An Asymptotic and Perturbative Expansion Approach in Finance

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Today's Talk

This talk will introduce recent development in an asymptotic expansion approach in finance, particularly, the following topics:

- **asymptotic expansion in a general diffusion setting for approximations of density functions and option prices**
- **asymptotic expansion for basket option pricing in a local-stochastic volatility (LSV) with jumps**
- **perturbative expansion method for backward stochastic differential equations (BSDEs)**
- **perturbation technique with interacting particle method for BSDEs**
- **(On an application to of the method to mean-variance hedging problems in partially observable markets with stochastic filtering, please see "Making Mean-Variance Hedging Implementable in a Partially Observable Market,"** *Quantitative Finance*, **published online: 20 Mar 2014.)**

An Asymptotic Expansion in a General Diffusion Setting

T (2009, 2014)

Setting

- (W, P): **a** r**-dimensional Wiener Space**
- $X^{(\epsilon)} = (X^{(\epsilon),1}, \cdots, X^{(\epsilon),d})$: d**-dimensional stochastic process with a perturbation parameter** $\epsilon \in (0,1]$**:**

$$X_t^{(\epsilon),j} = x_0 + \int_0^t V_0^j(X_s^{(\epsilon)}, \epsilon)ds + \epsilon \int_0^t V^j(X_s^{(\epsilon)})dW_s \tag{1}$$

  where
  $V_0 = (V_0^1, \cdots, V_0^d)$: $\mathbf{R}^d \times (0,1] \mapsto \mathbf{R}^d$**, and**
  $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$
  are smooth functions with bounded derivatives of all orders.
- **(Remark) An application to non diffusion Wiener functional models in finance: please see Kunitomo-T(2001,2003) and Matuoka-T (2004) for the Heath-Jarrow-Morton (HJM) model.**

- **Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders.**
- **Then, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;**

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \cdots$$

in L^p for every $p > 1$ as $\epsilon \downarrow 0$,

- **The coefficients in the expansion are obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals.**
- **Next, normalize $g(X_T^{(\epsilon)})$ to**

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then,

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p for every $p > 1$.

- g_{1T} **follows a normal distribution, whose density function is given by**

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right)$$

where

$$\begin{aligned} C &= (\partial g(X_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt, \\ \Sigma_T &= \int_0^T \hat{V}_{g_T}(t) \hat{V}_{g_T}(t)' dt, \\ \hat{V}_{g_T}(t) &= \left(\partial g\left(X_T^{(0)}\right)\right)' \left[Y_T Y_t^{-1} V\left(X_t^{(0)}\right)\right]. \end{aligned} \tag{2}$$

Here, Y **denotes the solution to a matrix valued differential equation:**

$$dY_t = \partial V_0(X_t^{(0)}, 0) Y_t dt; \;\; Y_0 = I_d.$$

(∂V_0 **denotes the** $d \times d$ **matrix whose** (j,k)**-element is** $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j **is the** j**-th element of** V_0**, and** I_d **denotes the** $d \times d$ **identity matrix.)**

- **Let us assume** $\Sigma_T > 0$**, which means that the distribution of** g_{1T} **does not degenerate. In applications, it is easy to check this condition.**

- **Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space.**
- **Next, take $\Phi \in \mathcal{S}'$. Then, the asymptotic expansion of $\Phi(G^{(\epsilon)})$ as $\epsilon \downarrow 0$ can be verified by Watanabe theory. (e.g. Watanabe(1987))**
- **In particular, if we take the delta function at $x \in \mathbf{R}$, δ_x as Φ, we obtain an asymptotic expansion of the density for $G^{(\epsilon)}$.**
- **We define the following notations:**

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} := \sum_{\beta=1}^{l} \sum_{\vec{l}_\beta \in L_{l,\beta}} \sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta} \text{(for } l \geq 1), \quad \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(0)} := \sum_{\beta=0} \sum_{\vec{l}_0=(\emptyset)} \sum_{\vec{d}_0=(\emptyset)} \text{(for } l = 0), \tag{3}$$

$$L_{l,\beta} := \left\{ \vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = l, (l, l_j, \beta \in \mathbf{N}) \right\}, \tag{4}$$

$$\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n} \sum_{\vec{k}_\delta \in L_{n,\delta}} . \tag{5}$$

Then, the expectation of $\Phi(G^{(\epsilon)})$ is expanded as follows.

- **The expansion of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:**

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) \prod_{j=1}^{\delta} g_{(k_j+1)T}\right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{-\infty}^{\infty} \Phi^{(\delta)}(x) \\
&\quad \times \mathbf{E}\left[X^{\vec{k}_\delta} | g_{1T} = x\right] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{-\infty}^{\infty} \Phi(x)(-1)^{\delta} \\
&\quad \times \frac{d^\delta}{dx^\delta}\left\{\mathbf{E}\left[X^{\vec{k}_\delta} | g_{1T} = x\right] f_{g_{1T}}(x)\right\} dx + o(\epsilon^N) \qquad (6)
\end{aligned}$$

where $\Phi^{(\delta)}(g_{1T}) = \frac{d^\delta \Phi(x)}{dx^\delta}\Big|_{x=g_{1T}}$**, and**

$$X^{\vec{k}_\delta} := \prod_{j=1}^{\delta} g_{(k_j+1)T}. \qquad (7)$$

(Comments on Computation Scheme)

- **To compute the asymptotic expansion (6), we need to evaluate the conditional expectations of the form**

$$E\left[X^{\vec{k}_\delta} \middle| g_{1T} = x \right]$$

  where $X^{\vec{k}_\delta}$ is represented by a product of multiple Wiener-Itô integrals.
- **T(1995,1999) and T-Takehara-Toda(2009) shows a general scheme for deriving the conditional expectation formulas for the third and the higher order expansions, respectively.**
- **T- Toda(2009) introduces an alternative but equivalent computational algorithm for an asymptotic expansion.**
  - **We compute the unconditional expectations instead of the conditional ones by deriving a system of ordinary differential equations which the expectations satisfy.**
  - **It enables us to derive high order approximation formulas in an automatic manner.**

- **The next theorem shows a general result for an asymptotic expansion of the density function for $G^{(\epsilon)}$.**
- **In particular, the coefficients in the expansion are obtained through the solution of a system of ordinary differential equations(ODEs).**
- **Each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. Hence, one can easily solve (analytically or numerically) the system of ODEs.**

Theorem 1: The asymptotic expansion of the density function

The asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g(X_T^{(0)})}{\epsilon}$ up to ϵ^N-order is given by

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= f_{g_{1T}}(x) \\ &\quad + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x - C, \Sigma_T) \right) f_{g_{1T}}(x) + o(\epsilon^N), \end{aligned} \tag{8}$$

where $H_n(x;\Sigma_T)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x;\Sigma_T) = (-\Sigma_T)^n e^{x^2/2\Sigma_T} \frac{d^n}{dx^n} e^{-x^2/2\Sigma_T}, \tag{9}$$

and

$$\begin{aligned} C_{nm} &= \frac{1}{\Sigma_T^m} \sum_{\vec{k}_\delta}^{(m)} \sum_{\vec{l}^1_{\beta_1}, \vec{d}^1_{\beta_1}}^{(k_1+1)} \cdots \sum_{\vec{l}^\delta_{\beta_\delta}, \vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)} \frac{1}{\delta!(m-\delta)!} \\ &\left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!} \partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)}) \right) \frac{1}{i^{m-\delta}} \frac{\partial^{m-\delta}}{\partial \xi^{m-\delta}} \eta^{\vec{d}^1_{\beta_1} \otimes \cdots \otimes \vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1} \otimes \cdots \otimes \vec{l}^\delta_{\beta_\delta}} (T;\xi) \Big|_{\xi=0}, \; \left(i = \sqrt{-1}\right). \end{aligned} \tag{10}$$

Theorem 1(continued)

$\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ are obtained as a solution to the following system of ODEs:

$$\frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} = \sum_{k=1}^{\beta}\frac{1}{l_k!}\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\partial_\epsilon^{l_k}V_0^{d_k}(X_t^{(0)},0) \tag{11}$$

$$+\sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{d}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{d}_\gamma}\partial_\epsilon^{l_k-l}V_0^{d_k}(X_t^{(0)},0)$$

$$+\sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{d}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{d}_\gamma\otimes\vec{d}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi)$$

$$\times\partial^\gamma_{\vec{d}_\gamma}V^{d_k}(X_t^{(0)})\partial^\delta_{\vec{d}_\delta}V^{d_m}(X_t^{(0)})$$

$$+(i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{d}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{d}_\gamma}V^{d_k}(X_t^{(0)})\hat{V}_{g_T}(t)$$

$$\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi) = 0 \text{ for } (\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset),\ \eta^{(\emptyset)}_{(\emptyset)}(t;\xi)=1 \text{ for } (\vec{l}_\beta,\vec{d}_\beta)=(\emptyset,\emptyset), \tag{12}$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial\epsilon^l}$, $\partial^\beta_{\vec{d}_\beta} = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$.

- **Here, we use the following notations:**

$$\vec{l}_{\beta/k} := (l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_\beta)$$
$$\vec{l}_{\beta/k,n} := (l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_{n-1}, l_{n+1}, \cdots, l_\beta),\ 1 \le k < n \le \beta$$
$$\vec{l}_\beta \otimes \vec{m}_\gamma := (l_1, \cdots, l_\beta, m_1, \cdots, m_\gamma) \tag{13}$$

for $\vec{l}_\beta = (l_1, \cdots, l_\beta)$ **and** $\vec{m}_\gamma = (m_1, \cdots, m_\gamma)$**.**

- **For an expansion above up to the** ϵ^2**-order, we need the Hermite polynomials** $H_n(x;\Sigma)$ **up to** $n = 6$**:**

$$H_0(x;\Sigma) = 1,\ H_1(x;\Sigma) = x,\ H_2(x;\Sigma) = x^2 - \Sigma,$$
$$H_3(x;\Sigma) = x^3 - 3\Sigma x,\ H_4(x;\Sigma) = x^4 - 6\Sigma x^2 + 3\Sigma^2,$$
$$H_5(x;\Sigma) = x^5 - 10\Sigma x^3 + 15\Sigma^2 x,$$
$$H_6(x;\Sigma) = x^6 - 15\Sigma x^4 + 45\Sigma^2 x^2 - 15\Sigma^3.$$

- **Expansions of multidimensional densities are obtained in a similar way. (e.g. T (1999), T-Toda (2012))**

Asymptotic Expansion of Option Price

- **We consider a plain vanilla call option on $g(X_T^{(\epsilon)})$.**
- **An asymptotic expansion of a call option price with maturity T and strike price $K = g(X_T^{(0)}) - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given as follows:**

$$
\begin{aligned}
C(K,T) &= P(0,T)\mathbf{E}[\max\{g(X_T^{(\epsilon)}) - K, 0\}] \\
&= \epsilon P(0,T)\mathbf{E}\left[\max\left\{\left(\frac{g(X_T^{(\epsilon)}) - g(X_T^{(0)})}{\epsilon}\right) + \left(\frac{g(X_T^{(0)}) - K}{\epsilon}\right), 0\right\}\right] \\
&= \epsilon P(0,T)\mathbf{E}\left[\max\left\{G^{(\epsilon)} + y, 0\right\}\right] \\
&= \epsilon P(0,T)\int_{-y}^{\infty}(x+y) f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}), \qquad (14)
\end{aligned}
$$

where

- $P(0,T)$ **: the price at time 0 of a zero coupon bond with maturity T**
- $f_{G^{(\epsilon)},N}$ **: the asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order.**

Asymptotic Expansion of Option Price

Theorem 2

An asymptotic expansion up to the $\epsilon^{(N+1)}$-order of a call option price with maturity T and strike price K where $K = g(X_T^{(0)}) - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given as follows:

$$
\begin{aligned}
C(K,T) &= \epsilon P(0,T)\left[\sqrt{\Sigma_T}\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + CN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + yN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] \qquad (15)\\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) C_{n0}\left[\sqrt{\Sigma_T}\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + CN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right]\\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) C_{n1}\left[\Sigma_T\, N\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) - \sqrt{\Sigma_T}\, y\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right]\\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) \sum_{m=2}^{3n} C_{nm}\left[-y\sqrt{\Sigma_T}\, H_{m-1}\left(-(y+C);\Sigma_T\right)\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right.\\
&\left.+\Sigma_T^{\frac{3}{2}}\, H_{m-2}\left(-(y+C);\Sigma_T\right)\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] + y\sum_{n=1}^{N} \epsilon^{n+1} P(0,T)\, C_{n0} N\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\\
&+ y\sum_{n=1}^{N} \epsilon^{n+1} P(0,T) \sum_{m=1}^{3n} C_{nm}\ \sqrt{\Sigma_T} H_{m-1}\left(-(y+C);\Sigma_T\right)\, n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + o(\epsilon^{(N+1)}).
\end{aligned}
$$

Asymptotic Expansion of Option Price

Theorem 2(continued)

Here, C_{nm} **is given by (10),** $H_m(x;\Sigma_T)$ **is the Hermite polynomial of degree** m **defined in (9),**

$$C = (\partial g(X_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt,$$

$P(0,T)$ **denotes the price at time** 0 **of a zero coupon bond with maturity** T**.** $N(x)$ **stands for the standard normal distribution function, and its density function is given by** $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$**.**

Numerical Example: SABR model

- **We consider European plain-vanilla call/put prices under the SABR model (Hagan-Kumar-Lesniewski-Woodward (2002)) (interest rate=0%, for simplicity) :**

$$
\begin{aligned}
dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta}dW_t^1, \\
d\sigma^{(\epsilon)}(t) &= \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2,
\end{aligned}
$$

where $\nu_1 = \rho\nu$**,** $\nu_2 = (\sqrt{1-\rho^2})\nu$**.**
Payoff: $\max\{S_T - K, 0\}$ **(Call);** $\max\{K - S_T, 0\}$ **(Put).**

Numerical Example: Plain-Vanilla Option (SABR model)

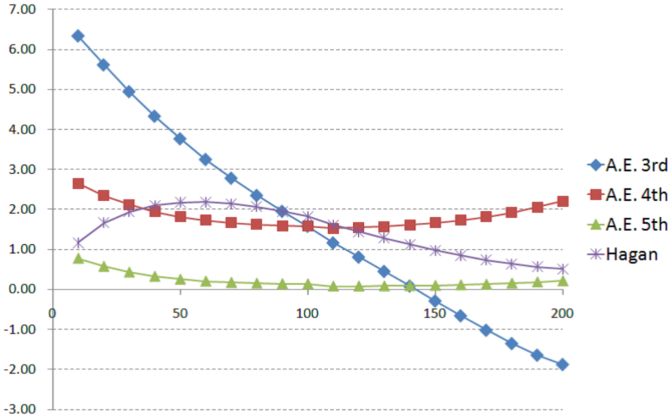


Figure: Approximation errors of ATM/OTM option prices $S_0 = 100$, $\beta = 0.5$ $\sigma_0 = 3$, $\nu = 0.3$, $\rho = -0.7$, $\epsilon = 1$, $T = 10$, $K = 10 \sim 200$.

