On Robust Utility Indifference Valuation

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Introduction

Robust Utility Maximization with a Claim

- Robust Utility Maximization
  - S: semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (càdlàg, locally bounded).
  - $U : \mathbb{R} \to \mathbb{R}$, utility function: strictly concave & Inada condition.
  - Θ: admissible strategies,
    e.g., $\Theta_{bb} := \{\theta \in L(S, \mathbb{P}) : \theta_0 = 0, \ \theta \cdot S \text{ is bounded below}\}$
  - $\mathcal{P}$: convex set of $P \ll \mathbb{P}$ ($\Rightarrow \mathcal{P} \subset L^1(\mathbb{P})$): $\sigma(L^1, L^\infty)$-compact.
  - $B \in L^0$: DO NOT assume $B \in L^\infty$, but "reasonable integrability".

$$\text{maximize} \quad \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)], \quad \text{among } \theta \in \Theta.$$

- Background:
  - Model uncertainty;
  - Robust utility indifference prices.
- Aim: develop a general duality theory with
  - general utility on $\mathbb{R}$;
  - unbounded B.

Crush Course on Convex Duality ($\mathcal{P} = \{\mathbb{P}\}$)

- Set $V(y) := \sup_x (U(x) - xy)$ ($\Rightarrow U(x) \le V(y) + xy$).

$$\sup_{\theta \in \Theta_{bb}} E[U(\theta \cdot S_T + B)] \le E[V(Y) + YB] + \underbrace{\sup_{\theta \in \Theta_{bb}} E[Y\theta \cdot S_T]}_{=0 \text{ or } +\infty}, \ \forall Y \ge 0.$$

$$\sup_{\theta \in \Theta_{bb}} E[Y\theta \cdot S_T] = 0 \ \Leftrightarrow \ Y = \lambda dQ/d\mathbb{P}, \ \exists \lambda \ge 0, \ \exists Q \in \mathcal{M}_{loc}$$

$$\Rightarrow \quad \underbrace{\sup_{\theta \in \Theta_{bb}} E[U(\theta \cdot S_T + B)]}_{\text{primal problem}} \le \overbrace{\inf_{\lambda \ge 0} \underbrace{\inf_{Q \in \mathcal{M}_V}}_{\mathcal{M}_V = \{Q \in \mathcal{M}_{loc} : E[V(\lambda dQ/d\mathbb{P})] < \infty, \exists \lambda > 0\}} E\left[V\left(\lambda \frac{dQ}{d\mathbb{P}}\right) + \lambda \frac{dQ}{d\mathbb{P}} B\right]}^{\text{dual problem}}$$

- "$\ge$" if B is suitably integrable.
- Dual optimizer $(\hat{\lambda}, \hat{Q})$ exists, $\hat{Q} \sim \mathbb{P}$, and $\exists \hat{\theta}$: S-integrable:
  - $\hat{\theta} \cdot S$ is $\mathcal{M}_V$-superMG and $\hat{Q}$-MG, and $V'(\hat{\lambda} d\hat{Q}/d\mathbb{P}) + B = -\hat{\theta} S_T$.
  - $E[U(\hat{\theta} \cdot S_T + B)] = \sup_{\theta \in \Theta_{bb}} E[U(\theta \cdot S_T + B)]$.

Duality for Robust Utility Maximization

$$\text{maximize} \quad \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)], \quad \text{among } \theta \in \Theta_{bb}.$$

Can we get a nice "duality"? How??

- A Common Strategy: Use minimax, then subjective duality:

$$\sup_\theta \inf_P E_P[U(\theta \cdot S_T + B)] \stackrel{?}{=} \inf_P \sup_\theta E_P[U(\theta \cdot S_T + B)]$$
$$\stackrel{?}{=} \inf_P \inf_{\lambda, Q} E_P\left[V\left(\lambda \frac{dQ}{dP}\right) + \lambda \frac{dQ}{dP} B\right]$$

  - Refs: Quenez04, Schied/Wu05, Schied07, Hernández²/Scheid06,07, Gundel05, Föllmer/Gundel06, Wittmüss08, O.10a,b

- Alternative view: Analysis of Robust Utility Functional:

$$u_{B,\mathcal{P}}(X) := \inf_{P \in \mathcal{P}} E_P[U(X + B)], \quad X \in L^\infty.$$

Abstract Duality: Fenchel's Th. (Rockafellar's ver)

- $f : E \to \mathbb{R} \cup \{+\infty\}$, convex, $g : E \to \mathbb{R} \cup \{-\infty\}$, concave.

