An Asymptotic Expansion Approach to Derivatives Pricing [1]

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Outlines

An Asymptotic Expansion Approach

- **An asymptotic expansion approach in finance is initiated by Kunitomo-T[1992], Yoshida[1992b] and T[1995,1999].**
  - **It provides us a unified methodology for valuation problems under a general diffusion model in finance, which is useful for evaluation of values/Greeks of European derivatives. So, it is used**
    - **in calibration.**
    - **in American option pricing through its value decomposition.**
    - **as an explanatory variable for the least square method (or as an estimator for excercise boundary) in Bermudan option pricing.**
    - **as a control variable for the control variate method in Monte Carlo simulation.**
    - **with known characteristic functions of jump processes to be applied to jump-diffusion models.**
  - **It is mathematically justified by Watanabe theory in Malliavin calculus. (Watanabe[1987], Yoshida[1992a], Kunitomo- T[2003])**

Today's Talk

- **This talk will introduce our results in the asymptotic expansion approach. In particular, I will**
  1. **briefly describe the outline of the asymptotic expansion approach, and list up its applications to option pricing.**
  2. **present a new approximation scheme (perturbation scheme) for solutions of forward backward stochastic differential equations(FBSDEs) with some applications.**
  3. **introduce a Monte Carlo implementation using the perturbation scheme for FBSDEs with interacting particle method.**

An Asymptotic Expansion in a General Diffusion Setting

Shiraya-Takahashi-Toda [2009]

Setting

- (W, P)**: a** r**-dimensional Wiener Space**
- $X^{(\epsilon)} = (X^{(\epsilon),1}, \cdots, X^{(\epsilon),d})$**:** d**-dimensional stochastic process dependent on a perturbation parameter** $\epsilon \in (0,1]$**:**

$$X_t^{(\epsilon),j} = x_0 + \int_0^t V_0^j(X_s^{(\epsilon)}, \epsilon)ds + \epsilon \int_0^t V^j(X_s^{(\epsilon)})dW_s \tag{1}$$

where
$V_0 = (V_0^1, \cdots, V_0^d)$**:** $\mathbf{R}^d \times (0,1] \mapsto \mathbf{R}^d$**, and**
$V = (V^1, \cdots, V^d)$**:** $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$
are smooth functions with bounded derivatives of all orders.

- **Next, suppose that a function $g : \mathbf{R}^N \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders.**
- **Then, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;**

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \cdots$$

  in L^p for every $p > 1$ as $\epsilon \downarrow 0$,
- **The coefficients in the expansion are obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals.**

- **Let A_{kt} the k-th expansion coefficient of $X_t^{(\epsilon)}$ ($A_{kt} = \frac{1}{k!} \left.\frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}\right|_{\epsilon=0}$), and A_{kt}^j, $j = 1, \cdots, d$ denote the j-th elements of A_{kt}.**
- **In particular, A_{1t} is represented by**

$$A_{1t} = \int_0^t Y_t Y_u^{-1} \left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u \right) \tag{2}$$

where Y denotes the solution to the differential equation;

$$dY_t = \partial V_0(X_t^{(0)}, 0) Y_t dt; \; Y_0 = I_d.$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j, k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

- **Note that A_{1t} follows a normal distribution.**

- **For $k \geq 2$, A^j_{kt}, $j = 1, \cdots, d$ is recursively determined by the following:** [2]

$$\begin{aligned} A^j_{kt} &= \frac{1}{k!}\int_0^t \partial^k_\epsilon V^j_0(X^{(0)}_u, 0)du \\ &\quad + \sum_{l=1}^{k}\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right)\partial^\beta_{\vec{d}_\beta}\partial^{k-l}_\epsilon V^j_0(X^{(0)}_u, 0)du \\ &\quad + \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right)\partial^\beta_{\vec{d}_\beta} V^j(X^{(0)}_u)dW_u \end{aligned} \tag{3}$$

where $\partial^l_\epsilon = \frac{\partial^l}{\partial \epsilon^l}$, $\partial^\beta_{\vec{d}_\beta} = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$,

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} := \sum_{\beta=1}^{l}\sum_{\vec{l}_\beta\in L_{l,\beta}}\sum_{\vec{d}_\beta\in\{1,\cdots,d\}^\beta}, \tag{4}$$

and

$$L_{l,\beta} := \left\{\vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = l; (l, l_j, \beta \in \mathbb{N})\right\}. \tag{5}$$

[2] They can be expressed as the finite sum of iterated multiple Wiener -$It\hat{o}$ integrals.

- **Then, g_{0T} and g_{1T} can be written as**

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{j=1}^{d} \partial_j g(X_T^{(0)}) A_{1T}^{j}. \end{aligned}$$

- **For $n \geq 2$, $g_{nT} = \frac{1}{n!} \left.\frac{\partial^n g(X_T^{(\epsilon)})}{\partial \epsilon^n}\right|_{\epsilon=0}$ is expressed as follows:**

$$g_{nT} = \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(n)} \frac{1}{\beta!} \partial_{\vec{d}_\beta}^{\beta} g(X_T^{(0)}) A_{l_1 T}^{d_1} \cdots A_{l_\beta T}^{d_\beta} \tag{6}$$

- **Next, normalize** $g(X_T^{(\epsilon)})$ **to**

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$**. Then,**

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p **for every** $p > 1$**.**

- **Moreover, let**

$$\hat{V}(x,t) = (\partial g(x))' [Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\textbf{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)}, t)\hat{V}(X_t^{(0)}, t)' dt > 0.$$

- **Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by**

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right)$$

where

$$C = (\partial g(X_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt.$$

- **Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate.**

- **Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing C^{∞}-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space.**
- **Next, take $\Phi \in \mathcal{S}'$. Then, the asymptotic expansion of a generalized Wiener functional $\Phi(G^{(\epsilon)})$ as $\epsilon \downarrow 0$ can be verified by Watanabe theory. (e.g. Watanabe(1987))**
- **In particular, if we take the delta function at $x \in \mathbf{R}$, δ_x as Φ, we obtain an asymptotic expansion of the density for $G^{(\epsilon)}$.**

- **The expectation of $\Phi(G^{(\epsilon)})$ is expanded as follows:**

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\mathbf{E}\left[\Phi^{(\delta)}(g_{1T})\prod_{j=1}^{\delta} g_{(k_j+1)T}\right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\int_{\mathbf{R}} \Phi^{(\delta)}(x) \\
&\quad \times \mathbf{E}\left[X^{\vec{k}_\delta} | g_{1T} = x\right] f_{g_{1T}}(x)dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\int_{\mathbf{R}} \Phi(x)(-1)^{\delta} \\
&\quad \times \frac{d^\delta}{dx^\delta}\left\{\mathbf{E}\left[X^{\vec{k}_\delta} | g_{1T} = x\right] f_{g_{1T}}(x)\right\} dx + o(\epsilon^N) \qquad (7)
\end{aligned}$$

where $\Phi^{(\delta)}(g_{1T}) = \left.\frac{d^\delta \Phi(x)}{dx^\delta}\right|_{x=g_{1T}}$**,** $\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n}\sum_{\vec{k}_\delta \in L_{n,\delta}}$**, and**

$$X^{\vec{k}_\delta} := \prod_{j=1}^{\delta} g_{(k_j+1)T}. \qquad (8)$$

Computation of the Asymptotic Expansion

- **To compute the asymptotic expansion (7), we need to evaluate the conditional expectations of the form**

$$E\left[X^{\vec{k}_\delta}\middle| g_{1T} = x\right]$$

where $X^{\vec{k}_\delta}$ is represented by a product of multiple Wiener-Itô integrals.