Extensions

- **Improvement Scheme for approximations of the tails of the densities and deep OTMs of option prices**

  New Improvement Scheme for Approximation Methods of Probability Density Functions **: T-Tsuzuki (2013)**

  Weak Approximation with Asymptotic Expansion and Multidimensional Malliavin Weights: **T-Yamada (2013)**

- **Different approximation formulas are obtained (e.g. the limiting distributions are given by log-normal, shifted log-normal and non-central χ^2) through change of variables of $X^{(\epsilon),j}$ or/and the different ways to setting the perturbation parameter ϵ: (e.g. $V_0^j(X_s^{(\epsilon)},)$, $\epsilon V_0^j(X_s^{(\epsilon)})$, $\epsilon^2 V_0^j(X_s^{(\epsilon)})$)**

  Please see T-Toda (2009, 2013) for the details and numerical examples.

(Weak Approximation with Asymptotic Expansion Method)

- $(\mathcal{W}, H, P)$: r-dimensional Wiener space
- We consider a SDE with parameter ε.

$$
\begin{aligned}
dX_t^{\varepsilon}(x) &= V_0(\varepsilon, X_t^{\varepsilon}(x))dt + \varepsilon \sum_{i=1}^{r} V_i(X_t^{\varepsilon}(x))dW_t^i, \\
X_0(x) &= x \in \mathbf{R}^N.
\end{aligned}
$$

- V_0: $C_b^{\infty}([0,1] \times \mathbf{R}^N; \mathbf{R}^N)$, uniformly bounded,
- $V_i \ i = 1, \cdots, r$: $C_b^{\infty}(\mathbf{R}^N; \mathbf{R}^N)$, uniformly bounded.
- Non-degeneracy of the Malliavin covariance matrix of $Y_t^{\varepsilon} = \frac{X_t^{\varepsilon}(x) - X_t^0(x)}{\varepsilon}$.
- $f : \mathbf{R}^N \to \mathbf{R}$ Lipschitz continuous function or bounded Borel function

Approximation of $E[f(X_t^\varepsilon(x))]$ with asymptotic expansion method:

$$E[f(X_t^\varepsilon(x))] = a_0(t,x) + \sum_{j=1}^{m} \varepsilon^j a_j(t,x) + o(\varepsilon^m), \qquad (16)$$

$a_0(t,x) = E\left[f(\bar{X}_t^0(x))\right]$**,** $\bar{X}_t^0(x)$**: a Gaussian variable,** $a_j(t,x) = E\left[f(\bar{X}_t^0(x))\Phi_t^j\right]$**,** Φ_t^j **are Malliavin weights obtained by IBP (integration-by-parts) in Malliavin Calculus.**

Let us define P_t **and** $Q_{(t)}^m$ **as follows:**

$$P_t f(x) \quad = \quad E[f(X_t^\varepsilon(x))]. \qquad (17)$$

$Q_{(t)}^m$**: an approximation of** P_t

$$\begin{aligned} Q_{(t)}^m f(x) &= a_0(t,x) + \sum_{i=1}^{m} \varepsilon^i a_i(t,x) \\ &= E[f(\bar{X}_t^0(x))\mathcal{M}^m(t,x,\bar{X}_t^0(x))], \end{aligned} \qquad (18)$$

where

$$\mathcal{M}^m(s,x,y) = 1 + \sum_{j=1}^{m} \varepsilon^j E[\Phi_t^j | \bar{X}_t^0(x) = y]. \qquad (19)$$

Error estimates of the asymptotic expansion:

Theorem

1. *For any* $s \in (0,1]$ *and Lipschitz continuous function* $f : \mathbf{R}^N \to \mathbf{R}$*, there exists* $C > 0$ *such that*

$$\|P_s f - Q^m_{(s)} f\|_\infty \leq C\varepsilon^{m+1} s^{(m+2)/2}. \tag{20}$$

2. *For any* $s \in (0,1]$ *and bounded Borel function* $f : \mathbf{R}^N \to \mathbf{R}$*, there exists* $C > 0$ *such that*

$$\|P_s f - Q^m_{(s)} f\|_\infty \leq C\varepsilon^{m+1} s^{(m+1)/2}. \tag{21}$$

($\|f\|_\infty = \sup_{x \in \mathbf{R}^N} |f(x)|$)

- **Let us divide $[0, T]$ into n equally time grids.**
- **We connect $Q^m_{(T/n)}$ by n times:**

$$(Q^m_{T/n})^n f(x) = Q^m_{T/n} \circ \cdots \circ Q^m_{T/n} f(x). \tag{22}$$

- **(Lipschitz f)** $\frac{1}{n^{(m+2)/2}} \times n$
  $\rightarrow$ **an approximation method of order** $\frac{1}{n^{m/2}}$
- **(bounded f)** $\frac{1}{n^{(m+1)/2}} \times n$
  $\rightarrow$ **an approximation method of order** $\frac{1}{n^{(m-1)/2}}$

Theorem

1. *For any Lipschitz continuous function* $f : \mathbf{R}^N \to \mathbf{R}$, *there exists* $C > 0$ *such that*

$$\|P_T f - (Q^m_{(T/n)})^n f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}.$$

2. *For any bounded Borel function* $f : \mathbf{R}^N \to \mathbf{R}$, *there exists* $C > 0$ *such that*

$$\|P_T f - (Q^m_{(T/n)})^n f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{(m-1)/2}}.$$

Remark This is an improvement scheme of the asymptotic expansion method.
(In fact, we don't need equally divided time grids.)

- **Local volatility model (LV) - CEV model -**

$$dX_t = \sigma X_t^{\beta} dW_t.$$

- **Stocastic volatility model (SV) - SABR model -**

$$\begin{aligned} dX_t &= \sigma_t X_t^{\beta} dW_t^1, \\ d\sigma_t &= \nu\sigma_t dW_t^2,\ dW_t^1 dW_t^2 = \rho dt. \end{aligned}$$

- **Call option price:** $P_T f(x) = E[(X_T - K)^+]$**,** $f(x) = (x-K)^+$**.**
- **Let us compute** $(Q^m_{(T/n)})^n f(x)$ **in local volatility (LV) and stochastic volatility models.**
  - **Discretization:** $n = 1, 2, 3$**, Order of the asymptotic expansion:** $m = 1, 2$
- **Comparison with Monte Carlo simulations**
  - **maturity** $T = 1, 2, 10$ **discretization:** $1,000$ $(T = 1)$**,** $2,000$ $(T = 2, 10)$**, number of sample paths** $10,000,000$

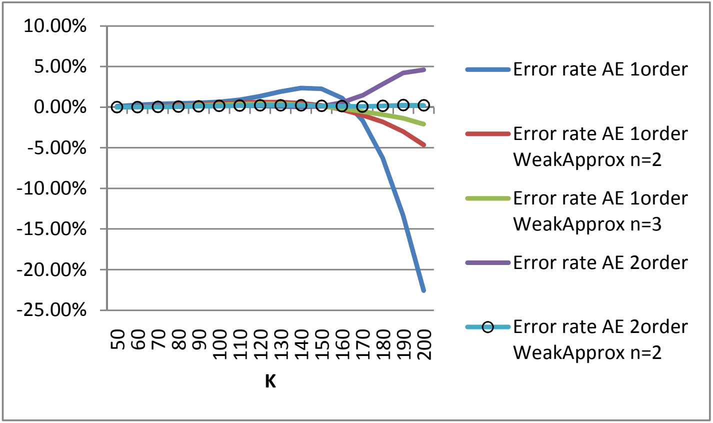


Figure: Error rate: $T = 1$, LV model (CEV with $\beta = 0.5$, $X_0 = 100$, $\sigma = 4$(initial vol. 40%)), the method with 1st & 2nd order AE

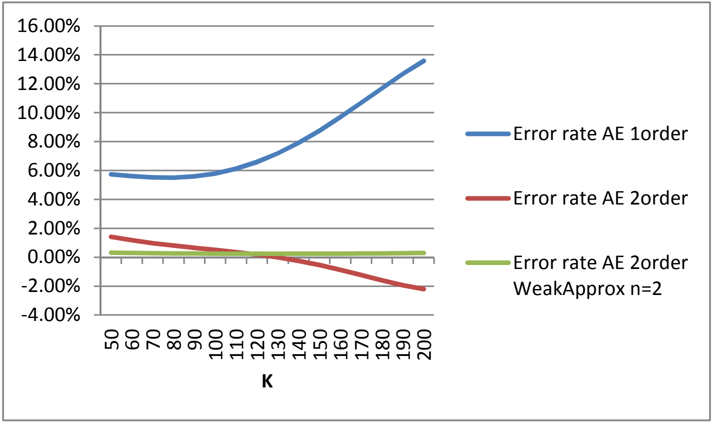


Figure: Error rate: $T = 10$, LV model (CEV with $\beta = 0.5$, $X_0 = 100$, $\sigma = 4$(initial vol. 40%)), the method with 1st & 2nd order AE

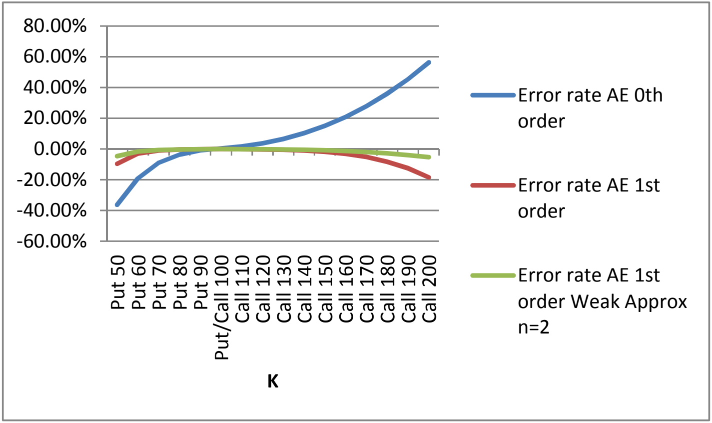


Figure: Error rate : $T = 1$: SV model(SABR with $\beta = 1\sigma = 0.3$, $\nu = 0.1$,$\rho = -0.5$), the method with 1st & 2nd order AE

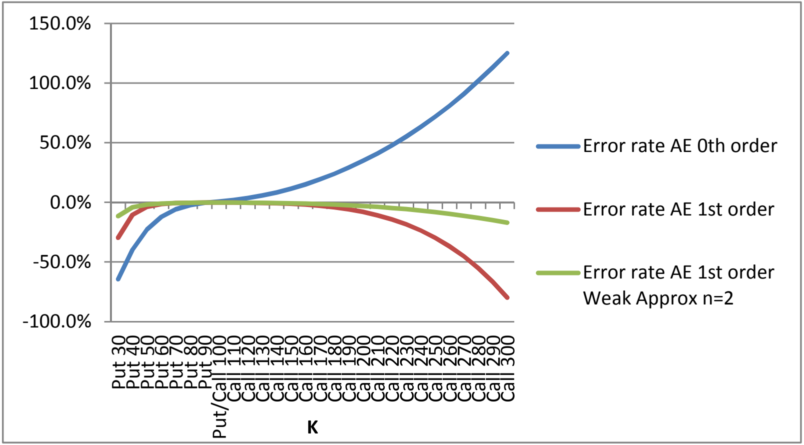


Figure: Error rate : $T = 2$: SV model(SABR with $\beta = 1$, $\sigma = 0.3$, $\nu = 0.1$,$\rho = -0.5$), the method with 1st & 2nd order AE

Basket Option

- **Basket options are one of popular exotic options in commodity and equity markets.**
  - **The payoff of a basket call option is expressed as** $(g(S_T) - K)^+$**.**
  - g **is a basket price function defined by** $g(S_T) := \sum_{i=1}^{d} w_i S_T^i$**, where** $S_T = (S_T^1, \cdots, S_T^d)$ **are asset prices, and** w_i **is a constant weight for each** i**.**
- **However, it is difficult to calculate a basket option price.**
  - **Numerical methods for PDE**
    - **It is difficult to solve high dimensional PDEs.**
  - **Monte Carlo method**
    - **It needs a large amount of computational time to obtain an accurate value.**

Preceding Studies on Basket Options

- **Black-Scholes model**
  - **e.g. Brigo, Mercurio, Rapisarda and Scotti (2004).**
- **Local volatility (LV) diffusion model**
  - **e.g. Bayer and Laurence (2012).**
- **Local volatility (LV) jump diffusion model**
  - **e.g. Xu and Zheng (2010).**
- **Local stochastic volatility (LSV) model**
  - **e.g. Shiraya -T (2014).**

Basket Option in LSV Model

Shiraya-T (2014) applied an asymptotic expansion method in general diffusions to basket option pricing.

