Periodically and Quasi-Periodically Driven Complex Systems
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Algebraic construction of quantum many-body scars

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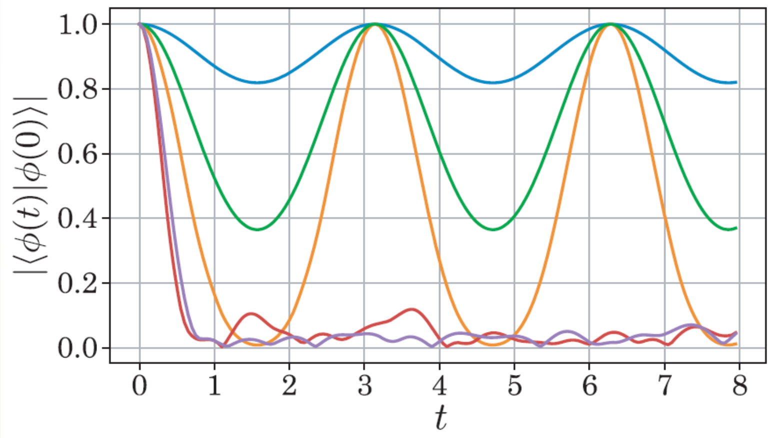


Institute for Physics of Intelligence

Trans-Scale Quantum Science Institute

Acknowledgments

- Onsager's scars
  - Naoyuki Shibata, Nobuyuki Yoshioka & HK, PRL **124**, 180604 (2020)
- Fermionic models
  - Hironobu Yoshida & HK, PRB **105**, 024520 (2022)
  - Kensuke Tamura & HK, PRB **106**, 144306 (2022)
- Integrable boundary states, scalar chirality, …
  - Kazuyuki Sanada, Yuan Miao & HK, arXiv:2304.13624.
- Dzyaloshinskii-Moriya + Zeeman model
  - Masaya Kunimi, Takafumi Tomita, HK & Yusuke Kato, arXiv:2306.05591

Outline

1. Introduction and Motivation

- Eigenstate thermalization hypothesis (ETH)
- Violation of ETH
- Rydberg-atom array & PXP model
- Quantum many-body scars (QMBS)

2. Onsager scars

3. Other scarred models

4. Summary

Eigenstate thermalization hypothesis (ETH)

- Setup

  H: Hamiltonian, $|E_n\rangle$: (normalized) energy eigenstate,
  O: macroscopic observable, ρ_{mc}: MC ensemble,
  Energy shell: $\mathrm{span}\{|E_n\rangle : H|E_n\rangle = E_n|E_n\rangle, E_n \in [E-\Delta E, E)\}$

- Thermal states

  A state $|E_n\rangle$ is said to be thermal if $\langle E_n|O|E_n\rangle \simeq \mathrm{Tr}[\rho_{\mathrm{mc}} O]$.

- Strong ETH: All $|E_n\rangle$ in the energy shell are thermal.

  Believed to be true for a large class of non-integrable systems

  Concept: von Neumann, Deutsch, Srednicki, Tasaki, …
  Numerical evidence: D'Alessio et al., Adv. Phys. **65** (2016).

- Weak ETH: Almost all $|E_n\rangle$ in the energy shell are thermal.

  Proved under certain conditions: translational sym., local interaction
  Biroli, Kollath, Lauchli, PRL **105** (2010);
  Iyoda, Kaneko, Sagawa, PRL **119** (2017)

Exceptions of strong ETH

1. Integrable systems
   Many conserved charges
   Strong ETH , Weak ETH

Ex.) S=1/2 Heisenberg chain

$$H_{\mathrm{Hei}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

2. Many-body localized (MBL) systems
   Emergent local integrals of motion
   Strong ETH , Weak ETH

$$H_{\mathrm{MBL}} = H_{\mathrm{Hei}} + \sum_{j=1}^{L} h_j S_j^z$$

3. Hilbert-space fragmentation
   Hilbert space splits into exp. many sectors
   Strong ETH , Weak ETH &

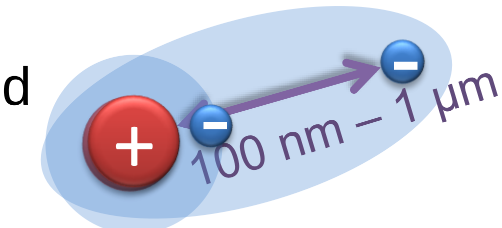
4. Quantum many-body scarred systems
   Strong ETH , Weak ETH
   Non-integrable but have *scarred* states which
   do not thermalize for an anomalously long time!