If either f or g is continuous at $\exists x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$,

$$\sup_{x \in E} (g(x) - f(x)) = \min_{y \in E^*} \underbrace{(f^*(y) - g_*(y))}_{f^*, g_*:\ \text{convex \& concave conjugates}}.$$

- $C \subset E$ convex cone, $f(x) = \delta_C(x) \Rightarrow$ convex.
- $f^*(y) = \sup_{x \in C} \langle x, y \rangle = \delta_{C^\circ}(y)$

$$C^\circ := \{y \in E^* : \langle x, y \rangle \le 0,\ \forall x \in C\} \quad \text{(polar cone)}.$$

- If g is continuous at $\exists x \in C$,

$$\sup_{x \in C} g(x) = \min_{y \in E^*} (-g_*(y) + \delta_{C^\circ}(y)) = \min_{y \in C^\circ} -g_*(y).$$

Our Idea

- Take $E = L^\infty \Rightarrow E^* = ba$.
- $\mathcal{C} := \{X \in L^\infty : \exists\theta \in \Theta_{bb} \text{ s.t. } X \le \theta \cdot S_T\}$: convex cone, $L^\infty_- \subset \mathcal{C}$,

$$\mathcal{C}^\circ \cap ba^\sigma = \{\lambda Q : \lambda \ge 0,\ Q \in \mathcal{M}_{loc}\}$$

$$u_{B,\mathcal{P}}(X) := \inf_{P\in\mathcal{P}} E_P[U(X + B)]$$

$$v_{B,\mathcal{P}}(\nu) := \sup_{X\in L^\infty} (u_{B,\mathcal{P}}(X) - \nu(X)) = -(u_{B,\mathcal{P}})_*(\nu).$$

$u_{B,\mathcal{P}}$ is conti. $\Rightarrow \sup_{X\in\mathcal{C}} u_{B,\mathcal{P}}(X) = \min_{\nu\in\mathcal{C}^\circ} v_{B,\mathcal{P}}(\nu) = \min_{\nu\in\mathcal{C}^\circ\cap\mathrm{dom}(v_{B,\mathcal{P}})} v_{B,\mathcal{P}}(\nu).$

- Remaining: $u_{B,\mathcal{P}}$ continuous? $v_{B,\mathcal{P}} = ??$
- Notations:

$$V(\nu|\mathcal{P}) := \inf_{P\in\mathcal{P}} V(\nu|P) := \inf_{P\in\mathcal{P}} \overbrace{E_P[V(d\nu/dP)]}^{:=+\infty \text{ if } \nu \not\ll P}$$

$$\mathcal{M}_V := \{Q \in \mathcal{M}_{loc} :\ V(\lambda Q|\mathcal{P}) < \infty,\ \exists\lambda > 0\}$$

- Assume $\mathcal{M}^e_V \neq \emptyset$.

Conjugate of Robust Utility Functional with B

Key Lemma

Under "suitable assumptions on B",

1. $u_{B,\mathcal{P}}$ is continuous on L^∞,
2. $v_{B,\mathcal{P}}(\nu) = \begin{cases} V(\nu|\mathcal{P}) + \nu(B) & \text{if } \nu\ \sigma\text{-additive},\ V(\nu|\mathcal{P}) < \infty \\ +\infty & \text{otherwise.} \end{cases}$

- Admitting the Key Lemma,

$$\sup_{X\in\mathcal{C}} u_{B,\mathcal{P}}(X) \overset{(1)}{=} \min_{\nu\in\mathcal{C}^\circ} v_{B,\mathcal{P}}(\nu) \overset{(2)}{=} \min_{\lambda>0, Q\in\mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]).$$

- Easy to verify:

$$\sup_{X\in\mathcal{C}} u_{B,\mathcal{P}}(X) \leq \sup_{\theta\in\Theta_{bb}} u_{B,\mathcal{P}}(\theta\cdot S_T) \leq \inf_{\lambda>0, Q\in\mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]).$$

$$\sup_{\theta\in\Theta_{bb}} \inf_{P\in\mathcal{P}} E_P[U(\theta\cdot S_T + B)] = \min_{\lambda>0, Q\in\mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]).$$

How to obtain the "Key Lemma"?

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$
  - $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$-measurable, $x \mapsto f(\omega, x)$ is convex, proper, lsc,
- Integral Functional

$$I_f(X) := E[f(\cdot, X)] = \int_\Omega f(\omega, X(\omega))\mathbb{P}(d\omega), \quad X \in L^\infty,$$

  - This is obviously a convex functional (whenever well-defined),
  - Regularity and the conjugate are obtained (Rockafellar's th.).
- Robust Version of Integral Functional

$$\mathcal{I}_{f,\mathcal{P}}(X) = \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)]$$

  - Still expected to be a nice convex function, but
  - How about the conjugate?
- Let $f(\omega, x) := -U(-x + B(\omega))$, then

$$u_{B,\mathcal{P}}(X) = -\mathcal{I}_{f,\mathcal{P}}(-X) \text{ and } v_{B,\mathcal{P}}(\nu) = (\mathcal{I}_{f,\mathcal{P}})^*(\nu).$$

Robust Version of Rockafellar Theorem

Integral Functionals and Rockafellar's Thm

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$-m'ble, $x \mapsto f(\omega, x)$ convex (lsc).

$$I_f(X) := E[f(\cdot, X)], \quad X \in L^0 \text{ (whenever make sense).}$$

- $f^*(\omega, y) := \sup_x (xy - f(\omega, x))$: ω-wise conjugate of f.
  - Automatically, $f^* \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$, lsc, proper, convex $\Rightarrow I_{f^*}$ makes sense.