(Example)

- **We consider the valuation of basket options with the following payoff ($C_B(K,T)$):**

$$C_B(K,T) = \max\left\{\hat{S}(T) - K, 0\right\},$$

where $\hat{S}(t) = \sum_{i=1}^{100} S_i(t)$.

- **Model of each S_i : SABR (λ-SABR(e.g. Labordere(2008)))**

$$dS^{(\epsilon)}(t) = \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta} dW_t^1,$$
$$d\sigma^{(\epsilon)}(t) = \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2,$$

where $\nu_1 = \rho\nu$, $\nu_2 = (\sqrt{1-\rho^2})\nu$.

($\beta = 0.5$. For the other parameters, please see Shiraya-T (2014) for the details.)

Numerical Example: Basket Option (LSV model)

Numerical Example: Basket Option with 100 Underlying Assets (Shiraya-T(2014))

Table: Basket Call Option ($T = 1$)

Strike(K)	8,000	9,000	10,000	11,000	12,000
Monte Carlo	2,037.1	1,167.5	517.6	160.8	31.7
AE3rd	2,037.4	1,167.6	517.6	160.5	31.5
Difference	0.3	0.2	-0.0	-0.2	-0.2
Relative Difference (%)	0.0%	0.0%	0.0%	-0.2%	-0.7%
MC Std Error	0.7	0.6	0.4	0.2	0.2

Monte Carlo: the number of trials is 3 million with the antithetic variable method.

Basket Option in LSV with Jumps

(Model) We introduce a local stochastic volatility with jumps model. (The model admits a local volatility function and jumps for not only the underlying asset price, but also its volatility process.)

$$\begin{aligned} S^i_T &= \int_0^T \alpha^i S^i_{t-} dt + \int_0^T \phi_{S^i}\left(\sigma^i_{t-}, S^i_{t-}\right) dW^{S^i}_t \\ &\quad + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S^i_{\tau_{j,l}-} - \int_0^T \Lambda_l S^i_{t-} \mathbf{E}[h_{S^i,l,1}] dt \right), \qquad (23) \\ \sigma^i_T &= \int_0^T \lambda^i(\theta^i - \sigma^i_{t-}) dt + \int_0^T \phi_{\sigma^i}\left(\sigma^i_{t-}\right) dW^{\sigma^i}_t \\ &\quad + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j} \sigma^i_{\tau_{j,l}-} - \int_0^T \Lambda_l \sigma^i_{t-} \mathbf{E}[h_{\sigma^i,l,1}] dt \right), \qquad (24) \end{aligned}$$

where $S^i_0 = s^i$, $\sigma^i_0 = \sigma^i$ ($i = 1, \cdots, d$).

Notations

- α^i **($i = 1, \cdots, d$) are constants.**
  λ^i **and** θ^i **($i = 1, \cdots, d$) are nonnegative constants.**
- $\phi_{S^i}(x, y)$ **and** $\phi_{\sigma^i}(x)$ **are some functions with appropriate regularity conditions.**
- W^{S^i} **and** W^{σ^i}**, ($i = 1, \cdots, d$) are correlated Brownian motions.**
- **Each** N_l**, ($l = 1, \cdots, n$) is a Poisson process with constant intensity** Λ_l**.** N_l**,** $l = 1, \cdots, n$ **are independent, and also independent of all** W^{S^i} **and** W^{σ^i}**.**
- $\tau_{j,l}$ **stands for the** j**-th jump time of** N_l**.**
- **For each** $l = 1, \cdots, n$ **and** $i = 1, \cdots, d$**, both** $\left(\sum_{j=1}^{N_{l,t}} h_{S^i,l,j}\right)_{t\geq 0}$ **and** $\left(\sum_{j=1}^{N_{l,t}} h_{\sigma^i,l,j}\right)_{t\geq 0}$ **are compound Poisson processes.**
  ($\sum_{j=1}^{N_{l,t}} \equiv 0$ **when** $N_{l,t} = 0$**)**

Notations (Continued)

- **For each l and x^i, $h_{x^i,l,j}$ ($j \in \mathbf{N}$) are i.i.d. random variables, where x^i stands for one of S^i and σ^i ($i = 1, \cdots, d$).**
  - **for the log-normal jump case, $h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1$, where $Y_{x^i,l,j} \sim N(m_{x^i,l}, \gamma^2_{x^i,l})$ (for all j).**
- **For the same l and j, $h_{S^i,l,j}$ and $h_{\sigma^{i'},l,j}$ ($i, i' = 1, \cdots, d$) are allowed to be dependent, that is $Y_{S^i,l,j}$ and $Y_{\sigma^{i'},l,j}$ ($i, i' = 1, \cdots, d$) are generally correlated.**
  ($h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ ($l \neq l'$) are independent. $h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ ($j \neq j'$) are independent. N_l and $h_{x^i,l',j}$ are independent.)

Remark

By specifying the functions ϕ_S and ϕ_σ, we can express various types of local-stochastic volatility models.

- **Quadratic Heston model**

$$\phi_S(\sigma, S) = (aS^2 + bS + c)\sqrt{\sigma}, \tag{25}$$

$$\phi_\sigma(\sigma) = \sqrt{\sigma}. \tag{26}$$

- **SABR(λ-SABR) model**

$$\phi_S(\sigma, S) = S^{\beta_S}\sigma, \tag{27}$$

$$\phi_\sigma(\sigma) = \sigma. \tag{28}$$

- **CEV-type volatility on volatility model**

$$\phi_S(\sigma, S) = S^{\beta_S}\sigma, \tag{29}$$

$$\phi_\sigma(\sigma) = \sigma^{\beta_\sigma}. \tag{30}$$

Perturbation

For a known parameter $\epsilon \in (0,1]$, we consider the following stochastic differential equation: using a $2d$-dimensional Brownian motion $Z = (Z^1, \cdots, Z^{2d})$ (in stead of the correlated ones, W^{S^i}, W^{σ^i}),

$$S_T^{i,(\epsilon)} = \int_0^T \alpha^i S_{t-}^{i,(\epsilon)} dt + \epsilon \sum_{j=1}^{2d} \int_0^T \Phi_{S^i,j}\left(\sigma_{t-}^{i,(\epsilon)}, S_{t-}^{i,(\epsilon)}\right) dZ_t^j + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j}^{(\epsilon)} S_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l S_{t-}^{i,(\epsilon)} \mathbf{E}\left[h_{S^i,l,1}^{(\epsilon)}\right] dt \right), \quad (31)$$

$$\sigma_T^{i,(\epsilon)} = \int_0^T \lambda^i(\theta^i - \sigma_{t-}^{i,(\epsilon)}) dt + \epsilon \sum_{j=1}^{2d} \int_0^T \Phi_{\sigma^i,j}\left(\sigma_{t-}^{i,(\epsilon)}\right) dZ_t^j + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j}^{(\epsilon)} \sigma_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l \sigma_{t-}^{i,(\epsilon)} \mathbf{E}\left[h_{\sigma^i,l,1}^{(\epsilon)}\right] dt \right), \quad (32)$$

$$h_{x^i,l,j}^{(\epsilon)} = e^{\epsilon Y_{x^i,l,j}} - 1 \text{ (log-normal jump case)}, \quad (33)$$

$$h_{x^i,l,j}^{(\epsilon)} = \epsilon H_{x^i,l} \text{ (for all } j\text{, constant jump case)}. \quad (34)$$

Expansion around $\epsilon = 0$

We assume the asymptotic expansions around $\epsilon = 0$ as follows:

$$S_T^{i,(\epsilon)} \sim S_T^{i,(0)} + \epsilon S_T^{i,(1)} + \frac{\epsilon^2}{2!} S_T^{i,(2)} + \cdots , \qquad (35)$$

$$\sigma_T^{i,(\epsilon)} \sim \sigma_T^{i,(0)} + \epsilon \sigma_T^{i,(1)} + \frac{\epsilon^2}{2!} \sigma_T^{i,(2)} + \cdots , \qquad (36)$$

$$h_{x^i,l,j}^{(\epsilon)} \sim h_{x^i,l,j}^{(0)} + \epsilon h_{x^i,l,j}^{(1)} + \frac{\epsilon^2}{2!} h_{x^i,l,j}^{(2)} + \cdots , \qquad (37)$$

where $S_t^{i,(\iota)} := \left.\frac{\partial^\iota S_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\right|_{\epsilon=0}$, $\sigma_t^{i,(\iota)} := \left.\frac{\partial^\iota \sigma_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\right|_{\epsilon=0}$, $h_{x^i,l,j}^{(\iota)} := \left.\frac{\partial^\iota h_{x^i,l,j}^{(\epsilon)}}{\partial \epsilon^\iota}\right|_{\epsilon=0}$.

Notations

For ease of exposition, we introduce the following notations.

Let us define: $\Phi_{S^i} := (\Phi_{S^i,1}, \cdots, \Phi_{S^i,2d})$, $\Phi_{\sigma^i} = (\Phi_{\sigma^i,1}, \cdots, \Phi_{\sigma^i,2d}) \in \mathbf{R}^{2d}$ $\Phi_S := (\Phi_{S^1}, \cdots, \Phi_{S^d})'$, $\Phi_\sigma := (\Phi_{\sigma^1}, \cdots, \Phi_{\sigma^d})'$ **(**$d \times 2d$ **matrices).**

We also define an operator "$*$".

- **For** $d \times 2d$ **matrices** A **and** B**,**

$$A * B := \begin{bmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d,1} & \cdots & (A)_{d,2d}(B)_{d,2d} \end{bmatrix} \quad (38)$$

- **For a** $d \times 2d$ **matrix** A **and a** d**-dimensional vector** B**,**

$$A * B = B * A := \begin{bmatrix} (A)_{1,1}(B)_{1} & \cdots & (A)_{1,2d}(B)_{1} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d} & \cdots & (A)_{d,2d}(B)_{d} \end{bmatrix} \quad (39)$$

Notations

- **For d-dimensional vectors A and B,**

$$A * B := \begin{bmatrix} (A)_1(B)_1 \\ \vdots \\ (A)_d(B)_d \end{bmatrix} \tag{40}$$

- **We define ∂_x($x = S$ or σ) for a $d \times 2d$ matrix $\Phi_{\hat{x}}$ ($\hat{x} = S$ or σ) as follows:**

$$\partial_x \Phi_{\hat{x}} := \begin{bmatrix} \frac{\partial}{\partial x_1}(\Phi_{\hat{x}})_{1,1} & \cdots & \frac{\partial}{\partial x_1}(\Phi_{\hat{x}})_{1,2d} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_d}(\Phi_{\hat{x}})_{d,1} & \cdots & \frac{\partial}{\partial x_d}(\Phi_{\hat{x}})_{d,2d} \end{bmatrix}. \tag{41}$$

- **Let us also introduce the following notations:**
$s_0 = (s_0^1, \cdots, s_0^d)$, $S_t = (S_t^1, \cdots, S_t^d)$, $\sigma_t = (\sigma_t^1, \cdots, \sigma_t^d)$,
$e^{\alpha t} = (e^{\alpha^1 t}, \cdots, e^{\alpha^d t})$, $e^{-\lambda t} = (e^{-\lambda^1 t}, \cdots, e^{-\lambda^d t})$.

