \vec{\sigma_i}^\top \underline{S}^\top W \vec{J}_{W\vec{Y}} + \lambda_1(1 - H_{1t-})\Theta_{i1} W J_W(W[1 + \sum_{i=1}^{(k+2)} b_i \Theta_{i1}], \vec{Y}, 1, H_2, t) \\
&+ \lambda_2(1 - H_{2t-})\Theta_{i2} W J_W(W[1 + \sum_{i=1}^{(k+2)} b_i \Theta_{i2}], \vec{Y}, H_1, 1, t).
\end{aligned}
$$

We set $b_i = 0,\ i = 1, 2, \cdots, (k+2)$ in equilibrium and obtain the relation which the drift term, the diffusion term and the coefficients of the jump terms must satisfy in equilibrium.

$$
\begin{aligned}
\beta_i - r &= \vec{a}^\top \underline{G} \vec{\sigma}_i \left(\frac{-WJ_{WW}}{J_W} \right) + \vec{\sigma}_i^\top \underline{S}^\top \left(\frac{-\vec{J}_{W\vec{Y}}}{J_W} \right) \\
&- \Theta_{i1}\lambda_1(1 - H_1) \left(\frac{J_W(W, \vec{Y}, 1, H_2, t)}{J_W(W, \vec{Y}, H_1, H_2, t)} \right) \\
&- \Theta_{i2}\lambda_2(1 - H_2) \left(\frac{J_W(W, \vec{Y}, H_1, 1, t)}{J_W(W, \vec{Y}, H_1, H_2, t)} \right)
\end{aligned}
$$

We define $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ by

$$
\tilde{\lambda}_1 = \lambda_1 \left(\frac{J_W(W, \vec{Y}, 1, H_2, t)}{J_W(W, \vec{Y}, H_1, H_2, t)} \right) \quad (3.30)
$$

and

$$\tilde{\lambda}_2 = \lambda_2 \left(\frac{J_W(W, \vec{Y}, H_1, 1, t)}{J_W(W, \vec{Y}, H_1, H_2, t)} \right). \tag{3.31}$$

Note $\tilde{\lambda}_j \geq 0, j = 1, 2$ as long as $J_W > 0$. Suppose $P_i = P_i(W, \vec{Y}, H_1, H_2, t), i = 1, 2, \cdots, (k+2)$. Then, we first replace $\vec{\sigma}_i, i = 1, 2, \cdots, (k+2)$ by the coefficients of the diffusion terms which are obtained by applying Ito's lemma to $P_i = P_i(W, \vec{Y}, H_1, H_2, t), i = 1, 2, \cdots, (k+2)$.

$$\vec{\sigma}_i P_i = \underline{G}^\top \vec{a} W P_{iW} + \underline{S}^\top \vec{P}_{i\vec{Y}}$$

Thus th excess returns of $P_i, i = 1, 2, \cdots, (k+2)$ are expressed as

$$\begin{aligned}
(\beta_i - r)P_i &= (\vec{a}^\top \underline{GG}^\top \vec{a} W P_{iW} + \vec{a}^\top \underline{GS}^\top \vec{P}_{i\vec{Y}}) \left(\frac{-W J_{WW}}{J_W} \right) \\
&+ (P_{iW} W \vec{a}^\top \underline{GS}^\top + \vec{P}_{i\vec{Y}}^\top \underline{SS}^\top) \left(\frac{-\vec{J}_{W\vec{Y}}}{J_W} \right) \\
&- \Theta_{i1} \tilde{\lambda}_1 (1 - H_1) P_i - \Theta_{i2} \tilde{\lambda}_2 (1 - H_2) P_i \\
&= \left[\vec{a}^\top \underline{GG}^\top \vec{a} \left(\frac{-W^2 J_{WW}}{J_W} \right) + \vec{a}^\top \underline{GS}^\top \left(\frac{-W \vec{J}_{W\vec{Y}}}{J_W} \right) \right] P_{iW} \\
&+ \vec{P}_{i\vec{Y}}^\top \left[\underline{SG}^\top \vec{a} \left(\frac{-W J_{WW}}{J_W} \right) + \underline{SS}^\top \left(\frac{-\vec{J}_{W\vec{Y}}}{J_W} \right) \right] \\
&- \Theta_{i1} \tilde{\lambda}_1 (1 - H_1) P_i - \Theta_{i2} \tilde{\lambda}_2 (1 - H_2) P_i \\
&\equiv \phi_W P_{iW} + \vec{\phi}_{\vec{Y}}^\top \vec{P}_{i\vec{Y}} - \Theta_{i1} \tilde{\lambda}_1 (1 - H_1) - \Theta_{i2} \tilde{\lambda}_2 (1 - H_2),
\end{aligned}$$