Cf.) One-body scars in a Bunimovich stadium
E. Heller, PRL **53** (1984)

Experiment on Rydberg atom arrays

Bernien *et al.*, Nature **551** (2017)

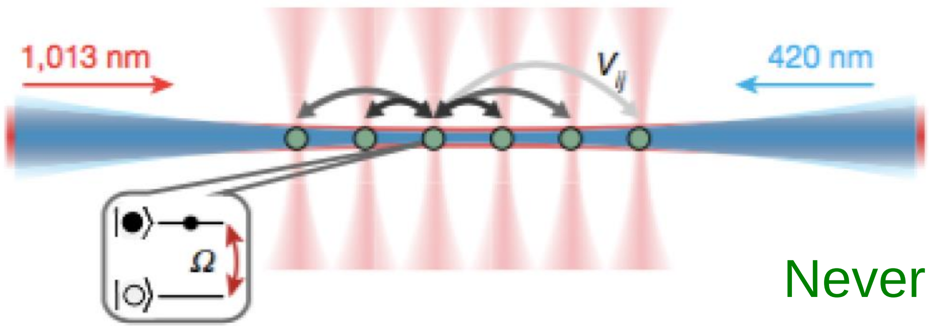
^{87}Rb: el. in 5s → 70s

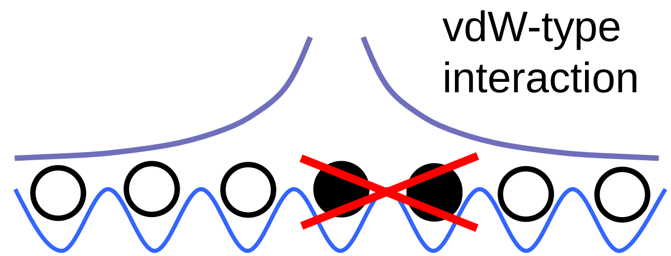


- Rydberg atoms

  Atoms in which one of the electrons is in an excited state with a very high principal quantum number.

- Rydberg blockade





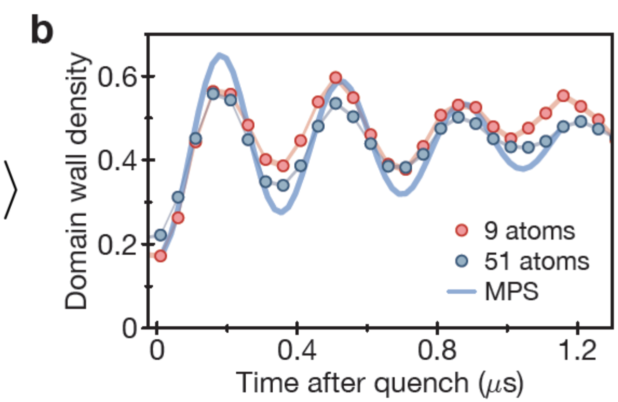
Never have adjacent excited states

- A surprising finding!

  Special initial states

$$|\mathbf{Z}_2\rangle = |\bullet \circ \bullet \circ \cdots\rangle, \ \ |\mathbf{Z}_2'\rangle = |\circ \bullet \circ \bullet \cdots\rangle$$

  Exhibit robust oscillations. Other initial states thermalize much more rapidly.

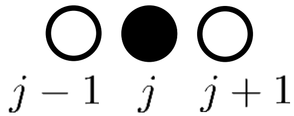


PXP model

- Hamiltonian

Turner *et al.*, Nat. Phys. **14**, 745 (2018)

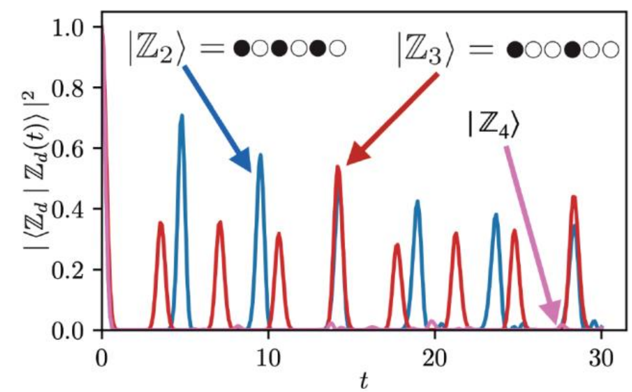
$$H_{\mathrm{PXP}} = \sum_j P_{j-1} X_j P_{j+1}, \qquad P = |\circ\rangle\langle\circ|,\ X = |\circ\rangle\langle\bullet| + |\bullet\rangle\langle\circ|$$



Particular case of:
Fendley, Sengupta & Sachdev, PRB **69** (2004);
Lesanovsky & Katsura, PRB **86** (2012)

- Properties
  1. Level statistics
     → Wigner-Dyson, non-integrable
  2. Long-time oscillations
  3. Energy (E) v.s. entanglement entropy (S)
     → Anomalously low S at high E

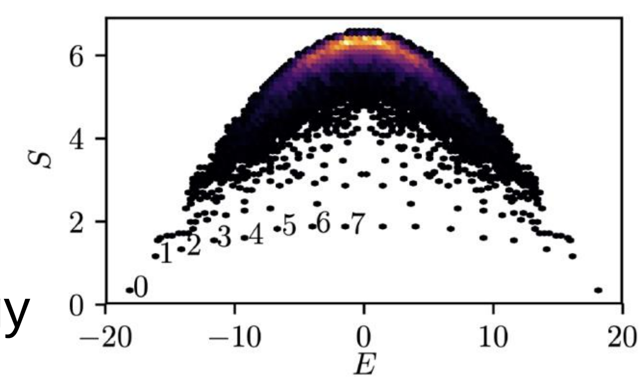
Revivals of fidelity



- Exact QMBS

  Lin and Motrunich, PRL **122**, 173401 (2019).

