Rigorous Results for Lattice Fermion Models with SU(N) Symmetry

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SU(N) physics in condensed matter and cold atoms

12th May. 2022

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- **Introduction**
- **Majorana reflection positivity in the attractive SU(N) Fermi-Hubbard model**
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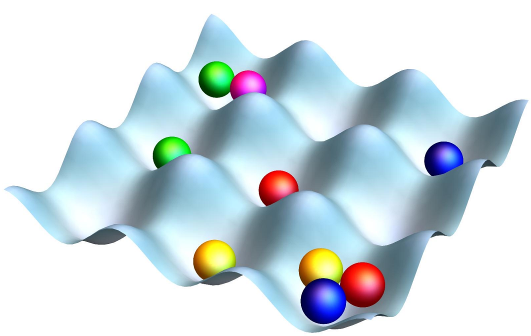
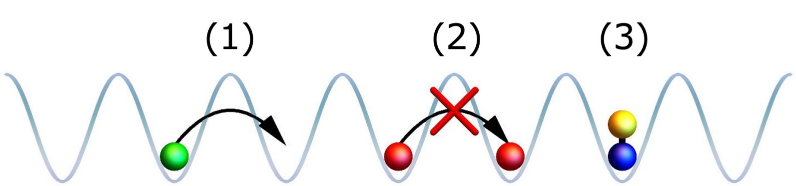
SU(N) Fermi-Hubbard model

$$\hat{H}_{\mathrm{S}} = \hat{H}_{\mathrm{hop}} + \hat{H}_{\mathrm{int}}$$

$$\hat{H}_{\mathrm{hop}} = \sum_{x,y\in\Lambda} \sum_{\sigma=1}^{N} t_{x,y} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{y,\sigma}$$

$$\hat{H}_{\mathrm{int}} = U \sum_{x\in\Lambda} \sum_{1\leq\sigma<\tau\leq N} \hat{n}_{x,\sigma} \hat{n}_{x,\tau}$$





N : the number of internal degrees of freedom

In **solids**:

N=2 (spins of electrons)

Ultracold atoms:

N=6

(nuclear spins of ^{173}Yb atoms)

N=10

(nuclear spins of ^{87}Sr atoms)

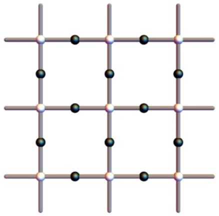
M. Cazalilla et al., NJP (2009),
A. Gorshkov et al., Nat. Phys (2010),

S. Taie et al., Nat. Phys. (2012),
F. Scazza, et al., Nat. Phys. (2014),
X. Zhang et al., Science (2014),......

Rigorous approaches to the SU(2) Fermi-Hubbard model

Theorems for the ground states

- Degeneracy, long-range order, ……
- Ferrimagnetism (repulsive interaction) E. H. Lieb, PRL **62**, 1201 (1989).
- Two-particle ODLRO (attractive interaction) S.-Q. Shen and Z.-M. Qiu, PRL **71**, 4238 (1993).
- Method: **spin reflection positivity**



e.g. Lieb lattice

η-pairing states

- exact eigenstates that exhibit the two-particle ODLRO
- Method: **η-SU(2) symmetry of the model** C. N. Yang, PRL **63**, 2144 (1989).

We extended these results to the SU(N) case.

Rigorous approaches to the SU(N) Fermi-Hubbard model

Rigorous results on the ground state (attractive interaction)

H. Yoshida and H. Katsura, PRL **126**, 100201 (2021)

- Degeneracy and charge order
- Applicable to a wide class of $N(\geq 3)$ -component lattice fermions in arbitrary dimension
  SU(N) symmetry is not (essentially) required
- Method: **Majorana reflection positivity**

Generalization of η-pairing states (η-clustering states)

H. Yoshida and H. Katsura, PRB **105**, 024520 (2022)

- Exact eigenstates of extended SU(N) Hubbard models
  with N-particle ODLRO
- Method: **Restricted spectrum generating algebra**

Contents

Majorana reflection positivity

- **Reflection positivity**: Inequality for expectation values of operators in the ground state or the Gibbs state

$$\langle \hat{A}\theta(\hat{A})\rangle \geq 0, \text{ etc.} \ldots \qquad \theta : \text{ reflection}$$

  - Reflection in real space F. J. Dyson et. al., J. Stat. Phys. **18**, 335 (1978).
  - Reflection in spin space (spin reflection positivity)
    - Theorems on the ground state of the SU(2) Fermi-Hubbard model (Lieb's theorem) E. H. Lieb, PRL **62**, 10 (1989).
  - Reflection between two sets of Majorana operators

    A. Jaffe and F. L. Pedrocchi, Ann. Henri Poincaré **16**, 189 (2015).

$$\hat{\gamma}^{(1)}_{x,\sigma} = \hat{c}_{x,\sigma} + \hat{c}^{\dagger}_{x,\sigma}, \quad \hat{\gamma}^{(2)}_{x,\sigma} = -i\left(\hat{c}_{x,\sigma} - \hat{c}^{\dagger}_{x,\sigma}\right)$$

$$\hat{\gamma}^{(1)}_{x,\sigma} \Longleftrightarrow \hat{\gamma}^{(2)}_{x,\sigma}$$

    - Used to solve sign problem in quantum Monte Carlo.
      Z. C. Wei, et.al., PRL **116**, 250601(2016), H. Xu, et.al., arXiv:1912.11233.
    - Applicable to SU(N) ($N > 2$) attractive Fermi-Hubbard models!

