Sine-square deformations of one-dimensional critical systems

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Outline

1. **Introduction**

- Many-body problem with inhomogeneities
- What is SSD (sine-square deformation)?
- What is special about SSD?

2. **Ground state of solvable models with SSD**

- Definitions and properties
- Free fermion systems with SSD
- Other results (spin chains, Dirac fermions, CFT, …)

3. **Excited states of solvable models with SSD**

- What about excited states of SSD?
- Further steps towards exact solution

4. **Summary**

Many-body problem with inhomogeneities

Disorder and inhomogeneity are unavoidable in real systems.
And they themselves exhibit a variety of interesting phenomena!

- Magnetic and electric fields in crystals
  Hofstadter problem, Wannier-Stark ladder, …
- Quasiperiodic and/or incommensurate systems
  Quasicrystal (Schechtman, Nobel Prize 2011)
- Randomness (potential, hopping, …)
  Anderson localization
- Impurity and boundary
  Orthogonality catastrophe, Kondo problem, …

The presence of inhomogeneity and/or boundary is usually an obstacle to solvability/integrability…

Main difficulty:
Single-particle problem is already nontrivial and hard to deal with…
(Single-particle eigenstates cannot be obtained in closed form.)
What happens when the interaction is switched on?

What is SSD (sine-square deformation)?

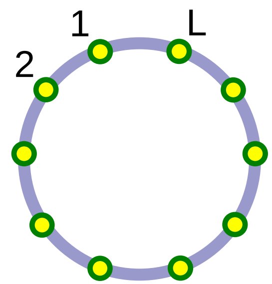
Chat with A. Kirillov and T. Nishino
Workshop “From DMRG to TNF”@YITP (Oct. 2010)

N: *“Hikihara-san found an interesting system.”*
K&K: *“Is that solvable or integrable?”*

Two `conventional’ boundary conditions

- Periodic chain

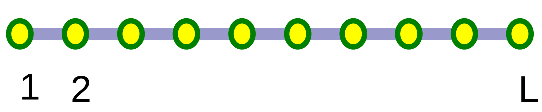
$$\mathcal{H}_0 = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



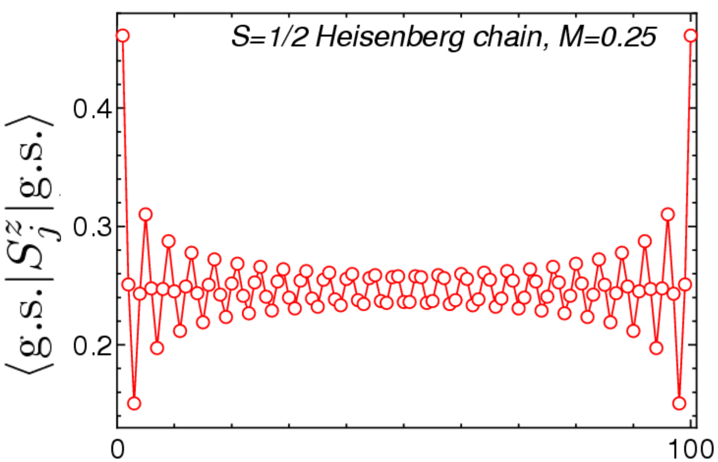
Any observable is translation invariant

- Open chain

$$\mathcal{H}_{\rm open} = \sum_{j=1}^{L-1} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



Friedel oscillation



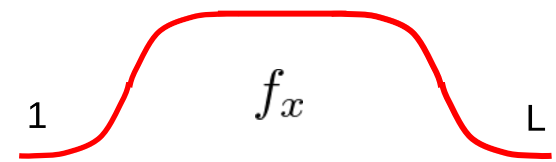
- Sine-square deformation??

What is SSD (sine-square deformation)? (contd.)

- Smooth boundary condition

  Vekic & White, *PRL* **71** (1993); *PRB* **53** (1996).

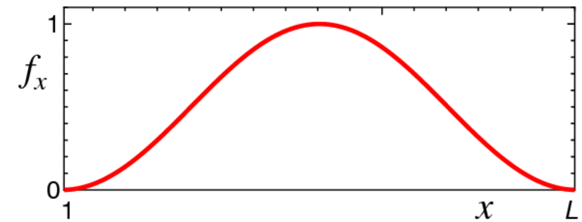
$$\mathcal{H}_f = \sum_{j=1}^{L-1} \underline{f_{j+1/2}}\, \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



Energy scale of local Hamiltonian at *x* is modified by *fx*.
This b.c., to some extent, reduces the boundary effect.

SSD = smooth b.c. with a specific fx.