  Exact eigenstates of H_{PXP} in the form of matrix product states (MPS)
  → Low entanglement states at high energy



Exact QMBS

Frustration-free system

$$H = \sum_j A_j^\dagger A_j, \quad A_j|\psi_0\rangle = 0 \ \forall j$$

$$\Rightarrow \quad H_{\rm new} = \sum_j A_j^\dagger C_j A_j$$

- Embedding method

  Shiraishi & Mori, PRL **119** (2017)

- AKLT models

  Moudgalya, Regnault & Bernevig, PRB **98** (2018)
  Mark, Lin & Motrunich, PRB **101** (2020)

- Ising and XY-like models

  Iadecola & Schecter, PRB **101** (2020)
  Chattopadhyay, Pichler, Lukin, Ho & PRB **101** (2020)

- Floquet scars

  Driven PXP: Sugiura, Kuwahara, Saito, PRR **3** (2021)
  Mizuta, Takasan & Kawakami, PRR **2** (2020)

- Recent reviews

  Serbyn, Abanin & Papic, Nat. Phys. **17** (2021)
  Moudgalya, Bernevig & Regnault, Rep. Prog. Phys. (2022)
  Chandran, Iadecola, Khemani & Moessner, ARCMP **14** (2023)

Today's subject

- Quantum many-body scars (QMBS)
  - ✓ Non-thermal eigenstates of non-integrable Hamiltonians
  - ✓ Finite-energy density
  - ✓ Entanglement entropy does not obey a volume law
- Constructing models with exact QMBS
  - ✓ Using Onsager algebra

    2d Ising model:
    Phys. Rev. 65 (1944)
  - ✓ Using integrable boundary states
  - ✓ Using (restricted) spectrum generating algebra
  - ✓ …

Outline

1. Introduction and Motivation

2. Onsager scars

- Strategy
- Perturbed S=1/2 XY chain
- Higher-spin models

3. Other scarred models

4. Summary

Strategy

1. Starting point:
   Integrable model with conserved charges $Q_1, Q_2, \ldots$
   They commute with the Hamiltonian H_{int}

2. Take a subalgebra $\{Q_1, Q_2, \ldots\}$

3. Find a reference eigenstate $H_{\mathrm{int}}|\psi_0\rangle = E_0|\psi_0\rangle$
   ψ_0: simple state, e.g., product state or MPS

4. Find a tower of eigenstates generated by acting with the subalgebra on the reference state:

   $(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle$ ← QMBS in non-integrable *H*

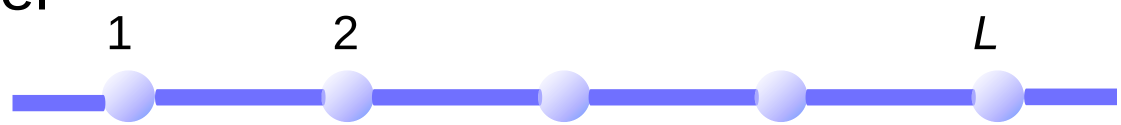
   They have the same energy with ψ_0

5. Add perturbations that break the integrability of H_{int} but do not hurt the tower of states

   $H = H_{\mathrm{int}} + H_{\mathrm{pert}},$ e.g., $H_{\mathrm{pert}}\,(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle = 0$

Example: *S*=1/2 XY chain

■ Model



S=1/2 at each site
L: even
Periodic chain

- Hamiltonian

$$H_{\mathrm{int}} = \sum_{j=1}^{L} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \qquad S_j^{\pm} := \frac{S_j^x \pm \mathrm{i} S_j^y}{2}$$

$$S_j^+ |\uparrow\rangle_j = 0$$
$$S_j^- |\uparrow\rangle_j = |\downarrow\rangle_j$$
$$S_j^+ |\downarrow\rangle_j = |\uparrow\rangle_j$$
$$S_j^- |\downarrow\rangle_j = 0$$

Can be mapped to free fermions via Jordan-Wigner

Lieb-Schultz-Mattis (1961), Katsura (1962)

- Conserved charges

Total S^z: $Q = \sum_{j=1}^{L} S_j^z$

Other charges are not very obvious in the spin basis...

"bi-magnon" operator: $Q^{\pm} = \sum_{j=1}^{L} (-1)^{j+1} S_j^{\pm} S_{j+1}^{\pm}, \quad [H_{\mathrm{int}}, Q^{\pm}] = 0$

An element of Onsager's algebra! Infinitely many such.

- Reference eigenstate

All down state: $|\Downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \quad H_{\mathrm{int}} |\Downarrow\rangle = 0$

Desired perturbations

- Tower of exact eigenstates (with fixed total S^z)

$$|\Downarrow\rangle,\ Q^+|\Downarrow\rangle, \ldots, (Q^+)^k|\Downarrow\rangle, \ldots, (Q^+)^{L/2}|\Downarrow\rangle \qquad ((Q^+)^{L/2+1} = 0)$$

- "Coherent state" $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle = \sum_{k=0}^{L/2} \frac{\beta^{2k}}{k!}(Q^+)^k|\Downarrow\rangle$

MPS with bond dim. 2

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[\begin{pmatrix} |\downarrow\rangle_1 & \beta|\uparrow\rangle_1 \\ \beta|\uparrow\rangle_1 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_2 & -\beta|\uparrow\rangle_2 \\ \beta|\uparrow\rangle_2 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_3 & -\beta|\uparrow\rangle_3 \\ \beta|\uparrow\rangle_3 & 0 \end{pmatrix}\cdots\right]$$