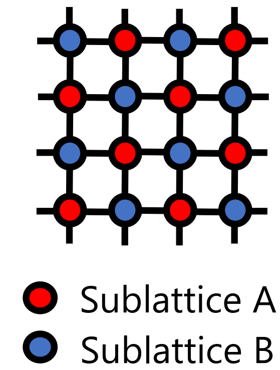
The Hamiltonian

$$\hat{H}_{\mathrm{S}} = \hat{H}_{\mathrm{hop}} + \hat{H}_{\mathrm{int}} \qquad N \geq 3$$

$$\hat{H}_{\mathrm{hop}} = \sum_{x,y\in\Lambda} \sum_{\sigma=1}^{N} t_{x,y} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{y,\sigma}$$

$$\hat{H}_{\mathrm{int}} = \sum_{x\in\Lambda} U_x \sum_{1\leq\sigma<\tau\leq N} \left(\hat{n}_{x,\sigma} - \frac{1}{2}\right)\left(\hat{n}_{x,\tau} - \frac{1}{2}\right)$$

Particle-hole symmetric interaction



- The SU(N) **attractive** Fermi-Hubbard model on connected bipartite lattice Λ
  - A, B : sublattices
  - $\hat{c}^{\dagger}_{x,\sigma}$ $(\hat{c}_{x,\sigma})$: creation (annihilation) operators of a fermion at site with the internal degree of freedom $\sigma = 1, \ldots, N$.
  - $\hat{n}_x = \sum_{\sigma=1}^{N} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{x,\sigma}$
  - $t_{x,y}$: hopping amplitude (real), $U_x < 0$: the strength of interaction

A. Rapp, G. Zaránd, C. Honerkamp, and W. Hofstetter, PRL **98**, 160405 (2007).

- Bipartite lattice: a lattice that allows neighboring sites to be painted in different colors when painted in two different colors.
  - Examples: d-dimensional cubic lattice, honeycomb lattice ......

Theorem 1 ：Basic properties of the ground state

$|A| = N_A, \quad |B| = N_B$: the number of sites on each sublattice

Theorem 1: When $N_A \neq N_B$, there are exactly **two ground states** in the whole Fock space. The two ground states are SU(N) singlets and their total fermion numbers are NN_A and NN_B , respectively.

When $N_A = N_B$, there are **at most two ground states**, each of which is an SU(N) singlet and whose total fermion number is $NN_A (= NN_B)$

- SU(N) singlet：a state invariant under SU(N) rotation of the internal degrees of freedom. i.e.,

$$\hat{F}^{\sigma,\tau} |\Phi_{\text{singlet}}\rangle = 0 \text{ for all } \sigma \neq \tau, \quad \hat{F}^{1,1} |\Phi_{\text{singlet}}\rangle = \cdots = \hat{F}^{N,N} |\Phi_{\text{singlet}}\rangle$$

where $\hat{F}^{\sigma,\tau} = \sum_{x\in\Lambda} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{x,\tau}$

Theorem 2 : An inequality about a density correlation

- Consider the following operator

$$\hat{S}_{x,y} = (-1)^x(-1)^y \left(\hat{n}_x - \frac{N}{2}\right)\left(\hat{n}_y - \frac{N}{2}\right)$$

$(-1)^x = 1 \text{ if } x \in A \text{ and } (-1)^x = -1 \text{ if } x \in B$

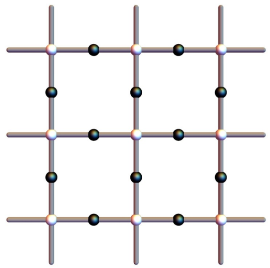
Theorem 2 : We have for any ground state $|\Phi_{\mathrm{GS}}\rangle$ and for $x, y \in \Lambda$

that $\left\langle \Phi_{\mathrm{GS}} \middle| \hat{S}_{x,y} \middle| \Phi_{\mathrm{GS}} \right\rangle > 0$.

Theorem 3: Existence of a density order

- Assume that $|N_A - N_B|$ is macroscopically large.

$$|N_A - N_B| = aN_{\mathrm{s}} \quad (0 < a < 1) \qquad N_{\mathrm{s}} = N_A + N_B$$



Lieb lattice (a=1/3)

- The order parameter

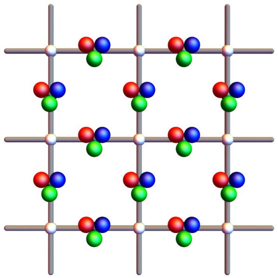
$$\hat{S} = \sum_{x\in\Lambda} (-1)^x \left(\hat{n}_x - \frac{N}{2}\right) \qquad (-1)^x = 1 \text{ if } x \in A \text{ and } (-1)^x = -1 \text{ if } x \in B$$

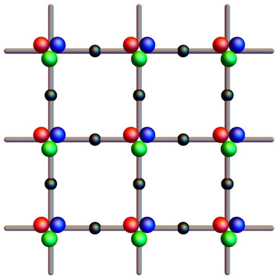
Theorem 3 : We have for any ground state $|\Phi_{\mathrm{GS}}\rangle$ that

$$\left\langle \Phi_{\mathrm{GS}} \middle| \hat{S}^2 \middle| \Phi_{\mathrm{GS}} \right\rangle > \left(\frac{aNN_{\mathrm{s}}}{2}\right)^2$$

The ground state has a density order.