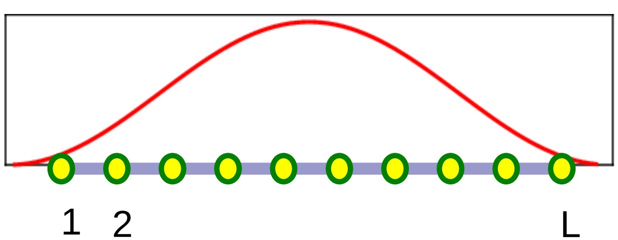
A. Gendiar *et al*., *PTP* **122** 953; **123** 393 (‘09-’10)



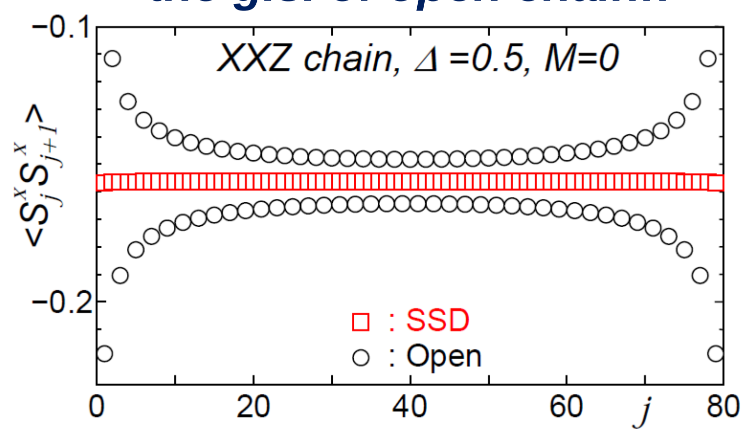
- Heisenberg chain with SSD

  Hikihara & Nishino, *PRB* **83** (2011)

$$\mathcal{H}_{\mathrm{SSD}} = \sum_{j=1}^{L-1} \sin^2(\pi j/L)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



No boundary effect in the g.s. of open chain!!



What is special about SSD?

- Suppression of the boundary effects in **1D critical** systems

  Friedel oscillation → negligible.

  g.s. correlations → position independent.

  ex) free-fermion, XXZ, 1D Hubbard (Gendiar *et al*., *PR*B **83** ('10)),

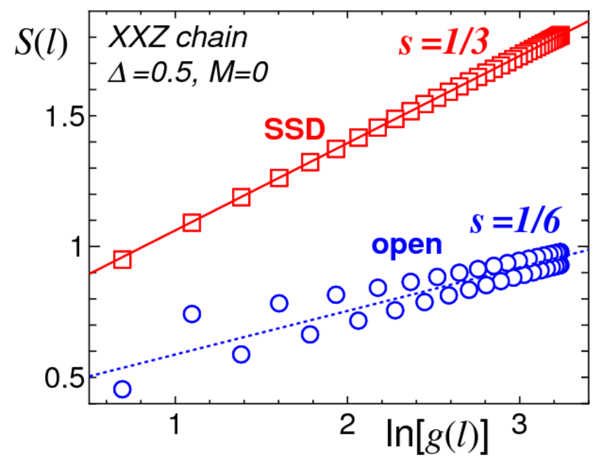
  Kondo lattice (Shibata-Hotta, *PR*B **84** ('11), …

- Scaling of entanglement entropy

$$\boxed{\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3}\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + s_1}$$

$\mathcal{S}^{\mathrm{SSD}}$

$$\mathcal{S}^{\mathrm{OBC}}(\ell, L) = \frac{c}{6}\ln\left[\frac{2L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + \frac{s_1}{2} + \ln(g)$$



- Wavefunction overlap

  The w.f. overlap between the ground states of systems with PBC and SSD is almost 1.

$$\langle \mathbf{v}_{\mathrm{SSD}} | \mathbf{v}_{\mathrm{PBC}} \rangle \approx 1$$

Motivation: to show that g.s. of $\mathcal{H}_{\mathrm{SSD}}$ ***= g.s. of*** $\mathcal{H}_{\mathrm{PBC}}$

for solvable models and conformal field theories.

→ Mechanism behind the success of SSD

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   - Definitions and properties
   - Free fermion systems with SSD
   - Other results (spin chains, Dirac fermions, CFT, …)
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   - What about excited states of SSD?
   - Further steps towards exact solution
4. **Summary**

Definitions

- **Uniform Hamiltonian with PBC**

Lattice model (L: total # of sites)

$$\mathcal{H}_0 = \sum_{j=1}^{L} h_j + \sum_{j=1}^{L} h_{j,j+1}$$

Field theory $(\ell = La)$

$$\mathcal{H}_0 = \int_0^{\ell} h(x)\, dx$$

- **Chiral Hamiltonian**

$$\mathcal{H}_\pm = \sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} h_j + \sum_{j=1}^{L} e^{\pm i\delta j} h_{j,j+1} \qquad \delta = \frac{2\pi}{L}$$

$$\mathcal{H}_\pm = \int_0^{\ell} e^{\pm i\delta x} h(x)\, dx \qquad \delta = \frac{2\pi}{\ell}$$

- **Sine-square deformed (SSD) Hamiltonian**

$$\mathcal{H}_{\rm SSD} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

ex) $$\mathcal{H}_{\rm SSD} = \frac{1}{2}\sum_{j=1}^{L}\left(1 - \frac{1}{2}e^{i\delta j} - \frac{1}{2}e^{-i\delta j}\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$

$$= \sum_{j=1}^{L}\sin^2\left(\frac{\pi}{L}j\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$

$$\boxed{\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}}$$

Properties

The ground state (vacuum) of $\mathcal{H}_0$ is

a) an exact eigenstate of $\mathcal{H}_{\rm SSD}$ (if $\mathcal{H}_\pm|0\rangle = 0$).