Rockafellar (1971, *Pacific J. Math.*)

(R1) $\exists X \in L^\infty$ s.t. $f(\cdot, X)^+ \in L^1$;

(R2) $\exists Y \in L^1$ s.t. $f^*(\cdot, Y)^+ \in L^1$.

$\Rightarrow I_f$ (resp. I_{f^*}) well-defined, lsc, proper convex on L^∞ (resp. L^1),

$$(I_f)^*(\nu) = I_{f^*}(d\nu_r/d\mathbb{P}) + \sup_{X \in \mathrm{dom}(I_f)} \nu_s(X), \ \forall \nu = \underbrace{\nu_r + \nu_s}_{\text{regular \& singular parts}} \in ba,$$

- $f(\cdot, X)^+ \in L^1$, $\forall X \in L^\infty \Rightarrow$ 2nd term $= 0$ or ∞ & I_f is conti.

Robust Ver. of Rockafellar's Theorem

- $\mathcal{P}$: set of prob's $\ll \mathbb{P}$: convex & weakly compact.

Robust Ver. of Integral Functionals

$$\mathcal{I}_{f,\mathcal{P}}(X) := \sup_{P\in\mathcal{P}} E_P[f(\cdot, X)], \quad X \in L^\infty := L^\infty(\mathbb{P}).$$

- Regular enough? Conjugate?

$$(\mathcal{I}_{f,\mathcal{P}})^*(\nu) \overset{??}{=} \text{“}\mathcal{I}_{f^*,\mathcal{P}}(d\nu_r/d\mathbb{P})\text{”} + \sup_{X\in\mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})} \nu_s(X)$$

- Another kind of conjugate integrand:

$$\tilde{f}(\cdot, y, z) := (zf)^*(\cdot, y) = \sup_x (xy - zf(\cdot, y)),\ y \in \mathbb{R},\ z \geq 0.$$

- $\tilde{f}(\cdot, y, z) = zf^*(\cdot, y/z)$ if $z > 0$.

$$\mathcal{J}_{\tilde{f},\mathcal{P}}(Y) := \inf_{P\in\mathcal{P}} E[\tilde{f}(\cdot, Y, dP/d\mathbb{P})]$$

- $\mathcal{P} = \{\mathbb{P}\} \Rightarrow \tilde{f}(\cdot, Y, d\mathbb{P}/d\mathbb{P}) = \tilde{f}(\cdot, Y, 1) = f^*(\cdot, Y)$, hence $\mathcal{J}_{\{\mathbb{P}\},\tilde{f}} = I_{f^*}$.

Rockafellar-Type Theorem

$\mathcal{D} := \{X \in L^\infty : \{f(\cdot, X)^+ dP/d\mathbb{P}\}_{P \in \mathcal{P}} \text{ is UI}\} \subset \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})$ and assume:

(I1) $\mathcal{D} \neq \emptyset$;

(I2) $\forall P \in \mathcal{P}, \exists Y \in L^1$ s.t. $\tilde{f}(\cdot, Y, dP/d\mathbb{P})^+ \in L^1$.

Then $\mathcal{I}_{f,\mathcal{P}}$ (resp. $\mathcal{J}_{\tilde{f},\mathcal{P}}$) proper, convex & lsc on L^∞ (resp. L^1), and

$$\mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu_r/d\mathbb{P}) + \sup_{X \in \mathcal{D}} \nu_s(X) \leq (\mathcal{I}_{f,\mathcal{P}})^*(\nu) \leq \mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu_r/d\mathbb{P}) + \sup_{X \in \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})} \nu_s(X)$$

- When $\mathcal{P} = \{\mathbb{P}\}$, $\mathcal{D} = \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})$, hence equality.
- In general, $\mathcal{D} \subset \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})$, but the inclusion can be strict.
- But, this estimate is still useful.