Expression of $S^{(1)}$

(Hereafter, we consider log-normal jump case.)
The expression of $S^{(1)}$ is given as follows:

$$\begin{aligned} S_T^{(1)} &= \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) dZ_t \\ &+ \sum_{l=1}^{n}\left(\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} - \Lambda_l T \mathbf{E}\left[h_{S,l,1}^{(1)}\right]\right) * S_T^{(0)}, \end{aligned} \quad (42)$$

where

$$S_t^{(0)} = e^{\alpha t} * s_0, \quad (43)$$

$$\sigma_t^{(0)} = \theta + (\sigma_0 - \theta) * e^{-\lambda t}, \quad (44)$$

$$h_{S,l,j}^{(1)} = Y_{S,l,j} := (Y_{S^1,l,j}, \cdots, Y_{S^d,l,j})'. \quad (45)$$

Expression of $S^{(2)}$

The expression of $S^{(2)}$ is given as follows:

$$\begin{aligned} S^{(2)}_T &= \int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S\left(\sigma^{(0)}_t, S^{(0)}_t\right) * S^{(1)}_{t-} dZ_t + \int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S\left(\sigma^{(0)}_t, S^{(0)}_t\right) * \sigma^{(1)}_{t-} dZ_t \\ &+ \sum_{l=1}^{n} \left(\left(\sum_{j=1}^{N_{l,T}} h^{(2)}_{S,l,j} - \Lambda_l T \mathbf{E}\left[h^{(2)}_{S,l,1}\right] \right) * S^{(0)}_T \right. \\ &\left. + \sum_{j=1}^{N_{l,T}} h^{(1)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * S^{(1)}_{\tau_{j,l}-} - \Lambda_l \mathbf{E}\left[h^{(1)}_{S,l,1}\right] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S^{(1)}_{t-} dt \right), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \sigma^{(1)}_T &= \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma^{(0)}_t\right) dZ_t + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h^{(1)}_{\sigma,l,j} * e^{-\lambda(T-\tau_{j,l})} * \sigma^{(0)}_{\tau_{j,l}} \right. \\ &\left. -\Lambda_l \mathbf{E}\left[h^{(1)}_{\sigma,l,1}\right] * e^{-\lambda T} * \int_0^T e^{\lambda t} * \sigma^{(0)}_t dt \right), \end{aligned} \tag{47}$$

$$h^{(1)}_{\sigma,l,j} = Y_{\sigma,l,j} := (Y_{\sigma^1,l,j}, \cdots, Y_{\sigma^d,l,j})', \tag{48}$$

$$h^{(2)}_{S,l,j} = Y_{S,l,j} * Y_{S,l,j}. \tag{49}$$

The higher order expansions are obtained in the similar way.

Expansion of Basket Call Payoff

Since $g(x) = \sum_{i=1}^{d} w_i x^i$ **is a linear function, we can obtain following expressions.**

$$g\left(S_T^{(\epsilon)}\right) = g\left(S_T^{(0)}\right) + \epsilon g\left(S_T^{(1)}\right) + \frac{\epsilon^2}{2} g\left(S_T^{(2)}\right) + o(\epsilon^2), \tag{50}$$

Then, for a strike price $K = g(S_T^0) - \epsilon y$ **for an arbitrary** $y \in \mathbf{R}$**, the payoff of the call option with maturity** T **is expanded as follows:**

$$\begin{aligned}
\left(g\left(S_T^{(\epsilon)}\right) - K\right)^+ &= \epsilon\left(\frac{g(S_T^{(\epsilon)}) - g(S_T^{(0)})}{\epsilon} + y\right)^+ \\
&= \epsilon\left(g(S_T^{(1)}) + \frac{\epsilon}{2} g(S_T^{(2)}) + y + o(\epsilon)\right)^+ \\
&= \epsilon\left(g(S_T^{(1)}) + y\right)^+ \\
&\quad + \frac{\epsilon^2}{2} 1_{\{g(S_T^{(1)}) > -y\}} g\left(S_T^{(2)}\right) + o(\epsilon^2).
\end{aligned} \tag{51}$$

Expression of $S^{(1)}$ on $\{N_l = k_l\}$

When the number of jumps is k_l ($l = 1, \cdots, n$), that is on $\{N_l = k_l\} := \{N_{1,T} = k_1, \cdots, N_{n,T} = k_n\}$,

$$S^{(1)}_T = \xi_{\{k_l\}} + \hat{S}_T, \quad (52)$$

$$\xi_{\{k_l\}} := \sum_{l=1}^{n} (k_l - \Lambda_l T) m_{S,l} * e^{\alpha T} * s_0 \ \textbf{(constant)}, \quad (53)$$

$$\hat{S}_T := \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma^{(0)}_t, S^{(0)}_t\right) dZ_t + \sum_{l=1}^{n}\left(\sum_{j=1}^{k_l} \gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha T} * s_0\right). \quad (54)$$

Here, $m_{S,l} = (m_{S^1,l}, \cdots, m_{S^d,l})$ denotes the mean vector, and $\gamma_{S,l} = (\gamma_{S^1,l}, \cdots, \gamma_{S^d,l})$ denotes the volatility vector of the jump sizes. $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$ is a random vector. Each $\zeta_{S^i,j,l} \sim N(0,1)$ with $\zeta_{S,j,l}$'s correlation matrix $\vartheta_{\zeta_{S,l}}$.

Density function of $\hat{S}_T$

The distribution of $g(\hat{S}_T)$ is $N\left(0, \Sigma_T^{\{k_l\}}\right)$, and its density function is expressed as

$$n\left(x; 0, \Sigma_T^{\{k_l\}}\right) := \frac{1}{\sqrt{2\pi\Sigma_T^{\{k_l\}}}} \exp\left\{\frac{-x^2}{2\Sigma_T^{\{k_l\}}}\right\}, \tag{55}$$

where

$$\begin{aligned}\Sigma_T^{\{k_l\}} \quad := \quad & \int_0^T \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)' \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right) dt \\ & + \sum_{l=1}^{n} k_l (w * \gamma_{S,l} * e^{\alpha T} * s_0)' \vartheta_{\zeta_{S,l}} (w * \gamma_{S,l} * e^{\alpha T} * s_0),\end{aligned} \tag{56}$$

where $\vartheta_{\zeta_{S,l}}$ is the correlation matrix of $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$, and $w = (w_1, \cdots, w_d)$.

Expansion of Basket Call Price

In this setting, a basket option price $E[(g(S_T) - K)^+]$ (with zero discount rate for simplicity) is approximated as follows:

$$\begin{aligned}
&\mathbf{E}\left[\left(g\left(S_T^{(\epsilon)}\right) - K\right)^+\right] \\
&= \epsilon \mathbf{E}\left[\mathbf{E}\left[\left(g(S_T^{(1)}) + y\right)^+ \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right] \\
&+ \frac{\epsilon^2}{2}\mathbf{E}\left[\mathbf{E}\left[1_{\{g(S_T^{(1)})>-y\}} g\left(S_T^{(2)}\right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right] \\
&+ o(\epsilon^2).
\end{aligned} \tag{57}$$

Coefficients of ϵ and ϵ^2 for the Call Price

The coefficient of ϵ is derived as:

$$\mathbf{E}\left[\mathbf{E}\left[\left(g\left(S_T^{(1)}\right)+y\right)^+ \middle| g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]$$
$$=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\left(x+(g(\xi_{\{k_l\}})+y\right)n(x;0,\Sigma_T^{\{k_l\}})dx, \quad (58)$$

and the coefficient of $\frac{\epsilon^2}{2}$ is expressed as:

$$\mathbf{E}\left[\mathbf{E}\left[1_{\{g(S_T^{(1)})>-y\}}g\left(S_T^{(2)}\right)\middle| g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]$$
$$=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\mathbf{E}\left[g\left(S_T^{(2)}\right)\middle| g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$
$$\times n(x;0,\Sigma_T^{\{k_l\}})dx, \quad (59)$$

where

$$p_{\{k_l\}}:=\prod_{l=1}^{n}\frac{(\Lambda_l T)^{k_l}e^{-\Lambda_l T}}{k_l!}, \quad (60)$$

which is a probability of $\{N_l=k_l\}:=\{N_{1,T}=k_1,\cdots,N_{n,T}=k_n\}$.

Approximate Basket Call Price

By applying conditional expectation formulas (Lemma 3.2 in Shiraya-T(2014)), we obtain an approximation formula for basket call option price as follows:

$$
\begin{aligned}
& E\left[(g(S_T) - K)^+\right] \\
& \approx \sum_{k=0}^{\infty} \sum_{\sum_{l=1}^{n} k_l = k} p_{\{k_l\}} \Bigg\{ y_{k_l} N\left(\frac{y_{k_l}}{\sqrt{\Sigma_T^{\{k_l\}}}} \right) + \Bigg(\Sigma_T^{\{k_l\}} \\
& \quad + C_1 \frac{H_1\left(y_{k_l}; \Sigma_T^{\{k_l\}}\right)}{\Sigma_T^{\{k_l\}}} + C_2 \frac{H_2\left(y_{k_l}; \Sigma_T^{\{k_l\}}\right)}{\left(\Sigma_T^{\{k_l\}}\right)^2} + C_3 \Bigg) n\left(y_{k_l}; 0, \Sigma_T^{\{k_l\}}\right) \Bigg\}.
\end{aligned}
\tag{61}
$$

- $N(x)$ **is the standard normal distribution function.**
- $n\left(x; 0, \Sigma_T^{\{k_l\}}\right) := \frac{1}{\sqrt{2\pi\Sigma_T^{\{k_l\}}}} \exp\left\{ \frac{-x^2}{2\Sigma_T^{\{k_l\}}} \right\}$
- **Coefficients** C_1**,** C_2 **and** C_3 **are constants.**
- $H_k\left(x; \Sigma_T^{\{k_l\}}\right)$ **is a** k**-th order Hermite polynomial,**
- $y_{\{k_l\}} := g(\xi_{\{k_l\}}) + y$.

The detailed and the higher order expressions are given in Shiraya-T (2014).

Numerical Examples: Model

- **We use the following model for numerical examples.**

$$
\begin{aligned}
S^i_T &= \int_0^T \alpha^i S^i_{t-} dt + \int_0^T \sigma^i_{t-}(S^i_{t-})^{\beta_{S^i}} dW^{S^i}_t \\
&\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S^i_{\tau_{j,l}-} - \int_0^T \Lambda_l S^i_{t-} \mathbf{E}\left[h_{S^i,l,1}\right] dt \right), \qquad (62) \\
\sigma^i_T &= \int_0^T \lambda^i(\theta^i - \sigma^i_{t-}) dt + \int_0^T \nu^i (\sigma^i_{t-})^{\beta_{\sigma^i}} dW^{\sigma^i}_t \\
&\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j} \sigma^i_{\tau_{j,l}-} - \int_0^T \Lambda_l \sigma^i_{t-} \mathbf{E}\left[h_{\sigma^i,l,1}\right] dt \right), \qquad (63)
\end{aligned}
$$

- $h_{x^i,l,j} = H_{x^i,l}$ **(for all j, constant jump case),**
- $h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1$ **(log-normal jump case),**
  $Y_{x^i,l,j} \sim N(m_{x^i,l}, \gamma^2_{x^i,l})$ **for all j.**
  (x^i: S^i or σ^i), (σ^i, ν^i: positive constants)

Numerical Examples: Model

- **We set the jumps to be systematic jumps.**
  (that is, all asset prices and volatilities jump at the same time. i.e. $n = 1$**, all the elements of** ϑ **are** 1**, where** ϑ **is defined to be the** $2d \times 2d$ **correlation matrix among** $\zeta_{S^i,j,l}$ **and** $\zeta_{\sigma^i,j,l}$**,** $i = 1, \cdots, d$**.)**
- **We use the previous formula up to the** ϵ^2**-order for the LSV jump model with** ϵ^3**-order corrections, which are obtained by the corresponding LSV model with no jumps.**
  - **Examples of the expansion for no jump models are obtained in Shiraya -T (2014).**

Numerical Examples: Parameters

We examine 5 assets basket call option prices.

- **Parameters of all assets are the same.**

S_0^i	σ_0^i	α^i	β_{S^i}	β_{σ^i}	λ_i	θ_i	ν_i	w_i	Λ	T	n
100	**2**	**0**	**0.5**	**0.5**	**1**	**2**	**0.5**	**0.2**	**1**	**1**	**1**

(initial vol. of S^i: 20%, vol. on vol.: around 35%)

- **Jump Parameters**

	No Jumps	Constant Jumps		Log-normal Jumps		Mixed Jumps	
	case I	case II	case III	case IV	case V	case VI	case VII
H_{S^i}	-	-5%	-10%	-	-	-	-
H_{σ^i}	-	5%	10%	-	-	10%	20%
m_{S_i}	-	-	-	-5%	-10%	-5%	-10%
γ_{S_i}	-	-	-	10%	20%	10%	20%

(Mixed jumps means log-normal jumps for asset prices and constant jumps for the volatilities.)

Numerical Examples: Parameters

- **Correlations**

	W^{S_1}	W^{S_2}	W^{S_3}	W^{S_4}	W^{S_5}	W^{σ_1}	W^{σ_2}	W^{σ_3}	W^{σ_4}	W^{σ_5}
W^{S_1}	1	0.5	0.5	0.5	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
W^{S_2}	0.5	1	0.5	0.5	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
W^{S_3}	0.5	0.5	1	0.5	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
W^{S_4}	0.5	0.5	0.5	1	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
W^{S_5}	0.5	0.5	0.5	0.5	1	-0.5	-0.5	-0.5	-0.5	-0.5
W^{σ_1}	-0.5	-0.5	-0.5	-0.5	-0.5	1	0.5	0.5	0.5	0.5
W^{σ_2}	-0.5	-0.5	-0.5	-0.5	-0.5	0.5	1	0.5	0.5	0.5
W^{σ_3}	-0.5	-0.5	-0.5	-0.5	-0.5	0.5	0.5	1	0.5	0.5
W^{σ_4}	-0.5	-0.5	-0.5	-0.5	-0.5	0.5	0.5	0.5	1	0.5
W^{σ_5}	-0.5	-0.5	-0.5	-0.5	-0.5	0.5	0.5	0.5	0.5	1

Numerical Examples: Results (LSV with No Jumps)

We show the numerical results compared with those of the Monte Carlo method where the number of time steps is 512, and the number of trials is 5 millions with antithetic variables.