b) ***a*** ground state of $\mathcal{H}_{\rm SSD}$ (if $\mathcal{H}_{\rm SSD}$ is positive semi-definite)

c) the ***unique*** ground state of $\mathcal{H}_{\rm SSD}$ (if Perron-Frobenius applies)

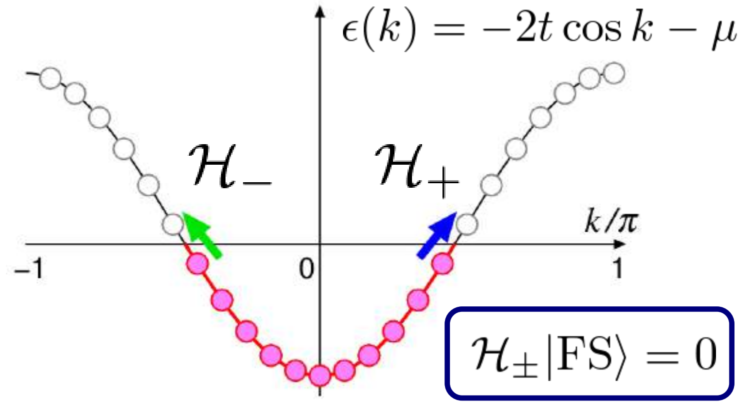
More limited…

Free fermion systems with SSD

- **Uniform Hamiltonian with PBC**

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} c_j^\dagger c_j$$

Fourier.tr. $\Rightarrow$ $\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k$



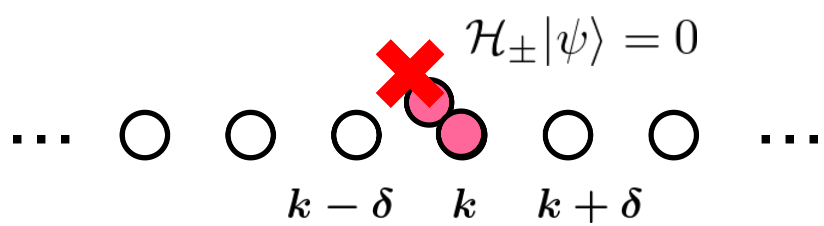
Ground state:
Fermi sea state ($\epsilon(k) < 0$ is occupied) $\mathcal{H}_0|\mathrm{FS}\rangle = E_g|\mathrm{FS}\rangle$

- **Chiral Hamiltonian** $\left(\delta = \frac{2\pi}{L}\right)$

$$\mathcal{H}_\pm = -t\sum_{j=1}^{L} e^{\pm i\delta j}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} c_j^\dagger c_j$$

Chiral Hamiltonian in *k*-space:

$$\mathcal{H}_\pm = e^{\mp i\delta/2}\sum_k \epsilon(k\mp\delta/2) c_k^\dagger c_{k\mp\delta}$$



If $\epsilon(k_\mathrm{F}+\delta/2) = \epsilon(-k_\mathrm{F}-\delta/2) = 0$ ***,***
then $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0$ ***due to the Pauli principle.*** $(c_k^\dagger)^2 = 0$

- **SSD Hamiltonian (OBC)**

(in real space)

$$\mathcal{H}_{\rm SSD} = -t\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L-1}\sin^2\left[\frac{\pi}{L}\left(j-\frac{1}{2}\right)\right]c_j^\dagger c_j$$

(in terms of uniform & chiral Hamiltonians)

$$\mathcal{H}_{\rm SSD} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

Chiral Hamiltonians do not disturb the Fermi sea! $\boxed{\mathcal{H}_\pm|{\rm FS}\rangle = 0}$

$$\mathcal{H}_{\rm SSD}|{\rm FS}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|{\rm FS}\rangle = \frac{E_g}{2}|{\rm FS}\rangle$$

Fermi sea is an exact eigenstate of SSD Hamiltonian!

- Abandon “from few to many” approach!

Solve many-body problem ***without*** using single-particle solutions.

Single-particle solutions of $\mathcal{H}_{\rm SSD}$ are not plane waves $\phi_k(j) = e^{\mathbf{i}kj}$
But slater determinants become identical when the states are occupied up to $E_{\rm F}$.

$$\det[\psi_k(j)]_{k,j=1,\ldots,N} = \det[\phi_k(j)]_{k,j=1,\ldots,N}$$

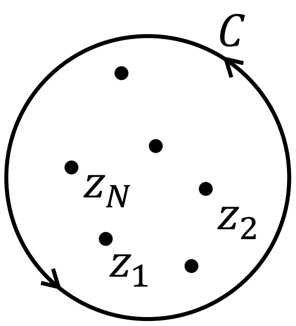
- **Fermi sea is the *unique* ground state of** $\mathcal{H}_{\mathrm{SSD}}$
  Jordan-Wigner tr.
  Free-fermion chain → XY spin chain in a field
  Perron-Frobenius thm.
  The g.s. of $\mathcal{H}_{\mathrm{SSD}}$ ($\mathcal{H}_0$) is nondegenerate.
  The g.s. of $\mathcal{H}_{\mathrm{SSD}}$ has nonvanishing overlap with the g.s. of $\mathcal{H}_0$.