Continued

$$\mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu_r/d\mathbb{P}) + \sup_{X\in\mathcal{D}} \nu_s(X) \leq (\mathcal{I}_{f,\mathcal{P}})^*(\nu) \leq \mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu_r/d\mathbb{P}) + \sup_{X\in \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}})} \nu_s(X)$$

- $x \mapsto f(\omega, x)$ convex & finite on the whole $\mathbb{R} \Rightarrow$ continuous.
  - f is non-random $\Rightarrow f(L^\infty) \subset L^\infty \Rightarrow \mathcal{D} = L^\infty$.
  - Whenever $\mathcal{D} = \mathrm{dom}(\mathcal{I}_{f,\mathcal{P}}) = L^\infty$, $\mathcal{I}_{f,\mathcal{P}}$ is conti. on L^∞, and

$$(\mathcal{I}_{f,\mathcal{P}})^*(\nu) = \begin{cases} \mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu/d\mathbb{P}) & \text{if } \nu \text{ is } \sigma\text{-additive} \\ +\infty & \text{otherwise.} \end{cases}$$

  - In general, the bound $\sup_{|x|\leq\|X\|_\infty} |f(\omega, x)|$ depends on ω.
- A sufficient condition for $\mathcal{D} = L^\infty$:
  (I1′) $\exists g \in C(\mathbb{R})$ & $\exists W \in L^0$ s.t. $\{WdP/d\mathbb{P}\}_{P\in\mathcal{P}}$ is UI &

$$f(\omega, x) \leq g(x) + W(\omega)$$

Analysis of Robust Utility Functional

What are "Suitable Conditions" on B?

- Let $f(\cdot, x) := -U(-x + B) \Rightarrow u_{B,\mathcal{P}}(X) = -\mathcal{I}_{f,\mathcal{P}}(-X)$ & $v_{B,\mathcal{P}} = \mathcal{I}^*_{f,\mathcal{P}}$.

$$f(\omega, x) \le -\frac{\varepsilon}{1+\varepsilon} U\left(-\frac{1+\varepsilon}{\varepsilon} x\right) - \frac{1}{1+\varepsilon} U(-(1+\varepsilon)B^-(\omega))$$

- (I1′) ($\Rightarrow \mathcal{D} = L^\infty$) if:

> (B^-) $\exists \varepsilon > 0$, $\{U(-(1+\varepsilon)B^-)dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ is UI.

- $\tilde{f}(\cdot, z, z) = zf^+(\cdot, 1) = z(V(1) + B)$, and

$$\begin{aligned}\frac{\varepsilon}{1+\varepsilon}(V(y) - V(1)) + U(-(1+\varepsilon)B^-) \\ \le V(y) + yB \le \frac{1+\varepsilon'}{\varepsilon'} V(y) - \frac{1}{\varepsilon'} U(-\varepsilon' B^+)\end{aligned} \qquad (*)$$

Taking $Y_P = dP/d\mathbb{P}$, $\tilde{f}(\cdot, Y_P, Y_P)^+ \in L^1$, $\forall P \in \mathcal{P}$ ($\Rightarrow$ (I2)) if:

> (B^+) $\forall P \in \mathcal{P}$, $\exists \varepsilon > 0$, $U(-\varepsilon B^+) \in L^1(P)$.

Robust Utility Functional

- $u_{B,\mathcal{P}}(X) = \inf_{P\in\mathcal{P}} E_P[U(X+B)]$: continuous?
- $v_{B,\mathcal{P}}(\nu) = \sup_{X\in L^\infty}(u_{B,\mathcal{P}}(X) - \nu(X)) = ??$

- Under (B^+) & (B^-):
  1. Rockafellar-Type Th. applies:

$$u_{B,\mathcal{P}} \text{ conti. } \& \ v_{B,\mathcal{P}}(\nu) = \begin{cases} \mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu/d\mathbb{P}) & \text{if } \nu \ \sigma\text{-additive}, \\ +\infty & \text{otherwise.} \end{cases}$$

  2. $\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P}) \in L^1 \Leftrightarrow V(\nu|P) < \infty$, hence $\forall \nu \in ba^\sigma_+$,

$$\mathcal{J}_{\tilde{f},\mathcal{P}}(d\nu/d\mathbb{P}) = \begin{cases} V(\nu|\mathcal{P}) + \nu(B) & \text{if } V(\nu|\mathcal{P}) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

Key Lemma (consequence of (B^-), (B^+)).