- **Previous works(e.g. T[1995,1999]) provide the conditional expectation formulas necessary for the expansions up to the third order.**
- **T-Takehara-Toda[2009] shows a general scheme for deriving formulas for the higher order expansions.**

Computation of the Asymptotic Expansion

- **Takahashi- Toda[2009] introduces an alternative but equivalent computational algorithm for an asymptotic expansion.**
  - **We compute the unconditional expectations instead of the conditional ones by deriving a system of ordinary differential equations which the expectations satisfy.**
  - **Thus, we are able to derive high order approximation formulas in an automatic manner.**

- **The next theorem shows a general result for an asymptotic expansion of the density function for $G^{(\epsilon)}$.**
- **In particular, the coefficients in the expansion are obtained through the solution of a system of ordinary differential equations(ODEs).**
- **Each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. Hence, one can easily solve (analytically or numerically) the system of ODEs.**

Theorem 1: The asymptotic expansion of the density function

The asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g(X_T^{(0)})}{\epsilon}$ up to ϵ^N-order is given by

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= f_{g_{1T}}(x) \\ &\quad + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x - C, \Sigma_T) \right) f_{g_{1T}}(x) + o(\epsilon^N), \end{aligned} \tag{9}$$

where $H_n(x;\Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x;\Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}, \tag{10}$$

and

$$\begin{aligned} C_{nm} &= \frac{1}{\Sigma_T^m} \sum_{\delta=1}^{m} \sum_{\vec{k}_\delta \in L_{n,\delta}} \sum_{\vec{l}^1_{\beta_1}, \vec{d}^1_{\beta_1}}^{(k_1+1)} \cdots \sum_{\vec{l}^\delta_{\beta_\delta}, \vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)} \frac{1}{\delta!(m-\delta)!} \\ &\quad \left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!} \partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)}) \right) \frac{1}{i^{m-\delta}} \frac{\partial^{m-\delta}}{\partial \xi^{m-\delta}} \eta^{\vec{d}^1_{\beta_1} \otimes \cdots \otimes \vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1} \otimes \cdots \otimes \vec{l}^\delta_{\beta_\delta}}(T;\xi) \Big|_{\xi=0}, \; \left(i = \sqrt{-1}\right). \end{aligned} \tag{11}$$

Theorem 1(continued)

$\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ **are obtained as a solution to the following system of ODEs:**

$$
\begin{aligned}
\frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\partial_\epsilon^{l_k}V_0^{d_k}(X_t^{(0)},0) \\
&\quad+\sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{d}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{d}_\gamma}\partial_\epsilon^{l_k-l}V_0^{d_k}(X_t^{(0)},0) \\
&\quad+\sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{d}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{d}_\gamma\otimes\vec{d}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi) \\
&\quad\times\partial^\gamma_{\vec{d}_\gamma}V^{d_k}(X_t^{(0)})\partial^\delta_{\vec{d}_\delta}V^{d_m}(X_t^{(0)}) \\
&\quad+(i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{d}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{d}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{d}_\gamma}V^{d_k}(X_t^{(0)})\hat{V}(X_t^{(0)},t) \\
\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi) &= 0,\ (\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset),\ \left(\eta^{(\emptyset)}_{(\emptyset)}(t;\xi)=1\right).
\end{aligned} \tag{12}
$$

Theorem 1(continued)

- **Here, we use the following notations:**

$$\begin{aligned}
\vec{l}_{\beta/k} &:= (l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_\beta) \\
\vec{l}_{\beta/k,n} &:= (l_1, \cdots, l_{k-1}, l_{k+1}, \cdots, l_{n-1}, l_{n+1}, \cdots, l_\beta), \ 1 \le k < n \le \beta \\
\vec{l}_\beta \otimes \vec{m}_\gamma &:= (l_1, \cdots, l_\beta, m_1, \cdots, m_\gamma)
\end{aligned}$$

for $\vec{l}_\beta = (l_1, \cdots, l_\beta)$ **and** $\vec{m}_\gamma = (m_1, \cdots, m_\gamma)$**.**

Some Comments

- **Due to the hierarchical structure of the ODEs with respect to $l = \sum_{j=1}^{\beta} l_j$ and $\eta^{(\emptyset)}_{(\emptyset)}(t;\xi) = 1$, one can easily solve these ODEs successively from lower order terms to higher order terms with initial conditions $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi) = 0$ for $(\vec{l}_\beta, \vec{d}_\beta) \neq (\emptyset, \emptyset)$.**
- **For instance, $\eta^{j}_{(1)}$, $\eta^{j,k}_{(1,1)}$ and $\eta^{j}_{(2)}$ are evaluated in the following order:**

$$\eta^{j}_{(1)} \to \eta^{j,k}_{(1,1)} \to \eta^{j}_{(2)}.$$

- **Different approximation formulas are obtained (T-Toda [2009,2012])**
  (e.g. the limiting distribution: Normal, Log-Normal, Shifted Log-Normal, Non-central Chi-square)
  through change of variables of $X^{(\epsilon),j}$
  or/and
  the different way to setting the perturbation parameter ϵ;
  $(V_0^j(X_s^{(\epsilon)},), \epsilon V_0^j(X_s^{(\epsilon)}), \epsilon^2 V_0^j(X_s^{(\epsilon)}) \cdots)$

Asymptotic Expansion of Option Price

- **We consider a plain vanilla call option on $g(X_T^{(\epsilon)})$.**
- **An asymptotic expansion of a call option price with maturity T and strike price $K = X_T^{(0)} - \epsilon y$ is given by**

$$C(K,T) = \epsilon P(0,T)\int_{-y}^{\infty}(x+y) f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}),$$

where,

  - $P(0,T)$ **: the price at time 0 of a zero coupon bond with maturity T**
  - $f_{G^{(\epsilon)},N}$ **: the asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order.**

- **Integrals are calculated by the formulas:**

$$\int_{-y}^{\infty}(x+y)H_k(x;\Sigma) f_{g_{1T}}(x)dx = \Sigma^2 H_{k-2}(-y;\Sigma) f_{g_{1T}}(y).$$

Applications

Some existing applications are as follows:

- **Calibration and pricing cap/floor and swaption under CEV-Heston LMM and Quadratic-Heston LMM.**
  (Shiraya-T-Yamazaki [2011])
- **Calibration and pricing under double-Heston model (T-Yamada [2012])**
- **Pricing continuous and discrete barrier options.**
  (Shiraya-T-Toda [2012]: pricing barrier options with continuous monitoring (by static hedging).
  Shiraya-T-Yamada [2011]: pricing barrier options with discrete monitoring.
  Kato-T-Yamada [2012]: pricing barrier options with continuous monitoring (PDE approach).)

Applications(Continued)

- **Pricing discrete average options.**
  (Shiraya-T [2011,2012].: pricing WTI average options based on calibration to the relevant futures (vanilla) option market.)
- **Pricing currency options under a market model of interest rates and a general diffusion stochastic volatility model with jumps of spot exchange rates. (T- Takehara [2010], T- Takehara-Toda [2009])**
- **Pricing currency basket and cross-currency average options based on calibration to the relevant (vanilla) option markets. (Shiraya-T [2012])**

Numerical Example: SABR/λ-SABR model

- **For numerical examples, we take SABR/λ-SABR model (Hagan-Kumar-Lesniewski-Woodward [2002], Henry-Labordère [2008]).**
- **We consider the European plain-vanilla, average and basket call/put prices under the following λ-SABR/SABR ($\lambda = 0$) model (interest rate=0%, for simplicity) :**

$$\begin{aligned} dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta} dW_t^1, \\ d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2, \end{aligned}$$

where $\nu_1 = \rho\nu$**,** $\nu_2 = (\sqrt{1-\rho^2})\nu$**.**

- **(Remark) We can modify the model to have smooth and bounded coefficients in order that regularity conditions for the expansion are satisfied: practically, it can be regarded as the original model, and its approximation formula can be the same.**

