Exact Analysis of Entanglement Entropy in Gapped Quantum Spin Chains

Hosho Katsura, Takaaki Hirano, Yasuhiro Hatsugai

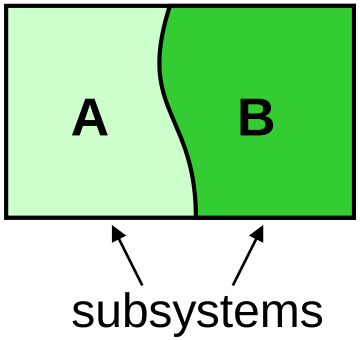
Department of Applied Physics, University of Tokyo, Japan

***Phys. Rev. B* 76, 012401(2007)**
(cond-mat/0702196)

Entanglement Entropy

Definition of Entanglement Entropy(EE)

- Many body ground state (assumed to be unique) : $|G\rangle$
- Density matrix : $\rho_{AB} = |G\rangle\langle G|$
- Reduced density matrix : $\rho_{\mathrm{A}} = \mathrm{Tr}_B \rho_{AB}$
- Entanglement Entropy : $S_A = -\mathrm{Tr}_A \rho_A \log_2 \rho_A$



Physical Meaning of the EE

- Direct product state

$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \Longrightarrow S_A = 0$$

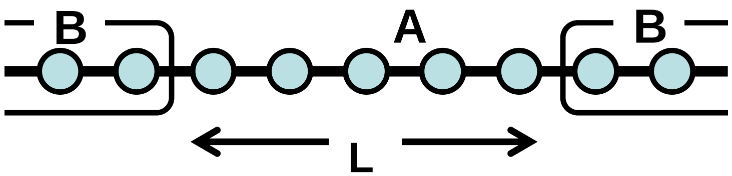
- Maximally entangled state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{D}} \sum_{j=1}^{D} |\Psi_A^j\rangle \otimes |\Psi_B^j\rangle \Longrightarrow S_A = \log_2 D$$

Entanglement Entropy in Spin Chains

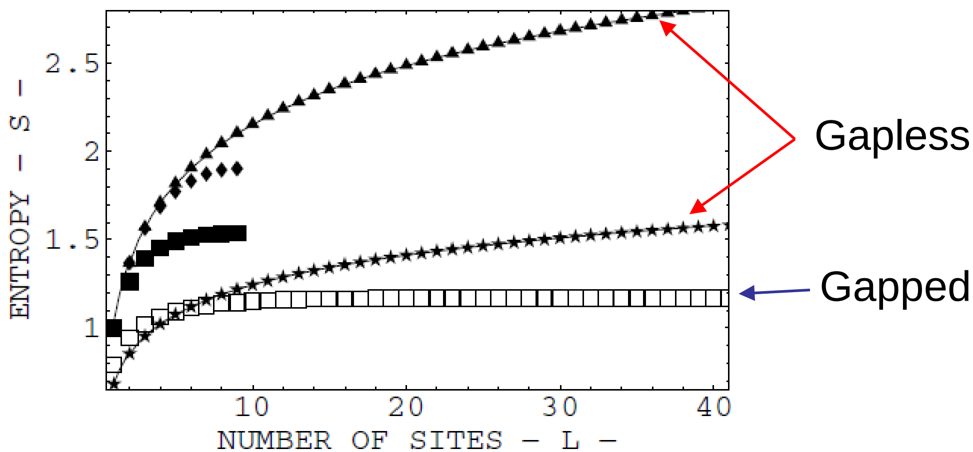
EE in the XXZ and the XY model with magnetic field

- Vidal *et al*., *PRL* **90** (2003)



$$H_{XXZ} = \sum_{l=0}^{N-1} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \sigma_l^z \sigma_{l+1}^z - \lambda \sigma_l^z),$$

$$H_{XY} = -\sum_{l=0}^{N-1} \left(\frac{a}{2} [(1+\gamma)\sigma_l^x \sigma_{l+1}^x + (1-\gamma)\sigma_l^y \sigma_{l+1}^y] + \sigma_l^z \right).$$



□ : XY(a=1.1, γ=1)

■ : XXZ(Δ=2.5, λ=0)

★ : XY(a=1, γ=1)

▲ : XY(a=∞, γ=0)