where

$$\phi_W = (\vec{a}^\top \underline{GG}^\top \vec{a}) \left(\frac{-W^2 J_{WW}}{J_W} \right) + (\vec{a}^\top \underline{GS}^\top) \left(-W \frac{\vec{J}_{W\vec{Y}}}{J_W} \right)$$

and

$$\vec{\phi}_{\vec{Y}} = \underline{SG}^\top \vec{a} \left(\frac{-W J_{WW}}{J_W} \right) + \underline{SS}^\top \left(\frac{-\vec{J}_{W\vec{Y}}}{J_W} \right). \tag{3.32}$$

Next, we substitute $\beta_i P_i$ by using Ito's formula. Then,

$$\begin{aligned}
\beta_i P_i &= \frac{1}{2} Var.(dW) P_{iWW} + \frac{1}{2} tr(\underline{SS}^\top P_{i\vec{Y}\vec{Y}}) + Cov.(dW, d\vec{Y})^\top P_{iW\vec{Y}} \\
&+ (\vec{a}^\top \vec{\alpha} W - c) P_{iW} + \vec{\mu}^\top P_{i\vec{Y}} + P_{it}.
\end{aligned}$$

Next comparing the jump terms which are also obtained by applying Ito's formula to $P_i = P_i(W, \vec{Y}, H_1, H_2, t)$ with those of the original process of P_i, we have

$$\begin{aligned} & (\Theta_{i1} - \varphi_{i1})P_i(t-)dH_{1t-} + (\Theta_{i2} - \varphi_{i2})P_i(t-)dH_{2t-} \\ = \;& P_i(W, \vec{Y}, H_1, H_2, t) - P_i(W, \vec{Y}, H_1, H_2, t-). \end{aligned}$$

Our objective is to derive the pricing equation of the pre-default value and hence set $H_{1t-} = H_{2t-} = 0$. Then,

$$\begin{aligned} & (\Theta_{i1} - \varphi_{i1})P_i(t-)H_{1t} + (\Theta_{i2} - \varphi_{i2})P_i(t-)H_{2t} \\ = \;& P_i(W, \vec{Y}, H_1, H_2, t) - P_i(W, \vec{Y}, 0, 0, t-). \end{aligned}$$

If $H_{1t} = 0$ and $H_{2t} = 0$, then the above equation identically holds since W and $\vec{Y}$ are continuous processes, and $P_i(W, \vec{Y}, 0, 0, t)$ is continuous in $W, \vec{Y}$ and t, that is $P_i(t) = P_i(t-)$. Moreover, we note that $H_{1t} = 1$ and $H_{2t} = 1$ do not occur simultaneously by assumption. Therefore, we have the relations.

$$\begin{aligned} (\Theta_{i1} - \varphi_{i1})P_i(t-) &= P_i(W, \vec{Y}, 1, 0, t) - P_i(W, \vec{Y}, 0, 0, t-) \\ &\text{and} \\ (\Theta_{i2} - \varphi_{i2})P_i(t-) &= P_i(W, \vec{Y}, 0, 1, t) - P_i(W, \vec{Y}, 0, 0, t-). \end{aligned}$$

Then, we also set $P_i(W, \vec{Y}, 1, 0, t) = 0$ and $P_i(W, \vec{Y}, 0, 1, t) = 0$ as the boundary conditions since the contracts are terminated once a default occurs. Hence, we obtain

$$\begin{aligned} \Theta_{i1}P_i(t-) &= -(1 - \varphi_{i1})P_i(t-) \\ &\text{and} \\ \Theta_{i2}P_i(t-) &= -(1 - \varphi_{i2})P_i(t-). \end{aligned}$$

Finally, using $rW = \vec{a}^\top \vec{\alpha} W - \phi_W$, we have the PDE for pre-default values of the defaultable securities.

