Supersymmetry breaking and Nambu-Goldstone fermions in lattice models

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Outline

Motivation and Introduction

- N=2 Supersymmetric (SUSY) QM
- Examples: free model, Nicolai model

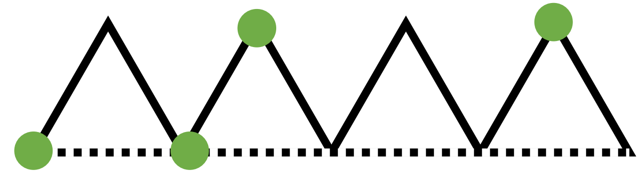
Part I: $\boldsymbol{Z}_2$ Nicolai model with N=2 SUSY

Part II: Majorana-Nicolai model with N=1 SUSY

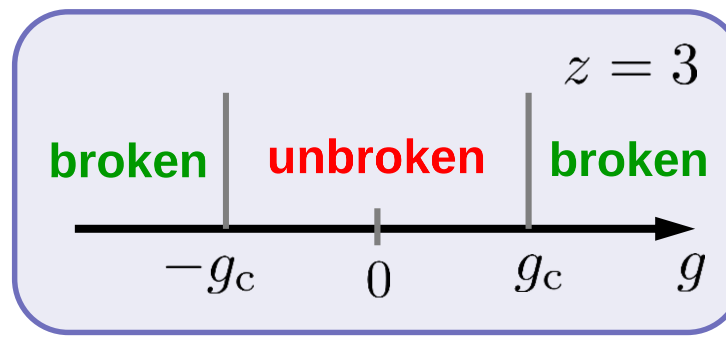
Summary

Today's talk

- Lattice models with exact supersymmetry
  - Super-weird quantum matter, never synthesized…
  - Mostly focus on 1 (spatial) dimension
    But can be defined in any dimension
  - Interaction is local, but as crazy as SYK
  - No AdS/CFT, but exotic dynamical exponent



- Results
  1. SUSY ***unbroken*** phase
     Highly degenerate g.s. with E=0
  2. SUSY ***broken*** phase
     Rigorous proof of SUSY breaking
     Nambu-Goldstone fermion with *cubic dispersion*



$1/g$ ~ int.

N=2 supersymmetric (SUSY) QM

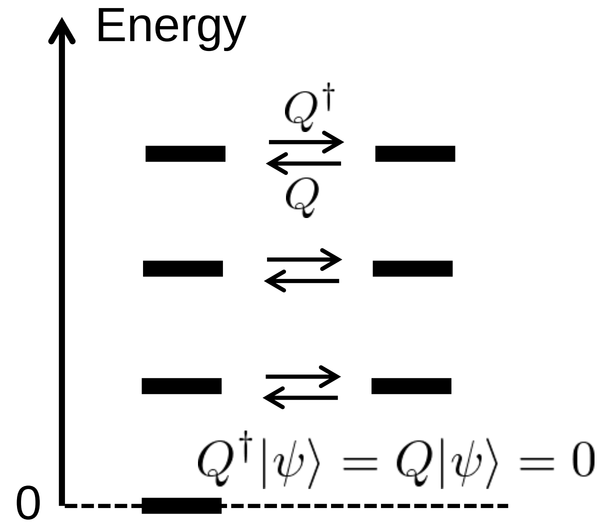
■ Algebra

- Supercharges: $Q,\ Q^\dagger, \quad Q^2 = 0,\ (Q^\dagger)^2 = 0$
- Fermionic parity: $\{Q, (-1)^F\} = \{Q^\dagger, (-1)^F\} = 0$
- Hamiltonian: $H = \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q$
- Symmetry: $[H, Q] = [H, Q^\dagger] = [H, (-1)^F] = 0$

■ Spectrum of *H*

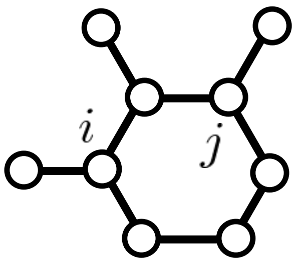
- $E \geqq 0$ for all states, as *H* is p.s.d
- $E > 0$ states come in pairs $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
- $E = 0$ iff a state is a SUSY singlet

G.S. energy = 0 → SUSY ***unbroken***
G.S. energy > 0 → SUSY ***broken***



Elementary example

■ Lattice bosons and fermions



- Lattice sites: *i*, *j* = 1,2, …, *N*
- Creation, annihilation ops.

$$[b_i, b_j^\dagger] = \delta_{i,j}, \quad \{c_i, c_j^\dagger\} = \delta_{i,j}, \quad [b_i, b_j] = \{c_i, c_j\} = 0.$$

(*b* and *f* are mutually commuting.)