1. $u_{B,\mathcal{P}}$ is continuous on all of L^∞, & $\forall \nu \geq 0$,
2. $$v_{B,\mathcal{P}}(\nu) = \begin{cases} V(\nu|\mathcal{P}) + \nu(B) & \text{if } \nu \text{ is } \sigma\text{-add.}, V(\nu|\mathcal{P}) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

Duality in Robust Utility Maximization

via Analysis of Robust Utility Functional

List of Assumptions & Admissible Classes

- $U \in C^1(\mathbb{R} \to \mathbb{R})$, strictly concave with (Inada) $\lim_{x\to-\infty} U'(x) = +\infty$ & $\lim_{x\to+\infty} U'(x) = 0$
- S: d-dim., càdlàg $\mathbb{P}$-locally bounded semimartingale.
- $\mathcal{P}$: convex & weakly compact set of prob's $P \ll \mathbb{P}$.
- $\mathcal{M}_V^e \neq \emptyset$. $\quad (\mathcal{M}_V := \{Q \in \mathcal{M}_{\mathrm{loc}} : V(\lambda Q|\mathcal{P}) < \infty, \exists \lambda > 0\})$

(B^-) $\exists \varepsilon > 0$ s.t. $\{U(-(1+\varepsilon)B^-)dP/d\mathbb{P}\}_{P\in\mathcal{P}}$ is UI.

(B^+) $\forall P \in \mathcal{P}$, $\exists \varepsilon > 0$ s.t. $U(-\varepsilon B^+) \in L^1(P)$.

- Admissible Strategies

$$\Theta_{bb} = \{\theta \in L(S,\mathbb{P}) : \theta_0 = 0, \theta \cdot S \text{ bounded below}\}$$
$$\Theta_V := \{\theta \in L(S,\mathbb{P}) : \theta_0 = 0, \theta \cdot S \text{ is a } Q\text{-superMG}, \forall Q \in \mathcal{M}_V\}$$

  - $\Theta_{bb} \subset \Theta_V$.
  - We take any $\Theta_{bb} \subset \Theta \subset \Theta_V$.

Duality in Robust Utility Maximization

Duality Theorem

Suppose $\Theta_{bb} \subset \Theta \subset \Theta_V$. Then

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] = \min_{\lambda > 0, Q \in \mathcal{M}_V} (V(\lambda Q | \mathcal{P}) + \lambda E_Q[B]).$$

A Couple of Variants:

(ELMM) "$\min_{Q \in \mathcal{M}_V}$" $\Rightarrow$ "$\inf_{Q \in \mathcal{M}_V^e}$": $\alpha P + (1-\alpha)\mathbb{P} \sim \mathbb{P}$, $\forall \alpha \in [0,1)$

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] = \inf_{\lambda > 0, Q \in \mathcal{M}_V^e} (V(\lambda Q | \mathcal{P}) + \lambda E_Q[B]).$$

(Initial capital) B satisfies (B^-) & (B^+) $\Rightarrow$ $x + B$ too ($x \in \mathbb{R}$).

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T + B)] = \inf_{\lambda > 0, Q \in \mathcal{M}_V^e} (V(\lambda Q | \mathcal{P}) + \lambda E_Q[B] + \lambda x).$$

Application: Robust Utility Indifference Prices

Comparing the maximal robust utility

- $\sup_{\theta\in\Theta_{bb}} u_{B,\mathcal{P}}(-p+\theta\cdot S_T)$: buy the claim B at the price p.
- $\sup_{\theta\in\Theta_{bb}} u_{0,\mathcal{P}}(\theta\cdot S_T)$: not buy.
- Indifference Price $p(B)$: maximal acceptable price:

$$p(B) := \sup\{p : \sup_{\theta\in\Theta_{bb}} u_{B,\mathcal{P}}(-p+\theta\cdot S_T) \geq \sup_{\theta\in\Theta_{bb}} u_{0,\mathcal{P}}(\theta\cdot S_T)\}$$

Corollary

$$p(B) = \inf_{Q\in\mathcal{M}_V^e} (E_Q[B] + \gamma(Q)),$$

$$\gamma(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\left(V(\lambda Q|\mathcal{P}) - \inf_{\lambda'>0, Q'\in\mathcal{M}_V} V(\lambda' Q'|\mathcal{P})\right).$$

Example

Example: exponential utility

- Let $U(x) = -e^{-x} \Rightarrow V(y) = y \log y - y$.
- $\mathcal{H}(Q|P) = \begin{cases} E_P[(dQ/dP)\log(dQ/dP)] & \text{if } Q \ll P \\ +\infty & \text{if } Q \not\ll P \end{cases}.$
- $V(\lambda Q|P) = \lambda \mathcal{H}(Q|P) + \lambda \log \lambda - \lambda$, hence

$$(V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]) = \lambda(\mathcal{H}(Q|\mathcal{P}) + E_Q[B]) + \lambda \log \lambda - \lambda$$
$$\xrightarrow{\min_{\lambda>0}} -\exp\left(-(\mathcal{H}(Q|\mathcal{P}) + E_Q[B])\right).$$

- Duality gets a simple form:

$$\sup_{\theta \in \Theta_{bb}} \inf_{P \in \mathcal{P}} E_P\left[-e^{-(\theta \cdot S_T + B)}\right] = -\exp\left(-\inf_{Q \in \mathcal{M}_V} (\mathcal{H}(Q|\mathcal{P}) + E_Q[B])\right).$$