Theorem 3.2 *The pre-default value of the defaultable securitie, P_i must satisfy the partial differential equation.*

$$
\begin{aligned}
&\frac{1}{2}Var.(dW)P_{iWW} + \frac{1}{2}tr(\underline{S}\underline{S}^\top P_{i\vec{Y}\vec{Y}}) + Cov.(dW, d\vec{Y})^\top P_{iW\vec{Y}} \\
&+(r^*W - c)P_{iW} + (\vec{\mu}^\top - \vec{\phi}_{\vec{Y}}^\top)P_{i\vec{Y}} + P_{it} \\
&-\{r + (1-\varphi_{i1})\tilde{\lambda}_1 + (1-\varphi_{i2})\tilde{\lambda}_2\}P_i = 0. \qquad (3.33)
\end{aligned}
$$

Hence, once a terminal boundary condition is specified by the payoff of a contract without default, we can evaluate the pre-default value of the defaultable financial asset. In particular, given the terminal condition, the solution of these PDE is represented by the conditional expectation under some probability measure.

Proposition 3.1 *The pre-default value of P_i at the time t is represented by*

$$P_i(t) = \mathbf{E}_t^*\left[e^{-\int_t^T R_i(u)du}X_i(T)\right] \qquad (3.34)$$

where $X_i(T)$ is the terminal boundary condition determined by the payoff of the security i without default and

$$R_i(t) = r(t) + \{1-\varphi_{i1}(t)\}\tilde{\lambda}_1(t) + \{1-\varphi_{i2}(t)\}\tilde{\lambda}_2(t). \qquad (3.35)$$

The conditional expectation is taken under the measure Q where W,$\vec{Y}$ and $H_j, j = 1,2$ follow the stochastic processes

$$
\begin{aligned}
dW &= (rW - c)dt + W\vec{a}^{*\top}\underline{G}d\vec{w}_t^* \\
d\vec{Y} &= (\vec{\mu} - \vec{\phi}_{\vec{Y}})dt + \underline{S}d\vec{w}_t^* \\
&\text{and} \\
dH_j &= \tilde{\lambda}_j(1-H_j)dt + dm_{jt}^* \quad j = 1,2,
\end{aligned}
$$

where $\vec{w}^*$ and m_j^*, $j = 1,2$ denote the $(n+k)$ dimensional Brownian motion and the jump martingales under Q respectively.

Finally, we make a comment on the recovery rate $\varphi_{ij}, j = 1,2$ of the security i when the payoff to both the parties in the contract can be positive or negative such as a swap. In this case, φ_{i1} and φ_{i2} are given respectively by

$$
\begin{aligned}
\varphi_{i1} &= \varphi_2^* 1_{\{P_i \geq 0\}} + \varphi_1 1_{\{P_i < 0\}} \\
&\text{and} \\
\varphi_{i2} &= \varphi_2 1_{\{P_i \geq 0\}} + \varphi_1^* 1_{\{P_i < 0\}}.
\end{aligned}
$$

As *the full payment rule* becomes common in practice, $\varphi_1^* = \varphi_2^* = 1$. Then, $R_i(t)$ in the case is given by

$$R_i(t) = r(t) + \{1 - \varphi_1(t)\}\tilde{\lambda}_1(t)1_{\{P_i<0\}} + \{1 - \varphi_2(t)\}\tilde{\lambda}_2(t)1_{\{P_i\geq 0\}}.$$

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3.2 An Example: A Two Factor Model

We next show a simple example which illustrates a two-factor model as a special case of the general model presented in the previous subsection. First, we fix the period of the economy as $[0, T^*]$ where T^* is a prespecified constant. Next, we specify the state variables, one of which governs the macro economy and the other of which is the default indicator function specific to the company **j**.

$$\begin{aligned} dY &= (\xi_1 Y + \xi_2)dt + \sigma\sqrt{Y}dw_t \\ dH_j &= \lambda(W, Y, t)(1 - H_j)dt + dm_j \end{aligned}$$

where w_t is a one-dimensional Brownian motion and m_j is a jump martingale. $Y(t)$ follows the square-root process and $H_j(t)$ follws a jump process whose intensity depends on the wealth and the interest rate in equilibrium.

There is a single agent in the economy where the representative agent's utility function is represented by a logarithmic utility.

$$\mathbf{E}\left[\int_0^{T^*} e^{-\rho t}\log c_t \; dt\right]$$

where ρ denotes the rate of the time preference which is a positive constant.