- **Mysterious identity** (H.K., *JPA **44**, 252001 ('11).)*
  First quantized picture:
  Fermi sea state = Vandermonde det.

$$\Delta(z_1, \ldots, z_N) = \prod_{1 \le i < j \le N} (z_i - z_j) \qquad z_j = \exp\left(i \frac{2\pi}{L} x_j \right)$$

The fact that $\mathcal{H}_{\pm} |\mathrm{FS}\rangle = 0$ translates into a remarkable identity!
For any set of {z_1, …, z_N} and t, we have

$$\sum_{j=1}^{N} z_j \prod_{k(\neq j)} \frac{z_j - t z_k}{z_j - z_k} = \sum_{j=1}^{N} z_j$$



$$f(z) = \prod_{1 \le j \le N} \frac{z - t z_j}{z - z_j}$$

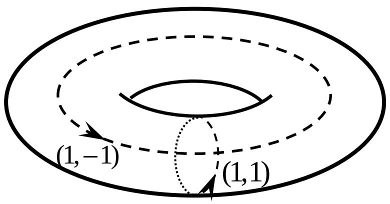
First nontrivial identity from the generating function of the Macdonald operators

Application to Hofstadter problem

- **Inhomogeneous tight-binding model**

$$-2\sin\left(\pi\frac{P}{Q}j\right)\varphi_{j+1} - 2\sin\left(\pi\frac{P}{Q}(j-1)\right)\varphi_{j-1} = \epsilon\varphi_j$$

$$p_+ = \pi/2$$



Hofstadter problem on a special line (**mid-band condition**)

(Functional) Bethe ansatz:
Wiegmann-Zabrodin, PRL **72** (‘94); NPB **422** (‘96).
Exact *E=0* (single-particle) state
Hatsugai-Kohmoto-Wu, PRL **73** (‘94); PRB **53** (‘96).

- **Many-body *E=0* state**

Hamiltonian (second quantization) → ***Sine deformation!***

$$\mathcal{H}_{\sin} = -2\sum_{j=1}^{2Q}\sin\left(\pi\frac{P}{Q}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \quad = -i(\mathcal{H}_+ - \mathcal{H}_-)$$

$$\mathcal{H}_\pm = -\sum_{j=1}^{L}\exp\left(i\frac{2\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

if P=1 and L=2Q
(Q should be odd).

Fermi sea (with N_e=Q) is an ***exact E=0 eigenstate*** of $H_{\sin}$
since $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0$. (NOTE: |FS> is not the ground state of $H_{\sin}$.)

Other results (1)

Anisotropic XY chain

- **Uniform Hamiltonian with PBC**

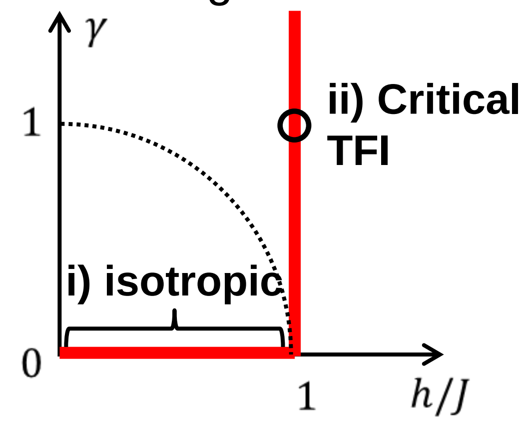
$$\mathcal{H}_0 = -J\sum_{j=1}^{L}[(1+\gamma)S_j^x S_{j+1}^x + (1-\gamma)S_j^y S_{j+1}^y] - h\sum_{j=1}^{L} S_j^z,$$

Jordan-Wigner,
Fourier, Bogoliubov. tr.

$$\mathcal{H}_0 = \sum_{k\in\mathcal{K}} \epsilon_0(k)\left(d_k^\dagger d_k - \frac{1}{2}\right)$$

$$\epsilon_0(k) = \sqrt{(h+J\cos k)^2 + (J\gamma\sin k)^2}$$

Phase diagram



Ground state: $d_k|0\rangle = 0$ for all *k*.