Numerical Example: SABR/λ-SABR model

- **Approximated prices by the asymptotic expansion method are calculated up to the fifth/fourth order.**
- **Note that all the solutions to differential equations are obtained analytically.**
- **Benchmark values are computed by Monte Carlo simulations with 10^8 samples.**
- **In the SABR ($\lambda = 0$) case, we also calculate the approximated price by Hagan-Kumar-Lesniewski-Woodward [2002] formula to compare with our method.**

Numerical Example: Plain-Vanilla Option (SABR model)

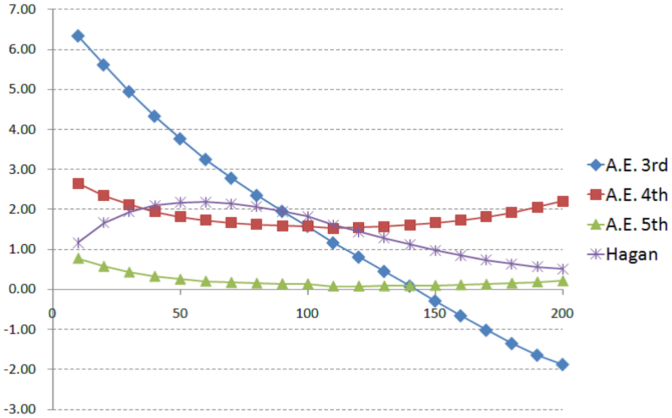


Figure: Approximation errors of ATM/OTM option prices with several strikes.
$S_0 = 100$, $K = 10 \sim 200$, $\beta = 0.5$ $\sigma_0 = 3$, $\nu = 0.3$, $\rho = -0.7$, $\epsilon = 1$, $T = 10$.

Numerical Example: Average Option (λ-SABR)

Shiraya-Takahashi-Toda Payoffs of Average Call($C_A^{(\epsilon)}(K,T)$) and Put($P_A^{(\epsilon)}(K,T)$) :

$$
\begin{aligned}
C_A^{(\epsilon)}(K,T) &= \mathbf{E}\left[\max\left\{\frac{1}{T}\int_0^T S^{(\epsilon)}(u)du - K, 0\right\}\right] \\
P_A^{(\epsilon)}(K,T) &= \mathbf{E}\left[\max\left\{K - \frac{1}{T}\int_0^T S^{(\epsilon)}(u)du, 0\right\}\right]
\end{aligned}
$$

Parameters :

Model	$S(0)$	β	$\sigma(0)$	λ	θ	ν	ρ	T
λ-SABR	100	1.0	0.3	1.0	0.3	0.3	-0.5	2

Numerical Example: Average Option (λ-SABR)

Asymptotic expansions for average option prices under the λ-SABR model up to the fourth order

Model	Strike(C/P)	MC Price(std. error)	A.E. Price(Difference) 1st	2nd	3rd	4th
λ-SABR	50 Put	0.024 (0.000)	0.162	-0.076	-0.001	0.001
	80 Put	2.251 (0.001)	0.609	0.060	0.004	0.003
	100 Call	9.685 (0.002)	0.088	0.088	0.001	0.001
	120 Call	3.348 (0.001)	-0.488	0.061	0.005	0.006
	150 Call	0.495 (0.000)	-0.309	-0.071	0.004	0.002

Numerical Example: Basket Option (λ-SABR)

- **We consider the valuation of basket options with the following payoff ($C_B^{(\epsilon)}(K,T)$):**

$$C_B^{(\epsilon)}(K,T) \quad = \quad \max\left\{\hat{S}^{(\epsilon)}(T) - K, 0\right\},$$

  where $\hat{S}^{(\epsilon)}(t) = \sum_{i=1}^{100} S_i^{(\epsilon)}(t)$.
- **Model of each S_i :λ-SABR with $\hat{S}^{(\epsilon)}(0) = 10,000$,** $\lambda_i = 1,\ \beta_i = 0.5,\ \rho_i = -0.4$. **(As for the other parameters, please see Shiraya-Takahashi [2012] for the details.)** [3]

[3] The average of $\sigma_i(0)$ and that of θ_i are 15%, and the average of ν_i is 30%. The correlation between two different asset prices(volatilities) is 0.8(0.3).

Numerical Example: Basket Option (λ-SABR)

Numerical Example: Basket Option with 100 Underlying Assets [4]

Table: Basket Call Option ($T = 1$)

Strike(K)	8,000	9,000	10,000(ATM)	11,000	12,000
Monte Carlo	2,037.1	1,167.5	517.6	160.8	31.7
AE3rd	2,037.4	1,167.6	517.6	160.5	31.5
Difference	0.3	0.2	-0.0	-0.2	-0.2
Relative Difference (%)	0.0%	0.0%	0.0%	-0.2%	-0.7%
MC Std Error	0.7	0.6	0.4	0.2	0.2

Monte Carlo: the number of trials is 3 million with the antithetic variable method.

[4] Please see Alos-Bayer-Laurence [2011] for other fast accurate analytical method for pricing basket options with 100 underlying assets, for instance.

FBSDE Approximation Scheme

- **The forward backward stochastic differential equations (FBSDEs) have been found particularly relevant for various valuation problems (e.g. pricing securities under the existence of relevant parties' credit risks, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets).**
- **Their financial applications are discussed in details for example, El Karoui, Peng and Quenez [1997], Ma and Yong [2000], a recent book edited by Carmona [2009], Crépey [2011], and references therein.**
- **The importance of FBSDEs is expected to increase in coming years since the new financial regulations will put significant constraints on available assets and trading strategies.**

FBSDE Approximation Scheme- Setup-

Fujii-Takahashi[2011]

- **The probability space is taken as $(\Omega, \mathcal{F}, P)$ and $T \in (0, \infty)$ denotes some fixed time horizon. $W_t = (W_t^1, \cdots, W_t^r)^*$, $0 \le t \le T$ is $\mathbb{R}^r$-valued Brownian motion defined on $(\Omega, \mathcal{F}, P)$, and $(\mathcal{F}_t)_{\{0 \le t \le T\}}$ stands for P-augmented natural filtration generated by the Brownian motion.**
- **We consider the following forward-backward stochastic differential equation (FBSDE):**

$$dV_t = -f(X_t, V_t, Z_t)dt + Z_t \cdot dW_t \tag{13}$$

$$V_T = \Phi(X_T), \tag{14}$$

where V takes the value in $\mathbb{R}$, and $X_t \in \mathbb{R}^d$ is assumed to follow a diffusion which is the solution to the (forward) SDE:

$$dX_t = \gamma_0(X_t)dt + \gamma(X_t) \cdot dW_t; \;\; X_0 = x \, . \tag{15}$$

FBSDE Approximation Scheme- Setup-

- $\Phi(X_T)$ **denotes the terminal payoff where** $\Phi(x)$ **is a deterministic function of** x**.**
- Z **and** γ **take values in** $\mathbb{R}^r$ **and** $\mathbb{R}^{d\times r}$ **respectively.**
- **We assume that the appropriate regularity conditions are satisfied for the necessary treatments.**

Perturbative Expansion for Non-linear Generator

- **In order to solve the pair of (V_t, Z_t) in terms of X_t, we extract the linear term from the generator f and treat the residual non-linear term as the perturbation to the linear FBSDE.**
- **We introduce the perturbation parameter ϵ, and then write the equation as**

$$dV_t^{(\epsilon)} = c(X_t)V_t^{(\epsilon)}dt - \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \quad (16)$$

$$V_T^{(\epsilon)} = \Phi(X_T)\,, \quad (17)$$

where $\epsilon = 1$ corresponds to the original model by [5]

$$f(X_t, V_t, Z_t) = -c(X_t)V_t + g(X_t, V_t, Z_t)\,. \quad (18)$$

[5] Or, one can consider $\epsilon = 1$ as simply a parameter convenient to count the approximation order. The actual quantity that should be small for the approximation is the residual part g.