◆ : XXZ(Δ=1, λ=0)

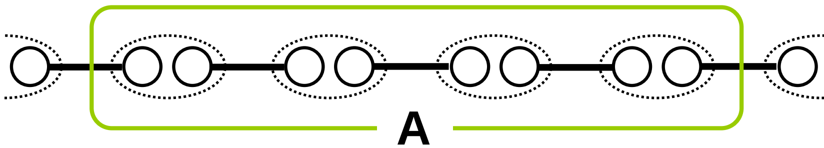
critical

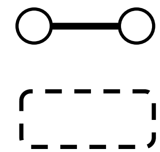
Gapless point : $S_L \sim \dfrac{c+\bar{c}}{6} \log_2 L + k$ (Conformal field theory prediction)

Conjecture: gapped→saturation, gapless→logarithmic divergence

EE in the *S*=1 valence-bond-solid state (Haldane gapped systems)

- *S*=1 valence-bond-solid state





: spin singlet

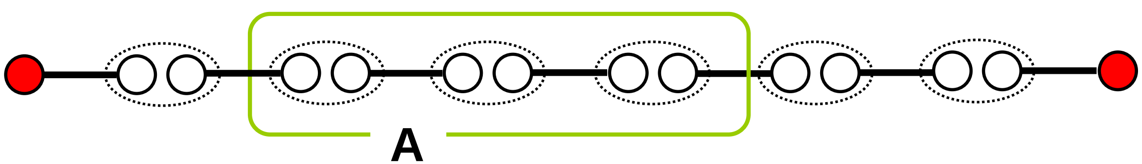
: symmetrization

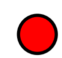
This state is a GS of the following Hamiltonian :

$$H = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2$$

Affleck-Kennedy-Lieb-Tasaki(AKLT) model (1987)

- Fan, Korepin and Roychowdhury, *PRL* **94** (2004)





: spin-1/2

1. exact expression for the EE.
2. $S_L \to \log_2 4 = 2$ in the thermodynamic limit.

$\rightarrow$ They 'partially' confirmed the conjecture proposed by Vidal et al.

Outline

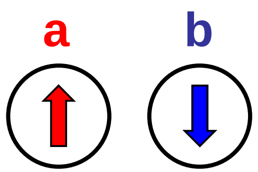
1. Generic Valence-Bond-Solid with arbitrary integer spin-*S*
2. Exact expression for the entanglement entropy of VBS
   a) Confirmation of the conjecture proposed by Vidal et al.
   b) Relationship between the EE and correlation functions
3. Edge state interpretation and its application
4. Numerical results
5. Summary

VBS with arbitrary integer-spin S

Preliminary Notions

Schwinger boson reps. of spin-operators

$$\begin{cases} S_j^+ = a_j^\dagger b_j,\ S_j^- = b_j^\dagger a_j, \\ S_j^z = (a_j^\dagger a_j - b_j^\dagger b_j)/2, \end{cases}$$



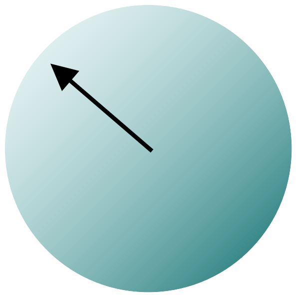
Constraint : $a_j^\dagger a_j + b_j^\dagger b_j = 2S.$

Boson *a/b* creates up/down spin.

Coherent state representation of Schwinger bosons

$$|\hat{\Omega}\rangle = \frac{(ua^\dagger + vb^\dagger)^{2S}}{\sqrt{(2S)!}}|0\rangle,$$

$$\hat{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



Spinor coordinate :

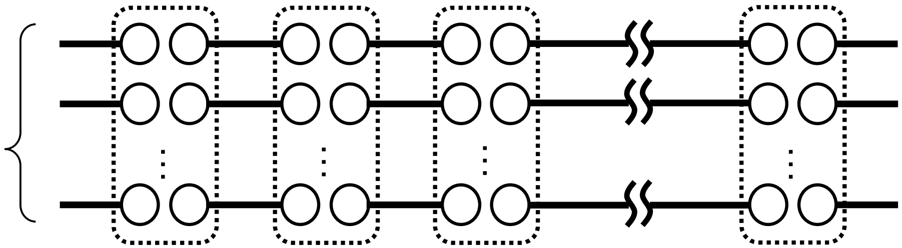
$$(u, v) = (\cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2})$$

Advantage : The constraint $a_j^\dagger a_j + b_j^\dagger b_j = 2S$ is satisfied automatically.