- **Chiral Hamiltonian** $\left(\delta = \frac{2\pi}{L}\right)$

$$\mathcal{H}_\pm = -J\sum_{j=1}^{L} e^{\pm i\delta j}[(1+\gamma)S_j^x S_{j+1}^x + (1-\gamma)S_j^y S_{j+1}^y] - h\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} S_j^z$$

$$\mathcal{H}_\pm = \frac{1}{2}e^{\mp i\delta/2}\sum_{k\in\mathcal{K}}\left[\epsilon_\pm(k)d_k^\dagger d_{k\mp\delta} - \cancel{i\eta_\pm(k)d_k^\dagger d_{-k\pm\delta}^\dagger} + i\eta_\pm(k)d_{-k}d_{k\mp\delta} - \epsilon_\pm(k)d_{-k}d_{k\mp\delta}^\dagger,\right]$$

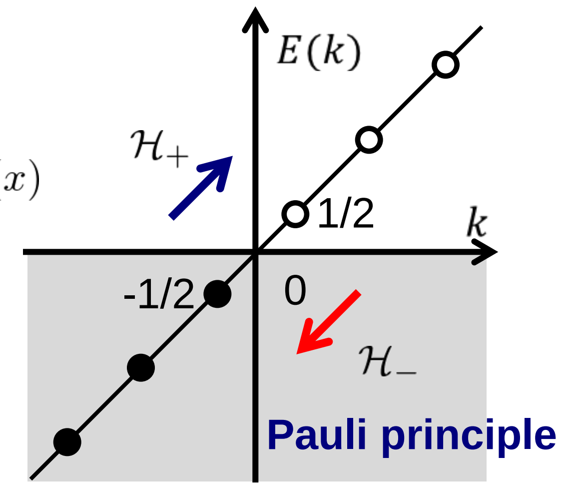
$\eta_\pm(k) = 0$ for all *k* when i) $\gamma = 0$, ii) $\gamma = 1,\ h/J = 1$

$\rightarrow$ $\mathcal{H}_\pm|0\rangle = 0$ $\qquad \mathcal{H}_{\rm SSD}|0\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|0\rangle = \frac{E_g}{2}|0\rangle$

The ground state $|0\rangle$ is the ***unique*** ground state of $\mathcal{H}_{\rm SSD}$ (**Property c**).
(From Perron-Frobenius thm.)

Other results (2)

Dirac fermions



- **Uniform Hamiltonian with APBC** $\psi_R(x+\ell) = -\psi_R(x)$

$$\mathcal{H}_0 = -i\frac{v_F}{2\pi}\int_0^\ell dx\ :\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x):$$

Fourier tr. $\Rightarrow$

$$\mathcal{H}_0 = \frac{2\pi}{\ell}v_F \sum_{n\in\mathbb{Z}+\frac{1}{2}} n :\psi_{R,n}^\dagger \psi_{R,n}:$$

Ground state: Dirac sea $\quad \mathcal{H}_0|\mathrm{DS}\rangle = 0$

- **Chiral Hamiltonian** $\left(\delta = \frac{2\pi}{\ell}\right)$ $\quad (\mathcal{H}_\pm)^\dagger = \mathcal{H}_\mp$

$$\mathcal{H}_\pm = -i\frac{v_F}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x} :\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x): \pm \frac{\pi v_F}{2\ell}\frac{1}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x} :\psi_R^\dagger(x)\psi_R(x):$$

$$\mathcal{H}_\pm = \frac{2\pi}{\ell}v_F \sum_{n\in\mathbb{Z}+\frac{1}{2}} \left(n \pm \frac{1}{2}\right) \psi_{R,n\pm 1}^\dagger \psi_{R,n} \quad \rightarrow \quad \mathcal{H}_\pm|\mathrm{DS}\rangle = 0$$

- **Positive semi-definiteness**

$$\mathcal{H}_{\mathrm{SSD}} = \sum_{n\in\mathbb{Z}+\frac{1}{2}, n>0} (\alpha_{R,n}^\dagger \alpha_{R,n} + \beta_{R,n}\beta_{R,n}^\dagger)$$

$$\alpha_{R,n} = \psi_{R,n} - \psi_{R,n+1}$$
$$\beta_{R,n} = \psi_{R,-n} - \psi_{R,-n-1}$$

$\rightarrow$ $\langle\Psi|\mathcal{H}_{\mathrm{SSD}}|\Psi\rangle \geq 0$ for any state.

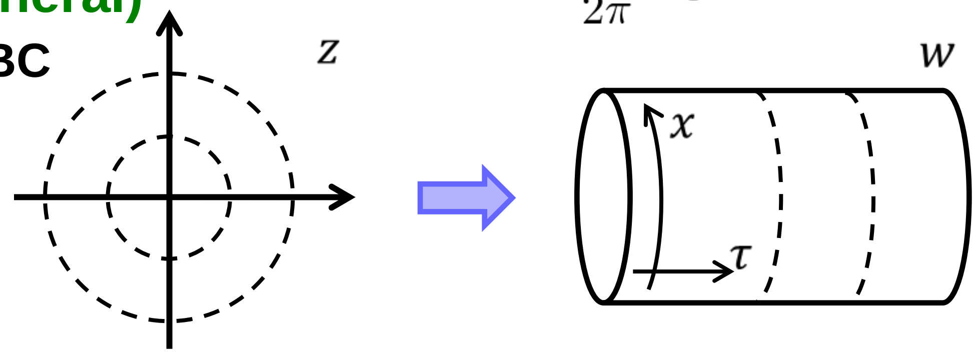
The Dirac sea is ***a*** ground state of $\mathcal{H}_{\mathrm{SSD}}$ (**Property b**).