(left) Two ground states
in the limit $|U_x| \gg |t_{x,y}|$
in the SU(3) attractive Hubbard model

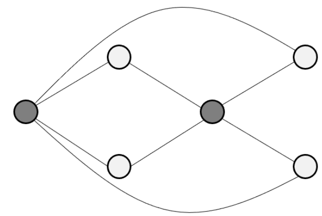




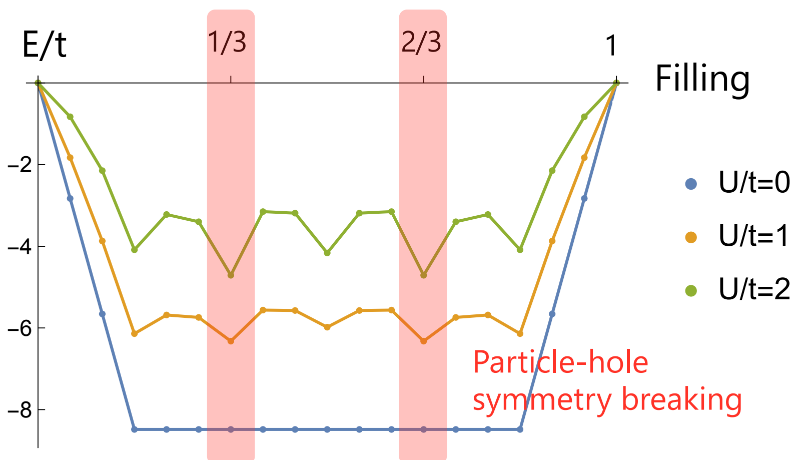
An Example

- We calculated the lowest energy of the attractive **SU(3)** Fermi-Hubbard model by exact diagonalization.
  - hopping amplitude: -t   the strength of interaction: -U (t, U > 0)

Lattice:



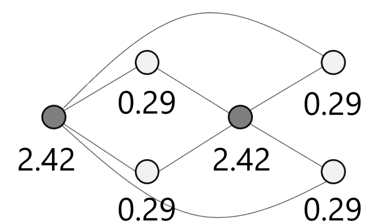
The lowest energy in each filling sector



In these two filling sectors, the lowest energy level is unique

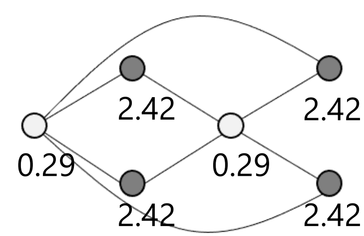
* The energy is shifted by a constant so that the energy of the vacuum is zero.

The expectation value of the particle number at each site (U/t=1)



1/3 filling

$$\left\langle \Phi_{\mathrm{GS}} \middle| \hat{S}^2 \middle| \Phi_{\mathrm{GS}} \right\rangle = 52.23 > \left(\frac{aNN_s}{2}\right)^2 = 9$$



2/3 filling

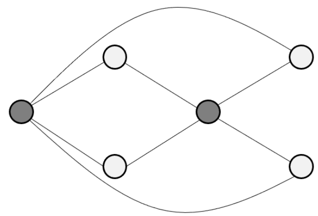
$$\left\langle \Phi_{\mathrm{GS}} \middle| \hat{S}^2 \middle| \Phi_{\mathrm{GS}} \right\rangle = 52.23 > \left(\frac{aNN_s}{2}\right)^2 = 9$$

Cf.) A similar behavior is numerically found in a spinless fermion model on a Lieb lattice.
M. Bercx, et.al., PRB **95**, 035108 (2017).

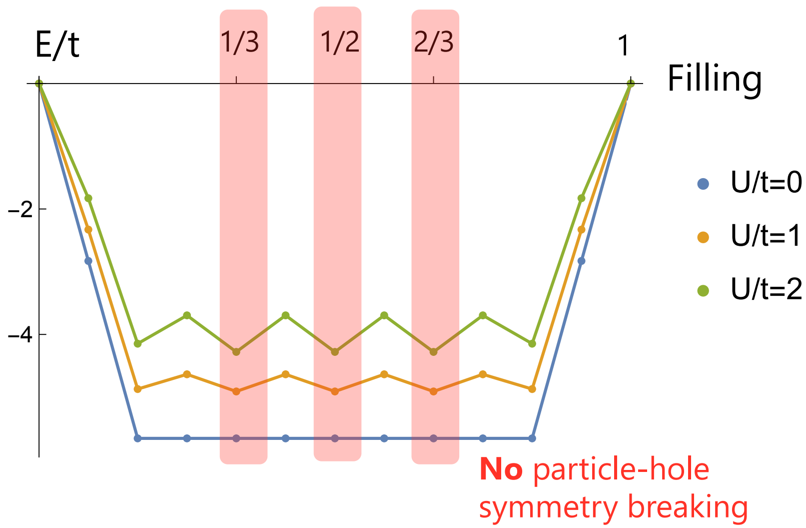
SU(2) case

- For the **SU(2)** attractive Fermi-Hubbard model, we performed the same calculation.
  - hopping amplitude: -t the strength of interaction: -U (t, U > 0)

Lattice:

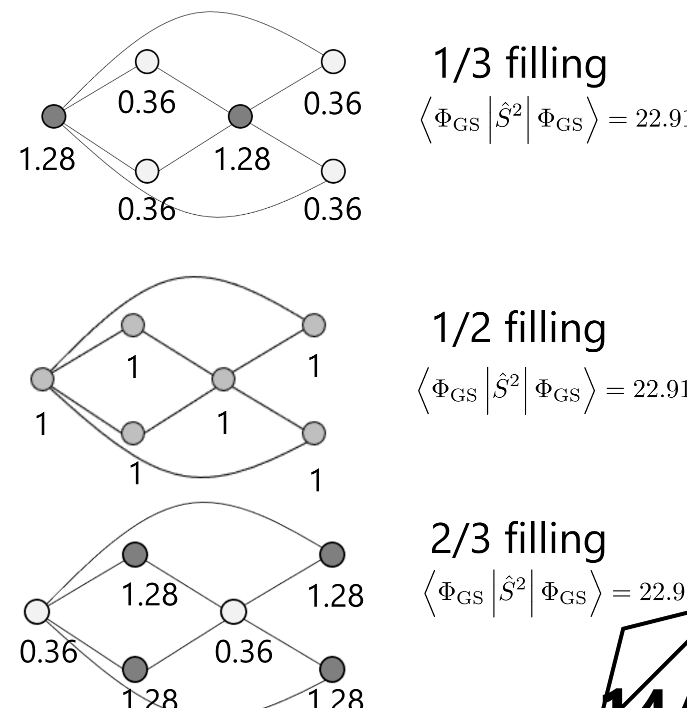


The lowest energy in each filling sector



* The energy is shifted by a constant so that the energy of the vacuum is zero.