- Vacuum state $b_i|\mathrm{vac}\rangle = c_i|\mathrm{vac}\rangle = 0, \forall i$

■ Supercharges and Hamiltonian

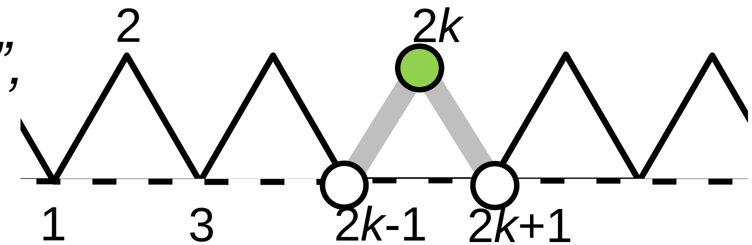
$$Q = \sum_j b_j^\dagger c_j, \quad Q^\dagger = \sum_j c_j^\dagger b_j$$

$$\Rightarrow \{Q, Q^\dagger\} = \sum_j b_j^\dagger b_j + \sum_j c_j^\dagger c_j$$

Just the total number of particles! |vac> is a SUSY singlet.

Nicolai model

"Supersymmetry and spin systems",
H. Nicolai, *JPA* **9**, 1497 (**1976**).
cf) Witten, *NPB* **202**, 253 (1982)



- Spinless fermion model in 1D (num. op.: $n_j = c_j^\dagger c_j$)

- Supercharge

$$Q = \sum_k c_{2k-1} c_{2k}^\dagger c_{2k+1}$$

- Hamiltonian

$$H = \sum_{k=1}^{N/2} (n_{2k} + n_{2k-1} n_{2k+1} - n_{2k-1} n_{2k} - n_{2k} n_{2k+1} + c_{2k}^\dagger c_{2k+3}^\dagger c_{2k-1} c_{2k+2} + \mathrm{h.c.})$$

- Highly degenerate *E*=0 g.s.

N	2	4	6	8	10	12
deg.	4	12	36	116	364	1172

- One-parameter extension $Q = \sum_k c_{2k-1} c_{2k}^\dagger c_{2k+1} + g \sum_k c_{2k-1}$

  ➢ Sannomiya, Katsura, Nakayama, *PRD*, **94**, 045014 (2016)

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Part I: $\boldsymbol{Z}_2$ Nicolai model with *N*=2 SUSY

- Supercharge and Hamiltonian
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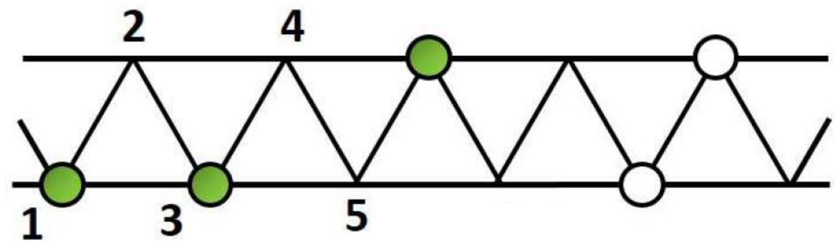
- SUSY broken phase
- Nambu-Goldston fermions

Part II: Majorana-Nicolai model with *N*=1 SUSY

Summary

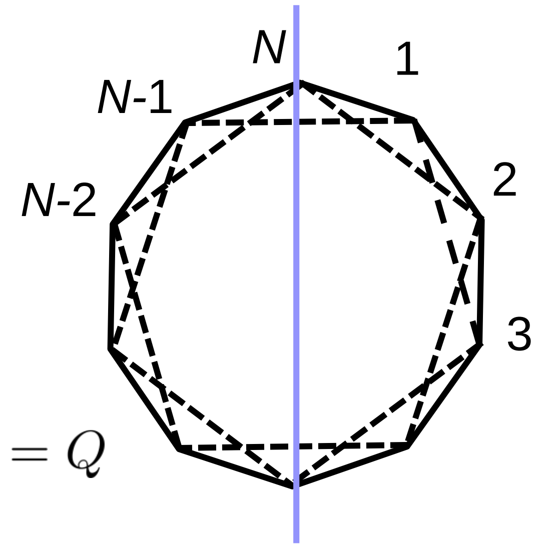
Z2 Nicolai model

■ Definition



- 1d periodic chain of length *N*
- Supercharge $(g \geq 0)$ $Q = \sum_{j=1}^{N} (g c_j + c_{j-1} c_j c_{j+1})$ ⇒ $Q^2 = 0$
- Hamiltonian $H = \{Q, Q^\dagger\}$

■ Symmetries



- SUSY $[H, Q] = [H, Q^\dagger] = 0$
- $\mathbf{Z}_2$ $[H, (-1)^F] = 0, \quad F = \sum_{j=1}^{N} n_j$
- Translation $T: \ c_j \to c_{j+1}$ $T^{-1} Q T = Q$
- Reflection-like sym.