Construction of *S*=integer valence-bond-solid state

Translationally invariant state

S



: spin singlet

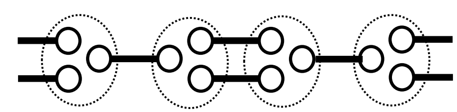
$$(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)$$

: symmetrization

Schwinger boson reps.

$$|\mathrm{VBS}\rangle = \prod_{j=0}^{L} (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S |\mathrm{vac}\rangle$$

We cannot construct a translationally invariant *S*=half-odd-integer VBS.



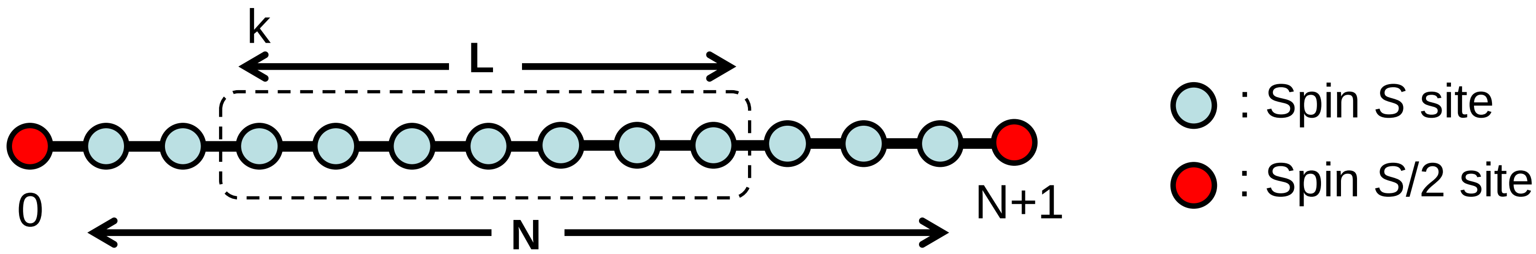
Corresponding Hamiltonian (Arovas-Auerbach-Haldane)

$$H = \sum_{j=1}^{N-1} \sum_{J=S+1}^{2S} A_J P_{j,j+1}^J + \pi_{0,1} + \pi_{N,N+1},$$

$P_{j,j+1}^J$ projects the bond spin onto the total spin *J* subspace.

$|\mathrm{VBS}\rangle$ is a zero energy GS of the above Hamiltonian.

Our setting



- Density matrix : $\rho = |\mathrm{VBS}\rangle\langle\mathrm{VBS}|/\langle\mathrm{VBS}|\mathrm{VBS}\rangle,$
- Reduced density matrix : $\rho_L = \mathrm{Tr}_{\overline{\mathcal{B}}_L}\rho$
- Entanglement Entropy : $\mathcal{S}_L = -\mathrm{Tr}_{\mathcal{B}_L}\rho_L \log_2 \rho_L$

Resolution of identity

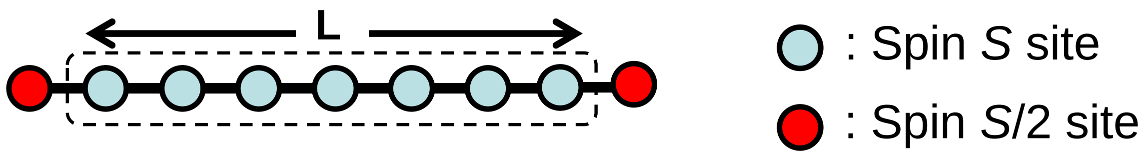
$$\mathrm{Tr}\mathcal{O} = \frac{2S+1}{4\pi}\int d\hat{\Omega}\langle\hat{\Omega}|\mathcal{O}|\hat{\Omega}\rangle$$

Integrating over $\hat{\Omega}_{k-l-1}$ $(l = 1, 2, ..., k-1)$ and $\hat{\Omega}_{k+L+m}$ $(m = 1, 2, ..., N-L-k+1)$ one can show the following

Theorem :

The reduced density matrix ρ_L does not depend on both the starting site *k* and the total length *N*.