- Case I (No jump)

Strike	80	90	100	110	120
MC	20.85	12.55	6.17	2.27	0.56
(StdErr)	(0.07)	(0.06)	(0.04)	(0.03)	(0.01)
AE	20.86	12.56	6.17	2.27	0.55
Diff	0.01	0.01	0.00	-0.01	-0.01

Numerical Examples: Results (LSV with Constant Jumps)

- **Case II ($H_S = -5\%, H_\sigma = 5\%$)**

Strike	80	90	100	110	120
MC	20.99	12.80	6.49	2.55	0.70
(StdErr)	(0.07)	(0.06)	(0.05)	(0.03)	(0.01)
AE	20.98	12.79	6.49	2.55	0.71
Diff	0.01	0.01	0.01	-0.00	-0.01

- **Case III ($H_S = -10\%, H_\sigma = 10\%$)**

Strike	80	90	100	110	120
MC	21.42	13.53	7.39	3.32	1.16
(StdErr)	(0.08)	(0.07)	(0.05)	(0.03)	(0.02)
AE	21.43	13.54	7.39	3.31	1.14
Diff	0.01	0.01	0.00	-0.01	-0.02

Numerical Examples: Results (LSV with Log-Normal Jumps)

- **Case IV ($m_S = -5\%, \gamma_S = 10\%$)**

Strike	80	90	100	110	120
MC	21.45	13.59	7.48	3.44	1.29
(StdErr)	(0.08)	(0.07)	(0.05)	(0.04)	(0.02)
AE	21.44	13.59	7.48	3.43	1.26
Diff	-0.01	-0.00	-0.00	-0.02	-0.03

- **Case V ($m_S = -10\%, \gamma_S = 20\%$)**

Strike	80	90	100	110	120
MC	23.17	15.91	10.02	5.72	3.00
(StdErr)	(0.11)	(0.09)	(0.08)	(0.06)	(0.05)
AE	23.10	15.85	9.98	5.70	2.98
Diff	-0.08	-0.06	-0.04	-0.02	-0.02

Numerical Examples: Results (LSV with Mixed Jumps)

- **Case VI ($m_S = -5\%, \gamma_S = 10\%, H_\sigma = 10\%$)**

Strike	80	90	100	110	120
MC	21.47	13.59	7.45	3.40	1.26
(StdErr)	(0.08)	(0.07)	(0.05)	(0.04)	(0.02)
AE	21.48	13.60	7.45	3.38	1.22
Diff	0.01	0.01	0.00	-0.02	-0.03

- **Case VII ($m_S = -10\%, \gamma_S = 20\%, H_\sigma = 20\%$)**

Strike	80	90	100	110	120
MC	23.23	15.92	9.97	5.64	2.94
(StdErr)	(0.10)	(0.09)	(0.08)	(0.06)	(0.05)
AE	23.14	15.85	9.91	5.59	2.89
Diff	-0.09	-0.07	-0.05	-0.04	-0.05

FBSDE Approximation Scheme

(Fujii-T(2012a))

- **The forward backward stochastic differential equations (FBSDEs) have been found particularly relevant for various valuation problems (e.g. pricing securities under asymmetric/imperfect collateralization, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets).**
- **Their financial applications are discussed in details for example, El Karoui, Peng and Quenez (1997), Ma and Yong (2000), a recent book edited by Carmona (2009), Crépey (2012(a,b)), T-Yamada (2012) and references therein.**
- **We will present a simple analytical approximation with perturbation scheme for the non-linear FBSDEs. (mathematical validity: T-Yamada(2013))**

FBSDE Approximation Scheme - Setup-

- **We consider the following FBSDE:**

$$dV_t = -f(X_t, V_t, Z_t)dt + Z_t \cdot dW_t \quad (64)$$

$$V_T = \Phi(X_T), \quad (65)$$

where V takes the value in $\mathbb{R}$, W is a r-dimensional Brownian motion, and $X_t \in \mathbb{R}^d$ is assumed to follow a diffusion which is the solution to the (forward) SDE:

$$dX_t = \gamma_0(X_t)dt + \gamma(X_t) \cdot dW_t; \; X_0 = x \; . \quad (66)$$

- **We assume that the appropriate regularity conditions are satisfied for the necessary treatments.**

Perturbative Expansion for Non-linear Generator

- **In order to solve the pair of (V_t, Z_t) in terms of X_t, we extract the linear term from the generator f and treat the residual non-linear term as the perturbation to the linear FBSDE.**
- **We introduce the perturbation parameter ϵ, and then write the equation as**

$$\begin{aligned} dV_t^{(\epsilon)} &= c(X_t)V_t^{(\epsilon)}dt - \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \\ V_T^{(\epsilon)} &= \Phi(X_T) \, , \end{aligned} \quad (67)$$

where $\epsilon = 1$ corresponds to the original model by [1]

$$f(X_t, V_t, Z_t) = -c(X_t)V_t + g(X_t, V_t, Z_t) \, . \quad (68)$$

[1] Or, one can consider $\epsilon = 1$ as simply a parameter convenient to count the approximation order. The actual quantity that should be small for the approximation is the residual part g.

Perturbative Expansion for Non-linear Generator

- **Choosing the linear term $c(X_t)V_t^{(\epsilon)}$ in such a way that the residual non-linear term g becomes as small as possible is expected to achieve good approximations.**
- **Now, we are going to expand the solution of BSDE (67) in terms of ϵ: that is, suppose $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as**

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{69}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots . \tag{70}$$

Perturbative Expansion for Non-linear Generator

- **Once we obtain the solution up to the certain order, say k for example, then by putting $\epsilon = 1$,**

$$\tilde{V}_t = \sum_{i=0}^{k} V_t^{(i)}, \qquad \tilde{Z}_t = \sum_{i=0}^{k} Z_t^{(i)} \tag{71}$$

is expected to provide a reasonable approximation for the original model as long as the residual term g is small enough to allow the perturbative treatment.

- **$V_t^{(i)}$ and $Z_t^{(i)}$, the corrections to each order can be calculated recursively using the results of the lower order approximations.**

Recursive Approximation

Zero-th Order

- **For the zero-th order of ϵ, one can easily see the following equation should be satisfied:**

$$dV_t^{(0)} = c(X_t)V_t^{(0)}dt + Z_t^{(0)} \cdot dW_t \tag{72}$$

$$V_T^{(0)} = \Phi(X_T)\,. \tag{73}$$

- **It can be integrated as**

$$V_t^{(0)} = E\left[e^{-\int_t^T c(X_s)ds}\Phi(X_T) \middle| \mathcal{F}_t \right] \tag{74}$$

which is equivalent to the pricing of a standard European contingent claim, and $V_t^{(0)}$ is a function of X_t.

- **Applying Itô's formula (or Malliavin derivative), we obtain $Z_t^{(0)}$ as a function of X_t, too.**

Recursive Approximation

First Order

- Now, let us consider the process $V^{(\epsilon)} - V^{(0)}$. One can see that its dynamics is governed by

$$\begin{aligned} d(V_t^{(\epsilon)} - V_t^{(0)}) &= c(X_t)(V_t^{(\epsilon)} - V_t^{(0)})dt \\ &- \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + (Z_t^{(\epsilon)} - Z_t^{(0)}) \cdot dW_t \\ V_T^{(\epsilon)} - V_T^{(0)} &= 0\,. \end{aligned} \tag{75}$$

- Now, by extracting the ϵ-first order term, we can once again recover the linear FBSDE

$$\begin{aligned} dV_t^{(1)} &= c(X_t)V_t^{(1)}dt - g(X_t, V_t^{(0)}, Z_t^{(0)})dt + Z_t^{(1)} \cdot dW_t \\ V_T^{(1)} &= 0\,, \end{aligned} \tag{76}$$

which leads to

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} g(X_u, V_u^{(0)}, Z_u^{(0)})du \middle| \mathcal{F}_t\right]. \tag{77}$$

Recursive Approximation

- **Because $V_u^{(0)}$ and $Z_u^{(0)}$ are some functions of X_u, we obtain $V_t^{(1)}$ as a function of X_t, and also $Z_t^{(1)}$ through Itô's formula (or Malliavin derivative).**
- **In exactly the same way, one can derive an arbitrarily higher order correction. Due to the ϵ in front of the non-linear term g, the system remains to be linear in the every order of approximation. e.g.**

$$\begin{aligned} dV_t^{(2)} &= c(X_t)V_t^{(2)}dt - \left(\frac{\partial}{\partial v}g(X_t, V_t^{(0)}, Z_t^{(0)})V_t^{(1)} \right. \\ &+ \left. \nabla_z g(X_t, V_t^{(0)}, Z_t^{(0)}) \cdot Z_t^{(1)}\right) dt + Z_t^{(2)} \cdot dW_t \\ V_T^{(2)} &= 0 \end{aligned}$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Suppose we have succeeded to express backward components (V_t, Z_t) in terms of X_t up to the $(i-1)$-th order. Now, in order to proceed to a higher order approximation, we have to give the following form of expressions with some deterministic function $G(\cdot)$ in terms of the forward components X_t, in general:**

$$V_t^{(i)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} G(X_u) du \middle| \mathcal{F}_t \right] \tag{78}$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Even if it is impossible to obtain the exact result, we can still obtain an analytic approximation for $(V_t^{(i)}, Z_t^{(i)})$.**
- **For instance, an asymptotic expansion method allows us to obtain the expression.**
  In fact, applying the method, Fujii-T (2012a) has provided some explicit approximations for $V_t^{(i)}$ and $Z_t^{(i)}$.
- **Also, Fujii-T (2012b) has explicitly derived an approximation formula for the dynamic optimal portfolio in an incomplete market and confirmed its accuracy comparing with the exact result by Cole-Hopf transformation. (Zariphopoulou (2001))**

Forward Agreement with Bilateral Default Risk

- As the first example, we consider a toy model for a forward agreement on a stock with bilateral default risk of the contracting parties, the investor (party-1) and its counter party (party-2). The terminal payoff of the contract from the view point of the party-1 is

$$\Phi(S_T) = S_T - K \tag{79}$$

  where T is the maturity of the contract, and K is a constant.

- We assume the underlying stock follows a simple geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{80}$$

  where the risk-free interest rate r and the volatility σ are assumed to be positive constants.

- The default intensity of party-i, h_i is specified as

$$h_1 = \lambda, \qquad h_2 = \lambda + h \tag{81}$$

  where λ and h are also positive constants.

Forward Agreement with Bilateral Default Risk

- **In this setup, the pre-default value of the contract at time t, V_t follows**

$$
\begin{aligned}
dV_t &= rV_t dt - h_1 \max(-V_t, 0)dt + h_2 \max(V_t, 0)dt + Z_t dW_t \\
&= (r+\lambda)V_t dt + h \max(V_t, 0)dt + Z_t dW_t \qquad (82) \\
V_T &= \Phi(S_T) \,. \qquad (83)
\end{aligned}
$$

- **Now, following the previous arguments, let us introduce the expansion parameter ϵ, and consider the following FBSDE:**

$$
\begin{aligned}
dV_t^{(\epsilon)} &= \mu V_t^{(\epsilon)} dt - \epsilon g(V_t^{(\epsilon)})dt + Z_t^{(\epsilon)} dW_t \qquad (84) \\
V_T^{(\epsilon)} &= \Phi(S_T) \qquad (85) \\
dS_t &= S_t(rdt + \sigma dW_t) \,, \qquad (86)
\end{aligned}
$$

where we have defined $\mu = r + \lambda$ and $g(v) = -hv\mathbf{1}_{\{v \geq 0\}}$.

Forward Agreement with Bilateral Default Risk

- The next figure shows the numerical results of the forward contract with bilateral default risk with various maturities with the direct solution from the PDE (as in Duffie-Huang [1996]).
- We have used

$$r = 0.02, \quad \lambda = 0.01, \quad h = 0.03, \tag{87}$$

$$\sigma = 0.2, \quad S_0 = 100\,, \tag{88}$$

  where the strike K is chosen to make $V_0^{(0)} = 0$ for each maturity.
- We have plot $V^{(1)}$ for the first order, and $V^{(1)} + V^{(2)}$ for the second order. (Note that we have put $\epsilon = 1$ to compare the original model.)

Forward Agreement with Bilateral Default Risk

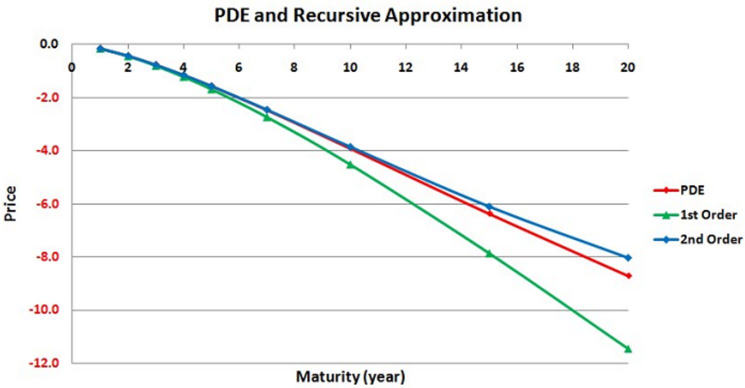


Figure: Numerical Comparison to PDE

Forward Agreement with Bilateral Default Risk

- **One can observe how the higher order correction improves the accuracy of approximation.**
- **In this example, the counter party is significantly riskier than the investor, and the underlying contract is volatile.**
- **Even in this situation, the simple approximation to the second order works quite well up to the very long maturity.**
- **In another example, [2], our second order approximation has obtained a fairly close value(2.953) to the one(2.95 with std 0.01) by a regression-based Monte Carlo simulation of Gobet-Lemor-Warin[2005].**

[2] a self-financing portfolio under the situation where there exists a difference between the lending and borrowing interest rates

Example: Density of Approximate CVA

Fujii-Shiraya-T(2012,2014)

- **When this technique is applied to evaluation of a pre-default contract value with bilateral counter party risk, Its first order approximation term can be regarded as CVA(credit value adjustment) [3].**
- **We present a simple example of an analytic approximation for this term by our 3rd order asymptotic expansion method.**
- **In particular, we consider a forex forward contract with bilateral counter party risk, where both parties post their collateral perfectly with the constant time-lag (Δ) by the same currency as the payment currency. We also assume the risk-free interest rate is equal to the collateral rate.**

[3] Our convention of CVA is different from other literatures by sign where it is defined as the “charge” to the clients. Thus, our CVA = -CVA.