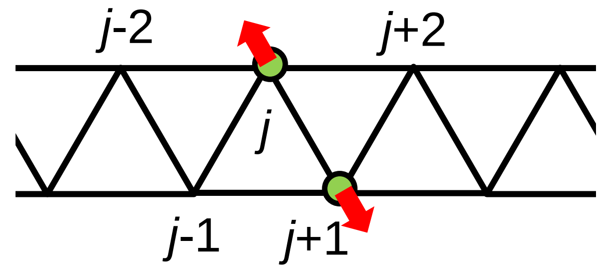
$$U: \ c_j \to \begin{cases} \mathrm{i}\, c_{N-j} & j = 1, \ldots, N-1 \\ \mathrm{i}\, c_N & j = N \end{cases} \qquad U^{-1} Q U = \mathrm{i} Q$$

Hamiltonian (explicit)

$$H = H_{\mathrm{free}} + H_1 + H_2 + g^2 N$$

1. Free (BdG) Hamiltonian

$$H_{\mathrm{free}} = g \sum_{j=1}^{N} (2c_j c_{j+1} - c_{j-1} c_{j+1} + \mathrm{H.c.})$$



2. Repulsive int. etc.

$$H_1 = \sum_{j=1}^{N} (1 - 3n_j + 2n_j n_{j+1} + n_j n_{j+2})$$

3. Pair-hopping term

$$H_2 = \sum_{j=1}^{N} (c_j^\dagger c_{j-1}^\dagger c_{j+2} c_{j+3} + \mathrm{H.c.}) + \sum_{j=1}^{N} \left[(n_{j-1} + n_j - 1) c_{j+1}^\dagger c_{j-2} + \mathrm{H.c.} \right]$$

- Large-g limit reduces to a free-fermion model

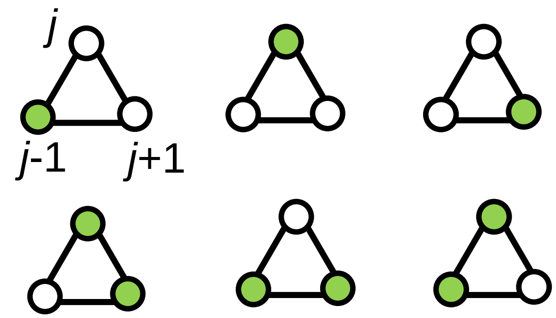
$$H \sim H_{\mathrm{free}} + g^2 N$$

1. SUSY is broken $\quad E_0/N \sim g^2$
2. gapless excitation $\quad E(p) \propto |p|^3$

SUSY is unbroken at *g* = 0

■ Classical ground states

$$Q = \sum_{j=1}^{N} q_j, \quad q_j = c_{j-1} c_j c_{j+1}$$



6 local states annihilated by q_j and $q_j^\dagger$

Global g.s. like …●○●○●○●○… → SUSY is ***unbroken!***

Transfer matrix can count such states, but miss entangled g.s.

■ Witten index

Q and $Q^\dagger$ preserves *F* modulo 3.

The index at each sector: $W_f = \mathrm{Tr}_{\mathcal{H}_f}[(-1)^F e^{-\beta H}], \qquad (f = 0, 1, 2)$

$$W = \sum_{f=0}^{2} |W_f| = \begin{cases} 2 \times 3^{\frac{N-1}{2}} & N: \text{ odd} \\ 4 \times 3^{\frac{N}{2}-1} & N: \text{ even} \end{cases}$$

W~$(1.73)^N$ gives a lower bound for the num. of g.s.

N	3	4	5	6	7	8	9	10	11	12	13
Z_{cl}	6	6	10	20	28	46	78	122	198	324	520
W	6	12	18	36	54	108	162	324	486	972	1458

Number of E=0 ground states

■ Open boundary chain

- Supercharge

$$Q = \sum_{j}^{N-2} c_j \, c_{j+1} \, c_{j+2} \qquad H = \{Q, Q^\dagger\}$$

N	3	4	5	6	7	8	9	10	11	12	13
Z	6	12	20	36	64	112	200	352	624	1104	1952

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®
founded in 1964 by N. J. A. Sloane

$$Z \geq W$$

They are generated by the recursion:

$$a_n = 2a_{n-2} + 2a_{n-3}, \qquad a_0 = 1,\ a_1 = 2,\ a_2 = 4$$

$$Z \sim (1.77)^N$$

■ Proof

- By homological perturbation lemma

  La, Schoutens, Shadrin, *JPA* **52**, 02LT01 (2019)

  Also studies the original Nicolai model with U(1)

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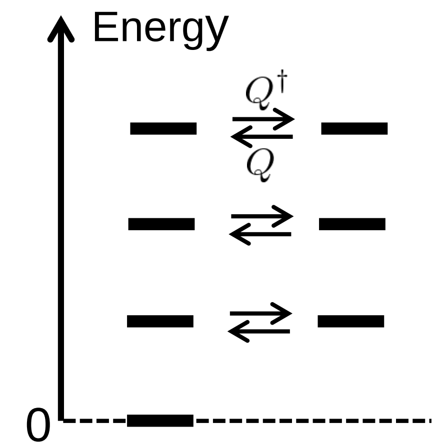
Summary

SUSY breaking



$$E(V) \to 0 \quad V \to \infty$$

■ Naïve definition

SUSY is unbroken ⇔ E=0 state exists
SUSY is broken ⇔ No E= 0 state

Subtle issue... (Witten, *NPB* **202** (1982))
"*SUSY may be broken in any finite volume yet restored in the infinite-volume limit*."