Perturbative Expansion for Non-linear Generator

- **Usually, $c(X_t)$ corresponds to the risk-free interest rate at time t, but it is not a necessary condition. One should choose the linear term in such a way that the residual non-linear term becomes as small as possible to achieve better convergence.**
- **Now, we are going to expand the solution of BSDE (16) and (17) in terms of ϵ: that is, suppose $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as**

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{19}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots . \tag{20}$$

Perturbative Expansion for Non-linear Generator

- **Once we obtain the solution up to the certain order, say k for example, then by putting $\epsilon = 1$,**

$$\tilde{V}_t = \sum_{i=0}^{k} V_t^{(i)}, \qquad \tilde{Z}_t = \sum_{i=0}^{k} Z_t^{(i)} \tag{21}$$

  is expected to provide a reasonable approximation for the original model as long as the residual term is small enough to allow the perturbative treatment.
- **As we will see, $V_t^{(i)}$ and $Z_t^{(i)}$, the corrections to each order can be calculated recursively using the results of the lower order approximations.**

Recursive Approximation

Zero-th Order

- **For the zero-th order of ϵ, one can easily see the following equation should be satisfied:**

$$dV_t^{(0)} = c(X_t)V_t^{(0)}dt + Z_t^{(0)} \cdot dW_t \tag{22}$$

$$V_T^{(0)} = \Phi(X_T)\,. \tag{23}$$

- **It can be integrated as**

$$V_t^{(0)} = E\left[e^{-\int_t^T c(X_s)ds}\Phi(X_T)\middle|\mathcal{F}_t\right] \tag{24}$$

  which is equivalent to the pricing of a standard European contingent claim, and $V_t^{(0)}$ is a function of X_t.

- **Applying Itô's formula (or Malliavin derivative), we obtain $Z_t^{(0)}$ as a function of X_t, too.**

Recursive Approximation

First Order

- Now, let us consider the process $V^{(\epsilon)} - V^{(0)}$. One can see that its dynamics is governed by

$$\begin{aligned} d(V_t^{(\epsilon)} - V_t^{(0)}) &= c(X_t)(V_t^{(\epsilon)} - V_t^{(0)})dt \\ &- \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + (Z_t^{(\epsilon)} - Z_t^{(0)}) \cdot dW_t \\ V_T^{(\epsilon)} - V_T^{(0)} &= 0\,. \end{aligned} \tag{25}$$

- Now, by extracting the ϵ-first order term, we can once again recover the linear FBSDE

$$\begin{aligned} dV_t^{(1)} &= c(X_t)V_t^{(1)}dt - g(X_t, V_t^{(0)}, Z_t^{(0)})dt + Z_t^{(1)} \cdot dW_t \\ V_T^{(1)} &= 0\,, \end{aligned} \tag{26}$$

which leads to

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} g(X_u, V_u^{(0)}, Z_u^{(0)})du \middle| \mathcal{F}_t\right]. \tag{27}$$

Recursive Approximation

First Order

- **Because $V_u^{(0)}$ and $Z_u^{(0)}$ are some functions of X_u, we obtain $Z_t^{(1)}$ as a function of X_t through Itô's formula (or Malliavin derivative).**
- **From these results, we can see that the required calculation is nothing more difficult than the zero-th order case as long as we have explicit expression for $V^{(0)}$ and $Z^{(0)}$.**

Recursive Approximation

Second and Higher Order Corrections

- **We can proceed the same way to the second order correction.**
- **By extracting the ϵ-second order terms from $V_t^{(\epsilon)} - (V_t^{(0)} + \epsilon V_t^{(1)})$, one can show that**

$$\begin{aligned} dV_t^{(2)} &= c(X_t)V_t^{(2)}dt - \left(\frac{\partial}{\partial v}g(X_t, V_t^{(0)}, Z_t^{(0)})V_t^{(1)} \right. \\ &+ \left. \nabla_z g(X_t, V_t^{(0)}, Z_t^{(0)}) \cdot Z_t^{(1)}\right) dt + Z_t^{(2)} \cdot dW_t \\ V_T^{(2)} &= 0 \end{aligned} \tag{28}$$

is a relevant FBSDE, which is once again linear in $V_t^{(2)}$.

Recursive Approximation

Second and Higher Order Corrections

- **As before, it leads to the following expression straightforwardly:**

$$
\begin{aligned}
V_t^{(2)} &= E\Bigg[\int_t^T e^{-\int_t^u c(X_s)ds}\left(\frac{\partial}{\partial v} g(X_u, V_u^{(0)}, Z_u^{(0)}) V_u^{(1)}\right. \\
&+ \left.\nabla_z g(X_u, V_u^{(0)}, Z_u^{(0)}) \cdot Z_u^{(1)}\right) du \Bigg| \mathcal{F}_t \Bigg].
\end{aligned}
$$

Also, $Z_t^{(2)}$ is obtained through It$\hat{o}$'s formula (or Malliavin derivative).

- **In exactly the same way, one can derive an arbitrarily higher order correction. Due to the ϵ in front of the non-linear term g, the system remains to be linear in the every order of approximation.**

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Suppose we have succeeded to express backward components (V_t, Z_t) in terms of X_t up to the $(i-1)$-th order. Now, in order to proceed to a higher order approximation, we have to give the following form of expressions with some deterministic function $G(\cdot)$ in terms of the forward components X_t, in general:**

$$V_t^{(i)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} G(X_u)du \middle| \mathcal{F}_t\right] \tag{29}$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Even if it is impossible to obtain the exact result, we can still obtain analytic approximation for $(V_t^{(i)}, Z_t^{(i)})$.**
- **In principle, The asymptotic expansion technique described before allows us to obtain the expression.**
- **Fujii-T [2011] have provided some explicit expressions for $V_t^{(i)}$ and $Z_t^{(i)}$.**
- **Fujii-T [2012] have explicitly derived an approximation formula for the dynamic optimal portfolio in an incomplete market and confirmed its accuracy comparing with the exact result by Cole-Hopf transformation. (Zariphopoulou [2001])**

Remark on Approximation of Coupled FBSDEs

- **Let us consider the following generic *coupled* non-linear FBSDE:**

$$\begin{aligned} dV_t &= -f(t, X_t, V_t, Z_t)dt + Z_t \cdot dW_t \\ V_T &= \Phi(X_T) \\ dX_t &= \gamma_0(t, X_t, V_t, Z_t)dt + \gamma(t, X_t, V_t, Z_t) \cdot dW_t; \;\; X_0 = x \,. \end{aligned}$$

- **We can treat this case in the similar way as before(decoupled case) by introducing the following perturbation to the forward process:**

$$\begin{aligned} d\tilde{V}_t &= c(t, \tilde{X}_t)\tilde{V}_t dt - \epsilon g(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)dt + \tilde{Z}_t \cdot dW_t \\ \tilde{V}_T &= \Phi(\tilde{X}_T) \\ d\tilde{X}_t &= \Big(r(t, \tilde{X}_t) + \epsilon\mu(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big)dt \\ &\qquad + \Big(\sigma(t, \tilde{X}_t) + \epsilon\eta(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big) \cdot dW_t \end{aligned}$$

- **We can also apply the same method under PDE(partial differential equation) formulation based on *four step scheme* (e.g. Ma-Yong [2000]).**

Please consult Fujii-T [2011] for the details.

A forward agreement with bilateral default risk

- **As the first example, we consider a toy model for a forward agreement on a stock with bilateral default risk of the contracting parties, the investor (party-1) and its counter party (party-2). The terminal payoff of the contract from the view point of the party-1 is**

$$\Phi(S_T) = S_T - K \tag{30}$$

where T is the maturity of the contract, and K is a constant.