FBSDE

We consider a forward contract on forex S^δ with strike K and maturity τ; the relevant FBSDE for the pre-default contract value is given as follows: ($h^{j,\delta}$ ($j=1,2$) is the each counter party's hazard rate process; ϵ, δ are expansion parameters.)

$$dV_t^\epsilon = rV_t^\epsilon dt - \epsilon f(h_t^{1,\delta}, h_t^{2,\delta}, V_t^\epsilon, V_{t-\Delta}^\epsilon)dt + Z_t^\epsilon dW_t;\ \ V_\tau = S_\tau^\delta - K,$$

$$f(h_t^{1,\delta}, h_t^{2,\delta}, V_t^\epsilon, V_{t-\epsilon}^\epsilon) = h_t^{1,\delta}(V_{t-\Delta}^\epsilon - V_t^\epsilon)^+ - h_t^{2,\delta}(V_t^\epsilon - V_{t-\Delta}^\epsilon)^+$$

$$dh_t^{j,\delta} = \alpha^j h_t^{j,\delta} dt + \delta\sigma_{h^j} h_t^{j,\delta}\left(\sum_{\eta=1}^{j} c_{j,\eta} dW_t^\eta\right);\ \ h_0^{j,\delta} = h_0^j, (j=1,2)$$

$$dS_t^\delta = \mu S_t^\delta dt + \delta v_t^\delta \left(S_t^\delta\right)^\beta \left(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta\right);\ \ S_0^\delta = s_0,\ \ \mu = r - r_f,$$

$$dv_t^\delta = \kappa(\theta - v_t^\delta)dt + \delta\xi v_t^\delta\left(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^\eta\right);\ \ v_0^\delta = v_0.$$

First order of ϵ

The first order equation is expressed as follows:

$$dV_t^1 = rV_t^1 dt - f(t, V_t^0, V_{t-\Delta}^0)dt + \sum_{\eta=1}^{4} Z_{t,\eta}^1 dW_t^\eta; \ V_\tau^1 = 0$$

Then, our CVA is represented by the following:

$$V_t^1 = \int_t^T e^{-r(u-t)} \mathbf{E}_t \left[f(u, V_u^0, V_{u-\Delta}^0) \right] du$$

$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\delta} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\delta} \cdot (V_u^0 - V_{u-\Delta}^0)^+,$$

where $V_{u-\Delta} = 0$ **when** $u < t + \Delta$**.**

$$\begin{aligned} V_u^0 &= e^{-r_f(\tau-u)} S_u^\delta - e^{-r(\tau-u)} K, \\ V_u^0 - V_{u-\Delta}^0 &= e^{-r_f(\tau-u)} S_u^\delta - e^{-r_f(\tau-u+\Delta)} S_{u-\Delta}^\delta - k(u;\Delta,r), \\ k(u;\Delta,r) &:= e^{-r(\tau-u)}(1-e^{-r\Delta})K. \end{aligned}$$

Numerical Example

- **We apply the asymptotic expansion method to evaluation of $c\hat{v}a(t,u) = e^{-r(u-t)}\mathbf{E}_t\left[f(u, V_u^0, V_{u-\Delta}^0)\right]$ up to the third order. Then, the value of CVA is approximated by**

$$CVA(t,\tau) = \int_t^\tau c\hat{v}a_{AE}(t,u)du + o(\delta^3). \quad (89)$$

- **Due to the analytical approximation of each $c\hat{v}a_{AE}(t,u)$, we have no problem in computation, which is very fast.**

Approximate density of CVA

We show the density function of approximate CVA by the asymptotic expansion method with Monte Carlo simulation. The maturity of forward contract is τ, T denotes the future time when CVA is evaluated, and Δ denotes the lag of collateral.

- **maturity (τ): 5 years, evaluation date (T): 2.5 years.**
- **strike:** $10,000$**.**
- **time step size:** $\frac{1}{400}$**year.**
- **the number of trials: 325,000 with antithetic variates.**

Procedure:

1. **implement Monte carlo simulation of the state variables (h^1, h^2, S, v) until T.**
2. **given each realization of the state variables, compute $c\hat{v}a_{AE}(T,u)$.**
3. **integrate $c\hat{v}a_{AE}(T,u)$ numerically with respect to the time parameter u from T to τ, and plot the values and their frequencies after normalization.**

Numerical Example

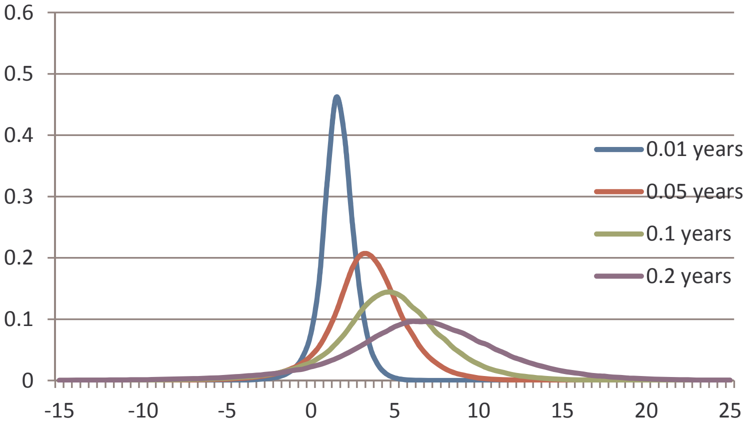
The parameters are set as follows:

- **parameters of h^1;**
  $h^1_0 = 0.02$, $\alpha^1 = -0.02$, $\sigma_{h1} = 20\%$.
- **parameters of h^2;**
  $h^2_0 = 0.01$, $\alpha^2 = 0.02$, $\sigma_{h2} = 30\%$.
- **parameters of S;**
  $S_0 = 10,000$, $r = \mu = 1\%$, $\beta = 1$.
- **parameters of ν;**
  $\nu_0 = 10\%$, $\kappa = 1$, $\theta = 20\%$, $\xi = 30\%$.
- **correlation matrix**

	h^1	h^2	S	ν
h^1	1	0.5	-0.3	0.2
h^2	0.5	1	0.1	0.1
S	-0.3	0.1	1	-0.8
ν	0.2	0.1	-0.8	1

Density of CVA

Figure: Density Functions of CVA with Different Time-Lags



Density of CVA

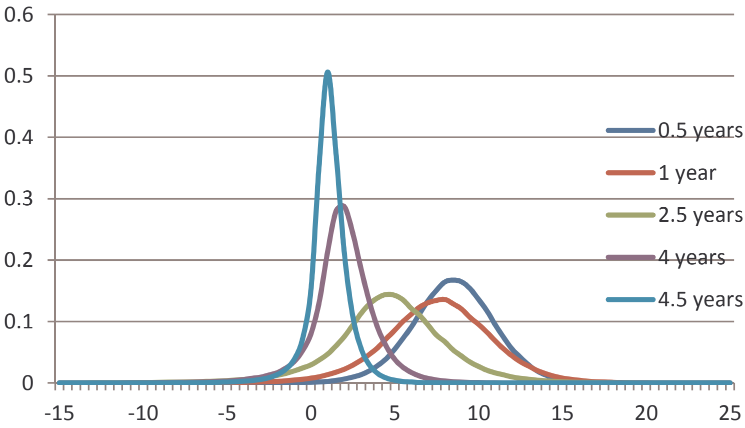
- **The longer the time lag is, the wider the density is.**
- **The mode (average) moves to the right when the time-lag becomes longer.**

$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\epsilon} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\epsilon} \cdot (V_u^0 - V_{u-\Delta}^0)^+.$$

- **When the first term increases, the CVA also increases.**
- **The hazard rate h^1 in the first term tends to be larger than h^2 in the second term in our parameterization.**

Density of CVA

Figure: Density Functions of CVA with Different Evaluation Dates



The shorter the time to maturity ($\tau - T$) becomes, CVA becomes smaller.

Perturbation Technique with Interacting Particle Method

(Fujii-T(2012))

- **We will provide a straightforward simulation scheme to solve nonlinear FBSDEs at each order of perturbative approximation.**
  - **Due to the convoluted nature of the perturbative expansion, it contains multi-dimensional time integrations of expectation values, which make standard Monte Carlo too time consuming.**
  - **To avoid nested simulations, we applied the particle representation inspired by the ideas of branching diffusion models(e.g. McKean (1975), Fujita (1966), Ikeda-Nagasawa-Watanabe (1965,1966,1968), Nagasawa-Sirao (1969)).**
  - **Comparing with the direct application of the branching diffusion method, our method is expected to be less numerically intensive since the interested system is already decomposed into a set of linear problems.**

Perturbation Technique with Interacting Particle Method

- **Again, let us introduce the perturbation parameter ϵ:**

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)})ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T), \end{cases} \tag{90}$$

where $X_t \in \mathbb{R}^d$ is assumed to follow a generic Markovian forward SDE

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s; \; X_t = x_t. \tag{91}$$

- **Let us fix the initial time as t. We denote the Malliavin derivative of X_u $(u \geq t)$ at time t as**

$$\mathcal{D}_t X_u \in \mathbb{R}^{r \times d}. \tag{92}$$

Perturbation Technique with Interacting Particle Method

- **Its dynamics in terms of the future time u is specified by $(Y_{t,u})^i_j = \partial_{x_t^j} X_u^i$:**

$$\begin{aligned} d(Y_{t,u})^i_j &= \partial_k \gamma_0^i(X_u)(Y_{t,u})^k_j du + \partial_k \gamma_a^i(X_u)(Y_{t,u})^k_j dW_u^a \\ (Y_{t,t})^i_j &= \delta^i_j \end{aligned} \quad (93)$$

where ∂_k denotes the differential with respect to the k-th component of X, and δ^i_j denotes Kronecker delta. Here, i and j run through $\{1, \cdots, d\}$ and $\{1, \cdots, r\}$ for a. Here, we adopt Einstein notation which assumes the summation of all the paired indexes.

- **Then, it is well-known that**

$$(\mathcal{D}_t X_u^i)_a = (Y_{t,u}\gamma(x_t))^i_a,$$

where $a \in \{1, \cdots, r\}$ is the index of r-dimensional Brownian motion.

Perturbation Technique with Interacting Particle Method

- **ϵ-0th order: For the zeroth order, it is easy to see**

$$V_t^{(0)} = \mathbb{E}\Big[\Psi(X_T)\Big|\mathcal{F}_t\Big] \tag{94}$$

$$Z_t^{a(0)} = \mathbb{E}\Big[\partial_i\Psi(X_T)(Y_{tT}\gamma(X_t))_a^i\Big|\mathcal{F}_t\Big]. \tag{95}$$

- **It is clear that they can be evaluated by standard Monte Carlo simulation. However, for their use in higher order approximation, it is crucial to obtain explicit approximate expressions for these two quantities. (e.g. Hagan et al.(2002), an asymptotic expansion method)**
- **In the following, let us suppose we have obtained the solutions up to a given order of asymptotic expansion, and write each of them as a function of x_t:**

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t). \end{cases} \tag{96}$$

Perturbation Technique with Interacting Particle Method

- **ϵ-1st order:**

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbb{E}\Big[f(X_u, V_u^{(0)}, Z_u^{(0)}) \Big| \mathcal{F}_t \Big] du \\ &= \int_t^T \mathbb{E}\Big[f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big) \Big| \mathcal{F}_t \Big] du \end{aligned} \qquad (97)$$

- **Next, define the new process for $(s > t)$:**

$$\hat{V}_{ts}^{(1)} = e^{\int_t^s \lambda_u du} V_s^{(1)}, \qquad (98)$$

where deterministic positive process λ_t (It can be a positive constant for the simplest case.).

Perturbation Technique with Interacting Particle Method

- Then, its dynamics is given by

$$d\hat{V}^{(1)}_{ts} = \lambda_s \hat{V}^{(1)}_{ts} ds - \lambda_s \hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) ds + e^{\int_t^s \lambda_u du} Z^{(1)}_s \cdot dW_s \ ,$$

where

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)).$$

- Since we have $\hat{V}^{(1)}_{tt} = V^{(1)}_t$, one can easily see the following relation holds:

$$V^{(1)}_t = \mathbb{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds} \lambda_u \hat{f}_{tu}(X_u, v^{(0)}(X_u), z^{(0)}(X_u)) du \middle| \mathcal{F}_t \right] \qquad (99)$$

- As in credit risk modeling (e.g. Bielecki-Rutkowski (2002)), it is the present value of default payment where the default intensity is λ_s with the default payoff at $s(> t)$ as $\hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, we obtain the following proposition.