■ Precise definition

- Ground-state energy density

$$\epsilon_0 := \frac{1}{V}\langle\psi_0|H|\psi_0\rangle$$

V= (# of sites) for lattice systems
ψ_0 : normalized g.s.

SUSY is said to be spontaneously broken if $\epsilon_0 > 0$.

Applies to both finite and infinite-volume systems!

SUSY breaking in finite chains

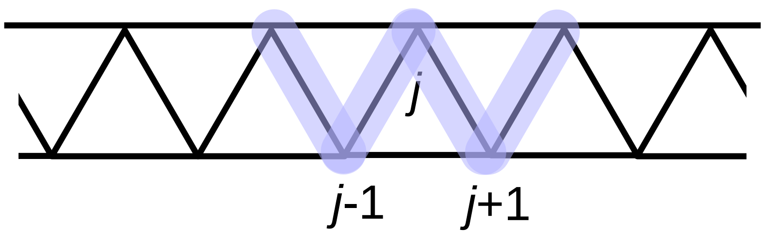
- Theorem 1

Consider the $\mathbf{Z}_2$ Nicolai model on a chain of length *N*.
If *g* > 0, then SUSY is spontaneously broken.

- Proof
  - Local operator s.t. $\{Q, \mathcal{O}_j\} = g$ well-defined for *g*>0

$$\mathcal{O}_j = c_j^\dagger \left[1 - \frac{1}{g}(c_{j+1}c_{j+2} - c_{j-1}c_{j+1} + c_{j-2}c_{j-1}) + \frac{2}{g^2} c_{j-2}c_{j-1}c_{j+1}c_{j+2} \right]$$

  - Proof by contradiction



Suppose ψ_0 is a normalized *E*=0 g.s. Then we have

$$g = \langle \psi_0 | \{Q, \mathcal{O}_j\} | \psi_0 \rangle = \langle \psi_0 | Q\mathcal{O}_j + \mathcal{O}_j Q | \psi_0 \rangle = 0$$

Contradiction. No *E*=0 state!

⇒ (g.s. energy/*N*) > 0 for any finite *N*.

SUSY breaking in the infinite chain

Theorem 2

Consider the $\mathbf{Z}_2$ Nicolai model on the infinite chain.
If $g > 0$, then SUSY is spontaneously broken.

Proof

- Previous work: H. Moriya, *PRD* **98**, 015018 (2018) [C^*-algebra]
- Physicist version

Locality: $\{Q, \mathcal{O}_j\} = g, \qquad \{\mathcal{O}_i, \mathcal{O}_j^\dagger\} = 0 \text{ if } |i-j| > 4$

Let ψ_0 be a ground state. Define $o = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \mathcal{O}_j$.
Then we have

$$0 < g = \langle\psi_0|\,\{o, Q\}\,|\psi_0\rangle = \langle\psi_0|\,oQ + Qo\,|\psi_0\rangle$$

$$\le 2\sqrt{\underbrace{\langle\psi_0|\,\{o, o^\dagger\}\,|\psi_0\rangle}_{\text{Order } 1/N}\underbrace{\langle\psi_0|\,\{Q, Q^\dagger\}\,|\psi_0\rangle}_{E_0(N)}}$$

Cauchy-Schwarz etc.

$$\le (\mathrm{Const.}) \times \sqrt{\epsilon_0(N)}$$

ε_0 cannot go to zero even when $N \to \infty$

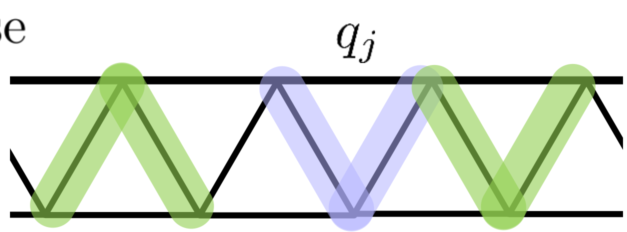
Nambu-Goldstone (type) Theorem

■ Variational state

- Local supercharge (g >0)

$$Q = \sum q_j, \quad q_j = g c_j + c_{j-1} c_j c_{j+1}$$

- Locality $\{q_j, q_\ell^\dagger\} = \begin{cases} \text{nonzero} & |j-\ell| \le 2 \\ 0 & \text{otherwise} \end{cases}$



- Fourier components

$$Q_p = \sum e^{-ipj} q_j, \quad (Q_p)^2 = 0$$

- Ansatz

ψ_0: SUSY broken g.s. Assume g.s. degeneracy is uniform in *N*.