We can set *N*=*L* without loss of generality.



Important Properties :

- Entanglement Entropy : $\mathcal{S}_L = \mathcal{S}_{\hat{L}}$ ($\because$ Schmidt decomposition)

(Reduced density matrix of two end spin-*S*/2's : $\rho_{\hat{L}}$)

$\Rightarrow$ We can focus on the RDM of two end spin-*S*/2's.

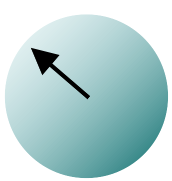
$$|\mathrm{vac}\rangle = |0,0\rangle_{0,L+1}$$

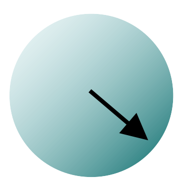
$$\rho_{\hat{L}} = \frac{\int \prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi} \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k\cdot\hat{\Omega}_{k+1}}{2}\right)^S P_0^\dagger Q_{L+1}^\dagger |\mathrm{vac}\rangle\langle\mathrm{vac}| P_0 Q_{L+1}}{(S!)^2 \int \left(\prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi}\right) \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k\cdot\hat{\Omega}_{k+1}}{2}\right)^S}$$

- Transformation property of spinor coordinates

$|\hat{\Omega}\rangle \Rightarrow |-\hat{\Omega}\rangle$

$(\theta,\phi) \Rightarrow (\pi-\theta, \phi+\pi)$





$(u,v) \Rightarrow (iv^*, -iu^*)$

$$\begin{aligned} P_0^\dagger|0\rangle &= (a_0^\dagger v_1^* - b_0^\dagger u_1^*)^S|0\rangle \\ &= (-i)^S\sqrt{S!}\,|-\hat{\Omega}_1\rangle \end{aligned}$$

($\langle 0|a^{S-l}b^{S+l}|\hat{\Omega}\rangle = \sqrt{(2S)!}u^{S-l}v^{S+l}$)

Then we can rewrite $\rho_{\hat{L}}$ as

$$\boxed{\rho_{\hat{L}} = \frac{\int \prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi} \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k \cdot \hat{\Omega}_{k+1}}{2}\right)^S |\hat{\Omega}_1\rangle_0 \langle \hat{\Omega}_1| \otimes |\hat{\Omega}_L\rangle_{L+1} \langle \hat{\Omega}_L|}{\int \left(\prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi}\right) \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k \cdot \hat{\Omega}_{k+1}}{2}\right)^S}.}$$

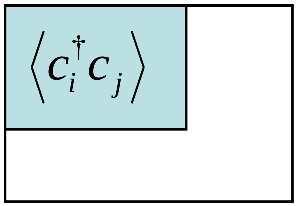
- This can be thought of as a correlation function between the density matrices $|\hat{\Omega}_1\rangle_0 \langle \hat{\Omega}_1|$ and $|\hat{\Omega}_L\rangle_{L+1} \langle \hat{\Omega}_L|$.

- Matrix elements of $\rho_{\hat{L}}$ are completely determined by

the two point correlation functions such as $\langle \vec{S}_1 \cdot \vec{S}_L \rangle$.

cf) Free fermionic models : correlation matrix $\langle c_i^\dagger c_j \rangle$

subsystem



Relation between EE and correlation functions

Eigenvalues of correlation matrix ($(S+1)^2 \times (S+1)^2$ matrix)

$\rightarrow$ **Entanglement entropy**

Eigenvalues of $\rho_{\hat{L}}$

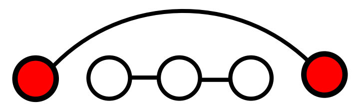
The eigenvalues of $\rho_{\hat{L}}$ can be obtaine using **transfer matrix**.