Perturbation Technique with Interacting Particle Method

Proposition

The $V_t^{(1)}$ in (97) can be equivalently expressed as

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau)\right)\Big|\mathcal{F}_t\right]. \tag{100}$$

Here τ is the interaction time where the interaction is drawn independently from Poisson distribution with an arbitrary deterministic positive intensity process λ_t. $\hat{f}$ is defined as

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s}e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) \,. \tag{101}$$

Perturbation Technique with Interacting Particle Method

- **Now, let us consider the component $Z^{(1)}$. It can be expressed as**

$$Z_t^{(1)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\right] du \tag{102}$$

Firstly, let us observe the dynamics of Malliavin derivative of $V^{(1)}$ follows

$$\begin{aligned} d(\mathcal{D}_t V_s^{(1)}) &= -(\mathcal{D}_t X_s^i)\nabla_i(X_s, v^{(0)}, z^{(0)}) f(X_s, v^{(0)}, z^{(0)}) + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s; \\ \mathcal{D}_t V_t^{(1)} &= Z_t^{(1)}, \end{aligned} \tag{103}$$

where

$$\nabla_i(x, v^{(0)}, z^{(0)}) \equiv \partial_i + \partial_i v^{(0)}(x)\partial_v + \partial_i z^{a(0)}(x)\partial_{z^a}, \tag{104}$$

$$f(x, v^{(0)}, z^{(0)}) \equiv f(x, v^{(0)}(x), z^{(0)}(x)). \tag{105}$$

Perturbation Technique with Interacting Particle Method

- **Define, for $(s > t)$,**

$$\widehat{\mathcal{D}_t V_s^{(1)}} = e^{\int_t^s \lambda_u du} (\mathcal{D}_t V_s^{(1)}). \tag{106}$$

Then, its dynamics can be written as

$$d(\widehat{\mathcal{D}_t V_s^{(1)}}) = \lambda_s(\widehat{\mathcal{D}_t V_s^{(1)}})ds - \lambda_s(\mathcal{D}_t X_s^i)\nabla_i(X_s, v^{(0)}, z^{(0)}) \hat{f}_{ts}(X_s, v^{(0)}, z^{(0)})ds$$
$$+ e^{\int_t^s \lambda_u du}(\mathcal{D}_t Z_s^{(0)}) \cdot dW_s. \tag{107}$$

We again have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)}. \tag{108}$$

- **Hence,**

$$Z_t^{(1)} = \mathbb{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds} \lambda_u(\mathcal{D}_t X_u^i)\nabla_i(X_u, v^{(0)}, z^{(0)}) \hat{f}_{tu}(X_u, v^{(0)}, z^{(0)}) du \middle| \mathcal{F}_t \right]. \tag{109}$$

Perturbation Technique with Interacting Particle Method

- **Thus, following the same argument for the previous proposition, we have the result below:**

Proposition

$Z_t^{(1)}$ in (102) is equivalently expressed as

$$Z_t^{a(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\Big[\mathbf{1}_{\{\tau<T\}}(Y_{t\tau}\gamma(X_\tau))_a^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\Big|\mathcal{F}_t\Big] \quad (110)$$

where the definitions of random time τ and the positive deterministic process λ are the same as those in the previous proposition.

Perturbation Technique with Interacting Particle Method

Monte Carlo Method

Now, we have a new particle interpretation of $(V^{(1)}, Z^{(1)})$ as follows:

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}, z^{(0)}\right)\middle|\mathcal{F}_t\right] \quad (111)$$

$$Z_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\mathcal{F}_t\right] \quad (112)$$

which allows efficient time integration with the following Monte Carlo scheme:

- **Run the diffusion processes of X and Y**
- **Carry out Poisson draw with probability $\lambda_s\Delta s$ at each time s and if "one" is drawn, set that time as τ.**
- **Then stores the relevant quantities at τ, or in the case of $(\tau > T)$ stores 0.**
- **Repeat the above procedures and take their expectation.**

Numerical Example

(Fujii-Shiraya-T (2012,2014)) An example for pre-default values with imperfect collateralization [4] :

- **The counter party sells OTC European options on WTI futures. [5]**
- **For simplicity, we consider a unilateral case, where counter party is defaultable, while the investor is default-free, and the collateral is posted as the same currency as the payment currency (that is, the currency is USD).**
- **We consider the following imperfect collateral cases:**
  - **No collateral**
  - **Cash collateral with time-lag**
  - **Asset collateral with time-lag**

[4] As for an application to American option pricing, please see Fujii-Sato-T (2012)
[5] Later, we will see a basket option on WTI and Brent futures.

Model

- **CIR model for the hazard rate process (h).**
- **SABR model for WTI futures price process (S and ν).**
- **Log-Normal model for a collateral asset price process (A).**

$$dh_t = \kappa(\theta - h_t)\,dt + \gamma\sqrt{h_t}c_1 dW_t^1;\ h_0 = \hat{h}_0 \tag{113}$$

$$dS_t = \nu_t (S_t)^\beta \left(\sum_{\eta=1}^{2} c_{2,\eta} dW_t^\eta\right);\ S_0 = s_0, \tag{114}$$

$$d\nu_t = \sigma_\nu \nu_t \left(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta\right);\ \nu_0 = \hat{\nu}_0, \tag{115}$$

$$dA_t = \mu_A A_t dt + \sigma_A A_t \left(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^\eta\right);\ A_0 = a_0. \tag{116}$$

Model

- The dynamics of pre-default value V can be described by a non-linear FBSDE:

$$\begin{cases} dV_t = rV_t dt - f(h_t, V_t, \Gamma_t)dt + Z_t \cdot dW_t \\ V_T = (S_T - K)^+ \text{ or } (K - S_T)^+, \end{cases} \tag{117}$$

where

- Γ_t : collateral process
  (e.g. cash collateral with a constant time lag Δ : $\Gamma_t = V_{t-\Delta}$)
- r(risk free rate), c(collateral rate), l(loss rate) : nonnegative constants for simplicity. [6]

We put ϵ in front of the driver, f to apply our perturbation technique with interacting particle method.

[6] Later, we will see a more general case, where a stochastic collateral cost is taken into account.

Model

- Counter party does not post collateral or posts collateral with the constant time-lag (Δ) by cash or an asset A.
- no collateral case:

$$f(h_t, V_t, \Gamma_t) = -lh_t(V_t)^+. \tag{118}$$

- time-lag collateral case
  - cash collateral:

$$\begin{aligned} f(h_t, V_t, \Gamma_t) &= (r-c)V_{t-\Delta} \\ &\quad -lh_t\left(V_t - V_{t-\Delta}\right)^+, \end{aligned} \tag{119}$$

  - asset collateral:

$$\begin{aligned} f(h_t, V_t, \Gamma_t) &= (r-c)V_{t-\Delta}\left(\frac{A_t}{A_{t-\Delta}}\right) \\ &\quad -lh_t\left(V_t - V_{t-\Delta}\left(\frac{A_t}{A_{t-\Delta}}\right)\right)^+. \end{aligned} \tag{120}$$

Parameters

- **We use the data of CME WTI option and futures prices.**
  The maturity of the underlying futures is DEC 15, and the maturity of WTI option is Nov 17, 2015.
- **Parameters of WTI futures are obtained by calibration to the market values of futures option prices on July 10, 2012.**
- **We assume that the risk free rate r is equal to collateral rate c.**
- **The discount rate is $c = 0.295\%$ which is calculated by OIS with the same maturity as the option maturity.**
- **The recovery rate is $R = 0$ (i.e. $l = 1$).**
- **We use the results of Denault et al., (2009) for the parameters of hazard rate processes and the results of Hull et al.(2005) for the initial values of hazard rates.**

Parameters/Monte Carlo

- **Calibrated parameters are as follows[7] (WTI price's initial vol.: about** 23**%):**

Table: Parameters of WTI DEC15 in SABR model

	s_0	β	$\hat{\nu}_0$	σ_ν	ρ
WTI DEC15	**84.48**	**0.5**	**2.117**	**0.410**	**-0.112**

We calculate the pre-default value of European option whose maturity is the same as that of futures option. (about 3.3 **years to the maturity)**
The details of Monte Carlo method simulation are as follows:

- **time step size is 1/200 years.**
- **the number of trial is 10 million.**
- **Hagan et al. formula (2002) is used for evaluation of default-free European options, that is** $V^{(0)}$**.**

[7] As futures options traded in CME(WTI) are American type, we calibrate to European option prices with the implied BS(log-normal) volatilities that are obtained by a binomial method.

Analysis

- **We check the following points.**
  - **correction effect: (S, h), (S, v), (h, v), (S, A), (v, A) and (h, A).**
  - **collateral effect: no collateral, cash collateral with constant time-lag or asset collateral with constant time-lag.**
  - **rating effect: from Aaa to B.**
  - **the second order value's effect.**
  - **maturity effect: from 2 years to 10 years.**

Correlation Effect

Firstly, we test the correlation effects among the hazard rates, the underlying asset price, its volatility and the collateral asset price.
In this example, we set the following assumptions.

- **the correlations which are not explicitly specified are set to be 0.**
- **parameters of the hazard rate processes are those of Baa rating.**
- **parameters of the collateral asset are $\mu_A = 0$ and $\sigma_A = 50\%$.**
- **the time-lag (Δ) of collateral is 0.1.**
- **strike price is ATM.**

Correlation Effect - No Collateral

Table: Pre-default values of call option contracts without collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.784	-0.987	-1.220	-1.465	-1.742
	2nd	0.027	0.043	0.065	0.091	0.123
	Total	13.890	13.704	13.492	13.273	13.029
S and ν	0th	13.789	14.338	14.648	14.719	14.553
	1st	-1.147	-1.192	-1.220	-1.231	-1.222
	2nd	0.061	0.063	0.065	0.066	0.065
	Total	12.703	13.210	13.492	13.554	13.397
h and ν	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.055	-1.134	-1.220	-1.312	-1.410
	2nd	0.050	0.057	0.065	0.074	0.085
	Total	13.642	13.570	13.492	13.410	13.322

- **When the correlation between S and h increases ($-0.7 \rightarrow +0.7$), the absolute values of the first and the second order become larger. (High correlation between S and h means that the default risk becomes high when the option value is high.)**

Correlation Effect - Cash Collateral

Table: Pre-default values of call option contracts with cash collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.116	-0.137	-0.160	-0.185	-0.211
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	14.532	14.511	14.488	14.463	14.436
S and ν	0th	13.789	14.338	14.648	14.719	14.553
	1st	-0.127	-0.144	-0.160	-0.174	-0.187
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	13.663	14.194	14.488	14.545	14.366
h and ν	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.130	-0.144	-0.160	-0.177	-0.195
	2nd	0.00004	0.00004	0.00004	0.00004	0.00004
	Total	14.518	14.503	14.488	14.471	14.452

- **The effect of the second order value seems negligible under collateralization with this level of time-lag.**

Correlation Effect - Asset Collatral

Table: Pre-default values of call option contracts with asset collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.128	-0.154	-0.183	-0.214	-0.249
	2nd	0.0001	0.0002	0.0003	0.0004	0.0006
	Total	14.520	14.494	14.465	14.433	14.399
S and ν	0th	13.789	14.338	14.648	14.719	14.553
	1st	-0.154	-0.169	-0.183	-0.194	-0.204
	2nd	0.0003	0.0003	0.0003	0.0003	0.0003
	Total	13.635	14.169	14.465	14.525	14.350
h and ν	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.152	-0.166	-0.183	-0.201	-0.220
	2nd	0.0002	0.0003	0.0003	0.0003	0.0004
	Total	14.496	14.481	14.465	14.447	14.428

- **The first order value with asset collateral is about 1.2 times as large as that with cash collateral.**
- **The effect of the second order value also seems negligible.**

Correlation Effect - Asset Collateral

Table: Pre-default values of call option contracts with asset collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.220	-0.202	-0.183	-0.160	-0.132
	2nd	0.0007	0.0005	0.0003	0.0001	0.0000
	Total	14.428	14.446	14.465	14.487	14.515
ν and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.192	-0.188	-0.183	-0.178	-0.172
	2nd	0.0004	0.0004	0.0003	0.0002	0.0002
	Total	14.455	14.460	14.465	14.470	14.475
h and A	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.192	-0.188	-0.183	-0.178	-0.174
	2nd	0.0004	0.0004	0.0003	0.0002	0.0002
	Total	14.456	14.460	14.465	14.470	14.474

- **Correlation effect between the underlying asset price and the collateral asset price seems similar order as the one between the underlying asset price and the hazard rate.**
- **When the correlation between S and A is negative, the increase in the option premium and the decrease in the collateral value occur simultaneously. (That is, it requires more collateral.)**

Rating Effect - No Collateral

Table: Pre-default values of call option contracts without collateral

Strike		70	80	85	90	100
Aaa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.474	-0.351	-0.300	-0.254	-0.182
	2nd	0.005	0.004	0.003	0.003	0.002
	Total	22.189	16.450	14.036	11.928	8.564
Baa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-1.879	-1.392	-1.186	-1.007	-0.720
	2nd	0.100	0.074	0.063	0.054	0.038
	Total	20.879	15.480	13.210	11.226	8.062
B	0th	22.658	16.798	14.333	12.179	8.744
	1st	-7.877	-5.833	-4.972	-4.219	-3.017
	2nd	2.155	1.595	1.359	1.153	0.823
	Total	16.936	12.560	10.720	9.113	6.551