The capital stock process is completely determined by the factor Y.

$$d\eta = \alpha Y\eta dt + G\sqrt{Y}\eta dw_t$$

Finally, we suppose that four financial assets in the economy are traded in the market as follows.

- The riskless money market account B_t with the riskless rate r which is endogenously determined.

$$dB = rBdt$$

- The equity S_t which may be interpreted as the market portfolio in and claims η_{T^*} at T^*

$$dS = \mu_S S dt + \sigma_S S dw_t$$

- The default-free zero-coupon bond $P(t,T)$ whose payoff at $T(< T^*)$ is $P(T,T) = 1$.

$$dP(t,T) = \mu_P P(t,T)dt + \sigma_P P(t,T)dw_t$$

- The defaultable zero-coupon bond $P_j(t,T)$ which is characterized by the payoff that the dividends $\varphi_j P_j$ is paid once a default occurs where φ_j is an exogenously given recovery rate, and $P_j(T,T) = 1$ at the maturity T without default.

$$dP_j = \beta_j P_j dt + \sigma P_j dw_t + (\Theta_j P_j - \varphi_j P_j) dH_{jt}$$

A simple calculation shows that in equilibrium, the spot rate r_t is a linear function of the factor Y_t and the market portfolio S_t is equivalent to the wealth W_t.

$$\begin{aligned} r_t &= (\alpha - G^2)Y_t \\ S_t &= \eta_t = W_t \end{aligned}$$

The zero coupon bond is easily given by an exponential function of r as in CIR(1985b).

$$P(t,T) = A(T-t)e^{B(T-t)r_t}$$

where $A(T-t)$ and $B(T-t)$ are the functions of $(T-t)$.

Now we turn to the main objective of pricing the defaultable bond. First we note that $\lambda_j(W,Y,t)$ may be rewritten by a functin of S, r and t since $S = W$ and r corresponds to Y by one-to-one. That is, we make the intensity depend on the market potfolio and the interest rate. The intuition behind this is that the possibility of default is closely related to the macro economy whose indicators may be the level of the market portfolio and that of the interest rate. For instance, when the interest rate as well as the price of the market portfolio is very low, the economy is usually in the recession and hence the possibility of default is relatively high. Alternatively, a company's profit may be positively or negatively related to the level of the interest rate or/and that of the market portfolio. For example, the banking sector is usually profitable when the short term interest rate is low and the yiled curve has a positive slope while the security company is profitable when the level of the market portfolio represented by a stock index rather high. Hence let $\lambda_j(W,Y,t) = \lambda_j(S,r,t)$. Then, by using the fundamental pricing PDE in the general model previously derived, we can easily show $P_j(t,T)$ satisfies the PDE

$$\frac{1}{2}\left(\frac{G^2}{\alpha - G^2}\right) rS^2 P_{jSS} + \frac{1}{2}\sigma^2(\alpha - G^2) rP_{jrr} + G\sigma rSP_{jSr} +$$
$$\left(r - \frac{\rho}{[1 - e^{-\rho(T^* - t)}]}\right) SP_{jS} + \left[(\xi_1 - G\sigma)r + (\alpha - G^2)\xi_2\right] P_{jr} + P_{jt} -$$
$$\{r + (1 - \varphi_j)\lambda_j\} P_j = 0.$$

The terminal boundary condition is obviously given by

$$P_j(T,T) = 1.$$

Equivalently, the solution of the PDE with the termianl boundary condtion is represented by a conditional expectation. That is,

$$P_j(t,T) = \mathbf{E}_t^* \left[e^{-\int_t^T \{r_u + (1-\varphi_j)\lambda_{ju}\} du} \right]$$

where the conditional expectation is taken under for $t \in [0, T^*)$,

$$\begin{aligned} dS &= \left(r - \frac{\rho}{[1 - e^{-\rho(T^* - t)}]}\right) Sdt + \sigma_S \sqrt{r} Sdw_t^* \\ dr &= \kappa(\theta - r)dt + \sigma_r \sqrt{r} dw_t^* \end{aligned}$$

where

$$\begin{aligned} \sigma_S &= \frac{G}{\sqrt{\alpha - G^2}} \\ \kappa &= -\xi_1 + G\sigma \\ \theta &= \frac{(\alpha - G^2)\xi_2}{(-\xi_1 + G\sigma)} \\ &\text{and} \\ \sigma_r &= \sigma\sqrt{\alpha - G^2}. \end{aligned}$$

For example, $\lambda_j(S,r,t)$ is specified by

$$\lambda_j(S,r,t) = \frac{1}{(aS+b)} + \frac{1}{(gr+h)}$$

or

$$\lambda_j(S,r,t) = \frac{1}{(aSr+b)}$$

where a, b, g and h are positive constants, which indicates that the default occurs more likely when the level of the market portfolio and the interest rate are low and vice versa.