The expectation value of the particle number at each site (U/t=1)



Outline of the proof(1/2)

H. Yoshida and H. Katsura, PRL **126**.100201 (2021)

1. Consider an energy **eigenoperator** $\hat{O}$.

$\{|E,j\rangle \mid j=1,\cdots,n_E\}$ n_E :degeneracy

$\hat{H}|E,j\rangle = E|E,j\rangle$: eigenvectors

⇨ $\{|E,j\rangle\langle E,k| | j,k=1,\cdots,n_E\}$

$\hat{H}\hat{O} = \hat{O}\hat{H} = E\hat{O}$: eigenoperators

2. We obtain a **matrix representation *W***

by expanding the eigenoperator $\hat{O}$ as

$$\hat{O}(W) = \sum_{\alpha,\beta} W_{\alpha,\beta}\hat{\Gamma}^{(1)}_{\alpha}\hat{\Gamma}^{(2)}_{\beta}$$

Z.-C. Wei, et al., PRB **92**, 161105(R) (2015).

$\hat{\Gamma}^{(1)}_{\alpha}, \hat{\Gamma}^{(2)}_{\beta}$: all possible products of $\hat{\gamma}^{(1)}_{x,\sigma}, \hat{\gamma}^{(2)}_{y,\tau}$ operators

$\alpha,\beta \subset \Lambda \times \{1,2,\ldots,N\}$

Outline of the proof(2/2)

H. Yoshida and H. Katsura, PRL **126**.100201 (2021)

3. Prove that the matrix W for the ground state is **positive or negative definite.**

Point. For a Hermitian matrix W , we define a new matrix $|W| := \sqrt{W^\dagger W}$.
Then, the following inequality holds (**Majorana reflection positivity**).

$$E(|W|) \leq E(W)$$

Here, $E(W)$ is the energy expectation value of the state represented by W.

$$E(W) = 2\,\mathrm{Tr}\left[KW^2\right] + \sum_{x\in\Lambda}\sum_{1\leq\sigma<\tau\leq N}^{N} U_x\,\mathrm{Tr}\left[WL_{xx,\sigma\tau}WL_{xx,\sigma\tau}\right]$$

$L_{xy,\sigma\tau}$ and K : matrix elements of $\hat{H}_{\mathrm{hop}}$, $\hat{H}_{\mathrm{int}}$ in the Majorana basis (Hermitian)

4. Using the property 3, we can prove

Theorem 1: Basic properties of the ground state,

Theorem 2: An inequality about a density correlation,

Theorem 3: Existence of a density order.

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- Introduction
- Majorana reflection positivity in the attractive SU(N) Fermi-Hubbard model
- **Generalized η-pairing states in the extended SU(N) Fermi-Hubbard model**
- Summary

Off-diagonal long-range order (ODLRO)

- **ODLRO characterizes BEC and superconductivity.**
- **Definitions:**

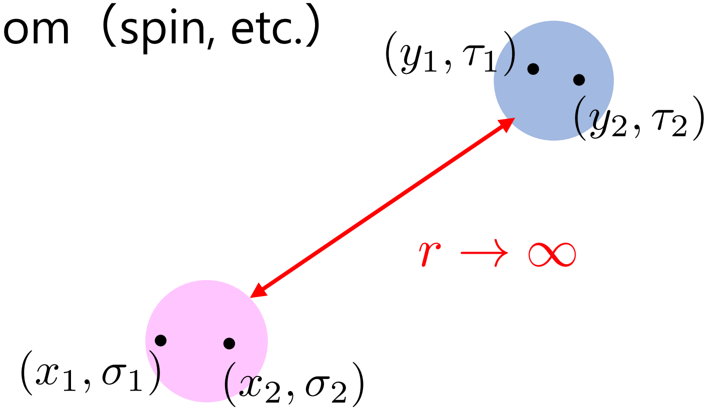
$\hat{c}^\dagger_{x,\sigma}$ $(\hat{c}_{x,\sigma})$: creation (annihilation) operator of fermions/bosons

x :a site σ : an internal degree of freedom (spin, etc.)

$(x_1, \ldots, x_n)$ $(y_1, \ldots, y_n)$: sets of sites

n sites $(x_1, \ldots, x_n)$ are close to each other

(So are $(y_1, \ldots, y_n)$)



- A state has n-particle ODLRO

$\Leftrightarrow$ In the limit $\min\limits_{1\le i,j\le n} |x_i - y_j| \to \infty$,

$$|\langle \hat{c}^\dagger_{x_n,\sigma_n} \ldots \hat{c}^\dagger_{x_1,\sigma_1} \hat{c}_{y_1,\tau_1} \ldots \hat{c}_{y_n,\tau_n} \rangle| \to \text{ const. } (>0)$$

Physical implications of ODLRO

- **Bosons : One-particle ODLRO is related to BEC.**

  O.Penrose and L.Onsager, Phys. Rev (1956).

- **Fermions :** One-particle ODLRO is prohibited.
  However, two-particle ODLRO is possible.