- **the worse is the rating, the more important the second order becomes.**
- **For the case of single B, if the second order value is not taken into account, the pre-default value is more than 10% different from the first order pre-default value.**

Rating Effect - Asset Collateral

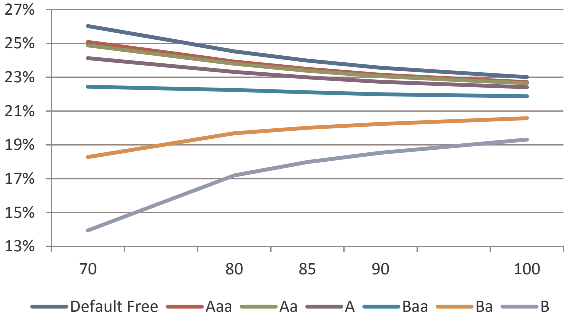
Table: Pre-default values of call option contracts with asset collateral

Strike		70	80	85	90	100
Aaa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.064	-0.051	-0.045	-0.040	-0.031
	2nd	0.00003	0.00002	0.00002	0.00001	0.00001
	Total	22.594	16.747	14.288	12.139	8.714
Baa	0th	22.658	16.798	14.333	12.179	8.744
	1st	-0.250	-0.199	-0.177	-0.156	-0.120
	2nd	0.00047	0.00035	0.00030	0.00025	0.00018
	Total	22.409	16.599	14.157	12.024	8.624
B	0th	22.658	16.798	14.333	12.179	8.744
	1st	-1.029	-0.822	-0.729	-0.644	-0.497
	2nd	0.00996	0.00737	0.00628	0.00533	0.00380
	Total	21.639	15.983	13.610	11.541	8.251

- **The effect of the second order value seems negligible under collateralization with this level of time lag, even if the rating is single B.**

Rating Effect - Implied Volatility (No Collateral)

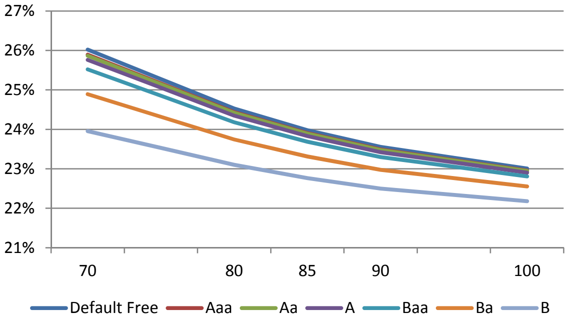
Figure: Implied volatilities of call options without collateral



- **The shape of the skew of rating B is different from that of rating Aaa. The difference of IV from the default-free case is larger for ITM, and the size of difference varies in rating.**

Rating Effect - Implied Volatility (Asset Collateral)

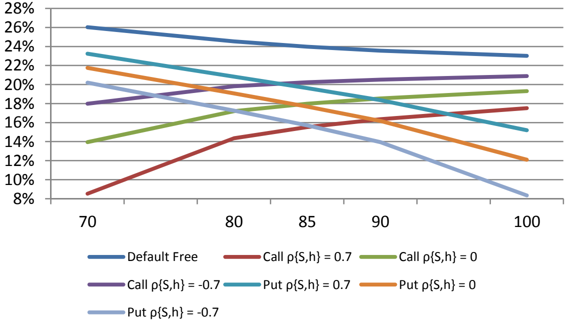
Figure: Implied volatilities of European call options with risky asset collateral



- **In this case, the shape of all ratings is similar.**
- **The level of implied volatility is different in rating.**

Correlation Effect (S, h) - Implied Volatility (Rating : B)

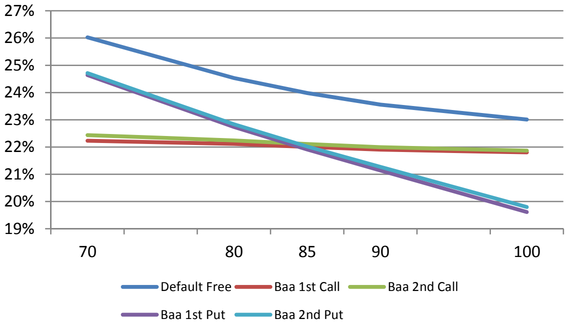
Figure: Implied volatilities of European call and put options without collateral



- **When the correlation between the underlying asset price and the hazard rate becomes high, a call option's implied volatility becomes low.**
- **This is because a default probability will increase if the price rises (that is, the option value rises).**
- **For the case of put options, the shape is reversed.**

The Second Order Effect - Implied Volatility (Baa)

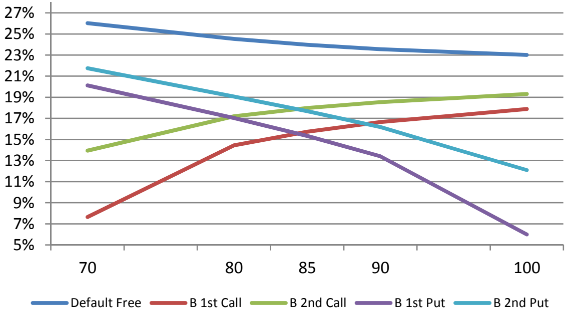
Figure: Implied volatilities of European call and put options without collateral



- **The difference between the first and the second is not so large in this case.**

The Second Order Effect - Implied Volatility (B)

Figure: Implied volatilities of European call and put options without collateral

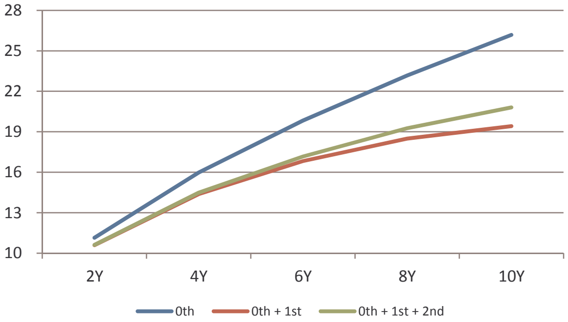


- **It seems better to take the second order value into account.**

Maturity Effect - No Collateral (Baa)

Next graph shows the values of 0th (default free), 1st and 2nd order price of the ATM option without collateral in Baa rating.

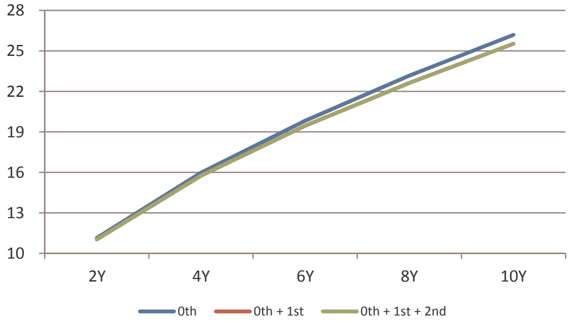
Figure: Pre-default values of call option contracts without collateral



- **For the long maturity case, the second order value has larger impact on the pre-default value.**
- **For the case of 10 years maturity, the 2nd order affects by more than 5%.**

Maturity Effect - Asset Collateral (Baa)

Figure: Pre-default values of call option contracts with asset collateral



- **When we post the collateral, the second order effect does not increase.**
- **The second order effect can be ignored even if the maturity is more than 10 years.**

Stochastic Collateral Cost

- **Next, we consider a more general case:**

$$dV_t = cV_t dt - f(y_t, \hat{y}_t, h_t, V_t, \Gamma_t)dt + Z \cdot dW_t, \quad (121)$$

where

$$f(y_t, \hat{y}_t, h_t, V_t, \Gamma_t) = \hat{y}_t \Gamma_t - y_t V_t - l h_t (V_t - \Gamma_t)^+ \quad (122)$$

$$y_t = r_t - c(\textbf{collateral cost of USD}) \quad (123)$$

$$\hat{y}_t = \hat{r}_t - \hat{c}_t(\textbf{collateral cost of } \Gamma_t) \quad (124)$$

- **For numerical examples, we set $\hat{y} \equiv 0$ and suppose $y_t = r_t - c$ where r follows a CIR process with a nonnegative constant c. Then, we put ϵ in front of f to apply our perturbation technique with interacting particle method.**

Stochastic Collateral Cost

- **CIR model for risk free rate process (r).**

$$dr_t \quad = \quad \kappa_r\,(\theta_r - r_t)\,dt + \gamma_r\,\sqrt{r_t}(\sum_{\eta=1}^{5} c_{5,\eta} dW_t^{\eta});\ r_0 = r(0). \qquad (125)$$

Table: Parameters of USD risk free rate process

	$r(0)$	κ_r	θ_r	γ_r
USD Risk Free Rate	**1%**	**0.2**	**1%**	**0.05**

- **The other parameters are the same as before.**
- **The rating of counter party is Baa.**
- **We check the following points.**
  - **correlation effect: (S, h), (S, y), and (h, y).**
  - **collateral effect: no collateral, asset collateral with constant time-lag 0.1.**

Correlation Effect - No Collateral

Table: Correlation Effects - No Collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.129	-1.335	-1.565	-1.821	-2.100
	2nd	0.051	0.072	0.099	0.131	0.170
	Total	13.570	13.385	13.181	12.958	12.717
S and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.418	-1.488	-1.565	-1.650	-1.742
	2nd	0.083	0.090	0.099	0.109	0.119
	Total	13.313	13.250	13.181	13.106	13.025
h and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-1.565	-1.565	-1.565	-1.565	-1.565
	2nd	0.093	0.096	0.099	0.102	0.105
	Total	13.176	13.179	13.181	13.184	13.187

Change in the correlation between S and y affects on the value by at most 2 %, while change in the correlation between S and h does by around 6%.

Correlation Effect - Asset Collateral

Table: Correlation Effects - Asset Collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.505	-0.532	-0.561	-0.593	-0.627
	2nd	0.007	0.008	0.008	0.009	0.010
	Total	14.150	14.124	14.095	14.064	14.030
S and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.397	-0.475	-0.561	-0.655	-0.757
	2nd	0.004	0.006	0.008	0.011	0.015
	Total	14.254	14.178	14.095	14.004	13.905
h and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.560	-0.561	-0.561	-0.561	-0.561
	2nd	0.008	0.008	0.008	0.009	0.009
	Total	14.095	14.095	14.095	14.095	14.095
A and y	0th	14.648	14.648	14.648	14.648	14.648
	1st	-0.561	-0.561	-0.561	-0.561	-0.561
	2nd	0.008	0.008	0.008	0.008	0.008
	Total	14.095	14.095	14.095	14.095	14.095

Change in the correlation between S and y has a larger effect than change in the correlation between S and h, (y_t is multiplied by V_t, whereas h_t is multiplied by $V_t - \Gamma_{t-\Delta}$.)

Basket Option

Next, we consider about a basket option of WTI and Brent: $\left(S_T^{(wti)} + S_T^{(brent)} - K\right)^+$

- **To calculate** $V^{(0)}$ **analytically, we use the asymptotic expansion method**
- **The maturity of the underlying futures is DEC 15.**
- **The maturity of basket option is Nov 10, 2015.**
- **The discount rate is** $c = 0.295\%$ **which is calculated by OIS with the same maturity as the option maturity.**
- **The parameters of the underlying asset prices are obtained by calibration to the market values of futures options on July 10, 2012. (around** 3.3 **years to the maturity)**

Calibrated parameters are follows. [8] **:**

Table: Parameters of Brent DEC15 in SABR model

	$S(0)$	β	$\nu(0)$	σ_ν	ρ
Brent DEC15	**90.14**	**0.5**	**2.184**	**0.446**	**-0.044**

[8] As futures options traded in ICE(Brent) are American type, we calibrate to European option prices with the implied BS(log-normal) volatilities that are obtained by a binomial method.

Basket Option

- **The correlation between WTI futures price (or Brent futures price) and Brent volatility (or WTI volatility) is set as the same value as the correlation between WTI futures price (or Brent futures Price) and WTI volatility (or Brent volatility).**
- **The correlations between WTI futures price (or volatility) and Brent futures price (or volatility) are calculated by using logarithmic historical price changes for the 30 days before July 10, 2012.**
- **The correlation between WTI future price and Brent future price is 0.980, and the correlation between WTI volatility and Brent volatility is 0.907.**

Basket Option

Table: Pre-default values of call option contracts without collateral

Strike		140	160	170	180	200
Aaa	0th	49.798	37.475	32.224	27.590	20.083
	1st	-1.036	-0.780	-0.671	-0.575	-0.418
	2nd	0.011	0.009	0.007	0.006	0.005
	Total	48.774	36.704	31.561	27.021	19.669
Baa	0th	49.798	37.475	32.224	27.590	20.083
	1st	-4.101	-3.089	-2.657	-2.276	-1.658
	2nd	0.217	0.164	0.141	0.121	0.088
	Total	45.915	34.550	29.708	25.435	18.513
B	0th	49.798	37.475	32.224	27.590	20.083
	1st	-17.176	-12.937	-11.130	-9.534	-6.946
	2nd	4.680	3.531	3.041	2.607	1.904
	Total	37.303	28.069	24.135	20.662	15.041

- **Moreover, applying the asymptotic expansion method, we are able to calculate pre-default values of various type of basket options. (Please see Shiraya-T (2014) for the detail.)**