Trial state (orthogonal to ψ_0) $|\psi_p\rangle = \dfrac{(Q_p + Q_p^\dagger)|\psi_0\rangle}{\|(Q_p + Q_p^\dagger)|\psi_0\rangle\|} \quad (p \neq 0)$

- Variational energy

$$\epsilon_{\rm var}(p) = \langle \psi_p | H | \psi_p \rangle - \langle \psi_0 | H | \psi_0 \rangle \le (\text{Const.}) \times |p|$$

Nambu-Goldstone (type) Theorem (contd.)

■ Proof

$$\epsilon_{\rm var}(p) = \frac{\langle\, [Q_p, [H, Q_p^\dagger]\,]\,\rangle_0}{\langle\, \{Q_p, Q_p^\dagger\}\,\rangle_0} \qquad (\langle \cdots \rangle_0 := \langle \psi_0 | \cdots | \psi_0 \rangle)$$

$[H, Q_p^\dagger]$ is a sum of local ops. But, $[Q_p, [H, Q_p^\dagger]]$ may not be so…

- Pitaevskii-Stringari inequality *JLTP* **85**, 377 (1991)

$$|\langle\psi|\,[\,A^\dagger, B\,]\,|\psi\rangle|^2 \le \langle\psi|\,\{A^\dagger, A\}\,|\psi\rangle\,\langle\psi|\,\{B^\dagger, B\}\,|\psi\rangle$$

Holds for any state ψ and any ops. A, B.

Local!

$$\Rightarrow \quad \left|\langle [Q_p, [H, Q_p^\dagger]\,]\rangle_0\right|^2 \le \langle\{Q_p, Q_p^\dagger\}\rangle_0\,\langle\{[Q_p,\, H],\, [H,\, Q_p^\dagger]\}\rangle_0$$

- Upper bound

For $|p|<<1$,

$$\epsilon_{\rm var}(p)^2 \le \frac{\langle\,\{\,[Q_p, H], [H, Q_p^\dagger]\,\}\,\rangle_0}{\langle\,\{Q_p, Q_p^\dagger\}\,\rangle_0} = \frac{f_{\rm n}(p)}{f_{\rm d}(p)} \quad \Rightarrow \quad \epsilon(p) \le ({\rm Const.}) \times |p|$$

$f_n(p)$: 1. Local, 2. $f_n(-p) = f_n(p)$, 3. $f_n(0) = 0$
$f_d(p)$: 1. Local, 2. $f_d(-p) = f_d(p)$, 3. $f_d(0) = E_0 > 0$

