Derivative Pricing under Collateralization *

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Introduction

New market realties after the Financial Crisis

- **Wide use of collateralization in OTC**
  Dramatic increase in recent years (ISDA Margin Survey 2011)
  - **30%(2003) → 70%(2010) in terms of trade volume for all OTC.**
  - **Coverage goes up to 80% (for all OTC) and 88% (for fixed income) among major financial institutions.**
  - **More than 80% of collateral is Cash.**
- **Persistently wide basis spreads:**
  Much more volatile CCS basis spread.
  Non-negligible basis spreads even in the single currency market. (e.g. Tenor swap spread, Libor-OIS spread)

Impact of Collateralization

Impact of collateralization :

- **Reduction of Counter-party Exposure**
- **Change of Funding Cost**
  - **Require new term structure model to distinguish discounting and reference rates.**
  - **Cost of collateral is different from currency to currency.**
  - **Choice of collateral currency ("cheapest-to-deliver" option).**
  - **Significant impact on derivative pricing.**

Topics of this talk

- **Derivative pricing under the perfect collateralization:**
    - **Symmetric collateralization and choice of collateral currency**
    - **Asymmetric collateralization and potential effects from the difference in collateral management**
- **Imperfect collateralization and CVA**
- **(CDS under the collateralization)**

(For details, please see the series of our papers ([8]- [12]).)

Setup

- Probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F}$ contains all the market information including defaults.
- Consider two firms, $i \in \{1, 2\}$, whose default time is $\tau^i \in [0, \infty]$, and $\tau = \tau^1 \wedge \tau^2$.
- τ^i (and hence τ) is assumed to be totally-inaccessible $\mathbb{F}$-stopping time.
- Indicator functions: $H^i_t = 1_{\{\tau^i \leq t\}}$, $H_t = 1_{\{\tau \leq t\}}$
- Assume the existence of absolutely continuous compensator for H^i:

$$A^i_t = \int_0^t h^i_s 1_{\{\tau^i > s\}} ds, \quad t \geq 0$$

- Assume no simultaneous defaults, and hence the hazard rate of H is

$$h_t = h^1_t + h^2_t \ .$$

- Money market account: $\beta_t = \exp\left(\int_0^t r_u du\right)$

Collateralization

- When party $i \in \{1,2\}$ has negative mark-to-market, it has to post collateral (cash) to party $j(\neq i)$, and it is assumed to be done continuously.
- collateral coverage ratio is $\delta_t^i \in \mathbb{R}_+$, and the value of collateral at time t is given by $\delta_t^i(-V_t^i)$.(V_t^i denotes the mark-to-market value of the contract from the view point of party i.)
  - δ_t^i effectively takes into account under- as well as over-collateralization. Thus $\delta_t^i < 1$, and also $\delta_t^i > 1$ are possible.
- party j has to pay the collateral rate c_t^i on the posted cash continuously.
- c_t^i is determined by the currency posted by party i.
  - market convention is to use overnight (O/N) rate at time t of corresponding currency.
    $\Rightarrow$ Traded through OIS (overnight index swap), which is also collateralized.
  - In general, $c_t^i \neq r_t^i$. (r_t^i is the risk-free interest rate of the same currency.) This is necessary to explain CCS swap market consistently.

Counterparty Exposure and Recovery Scheme

- Counterparty exposure to party j at time t from the view point of party i is given as:

$$\max(1-\delta_t^j,0)\max(V_t^i,0)+\max(\delta_t^i-1,0)\max(-V_t^i,0).$$

- Assume party-j's recovery rate at time t as $R_t^j \in [0,1]$.
- Then, the recovery value at the time of j's default is given as:

$$R_t^j\left([1-\delta_t^j]^+[V_t^i]^+ + [\delta_t^i-1]^+[-V_t^i]^+\right).$$

Pricing Formula

- Pricing from the view point of party 1.

$$S_t = \beta_t E^Q\Bigg[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}}\Big\{dD_u + (y_u^1 \delta_u^1 1_{\{S_u < 0\}} + y_u^2 \delta_u^2 1_{\{S_u \geq 0\}}) S_u du\Big\}$$
$$+ \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}}\Big(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2\Big)\Bigg| \mathcal{F}_t\Bigg]$$

- D: cumulative dividend to party 1.
- $y_t^i = r_t^i - c_t^i$, $(i \in \{1, 2\})$ denotes the instantaneous return at time t from the cash collateral posted by party i.
- Default payoff: Z^i when party i defaults.

$$Z^1(t, v) = \Big(1 - (1 - R_t^1)(1 - \delta_t^1)^+\Big) v 1_{\{v<0\}} + \Big(1 + (1 - R_t^1)(\delta_t^2 - 1)^+\Big) v 1_{\{v \geq 0\}}$$
$$Z^2(t, v) = \Big(1 - (1 - R_t^2)(1 - \delta_t^2)^+\Big) v 1_{\{v \geq 0\}} + \Big(1 + (1 - R_t^2)(\delta_t^1 - 1)^+\Big) v 1_{\{v<0\}}$$

Pricing Formula

Following the method in Duffie&Huang (1996), pre-default value of the contract $V_t 1_{\{\tau>t\}} = S_t$ is given by

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s \left(r_u - \mu(u, V_u)\right)\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T$$

where

$$\begin{aligned}\mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-R_t^i)(1-\delta_t^i)^+ h_t^i + (1-R_t^j)(\delta_t^i - 1)^+ h_t^j\end{aligned}$$

if some technical condition is met.