- $\inf_{\lambda>0, Q\in\mathcal{M}_V} V(\lambda Q|\mathcal{P}) = -e^{-\inf_{Q\in\mathcal{M}_V}\mathcal{H}(Q|\mathcal{P})}$;

$$\begin{aligned}\gamma(Q) &= \inf_{\lambda>0}\left(\mathcal{H}(Q|\mathcal{P}) + \log\lambda - 1 + \frac{1}{\lambda}e^{-\inf_{Q'}\mathcal{H}(Q'|\mathcal{P})}\right)\\ &= \mathcal{H}(Q|\mathcal{P}) - \inf_{Q'}\mathcal{H}(Q'|\mathcal{P}).\end{aligned}$$

- Thus, the indifference price is:

$$p(B) = \underbrace{\inf_{Q\in\mathcal{M}_V^e}\left(E_Q[B] + \mathcal{H}(Q|\mathcal{P})\right)}_{\text{dual with } B} - \underbrace{\inf_{Q'\in\mathcal{M}_V}\mathcal{H}(Q'|\mathcal{P})}_{\text{dual with } B=0}.$$

A 2-Brownian Setting

- $\mathbf{W} = (W_1, W_2)$: $\mathbb{P}$-BM, $\mathbb{F} = \mathbb{F}^{\mathbf{W}}$, and

$$X_t^1 = b_{1,t}dt + \sigma_{1,t}dW_{1,t}$$
$$X_t^2 = b_{2,t}dt + \sigma_{2,t}(\rho dW_{1,t} + \bar{\rho} dW_{2,t})$$

  - $b_1, b_2, \sigma_1, \sigma_2$ are bounded, $\sigma_1, \sigma_2 \geq \exists k > 0$, $\bar{\rho} = \sqrt{1-\rho^2}$, $|\rho| \leq 1$.
  - $S = \mathcal{E}(X^1)$.
- Uncertainty:
  - $(\omega, t) \mapsto C_t(\omega) \subset \mathbb{R}^2$, compact-valued pred., $\subset \exists \hat{C}$: compact, $\forall t, \omega$.

$$\mathcal{P} = \{\mathcal{E}_T(-\mathbf{p} \cdot \mathbf{W}) : \mathbf{p} = (p_1, p_2), \text{pred.}, \mathbf{p}_t(\omega) \in C_t(\omega)\}.$$

  - Convex, weakly compact,
  - This corresponds to the drift uncertainty:

$$dS_t = S_t \left((b_{1,t} + p_{1,t}\sigma_{1,t})dt + \sigma_{1,t} dW_{1,t}^P \right).$$

- ELMM's: $\lambda_t := b_{1,t}/\sigma_{1,t}$ $\longleftarrow$ bounded
  - Q is ELMM iff $dQ = \mathcal{E}_T(-(\lambda, q)\cdot \mathbf{W})d\mathbb{P}$ & $E[\mathcal{E}(-(\lambda, q)\cdot \mathbf{W})] = 1$.

$$\mathcal{H}(Q|P) = \frac{1}{2}E_Q\left[\int_0^T \|\mathbf{q}-\mathbf{p}\|^2 dt\right]$$

  - $\mathcal{H}(Q|\mathcal{P}) = \inf_{P\in\mathcal{P}} \mathcal{H}(Q|P) < \infty \Leftrightarrow \mathcal{H}(Q|\mathbb{P}) < \infty$.

$$\mathcal{M}_V^e = \left\{\mathcal{E}_T(-(\lambda, q)\cdot \mathbf{W});\ E[\mathcal{E}_T(-(\lambda, q)\cdot \mathbf{W})] = 1,\ E_Q[\int_0^T q_t^2 dt] < \infty\right\}$$

  - $e^{-(1+\varepsilon)B}, e^{\varepsilon B} \in L^1(\mathbb{P})\ \exists\varepsilon > 0 \Rightarrow$ all abstract results are OK.
- Dual Problem:

$$\text{minimize}\quad E_Q\left[\frac{1}{2}\int_0^T \| \underbrace{(\lambda_t, q_t)'}_{=:\mathbf{q}^\lambda} - \mathbf{p}_t\|^2 dt + B\right]$$

$$\text{over}\ \begin{cases} q: & \text{pred., } E[\mathcal{E}_T(-(\lambda, q)\cdot \mathbf{W})] = 1,\ E_Q[\int_0^T q_t^2 dt] < \infty \\ \mathbf{p}: & \text{pred., } \mathbf{p}_t(\omega) \in C_t(\omega) \end{cases}$$