Numerical results

- Exact diagonalization

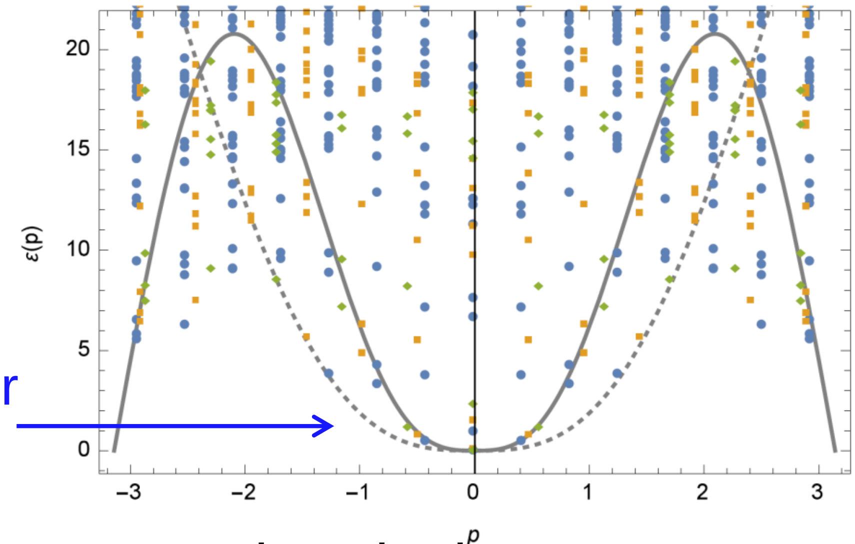
  N= 10, 11, …, 20

  4 g.s. for even N

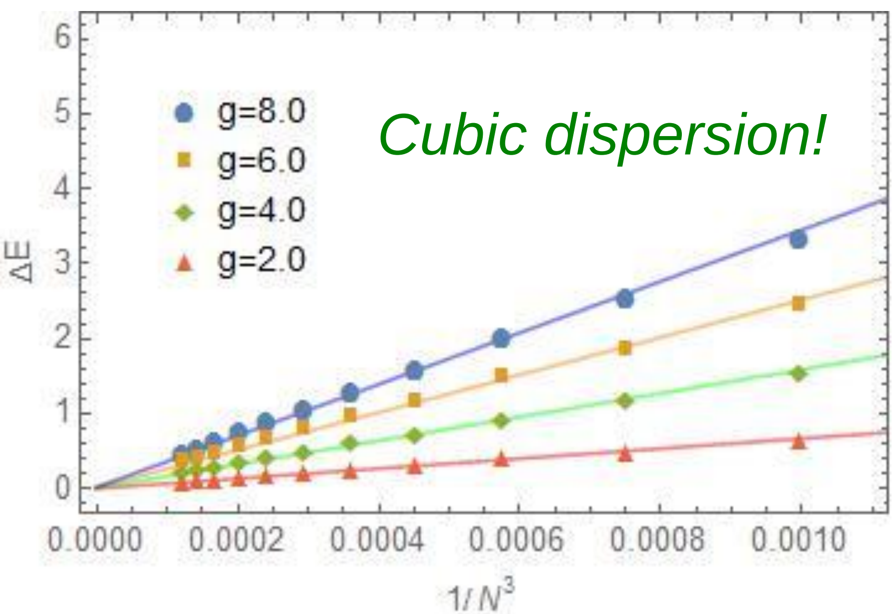
  2 g.s. for odd N

Excitation energies lower than linear dispersion!

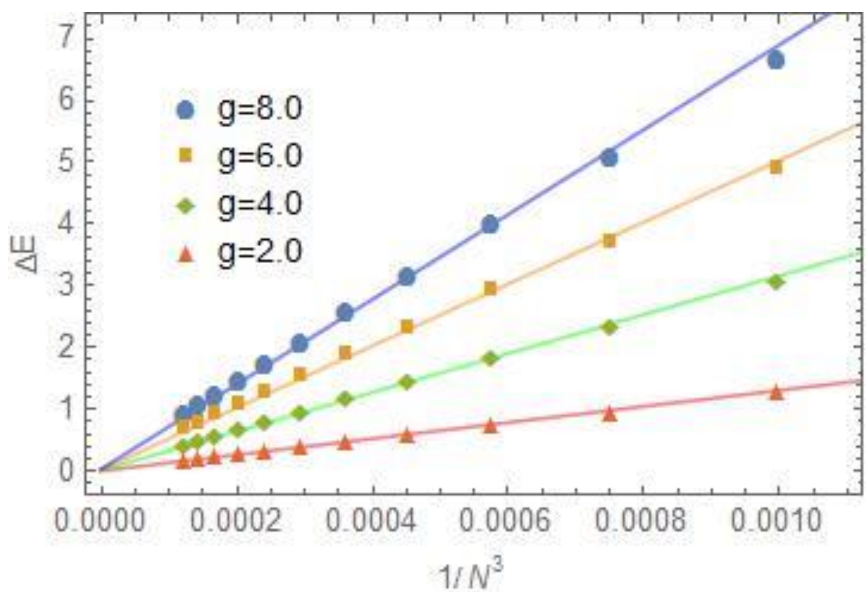
Dispersion (g=4)



• First excitation



• Second excitation



Large-*g* limit

- Non-interacting model $H \sim H_{\mathrm{free}} + g^2 N$
  - Spectrum

$$H_{\mathrm{free}} = g \sum (2c_j c_{j+1} - c_{j-1}c_{j+1} + \mathrm{H.c.})$$
$$= 2g \sum_{p>0} (c(p), c^{\dagger}(-p)) \begin{pmatrix} 0 & \mathrm{i}f(p) \\ -\mathrm{i}f(p) & 0 \end{pmatrix} \begin{pmatrix} c^{\dagger}(p) \\ c(-p) \end{pmatrix}$$

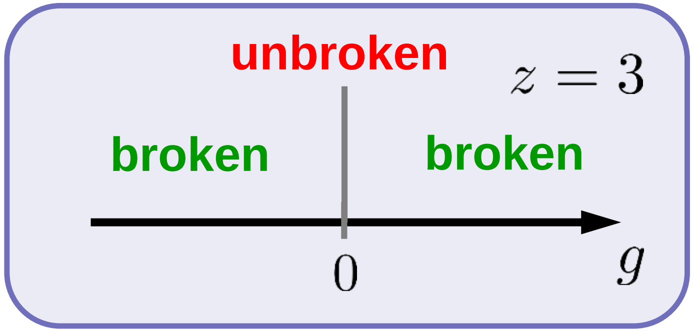
$$E(p) = \pm 2g|f(p)| \sim 2g|p|^3$$ *Cubic dispersion!*

Summary of Part I

Studied Z2 Nicolai model with *N*=2 SUSY

1. SUSY is ***unbroken*** at *g* = 0
   Highly degenerate g.s. with *E*=0

2. SUSY is ***broken*** for *g* ≠ 0
   Rigorous proof of SUSY breaking
   NG fermion with *cubic* dispersion
   Stability against SUSY perturbation



➢ Sannomiya, HK, Nakayama
PRD, **95**, 065001 (2017)

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- N=1 SUSY
- Supercharge and Hamiltonian
- SUSY unbroken & broken phases, NG fermion