- This technical condition becomes important when we consider credit derivatives.

Symmetric Case

Effective discount factor is non-linear:

$$r_t - \mu(t,v) = r_t - (\tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}}),$$

which makes the portfolio value non-additive.
If $\tilde{y}_t^1 = \tilde{y}_t^2 = \tilde{y}_t$, then we have

$$\mu(t,v) = \tilde{y}_t \ .$$

If $\tilde{y}$ is not explicitly dependent on V, we can recover the linearity.

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

Portfolio valuation can be decomposed into that of each payment.

$$\Downarrow$$

A good characteristic for market benchmark price.

Symmetric Perfect Collateralization

Special Cases

Case 1 : Benchmark for single currency product

- bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)
- both parties use the same currency (i) as collateral, which is also the payment (evaluation) currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

The valuation method for single currency swap adopted by LCH Swapclear (2010) is the same with this formula.

Symmetric Perfect Collateralization

Special Cases

Case 2 : Collateral in a Foreign Currency

- bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)
- both parties use the same currency (k) as collateral, which is different from the payment (evaluation) currency (i)

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

Funding Spread between the two currencies

$$y^{(i,k)} = y^{(i)} - y^{(k)} = \left(r^{(i)} - c^{(i)}\right) - \left(r^{(k)} - c^{(k)}\right)$$

- This is necessary to explain CCS basis spreads consistently.

Collateral Rate

Overnight Index Swap (OIS)

- exchange fixed rate(S) with compounded overnight rate periodically.
- collateralized by domestic currency ($\delta_T(\cdot)$denotes Dirac delta function at T.)

$$dD_s = \sum_{n=1}^{N} \delta_{T_n}(s)\left[\Delta_n S - \left\{\exp\left(\int_{T_{n-1}}^{T_n} c_u^{(i)} du\right) - 1\right\}\right]$$

- time t value of T_0 ($> t$)-start T_N-maturing OIS of currency (i);

$$V_t = \sum_{n=1}^{N} E^{Q^{(i)}}\left[e^{-\int_t^{T_n} c_u^{(i)} du}\left(\Delta_n S + 1 - e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du}\right)\middle|\mathcal{F}_t\right]$$

$$= \sum_{n=1}^{N} D^{(i)}(t,T_n)\Delta_n S - \left(D^{(i)}(t,T_0) - D^{(i)}(t,T_N)\right);\ (\Delta_n : \text{daycount fraction})$$

par rate at t:

$$\mathrm{OIS}_N(t) = S^{par}(t) = \frac{D^{(i)}(t,T_0) - D^{(i)}(t,T_N)}{\sum_{n=1}^{N} \Delta_n D^{(i)}(t,T_n)}$$

- $D^{(i)}(t,T) = E^{Q^{(i)}}\left[e^{-\int_t^T c_u^{(i)} du}\middle|\mathcal{F}_t\right]$ is a value of domestically collateralized zero-coupon bond.

Funding Spread

(i,j) **Mark-to-Market Cross Currency OIS**:
a funding spread(the difference of collateral costs) is directly linked to the corresponding CCOIS.

- compounded O/N rate of currency (i) is exchanged by that of currency (j) with additional spread periodically.
- notional of currency (j) is kept constant while that of currency (i) is refreshed at every reset time with the spot FX rate. (currency (i) is usually USD.)
- collateralized by currency (i) .
- payoff seen from the spread receiver($f_x^{(j,i)}(t)$ denotes FX rate at t that is, the price of the unit amount of currency i in terms of j):

$$dD_s = dD_s^{(j)} + f_x^{(j,i)}(s)dD_s^{(i)}$$

where

$$dD_s^{(j)} = -\delta_{T_0}(s) + \delta_{T_N}(s) + \sum_{n=1}^{N} \delta_{T_n}(s)\left[\left(e^{\int_{T_{n-1}}^{T_n} c_u^{(j)}du} - 1\right) + \delta_n B_N\right]$$

$$dD_s^{(i)} = \sum_{n=1}^{N} f_x^{(i,j)}(T_{n-1})\left[\delta_{T_{n-1}}(s) - \delta_{T_n}(s)e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du}\right]$$

Funding Spread

- in total, in terms of currency (j), we have

$$
\begin{aligned}
dD_s &= dD_s^{(j)} + f_x^{(j,i)}(s)dD_s^{(i)} \\
&= dD_s^{(j)} + \sum_{n=1}^{N}\left[\delta_{T_{n-1}}(s) - \delta_{T_n}(s)\frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})}e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du}\right] \\
&= \sum_{n=1}^{N}\delta_{T_n}(s)\left[e^{\int_{T_{n-1}}^{T_n} c_u^{(j)}du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})}e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du}\right]
\end{aligned}
$$

- time t value of spread receiver of (i,j)-MtMCCOIS:

$$
\begin{aligned}
V_t = \sum_{n=1}^{N} E^{Q^{(j)}}\Bigg[& e^{-\int_t^{T_n}(c_u^{(j)}+y_u^{(j,i)})du} \\
& \left\{e^{\int_{T_{n-1}}^{T_n} c_u^{(j)}du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})}e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du}\right\}\Bigg|\mathcal{F}_t\Bigg]
\end{aligned}
$$