- **We assume the underlying stock follows a simple geometric Brownian motion:**

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{31}$$

where the risk-free interest rate r and the volatility σ are assumed to be positive constants.

- **The default intensity of party-i h_i is specified as**

$$h_1 = \lambda, \qquad h_2 = \lambda + h \tag{32}$$

where λ and h are also positive constants.

A forward agreement with bilateral default risk

- In this setup, the pre-default value of the contract at time t, V_t, follows

$$\begin{aligned} dV_t &= rV_t dt - h_1 \max(-V_t, 0)dt + h_2 \max(V_t, 0)dt + Z_t dW_t \\ &= (r+\lambda)V_t dt + h \max(V_t, 0)dt + Z_t dW_t \qquad (33) \\ V_T &= \Phi(S_T) \,. \qquad (34) \end{aligned}$$

- Now, following the previous arguments, let us introduce the expansion parameter ϵ, and consider the following FBSDE:

$$\begin{aligned} dV_t^{(\epsilon)} &= \mu V_t^{(\epsilon)} dt - \epsilon g(V_t^{(\epsilon)})dt + Z_t^{(\epsilon)} dW_t \qquad (35) \\ V_T^{(\epsilon)} &= \Phi(S_T) \qquad (36) \\ dS_t &= S_t(rdt + \sigma dW_t) \,, \qquad (37) \end{aligned}$$

where we have defined $\mu = r + \lambda$ and $g(v) = -hv\mathbf{1}_{\{v \geq 0\}}$.

A forward agreement with bilateral default risk

- **The next figure shows the numerical results of the forward contract with bilateral default risk with various maturities with the direct solution from the PDE (as in Duffie-Huang [1996]).**
- **We have used**

$$r = 0.02, \quad \lambda = 0.01, \quad h = 0.03, \tag{38}$$

$$\sigma = 0.2, \quad S_0 = 100\,, \tag{39}$$

where the strike K is chosen to make $V_0^{(0)} = 0$ for each maturity.

- **We have plot $V^{(1)}$ for the first order, and $V^{(1)} + V^{(2)}$ for the second order. (Note that we have put $\epsilon = 1$ to compare the original model.)**

A forward agreement with bilateral default risk

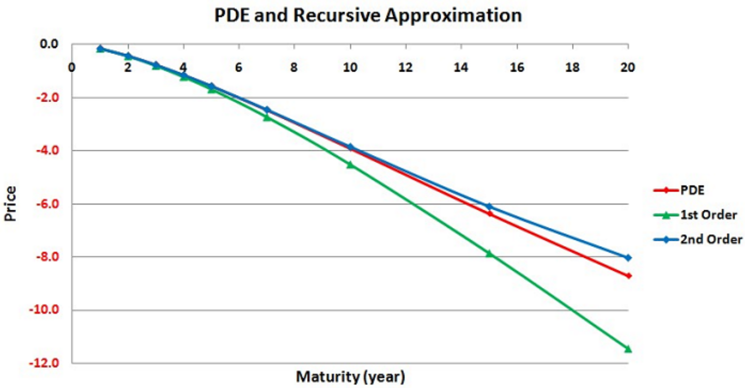


Figure: Numerical Comparison to PDE

A forward agreement with bilateral default risk

- **One can observe how the higher order correction improves the accuracy of approximation.**
- **In this example, the counter party is significantly riskier than the investor, and the underlying contract is quite volatile [6].**
- **Even in this situation, the simple approximation to the second order works quite well up to the very long maturity.**

[6] Of course, people rarely make such a risky contract to the counter party in the real market.

A self-financing portfolio with differential interest rates

- **As in Gobet-Lemor-Warin[2005], we consider the following valuation problem of self-financing portfolio under the situation where there exists a difference between the lending and borrowing interest rates. Here, we consider the problem under the physical measure.**

$$\begin{aligned} dV_t &= rV_t dt - \left\{ (R-r)\max\left(\frac{Z_t}{\sigma} - V_t, 0\right) - \theta Z_t \right\} dt + Z_t dW_t, \\ V_T &= \Phi(S_T) = \max(S_T - K_1, 0) - 2\max(S_T - K_2, 0), [7] \\ dS_t &= S_t\left(\mu dt + \sigma dW_t\right), \end{aligned} \tag{40}$$

where r and R are the lending and the borrowing rate, respectively; $\theta = (\mu - r)/\sigma$ denotes the market price of risk; r, R, μ and σ are all positive constants. Here, Z_t/σ represents the amount invested in the risky asset, i.e. stock S_t.

[7] This spread introduces both of the lending and borrowing activities.

A self-financing portfolio with differential interest rates

- **Let us introduce the expansion parameter as**

$$dV_t^{(\epsilon)} = rV_t^{(\epsilon)}dt - \epsilon g(V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)}dW_t \tag{41}$$

$$V_T^{(\epsilon)} = \Phi(S_T) , \tag{42}$$

where we have defined the non-linear perturbation function as

$$g(v, z) = (R - r)\max\left(\frac{z}{\sigma} - v, 0\right) - \theta z . \tag{43}$$

- **Now, we are going to expand $V_t^{(\epsilon)}$ in terms of ϵ.**

A self-financing portfolio with differential interest rates

- **Gobet-Lemor-Warin[2005] have carried out the detailed numerical study for the above problem using the regression-based Monte Carlo simulation.**
- **They have used**

$$\mu = 0.05, \quad \sigma = 0.2, \quad r = 0.01, \quad R = 0.06$$
$$T = 0.25, \quad S_0 = 100, \quad K_1 = 95, \quad K_2 = 105 \,.$$

After trying various sets of basis functions, they have obtained the price as $V_0 = 2.95$ with standard deviation 0.01.

A self-financing portfolio with differential interest rates

- **Now, let us provide the results from our perturbative expansion. Using the same model inputs, we have obtained**

$$\begin{aligned} V_0^{(0)} &= 2.7863 \\ V_0^{(1)} &= 0.1814 \\ V_0^{(2)} &= -0.0149. \end{aligned}$$

- **Thus, up to the first order, we have $V_0^{(0)} + V_0^{(1)} = 2.968$, which is already fairly close, and once we include the second order correction, we have $\sum_{i=0}^{2} V_0^{(i)} = 2.953$, which is perfectly consistent with their result of Monte Carlo simulation.**
- **Note that, we have derived analytic formulas with explicit expressions both for the contract value and its volatility.**

CVA

Fujii-Shiraya-Takahashi [2012]

- **When this technique is applied to evaluation of a pre-default contract value with bilateral counter party risk, Its first order approximation term can be regarded as CVA(credit value adjustment) [8].**
- **We present a simple example of an analytic approximation for this term by the asymptotic expansion method.**
- **In particular, we consider a forex forward contract with bilateral counter party risk, where both parties post their collateral perfectly with the constant time-lag (Δ) by the same currency as the payment currency. We also assume the risk-free interest rate is equal to the collateral rate.**

[8] Our convention of CVA is different from other literatures by sign where it is defined as the “charge” to the clients. Thus, our CVA = -CVA.