Field Theory

Conformal field theories (General)

- **Uniform Hamiltonian with PBC**

$$w = \frac{\ell}{2\pi}\log z$$



$$\mathcal{H}_0 = \int_0^{\ell} \frac{dx}{2\pi}\left(T_{\mathrm{cyl}}(w) + \overline{T}_{\mathrm{cyl}}(\overline{w})\right)$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Ground state: vacuum state $|\mathrm{vac}\rangle = |0\rangle \otimes |\overline{0}\rangle$

- **Chiral Hamiltonian** $\left(\delta = \frac{2\pi}{\ell}\right)$

$$\mathcal{H}_{\pm} = \int_0^{\ell} \frac{dx}{2\pi}\left(e^{\pm\delta w} T_{\mathrm{cyl}}(w) + e^{\mp\delta\overline{w}}\,\overline{T}_{\mathrm{cyl}}(\overline{w})\right)$$

***SL*(2,C) invariance**:*
$L_0|0\rangle = L_{\pm 1}|0\rangle = 0$

$\rightarrow$ $\mathcal{H}_{\pm}|\mathrm{vac}\rangle = 0$ $\qquad$ $\mathcal{H}_{\mathrm{SSD}}|\mathrm{vac}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|\mathrm{vac}\rangle = 0$

The vacuum is an exact eigenstate of $\mathcal{H}_{\mathrm{SSD}}$ (**Property a**)

- **Left-Right decomposition**

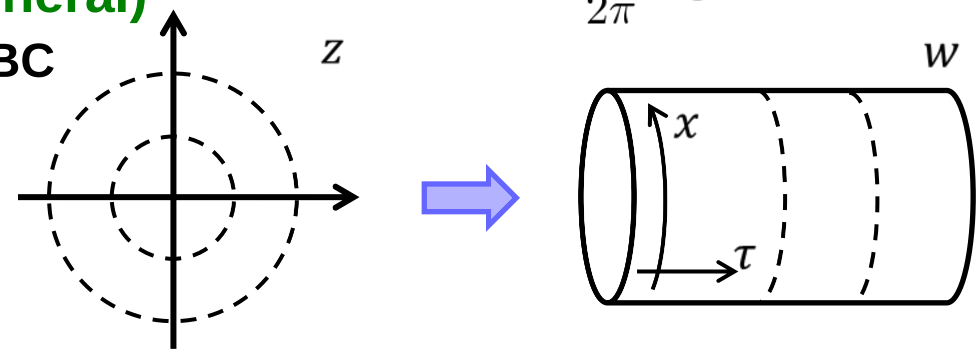
$$\mathcal{H}_{\mathrm{SSD}} = \mathcal{H}_L + \mathcal{H}_R - \underline{\frac{\pi c}{12\ell}} \qquad \mathcal{H}_L = \frac{\pi}{\ell}\left(L_0 - \frac{L_{+1} + L_{-1}}{2}\right)$$

Finite-size scaling of the g.s. energy of $\mathcal{H}_{\mathrm{SSD}}$

Field Theory

Conformal field theories (General)

$$w = \frac{\ell}{2\pi} \log z$$



- **Uniform Hamiltonian with PBC**

$$\mathcal{H}_0 = \frac{2\pi}{\ell}(L_0 + \overline{L}_0) - \frac{\pi c}{6\ell}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Ground state: vacuum state $|\mathrm{vac}\rangle = |0\rangle \otimes |\overline{0}\rangle$

- **Chiral Hamiltonian** $\left(\delta = \frac{2\pi}{\ell}\right)$

$$\mathcal{H}_\pm = \frac{2\pi}{\ell}(L_{\pm 1} + \overline{L}_{\mp 1})$$

SL(2,C) invariance:

$$L_0|0\rangle = L_{\pm 1}|0\rangle = 0$$

$\rightarrow$ $\mathcal{H}_\pm|\mathrm{vac}\rangle = 0$ $\qquad \mathcal{H}_{\mathrm{SSD}}|\mathrm{vac}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|\mathrm{vac}\rangle = 0$

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Finite-size scaling of the g.s. energy of $\mathcal{H}_{\mathrm{SSD}}$

c=1 CFT (free-boson theory)

- **Bosonization of Virasoro generators**

$$L_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} : a_m a_{n-m} :$$

Heisenberg algebra

$$[a_n, a_m] = n\delta_{n+m,0}$$

- **Positive semi-definiteness of H_{SSD}** (in charged bosonic Fock space)

$$\mathcal{H}_L = \frac{\pi}{2\ell}\sum_{n\geq 0}(a_{-n} - a_{-n-1})(a_n - a_{n+1}) \longleftarrow \quad a_n^\dagger = a_{-n}$$

The vacumm is ***a*** ground state of $\mathcal{H}_{\mathrm{SSD}}$ (**Property b**)