Funding Spread

- if $c^{(i)}$ and $y^{(j,i)}$ are independent,[1]

$$V_t = \sum_{n=1}^{N}\left[\delta_n B_N D^{(j,i)}(t,T_n) - D^{(j,i)}(t,T_{n-1})\left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right)\right]$$

$$D^{(j,i)}(t,T) = D^{(j)}(t,T)e^{-\int_t^T y^{(j,i)}(t,s)ds},$$

$$y^{(j,i)}(t,s) = -\frac{\partial}{\partial s}\ln E^{Q^{(j)}}\left[e^{-\int_t^s y_u^{(j,i)}du}\middle|\mathcal{F}_t\right]$$

- **MtMCCOIS basis spread:**

$$\begin{aligned} B_N &= \frac{\sum_{n=1}^{N} D^{(j,i)}(t,T_{n-1})\left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)}du\right)}{\sum_{n=1}^{N}\delta_n D^{(j,i)}(t,T_n)} \\ &\sim \frac{1}{T_N - T_0}\int_{T_0}^{T_N} y^{(j,i)}(t,u)du. \end{aligned}$$

[1] Except the CCS basis spread, y does not seem to have persistent correlations with other variables such as OIS and IRS.([9])

Modeling framework of Interest rates and FX

Symmetric perfectly collateralized price is becoming the market Benchmark, at least for standardized products.

"Term structure construction procedures": [2]

- **(1), OIS $\Rightarrow c^{(i)}(0,T)$(T-maturity instantaneous fwd rate at time 0)**
- **(2), results of (1) + IRS + TS $\Rightarrow B^{(i)}(0,T;\tau)$($i$-currency Libor-OIS spread for tenor τ)**
- **(3), results of (1),(2) +CCS $\Rightarrow y^{(i,j)}(0,T)$(funding spread)**

Given the initial term structures, no-arbitrage dynamics of $c^{(i)}(t,T)$,$B^{(i)}(t,T;\tau)$ and $y^{(i,j)}(t,T)$ in HJM-framework can be constructed.

(For the detail, please see Mercurio(2008) [16] and our paper [8].)

[2] Assume collateralization in domestic currency for OIS, IRS and TS. Assume collateralization in USD for CCS (USD crosses).

Curve Construction

- **Collateralized OIS**

$$\mathrm{OIS}_N(0)\sum_{n=1}^{N}\Delta_n D(0,T_n) = D(0,T_0) - D(0,T_N)$$

- **Collateralized IRS**

$$\mathrm{IRS}_M(0)\sum_{m=1}^{M}\Delta_m D(0,T_m) = \sum_{m=1}^{M}\delta_m D(0,T_m) E^{\mathcal{T}_m^c}[L(T_{m-1},T_m;\tau)]$$

- **Collateralized TS**

$$\sum_{n=1}^{N}\delta_n D(0,T_n)\left(E^{\mathcal{T}_n^c}\left[L(T_{n-1},T_n;\tau_S)\right] + TS_N(0)\right)$$

$$= \sum_{m=1}^{M}\delta_m D(0,T_m) E^{\mathcal{T}_m^c}\left[L(T_{m-1},T_m;\tau_L)\right]$$

Market quotes of collateralized OIS, IRS, TS, (and a proper spline method) allow us to determine all the relevant $\{D(0,T)\}$, and forward Libors $\{E^{\mathcal{T}_m^c}\left[L(T_{m-1},T_m,\tau)\right]\}$.

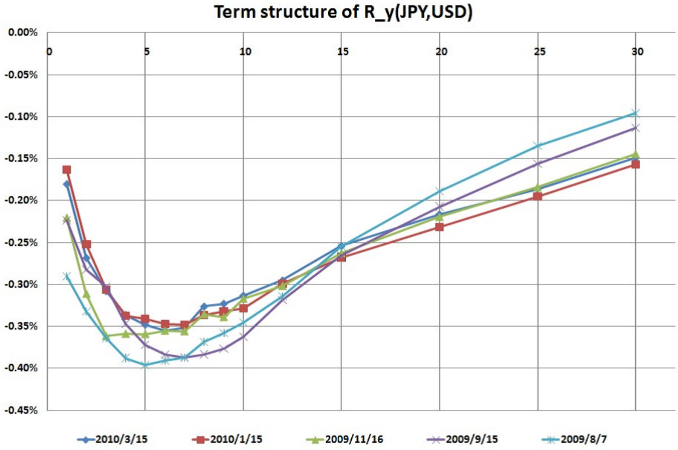
Curve Construction

MtMCCS basis spread($\to \{y^{(j,i)}(0,T)\}$ given $D_T^{(k)}$,$B_T^{(k)}$ $(k = i, j)$):

$$
\begin{aligned}
B_N = \;& \Bigg[\sum_{n=1}^{N} \delta_n^{(i)} D_{T_n}^{(i)} \left(\frac{D_{T_{n-1}}^{(j)}}{D_{T_{n-1}}^{(i)}} \right) e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} B_{T_n}^{(i)} \\
& - \sum_{n=1}^{N} \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} B_{T_n}^{(j)} \\
& - \sum_{n=1}^{N} D_{T_n}^{(j)} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} \left(e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(0,s)ds} - 1 \right) \Bigg] \\
& \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds}
\end{aligned}
$$

where we have used the notations: $D_{T_n}^{(k)} = D^{(k)}(0, T_n)$ and Libor-OIS spread,
$B_{T_n}^{(k)} = B^{(k)}(0, T; \tau) =$
$E^{\mathcal{T}^c_{n,(k)}}\left[L^{(k)}(T_{n-1}, T_n; \tau)\right] - \frac{1}{\delta_n^{(k)}}\left(\frac{D^{(k)}(0,T_{n-1})}{D^{(k)}(0,T_n)} - 1\right).$