  C. N. Yang, RMP (1962).

  - **Two-particle ODLRO is related to superconductivity**.
  - Examples of states with two-particle ODLRO
    : BCS wave function, **η-pairing states**

η-pairing states

C. N. Yang, PRL **63**, 2144 (1989).

- **Exact eigenstates of the SU(2) Fermi-Hubbard model with two-particle ODLRO**
  - We define η-operator $\hat{\tilde{\eta}}^\dagger$ as

$$\hat{\tilde{\eta}}^\dagger := \sum_{x=1}^{L} e^{i\pi x} \hat{\eta}_x^\dagger \qquad \hat{\eta}_x^\dagger := \hat{c}_{x,\uparrow}^\dagger \hat{c}_{x,\downarrow}^\dagger$$

  - **Definition: η-pairing states**

$$|\psi_M\rangle = \left(\hat{\tilde{\eta}}^\dagger\right)^M |0\rangle \qquad |0\rangle$$: The vacuum state

(On a general bipartite lattice)

$$e^{i\pi x} = \begin{cases} 1 & (x \in A) \\ -1 & (x \in B) \end{cases}$$

A, B: sublattice

η-pairing states

C. N. Yang, PRL **63**, 2144 (1989).

■ **Exact eigenstates of the SU(2) Fermi-Hubbard model with two-particle ODLRO**

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A, B: sublattice

- **Definition: η-pairing states**

$$|\psi_M\rangle = \left(\hat{\tilde{\eta}}^\dagger\right)^M |0\rangle \qquad |0\rangle \text{ : The vacuum state}$$

- The vacuum is an **eigenstate** of the Hamiltonian
- The η-operator **commutes** with the Hamiltonian

→ **η-pairing states are eigenstates of the SU(2) Fermi-Hubbard model.**

η-pairing scars
: Hubbard + perturbation

S. Moudgalya et.al., PRB (2020).
D. K. Mark and O. I. Motrunich, PRB (2020).
K. Pakrouski et.al., PRR (2021).

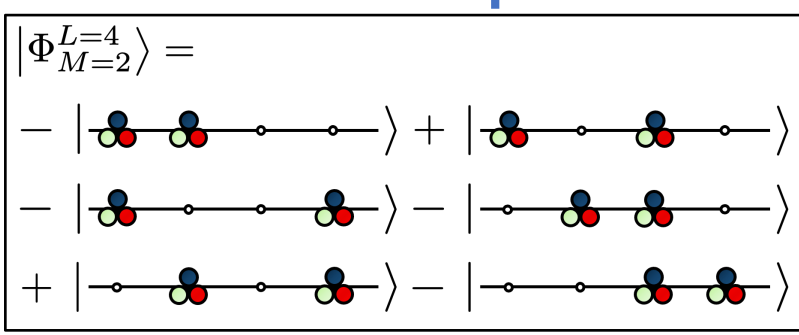
Motivation: generalization of η-pairing states

- **η-pairing states:** Exact eigenstates of the **SU(2) Fermi-Hubbard model** with **two-particle ODLRO**

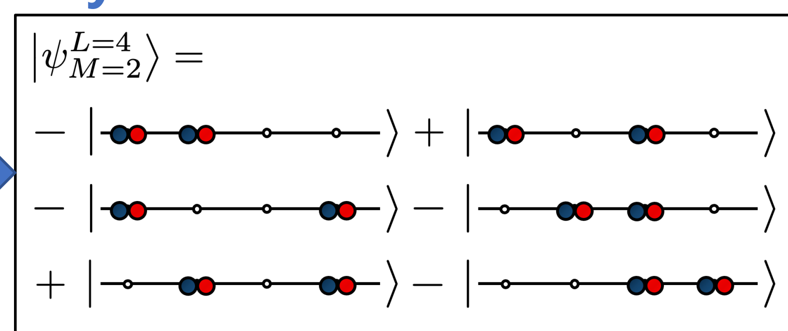
> **η-pairing scars**
> : Hubbard + perturbation
>
> S. Moudgalya et.al., PRB (2020).
> D. K. Mark and O. I. Motrunich, PRB (2020).

- Can we generalize η-pairing states to the **SU(N) Fermi-Hubbard model** ?
  - New type of **N-particle clustering states**
  - Candidate for **quantum many-body scars**

C. Wu, MPL B **20**, 1707 (2006).
A. Rapp, et al., PRL **98**, 160405 (2007).



An η-clustering state (N=3)



An η-pairing state (N=2)

HY and H.Katsura, PRB **105**, 024520 (2022)

η-clustering states: a natural extension

N: the number of internal degrees of freedom

- A straightforward extension→ $\left(\hat{\tilde{\eta}}^\dagger\right)^2 = 0$ when N is odd.

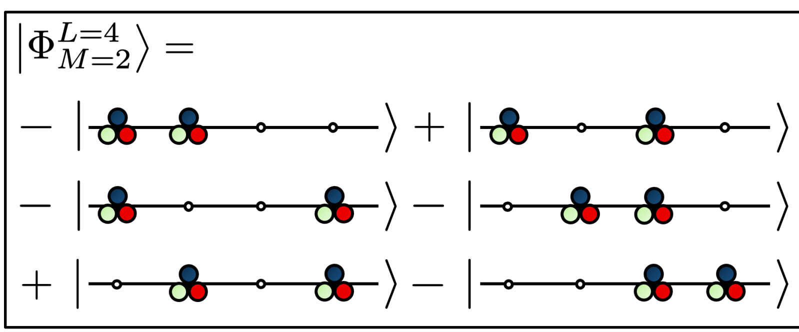
$$\hat{\tilde{\eta}}^\dagger := \sum_{x=1}^{L} e^{i\pi x} \hat{\eta}_x^\dagger \qquad \hat{\eta}_x^\dagger = \hat{c}_{x,1}^\dagger \hat{c}_{x,2}^\dagger \cdots \hat{c}_{x,N}^\dagger$$

- **η-clustering states:** nonvanishing **for all N**, defined on a chain.