$$T_{k,k+1} = \underline{\left(\frac{1-\hat{\Omega}_k\cdot\hat{\Omega}_{k+1}}{2}\right)^S} = \frac{4\pi}{S+1}\sum_{l=0}^{S}\lambda(l)\sum_{m=-l}^{l}Y_l^m(\hat{\Omega}_k)\overline{Y_l^m(\hat{\Omega}_{k+1})}$$

$$\rho_{\hat{L}}(J) = \frac{4\pi}{(S+1)^2}\sum_{l=0}^{S}\underline{\lambda(l)^{L-1}}\, I_l\left(\frac{1}{2}J(J+1)-\frac{S}{2}\left(\frac{S}{2}+1\right)\right) \xrightarrow{L\to\infty} \frac{1}{(S+1)^2}\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$$

$$\lambda(l) \equiv (-1)^l S!(S+1)!/[(S-l)!(S+l+1)!]$$

Total spin of the edge spins $J(=0,1,2,\ldots,S)$



j-th order polynomial $I_j(X)$ $\quad I_0(X) = \frac{1}{4\pi},\ I_1(X) = \frac{3}{4\pi}\frac{X}{(S/2+1)^2}$,⋯

Higher order polynomials are determined by the recursion relation :

$$I_{j+1}(X) = \frac{2j+3}{(S+j+2)^2}\left(\frac{4X}{j+1}+j\right)I_j(X) - \frac{j}{j+1}\cdot\frac{2j+3}{2j-1}\left(\frac{S-j+1}{S+j+2}\right)^2 I_{j-1}(X),$$

Entanglement Entropy

$$\mathcal{S}_L = -\sum_{J=0}^{S}(2J+1)\rho_{\hat{L}}(J)\log_2\rho_{\hat{L}}(J) \longrightarrow 2\log_2(S+1)$$

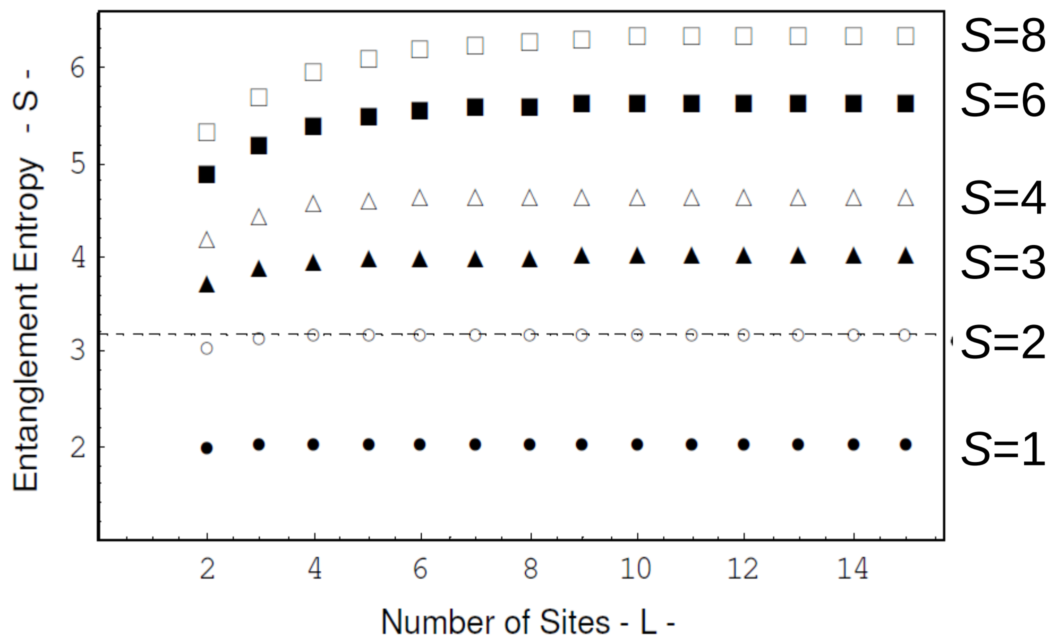
Saturation value

$\updownarrow$

Degrees of freedom

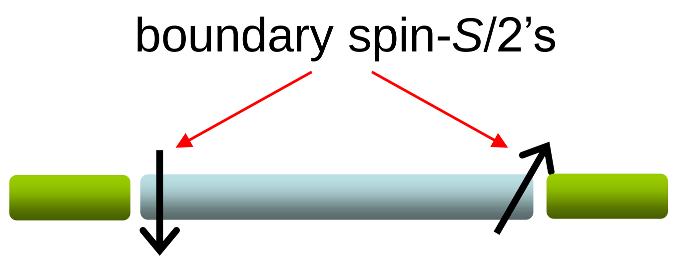
$$(2\cdot S/2+1)^2 = (S+1)^2$$

One can **exactly** calculate the finite size entanglement entropy by the recursion relation.