Summary

N=1 Supersymmetric (SUSY) QM

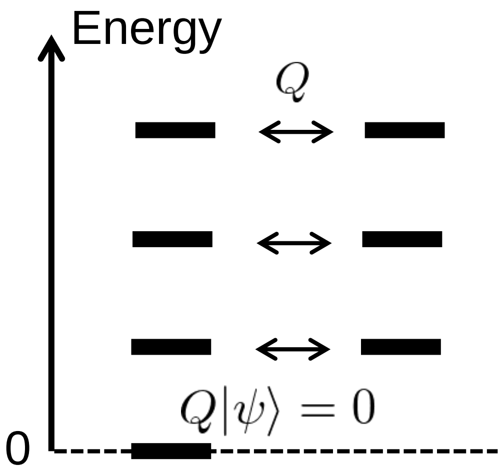
■ Algebra

- Fermionic parity: $(-1)^F$ (*F*: total fermion num.)
- Supercharge: Q $(Q^\dagger = Q)$ anti-commutes with $(-1)^F$
- Hamiltonian: $H = Q^2$
- Symmetry: $[H, (-1)^F] = [H, Q] = 0.$

■ Spectrum of *H*

- $E \geqq 0$ for all states, as *H* is p.s.d
- $E > 0$ states come in pairs $\{|\psi\rangle, Q|\psi\rangle\}$
- $E = 0$ state must be annihilated by Q

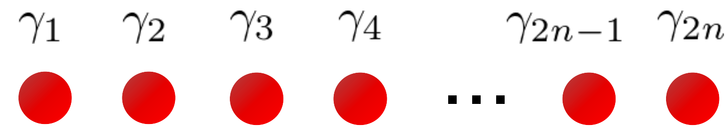
G.S. energy = 0 → SUSY ***unbroken***
G.S. energy > 0 → SUSY ***broken***



Lattice Majorana fermions

■ Definition

$$(\gamma_i)^\dagger = \gamma_i, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$



- Fermionic parity: $(-1)^F = \mathrm{i}^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$
- Complex fermions

$$c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - \mathrm{i}\gamma_{2j}) \qquad \{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = 0$$

■ Trivial example

$$Q = \sum_{j=1}^{2n} \gamma_j \quad \Rightarrow \quad H = Q^2 = \frac{1}{2}\sum_{i,j}\{\gamma_i, \gamma_j\} = 2n$$

Hamiltonian is constant. Trivially solvable.
E = 2*n* for any state. SUSY is broken. Too boring…

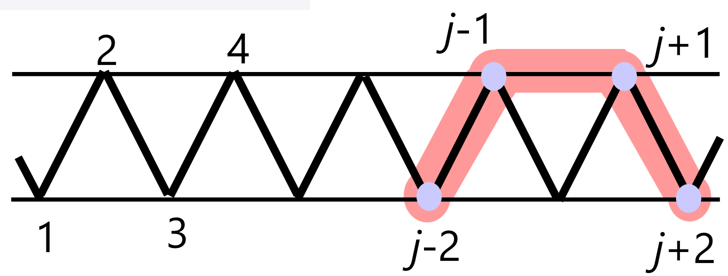
Majorana-Nicolai model

Definition

- Supercharge $$Q = \sum_{j=1}^{N} (g\gamma_j + \mathrm{i}\gamma_{j-1}\gamma_j\gamma_{j+1}), \qquad (g \in \mathbb{R})$$ with PBC
- Hamiltonian $$H = Q^2 = H_{\mathrm{free}} + H_{\mathrm{int}} + Ng^2$$

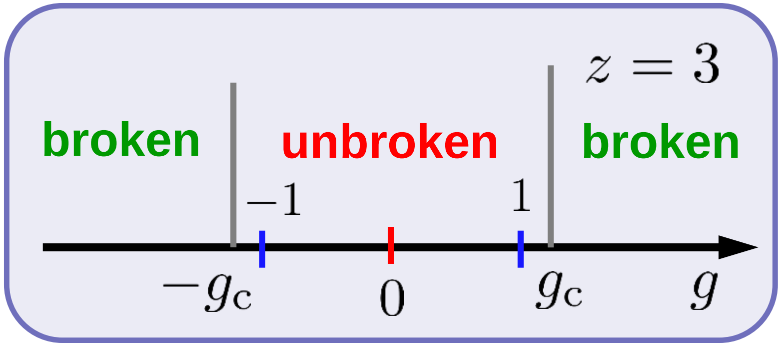
$$H_{\mathrm{free}} = 2\mathrm{i}g \sum (2\gamma_j\gamma_{j+1} - \gamma_j\gamma_{j+2})$$

$$H_{\mathrm{int}} = \sum (1 - 2\gamma_{j-2}\gamma_{j-1}\gamma_{j+1}\gamma_{j+2})$$



Phase diagram

Sannomiya, HK, *PRD* **99**, 045002 (2019)
O'Brien, Fendley, *PRL* **120**, 206403 (2018)



- Free-fermionic for g>>1. Rigorous upper bound on g_c.
- Integrable at g=0. [Fendley, arXiv:1901.08078]
  Super-frustration-free at g=±1.