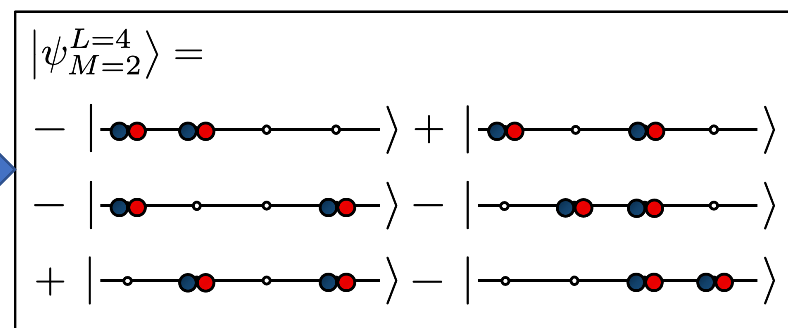
$$\left|\Phi_M^L\right\rangle := \frac{1}{M!}\left(\hat{\eta}^\dagger\right)^M |0\rangle \qquad \hat{\eta}^\dagger = \sum_{x=1}^{L} e^{i\pi x} \hat{U}_{1,\ldots,x-1} \hat{\eta}_x^\dagger$$

$$\hat{U}_{1,\ldots,x-1} = e^{i\pi \sum_{j=1}^{x-1} \hat{n}_j}$$

$\hat{n}_x$: particle number operator at site x



An η-clustering state（N=3）



An η-pairing state（N=2）

Extended SU(N) Hubbard model

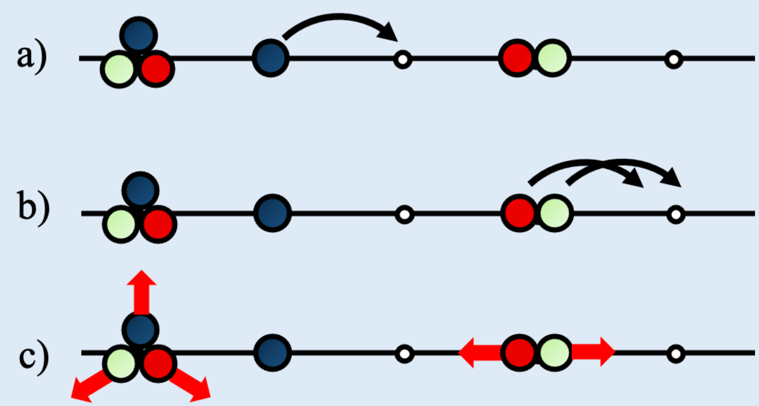
- η-clustering states are not eigenstates of the **SU(N) Hubbard model** when N>2. 😞
- By adding **(N-1) -body hopping terms**, we can construct a model where η-clustering states are exact eigenstates. 😃

$$\hat{H} = \hat{H}_1 + \hat{H}_{N-1} + \hat{H}_U$$

$$\hat{H}_1 = -t \sum_{x=1}^{L-1} \sum_{\sigma=1}^{N} \left(\hat{c}^\dagger_{x,\sigma} \hat{c}_{x+1,\sigma} + \text{ h.c. } \right)$$

$$\hat{H}_{N-1} = -t \sum_{x=1}^{L-1} \sum_{\sigma=1}^{N} \left(\bar{\hat{c}}^\dagger_{x,\sigma} \bar{\hat{c}}_{x+1,\sigma} + \text{ h.c. } \right)$$

$$\hat{H}_U = U \sum_{x=1}^{L} \hat{n}_x \left(\hat{n}_x - N \right)$$



where $\bar{\hat{c}}^\dagger_{x,\sigma} = \left[\hat{c}_{x,\sigma}, \hat{\eta}^\dagger_x \right]_\pm$ ($[\ ,\ ]_\pm$ is (anti-)commutator for N even (odd))

Connections to quantum many-body scars

- When N>2, $\hat{\eta}^\dagger$ **does not commute** with the Hamiltonian.
- However, they satisfy the **following relations**.

Restricted spectrum generating algebra (RSGA) of order 1.

(i) $\hat{H}|0\rangle = 0$

(ii) $[\hat{H}, \hat{\eta}^\dagger]|0\rangle = 0$

(iii) $[[\hat{H}, \hat{\eta}^\dagger], \hat{\eta}^\dagger] = 0$

【c.f.】 When N=2 (η-pairing scars), S. Moudgalya et.al, PRB (2020).

→ **η-clustering states are eigenstates of the Hamiltonian.**

Connections to quantum many-body scars

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【c.f.】 When N=2 (η-pairing scars),
S. Moudgalya et.al, PRB (2020).

η-clustering states are eigenstates of the Hamiltonian.

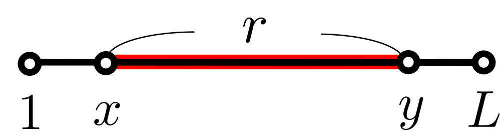
- The energy is in the **middle of the spectrum**
- The entanglement entropy scales **logarithmically** with the subsystem size.

Quantum many-body scars !