Edge State Interpretation and its application

	EE	Boundary spins	Degrees of freedom
1	$2\log_2 2$	Two spin-1/2's	$(2\cdot 1/2+1)^2=4$
2	$2\log_2 3$	Two spin-1's	$(2\cdot 1+1)^2=9$
S	$2\log_2(S+1)$	Two spin-*S*/2's	$(2\cdot S/2+1)^2=(S+1)^2$



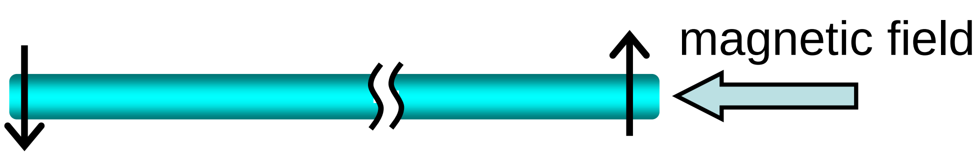
Degeneracy (open boundary)

cf) Kennedy triplet + GS

Application of the Edge states

Each edge state behaves as a free spin-*S*/2.

→ **This (*S*+1)-level system can be used as a qubit (qudit)!**



Open boundary condition

Numerical Results

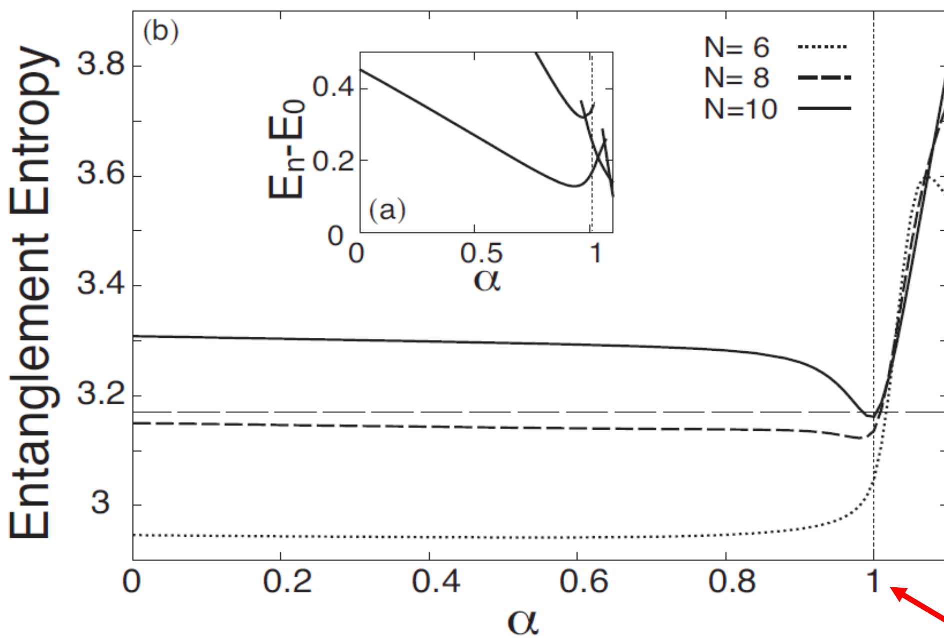
- ***S*=1 Spin Hamiltonian** (J.Phys.Soc.Jpn, **76** (2007) (Hirano and Hatsugai))
- **<u>*S*=2 Spin Hamiltonian</u>**

$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} + \alpha \left\{ \frac{2}{9}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{63}(\vec{S}_i \cdot \vec{S}_{i+1})^3 + \frac{10}{7} \right\},$$

$\alpha = 0$: Heisenberg, $\alpha = 1$: AKLT

(VBS g.s. is the exact g.s.)

EE from Exact Diagonalization



$\mathcal{S}_\infty(2)$ = $2\log_2 3$

AKLT

Minimum at the AKLT point

Summary

- Obtained the exact expression for the EE in VBS states with generic integer-Spin *S*.
- Confirmed the saturation of the EE in the generic VBS.
- Clarified the relation between the EE and the correlation function
- Proposed a novel application of the edge state as a qubit/qudit for quantum computation.