A Numerical Example

We will give a numerical example of $P_j(t,T)$ in the two-factor model. First, $\lambda_j(S,r,t)$ is specified by

$$\lambda_j(S,r,t) = \frac{1}{(aS+b)} + \frac{1}{(gr+h)}$$

where the parameters are given by $a = 1381.0, b = 3.3$ and $g = 2.0, h = 0.1$.

The other parameters are specified as follows.

$$\begin{aligned} T^* &= 30.00 \\ T &= 5.00 \\ \varphi &= 0.50 \\ S_0 &= 100.00 \\ \rho &= 0.07 \\ \sigma_S &= 0.80 \\ r_0 &= 0.03 \\ \kappa &= 0.25 \\ \theta &= 0.07 \\ \sigma_r &= 0.15 \end{aligned}$$

We compute $P_j(t,T)$ by the Monte Carlo simulation and the results are shown in Table 3.2. We list the spot yields of the five-year defaultable bonds, those of the default-free bonds with the same maturity and the spreads between them for different initial values of the spot rate and the market portfolio. In the first three rows, we fix the initial value of the market portfolio as 100 and compute the spreads for three different initial values of the spot rate, 3.00 %, 7.00 % and 10.00 %. As we expect, the lower is the initial value of the spot rate, the larger is the spread because in general, the intensity is larger when the initial value is smaller by the functional form of the intensity and consequently the default risk is higher. In fact, the spread is 162 basis point for the inital value of 3.00 % while the spread is 100 basis point for the inital value of 10.00 %. In the next two rows, we fix the initial value of the spot rate as 7.00 % and compute the spreads for two different initial values, 50 and 150 of the market portfolio. Then, the similar observation holds. That is, the spread for the inital value of 50 is 142 basis point and the spread for the inital value of 150 is 110 basis point. In the last two rows, we show the two extreme cases. In the first case, both the inital values are rather low; the inital value of the spot rate is 3.00 % and that of the market portfolio is 50. In another case, both are very high; the inital value of the spot rate is 10.00 % and that of the market portfolio is 150. Then, the spread is 186 basis point for the first case, which is the largest among all the spreads reported in this table, and the spread is 92 basis point for the second case, which is the smallest among all the spreads in this table.

4 Concluding Remarks

We present new models for pricing of the securities subject to default risks in a dynamic general equilibrium framework. We propose two types of the models, one of which is to utilize a predictable stopping time to chracterize the state of default, and the other of which is to introduce a totally inaccessible stopping time, a jump process to model the event of the default for each company. Both models are general enough to evaluate any defaultable securities. The state of default in the model may be related to macro economic varialbles such as the spot interest rate and the market portfolio which are determined in equilibrium inside the model as well as to the factors specific to each company such as a firm's asset. For instance, in the predictable stopping time model, the default occurs when the spot interest rate or/and the market stock index hit some low level which may vary among companies, and in the jump process model, the possibility of the default in the next instant of a company depends on the level of the spot interest rate and that of the wealth in the economy while the default itself occurs due to the firm's own reasons which are usually unobservable. To illustrate those features, we also present a set of simple examples such as one-facor or two-factor models.

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Table 3.1: The default boundaries and spreads(default-free 5y rate 6.73%)

S_{jL}	r_{jL}(%)	yield(%)	spread(bp)
60	0.5	7.87	114
50	0.5	7.57	85
40	0.5	7.00	27
50	0.25	7.36	63
50	0.75	7.76	102
50	1.00	7.85	111

Table 3.2: The defaultable spot yield(5Y)

r_0(%)	S_0	(1)defalutable yield(%)	(2)default-free yield(%)	(1)-(2)(bp)
3.00	100	6.18	4.56	162
7.00	100	7.91	6.73	118
10.00	100	9.36	8.36	100
7.00	50	8.15	6.73	142
7.00	150	8.83	6.73	110
3.00	50	6.42	4.56	186
10.00	150	9.28	8.36	92

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