D. K. Mark and O. I. Motrunich, PRB (2020).
K. Pakrouski et.al., PRR (2021).

Off-diagonal long-range order (ODLRO)

N: the number of internal degrees of freedom



■ **Singlet correlation function :** characterizes N-particle ODLRO

$$\left\langle \hat{\eta}_x^\dagger \hat{\eta}_y \right\rangle_M^L := \frac{\left\langle \Phi_M^L \middle| \hat{\eta}_x^\dagger \hat{\eta}_y \middle| \Phi_M^L \right\rangle}{\left\langle \Phi_M^L \mid \Phi_M^L \right\rangle}$$

$$\hat{\eta}_x^\dagger = \hat{c}_{x,1}^\dagger \hat{c}_{x,2}^\dagger \ldots \hat{c}_{x,N}^\dagger$$

$$\left|\Phi_M^L\right\rangle := \frac{1}{M!} \left(\hat{\eta}^\dagger\right)^M |0\rangle$$

L: system size

■ When N is **even**,

$$\left\langle \hat{\eta}_x^\dagger \hat{\eta}_y \right\rangle_M^L = (-1)^r \frac{M(L-M)}{L(L-1)}$$

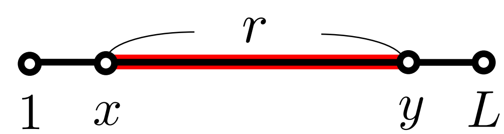
■ When N is **odd**,

$$\left\langle \hat{\eta}_x^\dagger \hat{\eta}_y \right\rangle_M^L = \frac{\sum_{j=j_{\min}}^{j_{\max}} (-1)^j \binom{L-r-1}{j} \binom{r-1}{M-j-1}}{(-1)^{M+r-1} \binom{L}{M}}$$

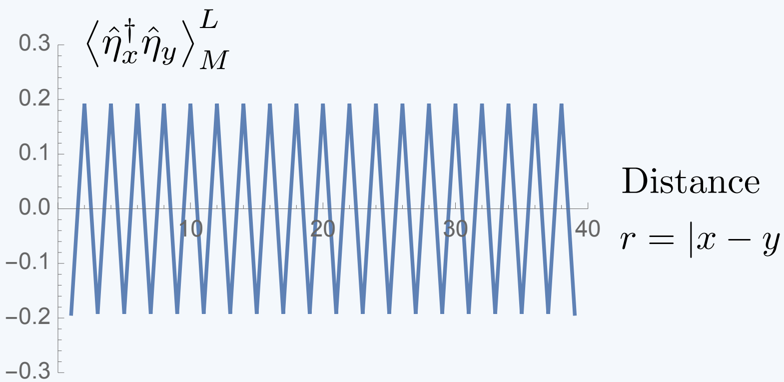
$$r = |x-y|,\ j_{\min} = \max\{0, M-r\},\ j_{\max} = \min\{L-r-1, M-1\}$$

Off-diagonal long-range order (ODLRO)

N: the number of internal degrees of freedom



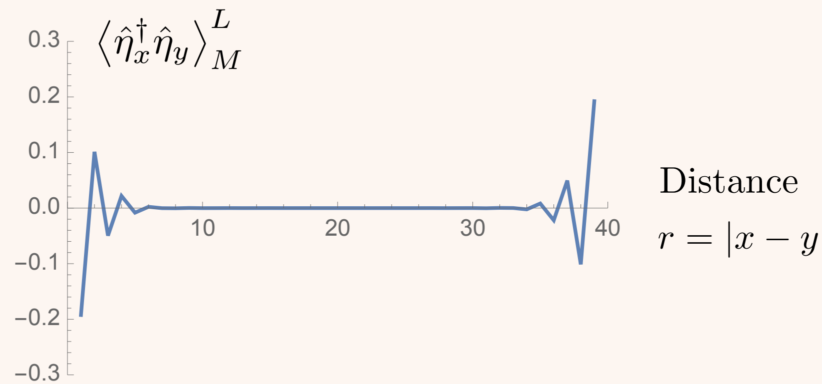
■ When N is **even**,



- The correlations do not decay with distance.

N-particle ODLRO in the bulk !

■ When N is **odd**,



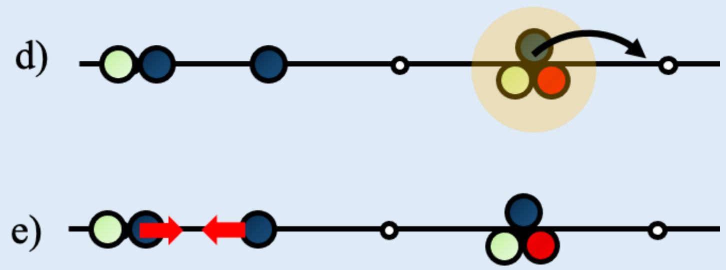
- The correlations decay exponentially with distance in the bulk.
- But **end-to-end correlations** do not vanish

Parent Hamiltonian for η-clustering states

- η-clustering states are **excited states** of the extended SU(N) Hubbard model.
- We constructed a model where they are the **unique ground states**.

$$\hat{H}_V = V \sum_{x=1}^{L-1} \left\{ \frac{1}{2} \left(\hat{\eta}_x^\dagger \hat{\eta}_{x+1} + \text{ h.c. } \right) - \frac{1}{N^2} \hat{n}_x \hat{n}_{x+1} + \frac{1}{2N} \left(\hat{n}_x + \hat{n}_{x+1} \right) \right\}$$

$$\hat{\eta}_x^\dagger = \hat{c}_{x,1}^\dagger \hat{c}_{x,2}^\dagger \dots \hat{c}_{x,N}^\dagger$$



Theorem: Consider the Hamiltonian $\hat{H} + \hat{H}_V$. If $U \leq 0$ and $V > 8N^2|t|$, then the ground states in the whole Fock space are exactly $(L+1)$ -fold degenerate and written as $|\Phi_M^L\rangle$ $(M = 0, \dots, L)$.

c.f. For N=2: J. de Boer and A. Schadschneider, PRL (1995).

Contents

Summary

H. Yoshida and H. Katsura, PRL **126**, 100201 (2021)

H. Yoshida and H. Katsura, PRB **105**, 024520 (2022)

- We proved theorems on the **SU(N) attractive Fermi-Hubbard** model using the **Majorana reflection positivity.**
  - Applications to **other multi-component fermions**?
  - Applications to **non-Hermitian systems**?