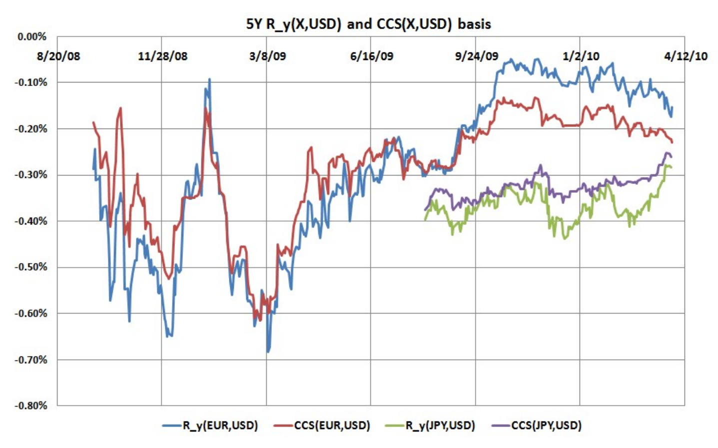
Curve Construction



posting USD as collateral tends to be expensive from the view point of collateral payers, which is particularly the case when the market is illiquid.

Curve Construction

Close relationship to CCS basis spread



A significant portion of CCS spreads movement stems from the change of $y^{(i,j)}$.

Choice of Collateral Currency

Special Cases

Case 3 : Multiple Eligible Collaterals

- bilateral perfect collateralization $(\delta^1 = \delta^2 = 1)$
- both parties choose the optimal currency from the eligible collateral set $\mathcal{C}$. Currency (i) is used as the evaluation currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}}[y_u^{(i,k)}])du\right) dD_s \middle| \mathcal{F}_t\right]$$

- The party who needs to post collateral has optionality.
- The cheapest collateral currency is chosen based on CCS information.
  To choose "strong" currency, such as USD,
  is expensive for the collateral payer.

Choice of Collateral Currency

Role of $y^{(i,j)}$

Optimal behavior of collateral payer can significantly change the derivative value.

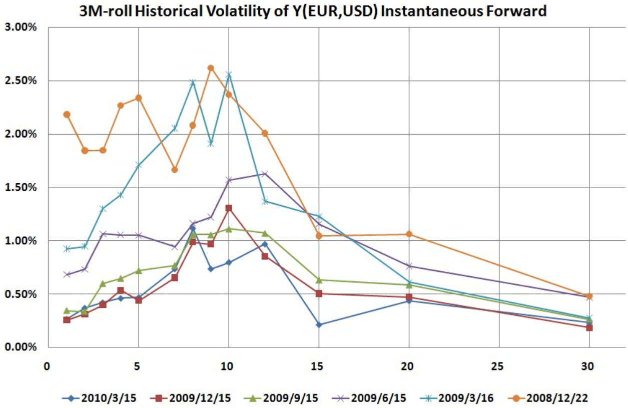
- **Payment currency and USD as eligible collateral is relatively common.**

$$D^{(i)}(t,T) \Rightarrow E_t^{Q_i}\left[e^{-\int_t^T \max\{y^{(i,USD)}(s),0\}ds}\right] D^{(i)}(t,T)$$

after neglecting small corrections from possible non-zero correlations. (under independence assumption between $y^{(i,j)}$ and $c^{(i)}$.)

- **Volatility of $y^{(i,j)}$ is an important determinant. (Embedded option change effective discounting factor, which crucially depends on the volatility of funding spread.)**

Choice of Collateral Currency



In a calm market, the vols tend to be 50 bps or so, but they were more than a percentage point just after the market crisis, which is reflecting a significant widening of the CCS basis spread to seek USD cash in the low liquidity market.

Choice of Collateral Currency

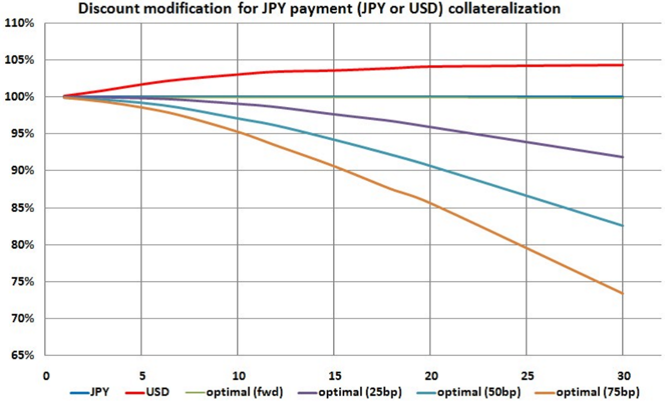


Figure: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16.

the effective discounting rate is increased by around 50 bps annually even when the annualized vol. of $y^{(JPY,USD)}$ is 50 bps.