- ***H_{SSD}* in real space**

$$\mathcal{H}_L + \mathcal{H}_R = \int_0^\ell dx\, f(x) : \left(\frac{d\phi_L}{dx}\right)^2 + \left(\frac{d\phi_R}{dx}\right)^2 :, \qquad f(x) = \sin^2\left(\frac{\pi x}{\ell}\right)$$

CFT associated with affine Lie (Kac-Moody) algebra

WZW model, Sutherland, Babudjan-Takhtajan, …

$$c = \frac{k\,\dim\mathfrak{g}}{k + h^\vee}$$

- **Sugawara construction**

$$L_n = \frac{1}{2(k+h^\vee)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m\in\mathbb{Z}} : J^a_m J^a_{n-m} :$$

Affine Lie algebra

$$[J^a_n, J^b_m] = i f^{ab}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}$$

$$\mathcal{H}_{\mathrm{L}} = \frac{\pi}{2\ell(k+h^\vee)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{n\geq 0}(J^a_{-n} - J^a_{-n-1})(J^a_n - J^a_{n+1})$$

(Positive semi-definite)

The vacuum is ***a*** ground state of $\mathcal{H}_{\mathrm{SSD}}$ (**Property b**)

N=2 SCFT with SSD (*J(z)*: Δ=1 U(1) current, *G*$^{\pm}$*(z)*: Δ=3/2 fermionic fields)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$
$$[L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right) G_{m+r}^{\pm},$$
$$[L_m, J_n] = -nJ_{m+n},$$
$$[J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm},$$
$$\{G_r^{\pm}, G_s^{\mp}\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},$$
$$\{G_r^{+}, G_s^{+}\} = \{G_r^{-}, G_s^{-}\} = 0,$$
$$[J_m, J_n] = \frac{c}{3} m\delta_{m+n,0},$$

$m, n \in \mathbb{Z},$

$r, s \in \mathbb{Z} + \alpha$

$\alpha = 0 \quad (\text{Ramond})$

$\alpha = \frac{1}{2} \quad (\text{Neveu} - \text{Schwarz})$

- ***H*$_0$ in the Ramond sector**

$$\frac{2\pi}{\ell}\left(L_0 - \frac{c}{24}\right) = \frac{\pi}{\ell}\{G_0^+, G_0^-\}$$

- ***H*$_{\text{SSD}}$ in the Neveu-Schwarz sector**

$$\mathcal{H}_{\mathrm{L}} = \frac{\pi}{\ell}\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) = \frac{\pi}{2\ell}\{Q, Q^{\dagger}\}$$

The vacuum is b) ***a*** ground state of $\mathcal{H}_{\mathrm{SSD}}$

Supersymmetry

$$Q^{\dagger} = \frac{G^+_{\frac{1}{2}} - G^+_{-\frac{1}{2}}}{\sqrt{2}}$$
$$Q^2 = (Q^{\dagger})^2 = 0$$

Outline

1. **Introduction**
   - Many-body problem with inhomogeneities
   - What is SSD (sine-square deformation)?
   - What is special about SSD?
2. **Ground state of solvable models with SSD**
   - Definitions and properties
   - Free fermion systems with SSD
   - Other results (spin chains, Dirac fermions, CFT, …)
3. **Excited states of solvable models with SSD**
   - What about excited states of SSD?
   - Further steps towards exact solution
4. **Summary**

What about excited states of SSD?

- Gapped or gapless?
  - **Lieb-Schultz-Mattis argument**
    LSM, *Ann. Phys*. **16** (‘61), Affleck & Lieb, *Lett. Math. Phys*. **12** (‘86).

G.S. of H_{SSD} (XY-chain) $\quad\Rightarrow\quad$ Trial state

$$|\Psi_0\rangle \quad\Longrightarrow\quad |\Psi_1\rangle := U|\Psi_0\rangle, \quad U = \exp\left(i\sum_{j=1}^{L}\frac{2\pi}{L}jS_j^z\right)$$

Orthogonality

$|\Psi_0\rangle$ is translation invariant.

$$\langle\Psi_0|\Psi_1\rangle = \langle\Psi_0|U|\Psi_0\rangle$$
$$= \langle\Psi_0|TUT^{-1}|\Psi_0\rangle = -e^{-2\pi iM/L}\langle\Psi_0|\Psi_1\rangle$$

$$\boxed{\langle\Psi_0|\Psi_1\rangle = 0}$$

if M (magnetization) is not $\pm L/2$.

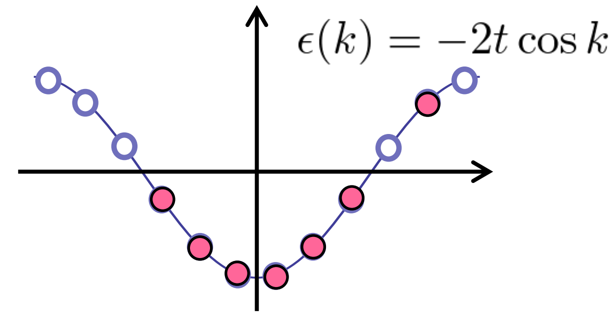
Upper bound on the gap

$$\Delta E = \langle\Psi_1|H_{\mathrm{SSD}}|\Psi_1\rangle - \langle\Psi_0|H_{\mathrm{SSD}}|\psi_0\rangle$$
$$= \langle\Psi_0|U^\dagger H_{\mathrm{SSD}}U - H_{\mathrm{SSD}}|\Psi_0\rangle$$
$$\leq \frac{\pi^2 J}{L} + O(1/L^2)$$

because $\sin^2(\pi j/L)$ is $O(1)$ for all j.