- Possible perturbations

$$\begin{pmatrix} |\downarrow\downarrow\downarrow\rangle - \beta^2(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) & \beta|\downarrow\downarrow\uparrow\rangle + \beta^3|\uparrow\uparrow\uparrow\rangle \\ \beta|\uparrow\downarrow\downarrow\rangle - \beta^3|\uparrow\uparrow\uparrow\rangle & \beta^2|\uparrow\downarrow\uparrow\rangle \end{pmatrix}_{1,2,3}$$

✓ We never have $|\downarrow\uparrow\downarrow\rangle$ or $(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)/\sqrt{2}$ in any three consecutive sites

✓ Identify Hermitian operators that annihilate $|\psi(\beta)\rangle$

$$H_{\mathrm{pert}}|\psi(\beta)\rangle = 0$$

Couplings can be random!

$$H_{\mathrm{pert}} = \sum_{j=1}^{L}\Big(c_j^{(1)}\,|\downarrow\uparrow\downarrow\rangle\langle\downarrow\uparrow\downarrow| + \frac{c_j^{(2)}}{2}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|) + c_j^{(3)}[|\downarrow\uparrow\downarrow\rangle(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|) + \mathrm{h.c.}]\Big)$$

Properties of the perturbed model

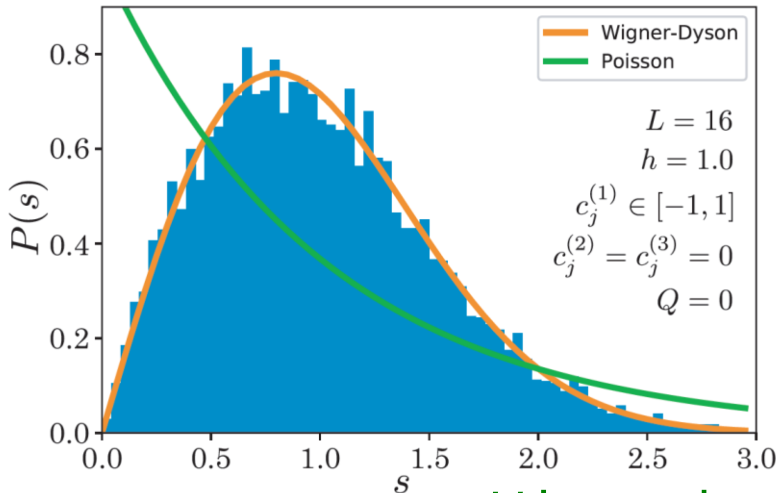
■ Level-spacing statistics

- Perturbed Hamiltonian

$$H = H_{\text{int}} + H_{\text{pert}} + hQ,$$

- System size: L=16
- Only diagonal perturbation
- Zero magnetization sector

Pal, Huse, PRB **82** (2010)



H is non-integrable!

■ Entanglement diagnosis

- Entanglement entropy (EE)

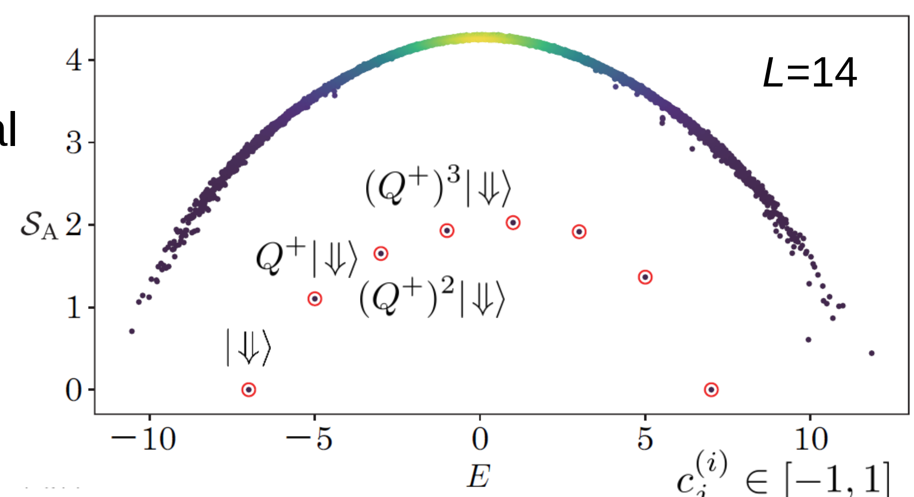
  Volume law → Thermal

  Sub-volume law → non-thermal

- QMBS states $(Q^+)^k|\Downarrow\rangle$

Rigorous result:
EE of QMBS ≦ O (ln *L*)

Half-chain EE



Dynamics

■ Initial state = coherent state

- Hamiltonian $H = H_{\text{int}} + H_{\text{pert}} + hQ,$
- Coherent state $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle$
- Time evolution

$$|\psi_t(\beta)\rangle = \exp(-\mathrm{i}Ht)|\psi(\beta)\rangle \propto |\psi(\beta e^{-\mathrm{i}ht})\rangle$$