More generic situations:

- If $\tilde{y}^1 \neq \tilde{y}^2$,

$$\begin{aligned} V_t &= E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))\right) dD_s \middle| \mathcal{F}_t\right] \\ \mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-R_t^i)(1-\delta_t^i)^+ h_t^i + (1-R_t^j)(\delta_t^i - 1)^+ h_t^j \end{aligned}$$

V follows a non-linear FBSDE.

Marginal Impact of asymmetry:

- Make use of Gateaux derivative as the first-order Approximation: Duffie&Skiadas (1994), Duffie&Huang (1996)

$$\lim_{\epsilon\downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta};\eta) - \frac{V_t(\bar{\eta}+\epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0$$

η and $\bar{\eta}$ are bounded and predictable

Approximation

Marginal Impact of Asymmetry

- We want to expand the price around symmetric benchmark price.

$$\begin{aligned}\mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ &= y_t + \Delta\tilde{y}_t^1 1_{\{v<0\}} + \Delta\tilde{y}_t^2 1_{\{v\geq 0\}} \\ \Delta\tilde{y}_t^i &= \tilde{y}_t^i - y_t\end{aligned}$$

- Calculate GD at symmetric $\mu = y$ point.

$$V_t(\mu) \simeq V_t(y) + \nabla V_t(y, \mu - y)$$

Asymmetric Collateralization

- Applying Gateaux Derivative at $\mu = y$ point:

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s \left(r_u - \mu(u, V_u)\right)\right) dD_s \middle| \mathcal{F}_t\right], \quad t \le T$$

is decomposed as $V_t = \overline{V}_t + \nabla V_t$, where

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\nabla V_t = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\overline{V}_s \left(\Delta\tilde{y}_s^1 1_{\{\overline{V}_s<0\}} + \Delta\tilde{y}_s^2 1_{\{\overline{V}_s\ge 0\}}\right) ds \middle| \mathcal{F}_t\right]$$

If y is chosen in such a way that it reflects the funding cost of the standard collateral agreements, $\overline{V}$ turns out to be the market benchmark price, and ∇V represents the correction for it.
For analytical approximation including higher order corrections for generic non-linear FBSDEs, see Fujii-Takahashi [13] for instance.

Asymmetric Collateralization

An example of asymmetric perfect collateralization

- party 1 choose optimal currency from the eligible collateral set $\mathcal{C}$, but the party 2 can only use currency (i) as collateral, either due to the asymmetric CSA or lack of easy access to foreign currency pool. The evaluation (payment) currency is (i).

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

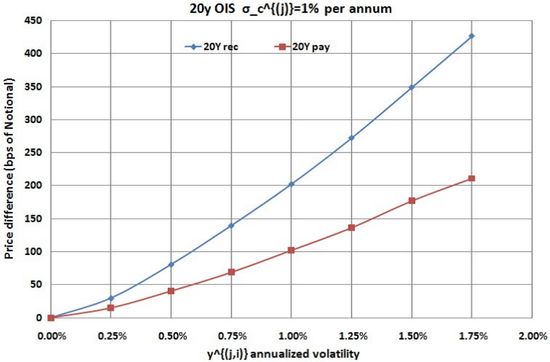
$$\nabla V_t = E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s c_u^{(i)} du\right) [-\overline{V}_s]^+ \max_{k\in\mathcal{C}}[y_s^{(i,k)}] \middle| \mathcal{F}_t\right]$$

$$V_t \simeq \overline{V}_t + \color{red}{\nabla V_t}$$

$\Rightarrow$ Expansion around the symmetric collateralization with currency (i).

Asymmetric Collateralization

- Numerical Example of ∇V for JPY-OIS.
- Eligible collateral are USD and JPY for party-1 but only JPY for party-2.



- OIS rate is set to make $\overline{V} = 0$.
- Difference between Receiver and Payer comes from up-ward sloping term structure. (the receiver's mark-to-market value tends to be negative in the long end of the contract, which makes the optionality larger.)

An Implication for Netting

Value with and without Netting

Assume perfect collateralization. Suppose that, for each party i, y_t^i is bounded and does not depend on the contract value directly. Let V^a, V^b, and V^{ab} be, respectively, the value process (from the view point of party 1) of contracts with cumulative dividend process D^a, D^b, and $D^a + D^b$. If $y^1 \geq y^2$, then $V^{ab} \geq V^a + V^b$, and if $y^1 \leq y^2$, $V^{ab} \leq V^a + V^b$.

- Proof can be done in the same way as Duffie&Huang(1996) using (stochastic) Gronwall-Bellman inequality.
- V^{ab} represents the value under netting agreement.
- A financial firm which can achieve lower funding cost y has an incentive to avoiding netting.