Dynamic Version and BSDE

$$
\begin{aligned}
J_t^{B,q,\mathbf{p}} &:= E_Q\left[\frac{1}{2}\int_t^T \|\mathbf{q}^\lambda - \mathbf{p}\|^2 ds + B \,\middle|\, \mathcal{F}_t\right] \\
&= E_Q\left[\frac{1}{2}\int_0^T \|\mathbf{q}^\lambda - \mathbf{p}\|^2 ds + B \,\middle|\, \mathcal{F}_t\right] - \frac{1}{2}\int_0^t \|\mathbf{q}^\lambda - \mathbf{p}\|^2 ds \\
&= \int_0^t Z_{1,s}^{q,\mathbf{p}} dW_{1,s}^q + \int_0^t Z_{2,s}^{q,\mathbf{p}} dW_{2,s}^q - \frac{1}{2}\int_0^t \|\mathbf{q}^\lambda - \mathbf{p}\|^2 ds \\
&= \int_0^t Z_{1,s}^{\mathbf{q},\mathbf{p}} dW_{1,s}^0 + \int_0^t Z_{2,s}^{q,\mathbf{p}} dW_{2,s}^0 - \int_0^t \underbrace{\left\{\frac{1}{2}\|\mathbf{q}^\lambda - \mathbf{p}\|^2 - qZ_{2,s}^{q,\mathbf{p}}\right\}}_{=: f_s(Z_{2,s}^{q,\mathbf{p}}, q, \mathbf{p})\ \leftarrow \text{driver}} ds
\end{aligned}
$$

$$
\hat{f}_t(z) := \inf_{q \in \mathbb{R}, \mathbf{p} \in C_t} f_t(z, q, \mathbf{p}) = \inf_{\mathbf{p} \in C_t} \left\{\frac{1}{2}(\lambda_t - p_1)^2 - p_2 z\right\} - \frac{1}{2}z^2 \quad \leftarrow \text{quadratic driver}
$$

- Comparison principle yields the BSDE for $J_t^B := \operatorname{ess\,inf}_{q,\mathbf{p}} J_t^{B,q,\mathbf{p}}$:

$$
J_t^B = B + \int_t^T \hat{f}_s(Z_{2,s}) ds - \int_t^T Z_{1,s} dW_{1,s}^0 - \int_t^T Z_{2,s} dW_{2,s}^0.
$$

- existence of sol. $(J^B, \mathbf{Z})$: Briand/Hu (06, *PTRF*)
- uniqueness: Briand/Hu (08, *PTRF*), Delbaen/Hu/Richou (09, *arXiv*).

Description by BSDE

- New BM: $d\bar{W}^{\rho} := \rho dW^{0,1} + \bar{\rho} dW^{0,2}$, $d\bar{W}^{\rho,\perp} = \bar{\rho} dW^{0,1} - \rho dW^{0,2}$
- If $b^1, b^2, \sigma^1, \sigma^2, C$: $\mathbb{F}^{\bar{W}^{\rho}}$-pred., & $B \in \mathcal{F}_T^{\bar{W}^{\rho}}$, BSDE simplifies to:

$$J_t^B = B + \int_t^T \widehat{f}_s(\bar{\rho}\bar{Z}_s)ds - \int_t^T \bar{Z}_s d\bar{W}_s^{\rho}$$

- Concrete example of C: $C_t = [\alpha_t - \delta_t, \alpha_t + \delta_t] \times [\beta_t - \varepsilon_t, \beta_t + \varepsilon_t]$.
  - $\alpha, \beta, \delta, \varepsilon$: bdd $\mathbb{F}^{\bar{W}^{\rho}}$-pred, $\delta, \varepsilon \geq 0$.

$$\widehat{f}_t(z) = \frac{(|\lambda_t - \alpha_t| - \delta_t)_+^2}{2} - \frac{z^2}{2} - \beta_t z - \varepsilon_t |z| =: \frac{k_t}{2} - \frac{1}{2}z^2 - \beta_t z - \varepsilon_t |z|.$$

  - $\widetilde{B} := B + \frac{1}{2}\int_0^T k_s ds$, $\widetilde{J}_t^{\widetilde{B}} := J_t^B + \frac{1}{2}\int_0^t k_s ds$, $Q^0 \to Q^{\beta}$,

$$\widetilde{J}_t^{\widetilde{B}} = \widetilde{B} + \int_t^T \underbrace{-\left\{\frac{1}{2}\bar{\rho}^2 Z_s^2 + \bar{\rho}\varepsilon_s |Z_s|\right\}}_{=:\widetilde{f}_t(Z_s)} (ds - \int_t^T Z_s dW_s$$

  - Absolute value term is disturbing!!