FBSDE

We consider a forward contract on forex S^{ϵ} with strike K and maturity τ; the relevant FBSDE for the pre-default contract value is given as follows: ($h^{j,\epsilon}$ ($j=1,2$): each counter party's hazard rate process; δ, ϵ: expansion parameters.)

$$dV_t^{\delta} = rV_t^{\delta}dt - \delta f(t, V_t^{\delta}, V_{t-\Delta}^{\delta})dt + Z_t^{\delta}dW_t;\ V_{\tau} = S_{\tau}^{\epsilon} - K, \tag{44}$$

$$f(t, V_t^{\delta}, V_{t-\Delta}^{\delta}) = h_t^{1,\epsilon}(V_{t-\Delta}^{\delta} - V_t^{\delta})^+ - h_t^{2,\epsilon}(V_t^{\delta} - V_{t-\Delta}^{\delta})^+ \tag{45}$$

$$\begin{aligned}
dh_t^{j,\epsilon} &= \phi_j(h_t^{j,\epsilon})dt + \epsilon\sigma_{h^j}g_j(h_t^{j,\epsilon})(\sum_{\eta=1}^{j} c_{j,\eta}dW_t^{\eta});\ h_0^{j,\epsilon} = h_0^j, (j=1,2) \\
dS_t^{\epsilon} &= \mu S_t^{\epsilon}dt + \epsilon g_4(v_t^{\epsilon})g_3(S_t^{\epsilon})S_t^{\epsilon}(\sum_{\eta=1}^{3} c_{3,\eta}dW_t^{\eta});\ S_0^{\epsilon} = s_0,\ \mu = r - r_f \\
dv_t^{\epsilon} &= \phi_3(v_t^{\epsilon})dt + \epsilon\xi g_4(v_t^{\epsilon})(\sum_{\eta=1}^{4} c_{4,\eta}dW_t^{\eta});\ v_0^{\epsilon} = v_0.
\end{aligned}$$

First order of δ

The first order equation is expressed as follows:

$$dV_t^1 = rV_t^1 dt - f(t, V_t^0, V_{t-\Delta}^0)dt + \sum_{\eta=1}^{4} Z_{t,\eta}^1 dW_t^\eta;\ V_\tau^1 = 0$$

Then, our CVA is represented by the following:

$$V_t^1 = \int_t^T e^{-r(u-t)} \mathbf{E}_t \left[f(u, V_u^0, V_{u-\Delta}^0) \right] du$$
$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\epsilon} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\epsilon} \cdot (V_u^0 - V_{u-\Delta}^0)^+,$$

where $V_{u-\Delta} = 0$ **when** $u < t + \Delta$**.**

$$\begin{aligned} V_u^0 &= e^{-r_f(\tau-u)} S_u^\epsilon - e^{-r(\tau-u)} K, \\ V_u^0 - V_{u-\Delta}^0 &= e^{-r_f(\tau-u)} S_u^\epsilon - e^{-r_f(\tau-u+\Delta)} S_{u-\Delta}^\epsilon - k(u; \Delta, r), \\ k(u; \Delta, r) &:= e^{-r(\tau-u)} (1 - e^{-r\Delta}) K. \end{aligned}$$

Example

Specifically, we apply the following local stochastic volatility model (S,v) to the underlying asset price that has correlations with both counter parties' hazard rate processes (h^1, h^2). ($\phi_j(x) = \alpha_t^j x$, $(j = 1,2)$, $\phi_3(x) = \kappa_t(\theta_t - x)$, $g_3(x) = \gamma_t x^{1-\beta}$ and $g_j(x) = x, (j = 1,2,4)$):

$$
\begin{aligned}
dh_t^{j,\epsilon} &= \alpha_t^j h_t^{j,\epsilon} dt + \epsilon\sigma_{h^j} h_t^{j,\epsilon} \Big(\sum_{\eta=1}^{j} c_{j,\eta} dW_t^{\eta}\Big);\ h_0^{j,\epsilon} = h_0^j, (j = 1,2) \\
dS_t^{\epsilon} &= \mu S_t^{\epsilon} dt + \epsilon\gamma_t v_t^{\epsilon} (S_t^{\epsilon})^{\beta} \Big(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^{\eta}\Big);\ S_0^{\epsilon} = s_0, \\
dv_t^{\epsilon} &= \kappa_t(\theta_t - v_t^{\epsilon}) dt + \epsilon\xi v_t^{\epsilon} \Big(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^{\eta}\Big);\ v_0^{\epsilon} = v_0.
\end{aligned}
$$

(ϵ is an expansion parameter.)

Example

- **We apply the asymptotic expansion method to evaluation of $IC(t,u) = e^{-r(u-t)}\mathbf{E}_t\left[f(u, V_u^0, V_{u-\Delta}^0)\right]$ up to the third order. Then, the value of CVA is approximated by**

$$CVA(t,\tau) = \int_t^\tau IC_{AE}(t,u)du + o(\epsilon^3). \tag{46}$$

- **Due to the analytical approximation of each $IC_{AE}(t,u)$, we have no problem in computation, which is very fast.**

Example

The parameters are set as follows:

- **parameters of h^1;**
  $h_0^1 = 0.02$, $\alpha^1 = -2\%$, $\sigma_{h1} = 20\%$.
- **parameters of h^2;**
  $h_0^2 = 0.01$, $\alpha^2 = 2\%$, $\sigma_{h2} = 30\%$.
- **parameters of S;**
  $S_0 = 10,000$, $r = \mu = 1\%$, $\beta = 1$, $\gamma = 1$.
- **parameters of ν;**
  $\nu_0 = 10\%$, $\kappa = 1$, $\theta = 20\%$, $\xi = 30\%$.
- **correlation matrix**

	h^1	h^2	S	ν
h^1	**1**	**0.5**	**-0.3**	**0.2**
h^2	**0.5**	**1**	**0.1**	**0.1**
S	**-0.3**	**0.1**	**1**	**-0.8**
ν	**0.2**	**0.1**	**-0.8**	**1**

Density of CVA

We show the density function of approximate CVA by the asymptotic expansion method with Monte Carlo simulation. The maturity of forward contract is τ, T denotes the future time when CVA is evaluated, and Δ denotes the lag of collateral.

- **maturity (τ): 5 years.**
- **strike: $10,000$.**
- **time step size: $\frac{1}{400}$year.**
- **the number of trials: 325,000 with antithetic variates.**

Procedure:

1. **implement Monte carlo simulation of the state variables (h^1, h^2, S, v) until T.**
2. **given each realization of the state variables, compute $IC_{AE}(T,u)$.**
3. **integrate $IC_{AE}(T,u)$ numerically with respect to the time parameter u from T to τ, and plot the values and their frequencies after normalization.**

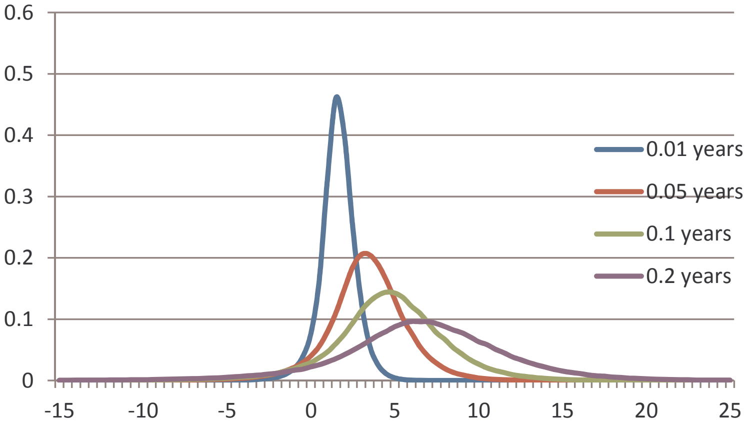
Density of CVA

The cases of simulations are

1. **Different Time-Lags**
   - **time-lag (Δ): 0.01, 0.05, 0.1 and 0.2 years.**
   - **evaluation date (T): 2.5 years.**
2. **Different Evaluation Dates**
   - **time-lag (Δ): 0.1 years.**
   - **evaluation date (T): 0.5, 1, 2.5, 4 and 4.5 years.**