Imperfect Collateralization

CVA as the Deviation from the Perfect Collateralization

- **Assume the both parties use the same currency for simplicity, and hence $y^1 = y^2 = y$.**

$$\mu(t,v) = y_t - \Big\{\Big((1-\delta_t^1)y_t + (1-R_t^1)(1-\delta_t^1)^+ h_t^1 - (1-R_t^2)(\delta_t^1 - 1)^+ h_t^2\Big) 1_{\{v<0\}} + \Big((1-\delta_t^2)y_t + (1-R_t^2)(1-\delta_t^2)^+ h_t^2 - (1-R_t^1)(\delta_t^2 - 1)^+ h_t^1\Big) 1_{\{v\geq 0\}}\Big\}$$

- **GD around $\mu = y$ decomposes the price into three parts:**
  - **Symmetric perfectly collateralized benchmark price**
  - **$(1-\delta^i)y1_{\{v \lesseqgtr 0\}} \Rightarrow$ Collateral Cost Adjustment (CCA)**
  - **Remaining h dependent terms $\Rightarrow$ Credit Value Adjustment (CVA)**

$$\begin{aligned} V_t &\simeq \overline{V}_t + \nabla V_t \\ &= \overline{V}_t + \mathrm{CCA} + \mathrm{CVA} \end{aligned}$$

Imperfect Collateralization

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du} y_s \left((1-\delta_s^1)[-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$

$$E^Q\Bigg[\int_t^T e^{-\int_t^s (r_u - y_u)du} (1-R_s^1)h_s^1 \left[(1-\delta_s^1)^+[-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right] ds$$

$$- \int_t^T e^{-\int_t^s (r_u - y_u)du} (1-R_s^2)h_s^2 \left[(1-\delta_s^2)^+[\overline{V}_s]^+ + (\delta_s^1 - 1)^+[-\overline{V}_s]^+\right] ds \Bigg| \mathcal{F}_t\Bigg]$$

- The discounting rate is different from the risk-free rate and reflects the funding cost of collateral, while the terms in CVA are pretty similar to the usual result of bilateral CVA.
- Dependence among y, δ and other variables such as $\overline{V}, h^i$ is particularly important. $\Rightarrow$ New type of Wrong (Right)-way Risk. (e.g. y is closely related to the CCS basis spread. Hence, y is expected to be highly sensitive to the market liquidity, and is also strongly affected by the overall market credit conditions.)

Collateral Thresholds

- **Thresholds: $\Gamma^i > 0$ for party-i: A threshold is a level of exposure below which collateral will not be called, and hence it represents an amount of uncollateralized exposure. Only the incremental exposure will be collateralized if the exposure is above the threshold.**

Case of perfect collateralization above the thresholds

$$S_t = \beta_t E^Q \left[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \{ dD_u + q(u, S_u) S_u du \} + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \left\{ Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right\} \middle| \mathcal{F}_t \right]$$

$$q(t, S_t) = y_t^1 \left(1 + \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + y_t^2 \left(1 - \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t > \Gamma_t^2\}}$$

$$Z^1(t, S_t) = S_t \left[\left(1 + (1 - R_t^1) \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + R_t^1 1_{\{-\Gamma_t^1 \leq S_t < 0\}} + 1_{\{S_t \geq 0\}} \right]$$

$$Z^2(t, S_t) = S_t \left[\left(1 - (1 - R_t^2) \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t \geq \Gamma_t^2\}} + R_t^2 1_{\{0 \leq S_t < \Gamma_t^2\}} + 1_{\{S_t < 0\}} \right]$$

Collateral Thresholds

Assume the domestic currency as collateral $y^1 = y^2 = y$.

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s c_u du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = -E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \overline{V}_s 1_{\{-\Gamma_s^1 \le \overline{V}_s < \Gamma_s^2\}} ds \middle| \mathcal{F}_t\right]$$

$$+E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \left\{\Gamma_s^1 1_{\{\overline{V}_s < -\Gamma_s^1\}} - \Gamma_s^2 1_{\{\overline{V}_s \ge \Gamma_s^2\}}\right\} ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$

$$E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{h_s^1(1-R_s^1)\left[-\overline{V}_s 1_{\{-\Gamma_s^1 \le \overline{V}_s < 0\}} + \Gamma_s^1 1_{\{\overline{V}_s < -\Gamma_s^1\}}\right]\right\} ds \middle| \mathcal{F}_t\right]$$

$$-E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{h_s^2(1-R_s^2)\left[\overline{V}_s 1_{\{0 < \overline{V}_s \le \Gamma_s^2\}} + \Gamma_s^2 1_{\{\overline{V}_s > \Gamma_s^2\}}\right]\right\} ds \middle| \mathcal{F}_t\right]$$

The terms in CCA reflect the fact that no collateral is posted in the range $\{-\Gamma_t^1 \le V_t \le \Gamma_t^2\}$, and that the posted amount of collateral is smaller than $|V|$ by the size of threshold.

The terms in CVA represent bilateral uncollateralized credit exposure, which is capped by each threshold.

Collateralized CDS

Contagion effect is a crucial factor to determine the fair price.

(Setup)

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$
- Relevant names $\mathcal{C} = \{0, 1, 2, \cdots, n\}$
- Default indicator function $H_t^i = 1_{\{\tau^i \le t\}}$
- $\mathbb{H}^i$ denote the filtration generated by H^i
- $\mathbb{G}$ denotes the background filtration generated by Brownian motions relevant for all the market risk factors except default indicators.
- Full filtration is assumed to be $\mathbb{F} = \mathbb{G} \vee \mathbb{H}^0 \vee \cdots \vee \mathbb{H}^n$.
- τ^i is $\mathbb{H}^i$ (and hence $\mathbb{F}$) stopping time.
- Assume no simultaneous default.
- Assume the existence of hazard rate process h^i where

  $$M_t^i = H_t^i - \int_0^t h_s^i 1_{\{\tau^i > s\}} ds$$

  is an $(Q, \mathbb{F})$-martingale.