á priori estimates

- Note: $-\frac{\bar\rho}{2}z^2 - \frac{1}{2}\frac{\bar\rho}{1-\bar\rho}\varepsilon_t^2 \le \tilde f_t(z) \le -\frac{\bar\rho^2}{2}z^2$
  - comparison principle suggests: $Y^{\text{low}} \le \tilde J^{\tilde B} \le Y^{\text{up}}$, where

$$
\begin{aligned}
Y_t^{\text{low}} &= \tilde B + \int_t^T -\frac{\bar\rho}{2}Z_s^2 - \frac{1}{2}\frac{\bar\rho}{1-\bar\rho}\varepsilon_s^2 ds - \int_t^T Z_s dW_s \\
Y_t^{\text{up}} &= \tilde B + \int_t^T -\frac{\bar\rho^2}{2}Z_s^2 ds - \int_t^T Z_s dW_s
\end{aligned}
\longrightarrow \text{solvable!!}
$$

$$
-\frac{1}{\bar\rho}\log E\left[e^{-\bar\rho\left(\tilde B - \frac{1}{2}\frac{\bar\rho}{1-\bar\rho}\int_t^T \varepsilon_s^2 ds\right)} \mid \mathcal{F}_t\right] \le \tilde J_t^{\tilde B} \le -\frac{1}{\bar\rho^2}\log E\left[e^{-\bar\rho^2 \tilde B} \mid \mathcal{F}_t\right]
$$

  - Removing "~",

$$
\begin{aligned}
&-\frac{1}{\bar\rho}\log E\left[e^{-\bar\rho\left(B + \frac{1}{2}\int_t^T k_s - \frac{\bar\rho}{1-\bar\rho}\varepsilon_s^2 ds\right)} \mid \mathcal{F}_t\right] \\
&\qquad \le J_t^B \le -\frac{1}{\bar\rho^2}\log E\left[e^{-\bar\rho^2\left(B + \frac{1}{2}\int_t^T k_s ds\right)} \mid \mathcal{F}_t\right]
\end{aligned}
$$

Markovian Case & Connection to PDE

- Suppose $X_2 =: Y$ is a diffusion with $\mathcal{A}_t := g\partial_y + \frac{1}{2}\gamma^2\partial_y^2$, i.e.,

$$dY_t = g(t, Y_t)dt + \gamma(t, Y_t)dW_t$$

  - Assume $\lambda, \alpha, \beta, \delta, \varepsilon$ all deterministic $\Rightarrow K_T := \int_0^T k_s ds$ is constant.
  - $B = h(Y_T) \Rightarrow \widetilde{B} = \widetilde{h}(Y_T)$, $\widetilde{h}(x) = h(x) + K_T$
  - "$\widetilde{J}_t^{\widetilde{B}} = v(t, Y_t)$" satisfies (if smooth),

$$\partial_t v(t, Y_t) = (\partial_t + \mathcal{A})v(t, Y_t)dt + \gamma(Y_t)\partial v(t, Y_t)dW_t.$$

  - Quadratic Cauchy associated to BSDE:

$$(\partial_t + \mathcal{A})v - \frac{1}{2}\bar{\rho}^2\gamma^2(\partial_y v)^2 - \bar{\rho}\varepsilon_t\gamma|\partial_y v| = 0, \quad v(T, y) = \widetilde{h}(y).$$

  - $\exists$smooth sol. $\Rightarrow (\widetilde{J}^{\widetilde{B}}, Z) = (v(\cdot, Y), \gamma(Y)\partial_y v(\cdot, Y))$ solves the BSDE.

$$(FBSDE) \quad \begin{aligned} dY^{t,y} &= g(s, Y_s^{t,y})ds + \gamma(s, Y_s^{t,y})dW_s,\ (s > t),\ \ Y_u^{t,y} = y,\ (u \le t) \\ \widetilde{J}^{t,y} &= \widetilde{h}(Y_T^{t,y}) + \int_s^T \widehat{f}_u(\bar{\rho}Z_u^{t,y})du - \int_s^T Z_u^{t,y}dW_u \end{aligned}$$

  - $v(t, y) := \widetilde{J}_t^{t,y}$ is a viscosity solution to the Cauchy problem.

Summary

- We have established the duality in robust utility maximization with unbounded claim:

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] = \min_{\lambda > 0, Q \in \mathcal{M}_V} (V(\lambda Q | \mathcal{P}) + \lambda E_Q[B])$$

- The key step is the analysis of robust utility functional:

$$u_{B,\mathcal{P}}(X) := \inf_{P \in \mathcal{P}} E_P[U(X + B)]$$

  via a Robust Extension of Rockafellar's theorem.
- Then everything else is cleared by standard methodology via Fenchel's theorem.
- Based on:

Owari (2010) Duality in Robust Utility Maximization with Unbounded Claim via a Robust Extension of Rockafellar's Theorem, Preprint.

Thank You for Your Attention !!

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