SUSY is unbroken at *g* = 1

Super-frustration-free systems

- Supercharge: $Q = \sum_j q_j, \quad \{q_j, (-1)^F\} = 0, \forall j$

Definition. $Q = \sum_j q_j$ is said to be *super-frustration-free* if there exists a state $|\psi\rangle$ such that $q_j|\psi\rangle = 0$ for all *j*.

- Corollary: Such ψ is a g.s. of $H=Q^2$.

Exact ground states

$$Q = \sum_{l=1}^{N/2} (\gamma_{2l-2} + \gamma_{2l+1}) \underbrace{(1 + \mathrm{i}\gamma_{2l-1}\gamma_{2l})}_{h_{2l-1}} = \sum_{l=1}^{N/2} (\gamma_{2l-1} + \gamma_{2l+2}) \underbrace{(1 + \mathrm{i}\gamma_{2l}\gamma_{2l+1})}_{h_{2l}}$$

- h_{2l-1} : Local *H* of Kitaev chain in a trivial phase
- h_{2l} : Local *H* of Kitaev chain in a topo. phase
- The g.s. of *H* are the same as those of Kitaev chains.
  They are annihilated by all local *q*. (2 other g.s. for *N* = 0 mod 8.)
  Consistent with Hsieh *et al*., *PRL* **117** (2016)?

SUSY broken phase

■ SUSY breaking

$$H = Q^2 = H_{\text{free}} + H_{\text{int}} + Ng^2$$

Since H_{int} is p.s.d., the g.s. energy is bounded as $E_0 \geq Ng^2 + E_0^{\text{free}}$.

$E_0^{\text{free}}/N \geq -8g/\pi$ → SUSY is broken for $g > 8/\pi = 2.546\ldots$

■ NG fermions

- Variational argument

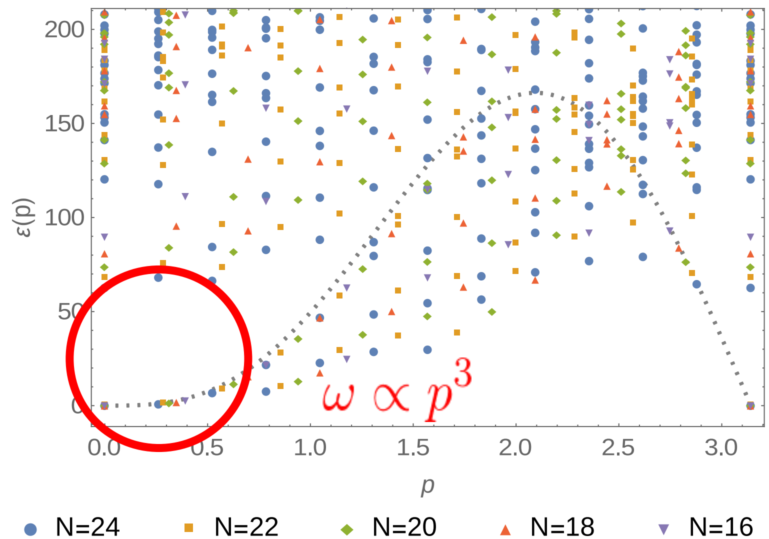
$$\Delta E_1 \leq \epsilon_{\text{var}}(p) \leq \sqrt{\frac{C}{e_0}}|p| + \mathcal{O}(p^3)$$

e_0 : g.s energy density

- Numerical result

g = 8.0, N= 16, 18, …, 24

Cubic dispersion around p=0



- Large-g limit

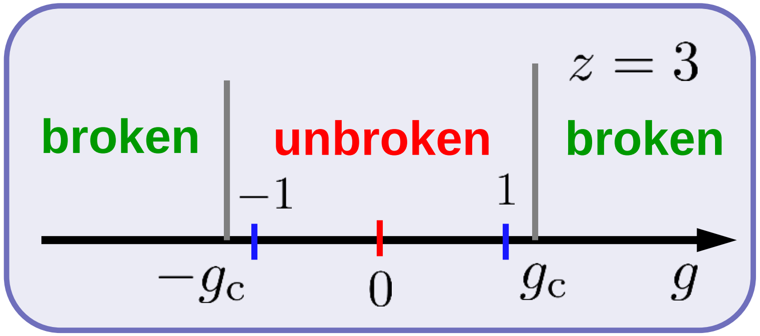
$$H_{\text{free}} = 8g \sum_{p>0} f(p)\gamma_p^\dagger \gamma_p + \text{const.}$$

$$f(p) = 2\sin p - \sin 2p = 2p^3 + \mathcal{O}(p^5)$$

Summary of Part II

Studied Majorana Nicolai model with *N*=1 SUSY

1. SUSY is ***unbroken*** for $|g| < g_c$
   Exact E=0 g.s. at g=1.
2. SUSY is ***broken*** for $|g| > g_c$
   Rigorous proof for $|g| > 8/\pi$
   NG fermion with *cubic* dispersion



➢ Sannomiya, HK, *PRD*, **99**, 045002 (2019)

What I did not touch on

- Majorana-Nicolai model in higher dim.
- SUSY Kitaev-honeycomb model
- SUSY SYK without disorder