It is plausible that 1d critical system with SSD is still critical.
But is this upper bound really optimal? → NO!

Free fermion systems with SSD



Ansatz state: $|\Psi\rangle = \left(\sum_{k\in\circ} \psi_k c_k^\dagger\right)|\mathrm{FS}\rangle$

Fermi sea + one extra fermion

Using $(\mathcal{H}_{\mathrm{SSD}} - E_g/2)|\mathrm{FS}\rangle = 0$,
we get Harper-like eq. in *k*-space (*m*=0,1,…,*L*/2-1):

$$-\sin\left(\frac{2\pi}{L}m\right)\psi_{m-1} + 2\sin\left[\frac{2\pi}{L}\left(m+\frac{1}{2}\right)\right]\psi_m - \sin\left[\frac{2\pi}{L}(m+1)\right]\psi_{m+1} = \varepsilon\psi_m$$

$$\forall\psi_m = \sqrt{\frac{2}{L}}$$

- **Scaling of excitation energy**

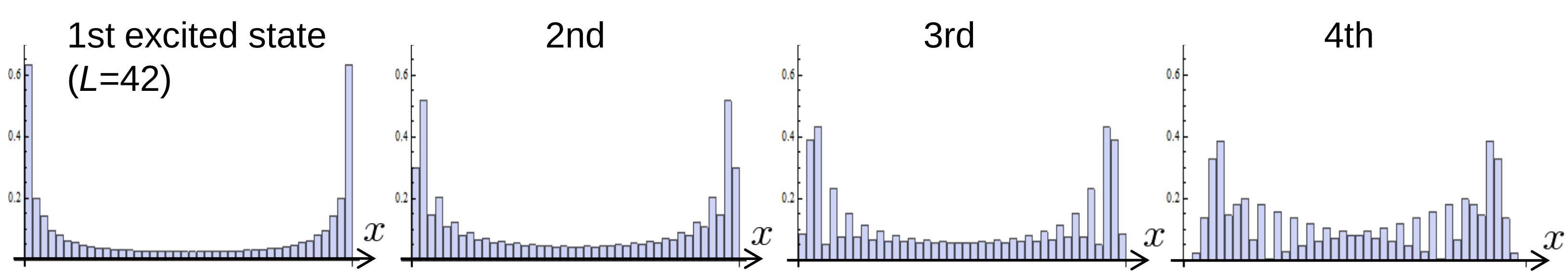
  A simple variational argument shows that $\varepsilon \leq \frac{2\pi}{L^2} + O\left(\frac{1}{L^3}\right)$.

  Very low-energy states!

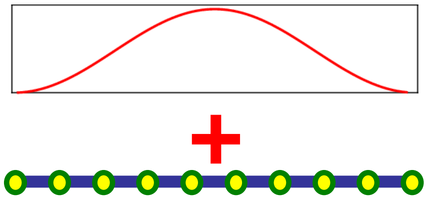
  Usual CFT scaling (1/*L*) breaks down.

- **Spatial profile of excited state**

Low-energy states ~ edge states



Summary



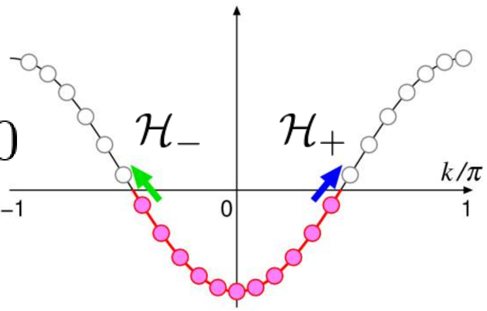
Hamiltonian with Sine-Square Deformation (SSD)
shares the same ground state with the periodic chain.
→ ***extremely efficient smooth boundary condition***

Mechanism of SSD:

Chiral Hamiltonians annihilate the periodic g.s.

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

$$\mathcal{H}_\pm|0\rangle = 0$$



ex) Free-fermion chain, anisotropic XY, Dirac fermions

CFT interpretation:

- Chiral Hamiltonians are $L_{\pm 1}$ in CFT
- SL(2, **C**) invariance → $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$
- Positive semi-definite H_{SSD} → The vacuum is still the vacuum.

Unsolved mysteries:

- SSD for unitary minimal CFTs, e.g. Potts chain (c=4/5)
- Uniqueness of the g.s. for affine Lie CFT, SCFT, …
- Exact results for lattice SSD not reducible to free fermions
- Excited states of free fermions with SSD (Bethe ansatz?)