(Please see [12] for details and numerical examples.)

Collateralized CDS

Continuously Collateralized CDS with Cash:

Reference entity: party-0 Investor: party-1 Counter party: party-2

$$S_t = \beta_t E^Q \left[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \Big(dD_u + q(u, S_u) S_u du \Big) \right.$$

$$\left. + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \Big(Z_u^0 dH_u^0 + Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \Big) \middle| \mathcal{F}_t \right]$$

$$q(t,v) = \delta_t^1 y_t^1 1_{\{v<0\}} + \delta_t^2 y_t^2 1_{\{v \geq 0\}}$$

$$Z_t^0 = (1 - R_t^0)$$

$$Z^1(t,v) = \Big(1 - (1 - R_t^1)(1 - \delta_t^1)^+\Big) v 1_{\{v<0\}} + \Big(1 + (1 - R_t^1)(\delta_t^2 - 1)^+\Big) v 1_{\{v \geq 0\}}$$

$$Z^2(t,v) = \Big(1 - (1 - R_t^2)(1 - \delta_t^2)^+\Big) v 1_{\{v \geq 0\}} + \Big(1 + (1 - R_t^2)(\delta_t^1 - 1)^+\Big) v 1_{\{v<0\}}$$

where

$$\tau = \tau^0 \wedge \tau^1 \wedge \tau^2$$

Survival Measure

- "no-jump" condition on the pre-default value process, which is required on the work of Duffie-Huang (1996) is violated in general when the contagious effects induce jumps to variables contained in pre-default value process.
    - Duffie-Huang carried out the comparison of BSDE each for S_t and $1_{\{\tau>t\}}V$. The result holds only when $\Delta V_\tau = 0$.
- Schönbucher (2000), Collin-Dufresne et.al. (2004) introduced "survival measure" as a way around the difficulty.

Survival Measure

Pre-default Value of a Continuously Collateralized CDS

Under the perfect and symmetric collateralization with domestic currency:

$$\delta^1 = \delta^2 = 1 \qquad y^1 = y^2 = y,$$

then we have its pre-default value as

$$V_t = E^{Q'}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u + h_u^0)du\right)\left(dD_s + Z_s^0 h_s^0 ds\right)\middle|\mathcal{F}_t'\right],$$

where the survival measure Q' is defined by

$$\begin{aligned}\eta_t &= \left.\frac{dQ'}{dQ}\right|_{\mathcal{F}_t} = 1_{\{\tau>t\}}\Lambda_t,\ \tau = \tau^0\wedge\tau^1\wedge\tau^2\\ \Lambda_t &= \exp\left(\int_0^t \tilde{h}_s ds\right), \qquad \tilde{h} = h^0 + h^1 + h^2\end{aligned}$$

and $\mathbb{F}' = (\mathcal{F}_t')_{t\geq 0}$ denotes the augmentation of $\mathbb{F}$ under Q'.
The measure change into Q' puts zero weight on the events where the parties $\{0,1,2\}$ default.

Financial Implications of Survival Measure

What does h^i really mean under the measure Q' ?

- If the investor 1 and the counter party 2 are default-free, then, h^0 is trivially the default intensity of the reference name.
- If this is not the case, interpretation of h^i is more difficult.

Define Π as the set of all subgroups of $\mathcal{C} = \{0, 1, 2, \cdots n\}$ and the empty set, then...

$$h_t^i = \sum_{\{D \in \Pi; i \notin D\}} \left(\prod_{j \in D} 1_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C} \setminus D} 1_{\{\tau^k > t\}} \right) h_D^i(t, \vec{\tau}_D)$$

Define the survival set $\mathcal{S} = \{0, 1, 2\}$, $\mathcal{C}' = \mathcal{C} \setminus \mathcal{S}$, and Π' as the set of all subgroups of $\mathcal{C}'$ and the empty set. Then, under Q', h^i is equal to

$$h_t'^i = \sum_{\{D \in \Pi'; i \notin D\}} \left(\prod_{j \in D} 1_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C}' \setminus D} 1_{\{\tau^k > t\}} \right) h_D^i(t, \vec{\tau}_D)$$

Financial Implications of Survival Measure

Simple Cases

- **3-party Case**
  - $\mathcal{C} = \mathcal{S} = \{0, 1, 2\}$ **and hence** $\Pi' = \{\emptyset\}$
  - $h_t'^i = h^i_{\{\emptyset\}}(t)$ **and in particular** $h_t'^0 = h^0_{\{\emptyset\}}(t)$
  - **Replacing by the default intensity conditioned on no-default in the pricing formula works.**
- **4-party Case**
  - $\mathcal{C} = \{0, 1, 2, 3\}$, $\mathcal{S} = \{0, 1, 2\}$ **and hence** $\Pi' = \{\emptyset, 3\}$.

$$\begin{aligned} h_t'^0 &= 1_{\{\tau^3 > t\}} h^0_{\{\emptyset\}} + 1_{\{\tau^3 \le t\}} h^0_{\{3\}}(t, \tau^3) \\ h_t'^3 &= h^3_{\{\emptyset\}}(t) \end{aligned}$$

  - **If we choose the counter party 3 instead of 2, then** $\mathcal{S} = \{0, 1, 3\}$, **which leads to different** h'^0 **and also the price.**

Financial Implications of Survival Measure

We find there is no contribution from the scenarios in which the names contained in $\mathcal{S}$ default.

$$\Downarrow$$

The protection buyer cannot obtain the protection for the contagious effects from the seller to the reference name. Same as the contagion from the buyer's default to the reference name. These contributions should be extracted from the protection value.