    T. Hayata and A. Yamamoto, arXiv:2106.06192

- We constructed the **N-component generalization of η-pairing states** and examined the **N-particle ODLRO**.
  - Experimental realization?

Key observation

- A representation of the Hamiltonian with Majorana operators

$$\hat{H} = \sum_{x\in A, y\in B}\sum_{\sigma=1}^{N} t_{x,y}\left(\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(1)}\hat{\gamma}_{y,\sigma}^{(1)} - \frac{i}{2}\hat{\gamma}_{x,\sigma}^{(2)}\hat{\gamma}_{y,\sigma}^{(2)}\right) + \sum_{x\in\Lambda}\sum_{1\le\sigma<\tau\le N} U_x\left(\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(1)}\hat{\gamma}_{x,\tau}^{(1)}\right)\left(-\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(2)}\hat{\gamma}_{x,\tau}^{(2)}\right)$$

- We define an anti-linear map θ as follows.

$$\theta\left(\hat{\gamma}_{x,\sigma}^{(1)}\right) = \hat{\gamma}_{x,\sigma}^{(2)}, \quad \theta\left(\hat{\gamma}_{x,\sigma}^{(2)}\right) = \hat{\gamma}_{x,\sigma}^{(1)}, \quad \theta(i) = -i$$

- Then, the Hamiltonian is written as

$$\hat{H} = \hat{H}^{(1)} + \theta\left(\hat{H}^{(1)}\right) - \sum_{x\in\Lambda}\sum_{1\le\sigma<\tau\le N} |U_x|\left(\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(1)}\hat{\gamma}_{x,\tau}^{(1)}\right)\theta\left(\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(1)}\hat{\gamma}_{x,\tau}^{(1)}\right)$$

$$\hat{H}^{(1)} = \sum_{x\in A, y\in B}\sum_{\sigma=1}^{N} t_{x,y}\left(\frac{i}{2}\hat{\gamma}_{x,\sigma}^{(1)}\hat{\gamma}_{y,\sigma}^{(1)}\right)$$

Effect of boundary conditions

TABLE I. The boundary conditions for which $\left|\Phi_M^L\right\rangle$ is an eigenstate with zero energy. We denote by O and (A)P the open boundary conditions and (anti-)periodic boundary conditions. The case of $M = 1, \ldots, L-1$ is shown. When $M = 0, L$, $\left|\Phi_M^L\right\rangle$ is an eigenstate for any boundary conditions.

	N: even		**N: odd**	
	L: even	L: odd	L: even	L: odd
M: even	O, P, AP	O	O, AP	O, P
M: odd	O, P, AP	O	O, P	O, AP

The behavior of singlet correlation function

- **η-clustering states** are mapped to a **ferromagnetic state** with $S^z_{\mathrm{tot}} = M/2$ as follows.

$$\hat{S}^+_x := e^{i\pi x}\hat{P}\hat{U}_{1,\dots,x-1}\hat{\eta}^\dagger_x\hat{P}$$

$$\hat{S}^-_x := e^{i\pi x}\hat{P}\hat{U}_{1,\dots,x-1}\hat{\eta}_x\hat{P}$$

$$\hat{S}^z_x := \hat{P}\left(\hat{\eta}^\dagger_x\hat{\eta}_x - \frac{1}{2}\right)\hat{P}$$

$$|0\rangle \mapsto |\downarrow,\downarrow,\dots,\downarrow\rangle$$

$\hat{P}$ ...... Projection operator to the subspace where $n_x = 0 \text{ or } N$ for all sites.

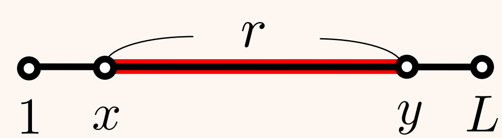
- **The singlet correlation function is mapped to**

$$\langle\hat{\eta}^\dagger_x\hat{\eta}_y\rangle \mapsto \begin{cases} (-1)^{x+y}\langle\hat{S}^+_x\hat{S}^-_y\rangle & \text{when } N \text{ is even .} \\ \langle\hat{S}^+_x e^{i\pi\sum_{j=x}^{y-1}(\hat{S}^z_j - \frac{1}{2})}\hat{S}^-_y\rangle & \text{when } N \text{ is odd.} \end{cases}$$

LRO in the bulk (for $\langle\hat{S}^+_x\hat{S}^-_y\rangle$)

Constant if S^z_{tot} is fixed and r=L (for $\hat{S}^+_x e^{i\pi\sum_{j=x}^{y-1}(\hat{S}^z_j - \frac{1}{2})}$)

→ **edge-to-edge correlations**



Relation with the Heisenberg model

- **η-clustering states** are mapped to a **ferromagnetic state** with $S^z_{\mathrm{tot}} = M/2$ as follows.

$$\hat{S}^+_x := e^{i\pi x}\hat{P}\hat{U}_{1,\ldots,x-1}\hat{\eta}^\dagger_x\hat{P}$$

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$$\hat{S}^z_x := \hat{P}\left(\hat{\eta}^\dagger_x\hat{\eta}_x - \frac{1}{2}\right)\hat{P}$$

$$|0\rangle \mapsto |\downarrow,\downarrow,\ldots,\downarrow\rangle$$

$\hat{P}$ ...... Projection operator to the subspace where $n_x = 0$ or N for all sites.

- $\hat{H}_V$ is mapped as follows.

$$\hat{P}\hat{H}_V\hat{P} \mapsto V\sum_{x=1}^{L-1}\left(-\hat{\boldsymbol{S}}_x\cdot\hat{\boldsymbol{S}}_{x+1} + \frac{1}{4}\right) \qquad V > 0$$

- This is the **ferromagnetic Heisenberg model**.

  → The energy of the **ferromagnetic states** are **lowered** by the Hamiltonian.