Density of CVA

Figure: Density Functions of CVA with Different Time-Lags



Density of CVA

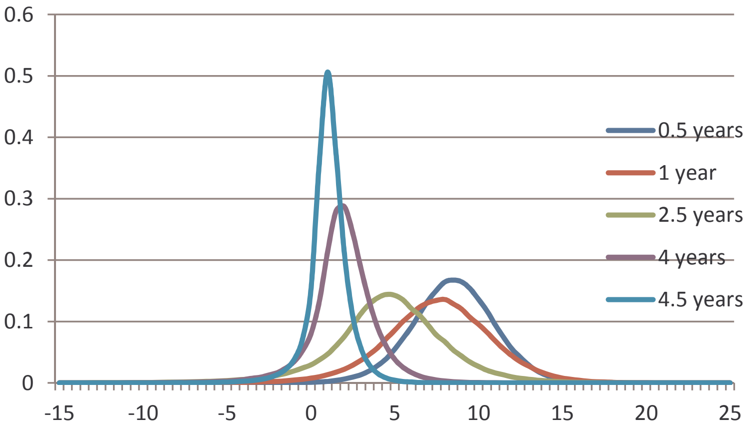
- **The longer the time lag is, the wider the density is.**
- **The mode (average) moves to the right when the time-lag becomes longer.**

$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\epsilon} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\epsilon} \cdot (V_u^0 - V_{u-\Delta}^0)^+.$$

  - **When the first term increases, the CVA also increases.**
  - **The hazard rate h^1 in the first term tends to be larger than h^2 in the second term in our parameterization.**

Density of CVA

Figure: Density Functions of CVA with Different Evaluation Dates



The shorter the time to maturity ($\tau - T$) becomes, CVA becomes smaller.

Perturbation Technique with Interacting Particle Method

Fujii-Takahashi[2012]

- **We consider the following forward-backward stochastic differential equation (FBSDE):**

$$\begin{aligned} dV_s &= -f(X_s, V_s, Z_s)ds + Z_s \cdot dW_s; \quad (47) \\ V_T &= \Psi(X_T), \quad (48) \end{aligned}$$

where V takes the value in $\mathbb{R}$, and $X_t \in \mathbb{R}^d$ is assumed to follow a generic Markovian forward SDE

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s; \;\; X_t = x_t \quad (49)$$

- **Again, let us introduce the perturbation parameter ϵ:**

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)})ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T). \end{cases} \quad (50)$$

Perturbation Technique with Interacting Particle Method

- **Let us fix the initial time as t. We denote the Malliavin derivative of X_u $(u \geq t)$ at time t as**

$$\mathcal{D}_t X_u \in \mathbb{R}^{r\times d} \tag{51}$$

- **Its dynamics in terms of the future time u is specified by *stochastic flow*, $(Y_{t,u})^i_j = \partial_{x_t^j} X^i_u$**

$$\begin{aligned} d(Y_{t,u})^i_j &= \partial_k \gamma^i_0(X_u)(Y_{t,u})^k_j du + \partial_k \gamma^i_a(X_u)(Y_{t,u})^k_j dW^a_u \\ (Y_{t,t})^i_j &= \delta^i_j \end{aligned} \tag{52}$$

where ∂_k denotes the differential with respect to the k-th component of X, and δ^i_j denotes Kronecker delta. Here, i and j run through $\{1, \cdots, d\}$ and $\{1, \cdots, r\}$ for a. Here, we adopt Einstein notation which assumes the summation of all the paired indexes.

- **Using the chain rule of Malliavin derivative, one sees**

$$\begin{aligned} (\mathcal{D}_t X^i_u)_a &= (Y_{t,u})^i_j \gamma^j_a(x_t) \\ &= (Y_{t,u}\gamma(x_t))^i_a, \end{aligned} \tag{53}$$

Perturbation Technique with Interacting Particle Method

- ϵ-0th order: For the zeroth order, it is easy to see

$$V_t^{(0)} = \mathbb{E}\Big[\Psi(X_T)\Big|\mathcal{F}_t\Big] \tag{54}$$

$$\begin{aligned} Z_t^{a(0)} &= \mathbb{E}\Big[\partial_i\Psi(X_T)(\mathcal{D}_t^a X_T^i)\Big|\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[\partial_i\Psi(X_T)(Y_{tT}\gamma(X_t))_a^i\Big|\mathcal{F}_t\Big] \end{aligned} \tag{55}$$

- It is clear that they can be evaluated by standard Monte Carlo simulation. However, for their use in higher order approximation, it is crucial to obtain explicit approximate expressions for these two quantities. We apply asymptotic expansion technique as before.
- In the following, let us suppose we have obtained the solutions up to a given order of asymptotic expansion, and write each of them as a function of x_t:

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t) \end{cases} \tag{56}$$

Perturbation Technique with Interacting Particle Method

- **ϵ-1st order:**

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbb{E}\Big[f(X_u, V_u^{(0)}, Z_u^{(0)}) \Big| \mathcal{F}_t \Big] du \\ &= \int_t^T \mathbb{E}\Big[f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big) \Big| \mathcal{F}_t \Big] du \end{aligned} \tag{57}$$

- **Next, define the new process for $(s > t)$:**

$$\hat{V}_{ts}^{(1)} = e^{\int_t^s \lambda_u du} V_s^{(1)}, \tag{58}$$

where deterministic positive process λ_t (It can be a positive constant for the simplest case.).

Perturbation Technique with Interacting Particle Method

- **Then, its dynamics is given by**

$$d\hat{V}^{(1)}_{ts} = \lambda_s \hat{V}^{(1)}_{ts} ds - \lambda_s \hat{f}_t(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) ds + e^{\int_t^s \lambda_u du} Z^{(1)}_s \cdot dW_s \ ,$$

where

$$\hat{f}_t(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)).$$

- **Since we have $\hat{V}^{(1)}_{tt} = V^{(1)}_t$, one can easily see the following relation holds:**

$$V^{(1)}_t = \int_t^T \mathbb{E}\left[e^{-\int_t^u \lambda_s ds} \lambda_u \hat{f}_t(X_u, v^{(0)}(X_u), z^{(0)}(X_u)) \middle| \mathcal{F}_t \right] du \qquad (59)$$

- **It is clear for those familiar with credit risk modeling (e.g. Bielecki-Rutkowski [2002]), it is nothing but the present value of default payment where the default intensity is λ with the default payoff at s as $\hat{f}_t(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, we obtain the following proposition.**

Perturbation Technique with Interacting Particle Method

Proposition

The $V_t^{(1)}$ in (57) can be equivalently expressed as

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_t\Big(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau)\Big)\Big|\mathcal{F}_t\right]. \tag{60}$$

Here τ is the interaction time where the interaction is drawn independently from Poisson distribution with an arbitrary deterministic positive intensity process λ_t. $\hat{f}$ is defined as

$$\hat{f}_t(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) \,. \tag{61}$$

Perturbation Technique with Interacting Particle Method

- **Now, let us consider the martingale component $Z^{(1)}$. It can be expressed as**

$$Z_t^{(1)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\right] du \tag{62}$$

Firstly, let us observe the dynamics of Malliavin derivative of $V^{(1)}$ follows

$$d(\mathcal{D}_t V_s^{(1)}) = -(\mathcal{D}_t X_s^i)\nabla_i(x, v^{(0)}, z^{(0)}) f(x, v^{(0)}, z^{(0)}) + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s; \tag{63}$$

$$\mathcal{D}_t V_t^{(1)} = Z_t^{(1)}, \tag{64}$$

where

$$\nabla_i(x, v^{(0)}, z^{(0)}) \equiv \partial_i + \partial_i v^{(0)}(x)\partial_v + \partial_i z^{a(0)}(x)\partial_{z^a}, \tag{65}$$

$$f(x, v^{(0)}, z^{(0)}) \equiv f(x, v^{(0)}(x), z^{(0)}(x)). \tag{66}$$

Perturbation Technique with Interacting Particle Method

- **Define, for $(s > t)$,**

$$\widehat{\mathcal{D}_t V_s^{(1)}} = e^{\int_t^s \lambda_u du}(\mathcal{D}_t V_s^{(1)}). \tag{67}$$