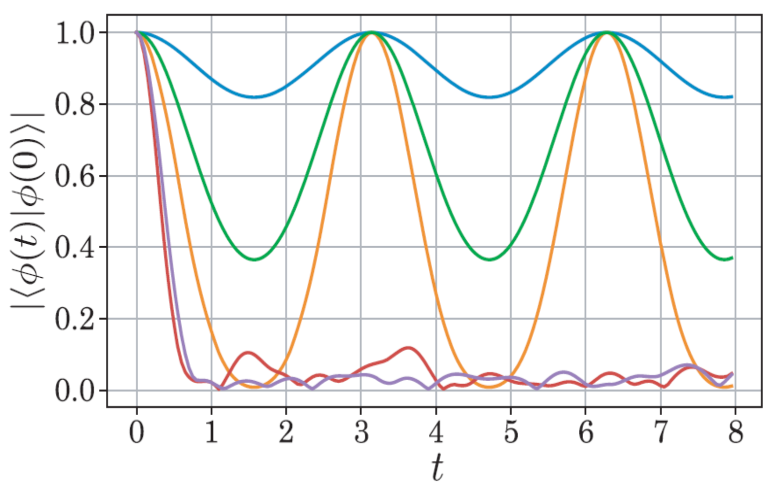
Revival at

$$t = t_k = \frac{\pi k}{h}, \quad k \in \mathbb{N}$$

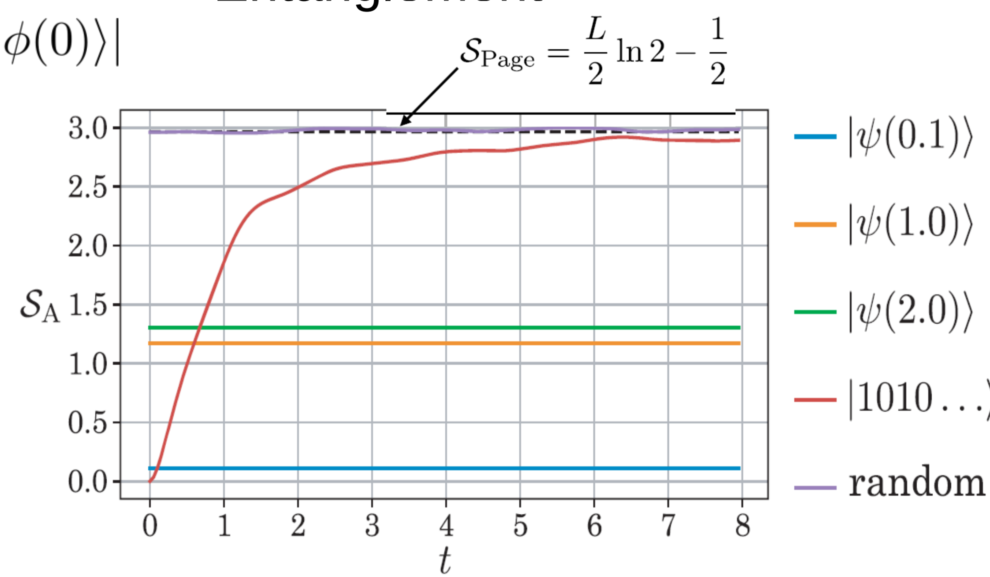
■ Numerical results $L = 10,\ h = 1.0, c_j^{(i)} \in [-1, 1]$ (random)

- Fidelity

$$|\langle\phi(t)|\phi(0)\rangle| = |\langle\phi(0)|\exp(\mathrm{i}Ht)|\phi(0)\rangle|$$



- Entanglement



What about *S* >1/2 ?

■ Self-dual U(1)-invariant clock model

Vernier, O'Brien & Fendley, J. Stat. Mech. (2019)

- Matrices $\omega = \exp(2\pi\mathrm{i}/n)$

$$\tau = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}, \quad S^{+} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad S^{-} = (S^{+})^{\dagger}$$

- Hamiltonian

Truly interacting for *n*>2!

$$H_n = \mathrm{i}\sum_{j=1}^{L}\sum_{a=0}^{n-1}\frac{1}{1-\omega^{-a}}[(2a-n)\tau_j^a + n(S_j^+S_{j+1}^-)^{n-a} - n(S_j^-S_{j+1}^+)^a]$$

H_2 boils down to (twisted) XY, H_3 → *S*=1 Fateev-Zamolodchikov

- U(1) symmetry $[H_n, Q] = 0, \quad Q = \sum_{j=1}^{L} S_j^z$
- Self-duality (in the $\sigma-\tau$ rep.)
- Onsager algebra! $Q^{+} = \sum_{j=1}^{L}\sum_{a=1}^{n-1}\frac{1}{1-\omega^{-a}}(S_j^+)^a(S_{j+1}^+)^{n-a}, \quad [H_n, Q^+] = 0$

S=1 (n=3) model

- Integrable Hamiltonian

$$H_{\text{int}} = \sqrt{3}\sum_{j=1}^{L}\left[S_j^+S_{j+1}^- + S_j^-S_{j+1}^+ - (S_j^+S_{j+1}^-)^2 - (S_j^-S_{j+1}^+)^2 - (S_j^z)^2 + \frac{2}{3}\right]$$