Financial Implications of Survival Measure

4-party Case: $\mathcal{S} = \{0, 1, 2\}$

$$V_t \quad = \quad 1_{\{\tau^3 \le t\}} V_{\{3\}}(t) + 1_{\{\tau^3 > t\}} V_{\{\emptyset\}}(t)$$

$$\begin{aligned}
V_{\{3\}}(t) &= E^{Q'}\left[\int_{]t,T]} e^{-\int_t^s (c_u + h^0_{\{3\}}(u,\tau^3))du}\left(dD_s + Z^0_s h^0_{\{3\}}(s,\tau^3)ds\right)\Big|\mathcal{F}'_t\right] \\
V_{\{\emptyset\}}(t) &= E^{Q'}\left[\int_{]t,T]} e^{-\int_t^s (c^0_u + h^0_{\{\emptyset\}} + h^3_{\{\emptyset\}}(u))du}\left(dD_s + Z^0_s h^0_{\{\emptyset\}}(s)ds\right)\right. \\
&+ \int_{]t,T]} e^{-\int_t^s c_u du}\left(\int_t^s e^{-\left(\int_t^v h^0_{\{\emptyset\}}(u) + \int_v^s h^0_{\{3\}}(u,v)\right)du}\left[e^{-\int_t^v h^3_{\{\emptyset\}}(x)dx} h^3_{\{0\}}(v)\right]dv\right) dD_s \\
&+ \int_t^T e^{-\int_t^s c_u du}\left(\int_t^s e^{-\left(\int_t^v h^0_{\{\emptyset\}}(u) + \int_v^s h^0_{\{3\}}(u,v)\right)du}\right. \\
&\qquad \times \left.\left.\left[e^{-\int_t^v h^3_{\{\emptyset\}}(x)dx} h^3_{\{\emptyset\}}(v)\right] h^0_{\{3\}}(s,v)dv\right) Z^0_s ds\Big|\mathcal{F}'_t\right]
\end{aligned}$$

Financial Implications of Survival Measure

if the investor enters a back-to-back trade with the counter party 3, the pre-default value of this offsetting contract is given in the same way: [3]

$$V_t^{B2B} = -\left(1_{\{\tau^2 \le t\}} V_{\{2\}}(t) + 1_{\{\tau^2 > t\}} V_{\{\emptyset\}}(t)\right)$$

$$V_{\{2\}}(t) = E^{Q''}\left[\int_{]t,T]} e^{-\int_t^s (c_u + h^0_{\{2\}}(u,\tau^2))du} \left(dD_s + Z_s^0 h^0_{\{2\}}(s,\tau^2)ds\right) \Big| \mathcal{F}_t''\right]$$

$$\begin{aligned} V_{\{\emptyset\}}(t) = {} & E^{Q''}\left[\int_{]t,T]} e^{-\int_t^s (c_u + h^0_{\{\emptyset\}}(u))du} \right. \\ & \left. \times \left(e^{-\int_t^s h^2_{\{\emptyset\}}(u)du}\right) \left(dD_s + Z_s^0 h^0_{\{\emptyset\}}(s)ds\right) \Big| \mathcal{F}_t''\right] \\ & + E^{Q''}\left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h^0_{\{\emptyset\}}(u) + \int_v^s h^0_{\{2\}}(u,v)\right)du} \right.\right. \\ & \left.\left. \times \left[e^{-\int_t^v h^2_{\{\emptyset\}}(x)dx} h^2_{\{\emptyset\}}(v)\right] dv\right) dD_s \Big| \mathcal{F}_t''\right] \\ & + E^{Q''}\left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h^0_{\{\emptyset\}}(u) + \int_v^s h^0_{\{2\}}(u,v)\right)du} \right.\right. \\ & \left.\left. \times \left[e^{-\int_t^v h^2_{\{\emptyset\}}(x)dx} h^2_{\{\emptyset\}}(v)\right] h^0_{\{2\}}(s,v)dv\right) Z_s^0 ds \Big| \mathcal{F}_t''\right]. \end{aligned}$$

[3] Q'' and $(\mathcal{F}_t'')_{t\ge 0}$ are defined for the new survival set $\mathcal{S}^{B2B} = \{0,1,3\}$.

Financial Implications of Survival Measure

- $V_0 + V_0^{B2B}$ is not zero in general. and does depend on the default intensities of party-2 and -3, and also their contagious effects to the reference entity.
- Suppose that the investor is a CCP just entered into the back-to-back trade with the party-2 and -3 who have the same marginal default intensities.
- Even under the perfect collateralization, if the CCP applies the same CDS price (or premium) to the two parties, it has, in general, the mark-to-market loss or profit even at the inception of the contract.

Conclusions

We have seen:

- **Collateralized derivative pricing:**
  - **Funding spreads and their linkage to CCS are crucial.**
  - **Choice of collateral has important effects on valuation.**
  - **Imperfect collateralization leads to collateral cost adjustment and credit value adjustment.**
- **Collateralized CDS:**
  - **Even under the perfect collateralization, the contagious effects cannot be recovered.**
  - **A simple back-to-back trade has non-zero value.**

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