Then, its dynamics can be written as

$$\begin{aligned} d(\widehat{\mathcal{D}_t V_s^{(1)}}) &= \lambda_s(\widehat{\mathcal{D}_t V_s^{(1)}})ds - \lambda_s(\mathcal{D}_t X_s^i)\nabla_i(X_s, v^{(0)}, z^{(0)})\hat{f}_t(X_s, v^{(0)}, z^{(0)})ds \\ &\quad + e^{\int_t^s \lambda_u du}(\mathcal{D}_t Z_s^{(0)}) \cdot dW_s. \end{aligned} \tag{68}$$

We again have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)}. \tag{69}$$

- **Hence,**

$$Z_t^{(1)} = \int_t^T \mathbb{E}\left[e^{-\int_t^u \lambda_s ds} \lambda_s(\mathcal{D}_t X_u^i)\nabla_i(X_u, v^{(0)}, z^{(0)})\hat{f}_t(X_u, v^{(0)}, z^{(0)}) \middle| \mathcal{F}_t \right]. \tag{70}$$

Perturbation Technique with Interacting Particle Method

- **Thus, following the same argument for the proposition 1, we have the result below:**

Proposition

$Z_t^{(1)}$ in (62) is equivalently expressed as

$$Z_t^{a(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t\tau}\gamma(X_\tau))_a^i \nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_t(X_\tau, v^{(0)}, z^{(0)})\Big|\mathcal{F}_t\right] \tag{71}$$

where the definitions of random time τ and the positive deterministic process λ are the same as those in proposition 1.

Perturbation Technique with Interacting Particle Method

Monte Carlo Method

Now, we have a new particle interpretation of $(V^{(1)}, Z^{(1)})$ as follows:

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_t\Big(X_\tau, v^{(0)}, z^{(0)}\Big)\Big|\mathcal{F}_t\right] \tag{72}$$

$$Z_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_t(X_\tau, v^{(0)}, z^{(0)})\Big|\mathcal{F}_t\right] \tag{73}$$

which allows efficient time integration with the following Monte Carlo scheme:

• Run the diffusion processes of X and Y

• Carry out Poisson draw with probability $\lambda_s\Delta s$ at each time s and if "one" is drawn, set that time as τ.

• Then stores the relevant quantities at τ, or in the case of $(\tau > T)$ stores 0.

• Repeat the above procedures and take their expectation.

$Z_t^{(2)}$

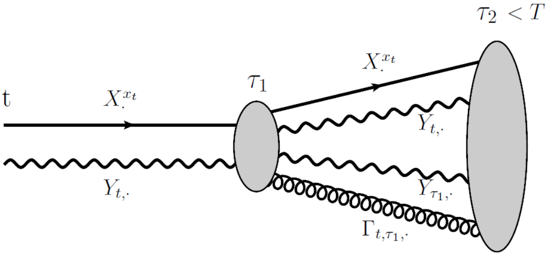


Figure 1: A particle interpretation for $Z_t^{(2)}$.

$$(\Gamma_{t,s,u})^i_{jk} \quad := \quad \frac{\partial^2}{\partial x_t^j \partial x_s^k} X_u^i; \ (t < s < u) \tag{74}$$

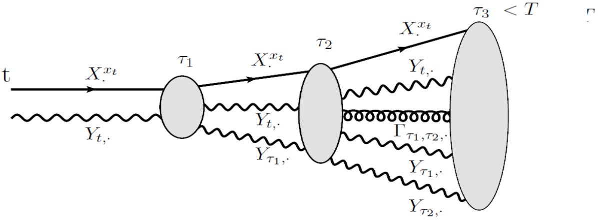


Figure 2: A particle interpretation for the first half of $V_t^{(3)}$.

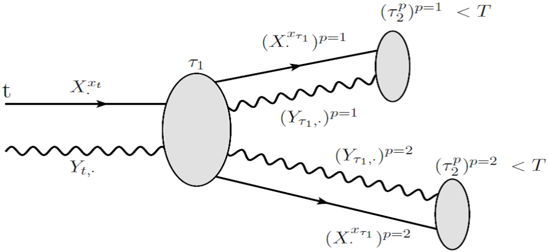


Figure 3: A particle interpretation for the second half of $V_t^{(3)}$.

Pricing American Option

Fujii-Sato-Takahashi [2012]

- **Let us take the probability space as $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is a risk-neutral probability measure. We consider a generic process for the relevant stock price as**

$$dS_t = (r_t - y_t)S_t dt + S_t \sigma_t \cdot dW_t, \tag{75}$$

where W is a d-dimensional $\mathbb{Q}$-Brownian motion. All the stochastic processes are $\mathcal{F}_t$ adapted. Here, r and y are processes for a risk-free interest rate and a dividend yield, respectively.

- **Based on the previous works for the early excercise premium of an American option (e.g. Kim [1990], Jacka [1991], Carr-Jarrow-Myneni [1992], Rutkowski [1994], Karatzas-Shreve [1998], Benth-Karlsen-Reikvam [2003]), we can express the dynamics of an American option value as the following BSDE:**

$$\begin{cases} dV_t = r_t V_t dt - C(S_t)\mathbf{1}_{\{V_t \leq \Psi(S_t)\}} dt + Z_t \cdot dW_t \\ V_T = \Psi^+(S_T) \end{cases} \tag{76}$$

Pricing American Option

- **Here, $Z \in \mathbb{R}^d$ is an appropriate $\mathcal{F}_t$-adapted process that should be solved at the same time with V. Here, we have replaced Ψ^+ by Ψ in the indicator function since V should be clearly positive.**
- $\Psi^+(x) = [\Psi(x)]^+$ **denotes a payoff function:**

$$\Psi(x) = \begin{cases} x - K & \textbf{for a Call} \\ K - x & \textbf{for a Put} \end{cases}$$

$C(S_t)$ **is a process denoting an instantaneous early exercise premium:**

$$C(S_t) = \begin{cases} (y_t S_t - r_t K)^+ & \textbf{for a Call} \\ (r_t K - y_t S_t)^+ & \textbf{for a Put} \end{cases}$$

Pricing American Option

- **Let us introduce perturbation parameter ϵ as**

$$\begin{cases} dV_t^{(\epsilon)} = r_t V_t^{(\epsilon)} dt - \epsilon C(S_t)\theta(\Psi(S_t) - V_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \\ V_T^{(\epsilon)} = \Psi^+(S_T) \end{cases} \tag{77}$$

where $\theta(\cdot)$ is the Heaviside step function. Since the dynamics of the stock price S and the other possible state processes are not affected by $(V^{(\epsilon)}, Z^{(\epsilon)})$, we have a decoupled non-linear FBSDE system to solve.

Pricing American Option

- **In the Black-Scholes model, the dynamics of stock is given by**

$$dS_t/S_t = (r - y)dt + \sigma dW_t \tag{78}$$

  where r, y and σ are all constants, the transition density is explicitly known.
- **We give the approximated prices of American options by our Monte Carlo method of $V^{(i)}$ up to the 2nd order. ($V^{(0)}$ is the corresponding Black-Scoles European price.)**
- **"Benchmark" denotes the American option price taken from Ju-Zhong [1999], which is obtained by $10,000$ time steps binomial tree methods.**

Pricing American Option -Monte Carlo-

S_0	Benchmark	ϵ-0th	ϵ-1st	ϵ-2nd
80	2.69	2.65	2.71	2.70
90	5.72	5.62	5.78	5.73
100	10.24	10.02	10.37	10.26
110	16.18	15.77	16.44	16.22
120	23.36	22.65	23.79	23.40

Table: American Calls with $T = 0.5$, $K = 100$, $r = 0.03$, $y = 0.07$, $\sigma = 0.40$. S_0 denotes the initial value of the stock. one million sample paths, 500 time steps.

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