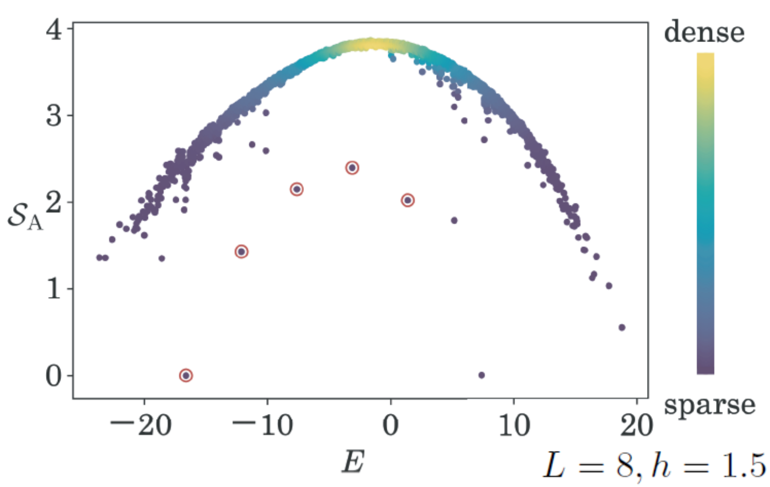
- Coherent state

$$Q^+ = \frac{2}{\sqrt{3}}\sum_{j=1}^{L} S_j^+(S_j^+ - S_{j+1}^+)S_{j+1}^+, \quad |\psi(\beta)\rangle = \exp(\beta^2 Q^+)|-,-,\cdots,-\rangle$$

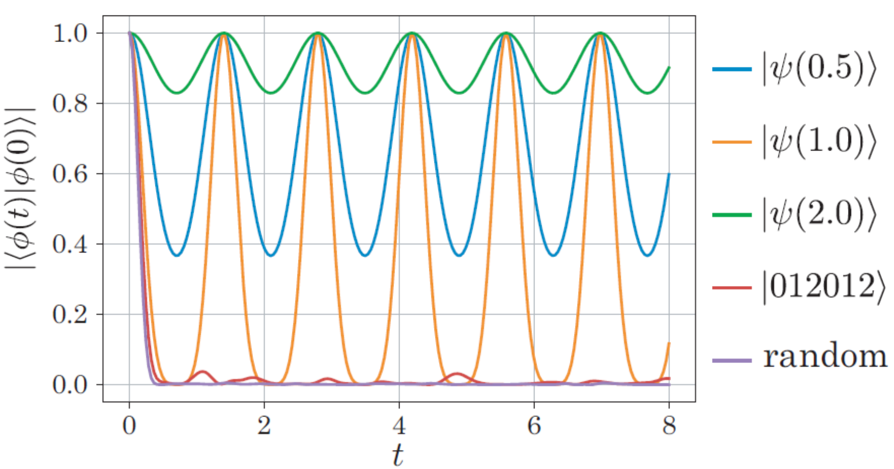
Again a matrix product state (MPS). The bond dimension is 3.
Desired perturbations can be identified from this MPS.

$$H = H_{\text{int}} + H_{\text{pert}} + hQ,$$

- Half-chain entanglement



- Fidelity



Outline

1. Introduction and Motivation
2. Onsager scars
3. Other scarred models

- Boundary scars & scalar chirality
- Dzyaloshinskii-Moriya int.+Zeeman

4. Summary

Integrable boundary states

- Integrable Hamiltonian: $H_{\text{int}} = \sum_j H_j$ (1d nearest neighbor int.)
- Boost operator: $B = \sum_j j H_j$
- Conserved charges: $Q_{n+1} = [B, Q_n], \quad Q_2 \propto H_{\text{int}}$

  Q_{2k}/Q_{2k+1} is even / odd under parity $\mathcal{I}$:

  $$\mathcal{I}|\sigma_1, \sigma_2, \cdots, \sigma_{L-1}, \sigma_L\rangle = |\sigma_L, \sigma_{L-1}, \cdots, \sigma_2, \sigma_1\rangle$$

  ➢ Example: *S* =1/2 Heisenberg chain

  Scalar chirality

  $$H_{\text{int}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \quad \Rightarrow \quad Q_3 \propto C_{\text{SC}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot (\boldsymbol{S}_{j+1} \times \boldsymbol{S}_{j+2})$$

- Integrable boundary states: Piroli, Pozsgay & Vernier, NPB **925** (2017)

$$|\Psi_0\rangle \text{ such that } Q_{2k+1}|\Psi_0\rangle = 0 \text{ for all } k = 1, 2, 3, \ldots$$

Lattice version of boundary states in integrable QFT:
Ghoshal & Zamolodchikov, IJMP **A9**, 3841 (1994)

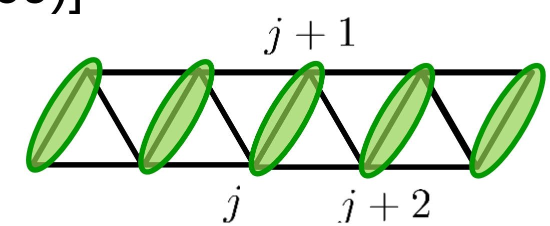
Boundary scars

- If $|\Psi_0\rangle$ is an eigenstate of a non-integrable Hamiltonian H_0,

then it is an eigenstate of $H_0 + \sum_{k=1}^{\infty} t_k Q_{2k+1} \quad (t_k \in \mathbb{R})$

- Example of a scarred model
  - H_0 : Majumdar-Ghosh model [JMP **10** (1969)]

$$H_{\mathrm{MG}} = \sum_{j=1}^{L} \left[(\boldsymbol{S}_j + \boldsymbol{S}_{j+1} + \boldsymbol{S}_{j+2})^2 - \frac{3}{4} \right]$$



Dimer g.s. are annihilated by C_{SC}!

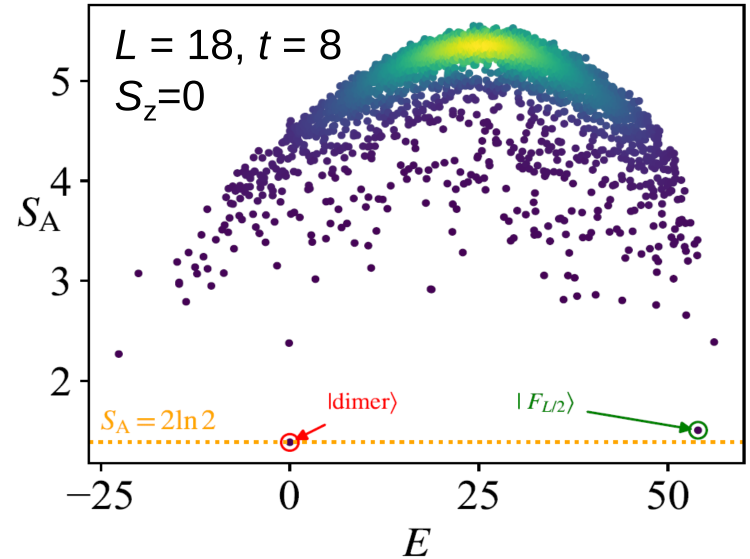
  - Hamiltonian

$$H(t) = H_{\mathrm{MG}} + tC_{\mathrm{SC}}$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot

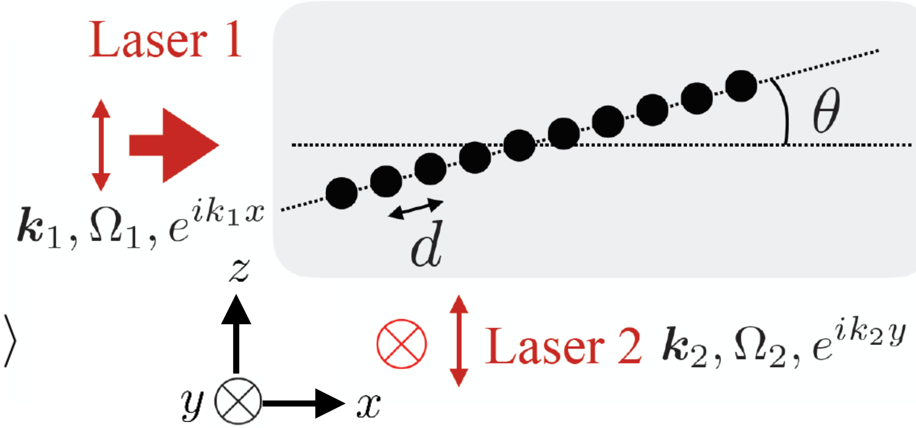
✓ Dimer g.s. is a scar!



Toward realization of spin models

■ Experimental setup

- 1d array of Rb atoms



- Effective spin states

$$|\downarrow\rangle \leftrightarrow |n_1 S_{1/2}\rangle, \ |\uparrow\rangle \leftrightarrow |n_2 S_{1/2}\rangle$$

- Effective Hamiltonian
→ *S*=1/2 XXZ chain in a rotating magnetic field

$$-\Omega_{\mathrm{eff}}[\cos(qj)S_j^x + \sin(qj)S_j^y] - \tilde{\Delta}S_j^z, \quad q = k_1 d\cos\theta$$

■ Hamiltonian in spin-rotating frame

$$H_{\mathrm{eff}} = J\cos q\sum_j (S_j^z S_{j+1}^z + S_j^x S_{j+1}^x) + J\delta\sum_j S_j^y S_{j+1}^y - \tilde{\Delta}\sum_j S_j^y$$

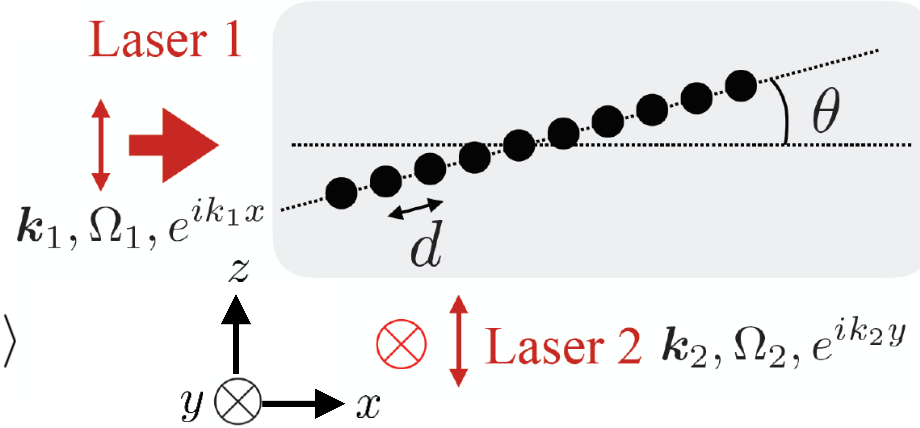
$$- J\sin q\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - \Omega_{\mathrm{eff}}\sum_j S_j^z$$

DH model

- Tuning *q*, *δ*, etc. → Model with only Dzyaloshinskii-Moriya int. and field in the *z*-direction [Kodama, Kato & Tanaka, PRB **107** (2023)]

Toward realization of spin models

■ Experimental setup



- 1d array of Rb atoms
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■ Hamiltonian in spin-rotating frame

$$H_{\mathrm{eff}} = \cancel{J\cos q\sum_j (S_j^z S_{j+1}^z + S_j^x S_{j+1}^x)} + \cancel{J\delta\sum_j S_j^y S_{j+1}^y} - \cancel{\tilde{\Delta}\sum_j S_j^y}$$

$$\boxed{-J\sin q\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - \Omega_{\mathrm{eff}}\sum_j S_j^z}$$

DH model

- Tuning *q*, *δ*, etc. → Model with only Dzyaloshinskii-Moriya int. and field in the *z*-direction [Kodama, Kato & Tanaka, PRB **107** (2023)]

QMBS states in DH model

- Hamiltonian $H_{\mathrm{DH}} = D\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - H\sum_j S_j^z$ PBC or special OBC
- Raising operator $Q^\dagger = \sum_j P_{j-1} S_j^+ P_{j+1}$ Similar to $Q^\dagger$ in Schecter & Iadecola, PRL **123** (2019)
- They satisfy a restricted spectrum generating algebra (SGA)

$$H_{\mathrm{DH}}|\Downarrow\rangle = E_0|\Downarrow\rangle \quad (|\Downarrow\rangle = |\downarrow\cdots\downarrow\rangle)$$

$$[H_{\mathrm{DH}}, Q^\dagger]|\Downarrow\rangle = -HQ^\dagger|\Downarrow\rangle$$

$$[\,[\,H_{\mathrm{DH}}, Q^\dagger\,], Q^\dagger\,] = 0$$

See e.g., Moudgalya *et al*., PRB **102**, 085140 (2020)

- Exact eigenstates

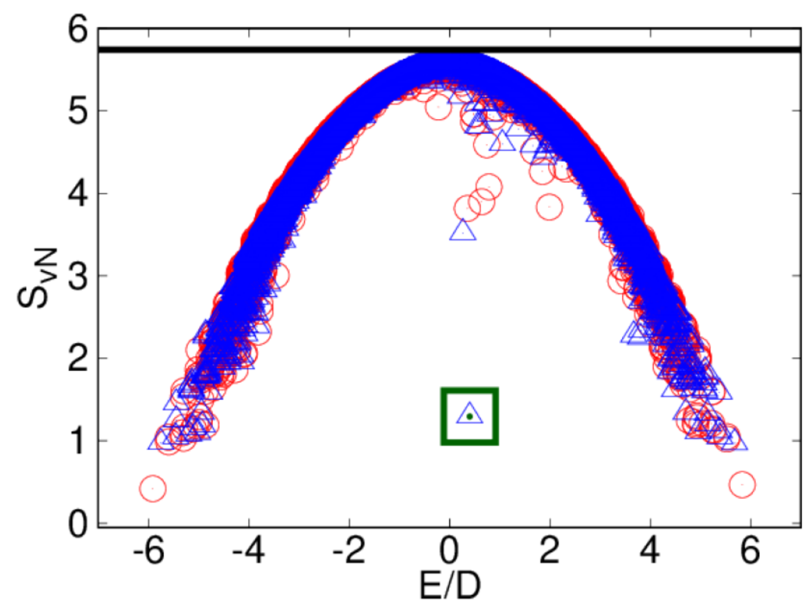
$$|S_n\rangle = (Q^\dagger)^n|\Downarrow\rangle$$

$$H_{\mathrm{DH}}|S_n\rangle = (E_0 - nH)|S_n\rangle$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot, fidelity

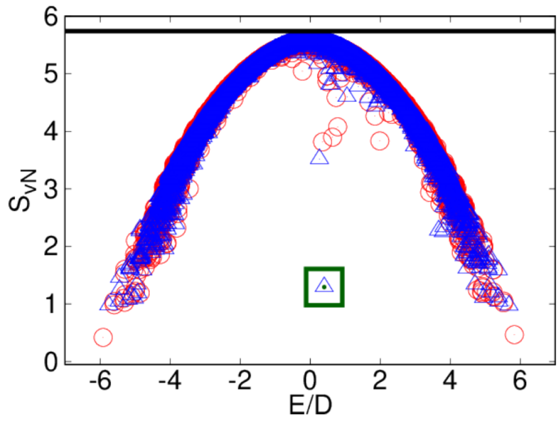
✓ They are scars!



OBC, L=18, H=0.1D, Soliton num. = 5

Summary

- **Constructing models with QMBS**
  - Using Onsager algebra
    Perturbed S=1/2 XY chain, higher-spin models
  - Using integrable boundary states
    Majumdar-Ghosh + scalar chirality
  - Using restricted SGA
    Dzyaloshinskii-Moriya + Zeeman
    Proposal for an experiment



- **Other models**
  - Correlated hopping model: Tamura & HK, PRB **106** (2022)
  - Generalization of eta-pairing: Yoshida & HK, PRB **105** (2022)
  - S=1 AKLT + SU(3) scalar chirality
  - Perturbed S=1 scalar chirality in 1d and 2d