Some recent results on ranking patterns and arrangements

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References for this talk

This talk is based on the following joint works after the Sapporo conference in August 2009.

- "Ranking patterns of unfolding models of codimension one" by Hidehiko Kamiya, A. Takemura, Hiroaki Terao. *Advances in Applied Mathematics*, **47** (2011) 379-400.
- "Application of arrangement theory to unfolding models" by Hidehiko Kamiya, A. Takemura, Norihide Tokushige. arXiv:1004.0043v1. To appear in Advanced Studies in Pure Mathematics series.
- "On intersection lattices of hyperplane arrangements generated by generic points" by Hiroshi Koizumi, Yasuhide Numata, A. Takemura. arXiv:1009.3676v1. To appear in *Annals of Combinatorics*.
- "Arrangements stable under the Coxeter groups" by Hidehiko Kamiya, A. Takemura, Hiroaki Terao. arXiv:1103.5179v1

Introduction to ranking models and arrangements

- The **unfolding model** (Coombs (1950), Psychol. Rev.) also known as the **ideal point model**.
- A model for preference rankings.
- **Objects** $1, 2, \ldots, m$.
- An **individual** ranks these m objects.
- The objects $1, 2, \ldots, m$ are represented by

$$\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^n.$$

- The individual is represented by

$$\boldsymbol{y} \in \mathbb{R}^n \ \ \text{(his/her \textbf{ideal point})}.$$

- $\mathbb{R}^n$: the **joint space**.

Introduction to ranking models and arrangements

- Preference: The nearer, the more preferred, i.e.,

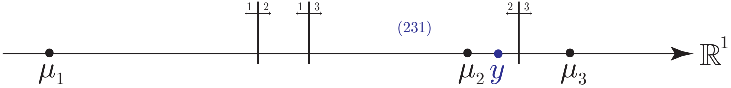
$$\boldsymbol{y} \text{ prefers } i \text{ to } j \iff \|\boldsymbol{y} - \boldsymbol{\mu}_i\| < \|\boldsymbol{y} - \boldsymbol{\mu}_j\|.$$

- $\boldsymbol{y}$ has ranking $(\underbrace{i_1}_{\text{best}}\ \underbrace{i_2}_{\text{2nd best}}\ \cdots\ \underbrace{i_m}_{\text{worst object}})$ iff

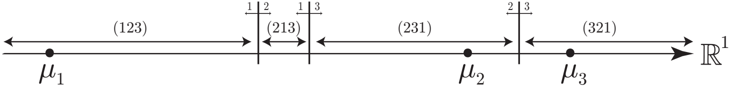
$$\|\boldsymbol{y} - \boldsymbol{\mu}_{i_1}\| < \|\boldsymbol{y} - \boldsymbol{\mu}_{i_2}\| < \cdots < \|\boldsymbol{y} - \boldsymbol{\mu}_{i_m}\|.$$

Example

- $n = 1,\ m = 3$:



- Now, let $\boldsymbol{y}$ run through $\mathbb{R}^1$.



- Then, $(123), (213), (231), (321)$ appear.
  (Four out of $3! = 6$ rankings.)

Ranking pattern

- In general (for general n, m), we say

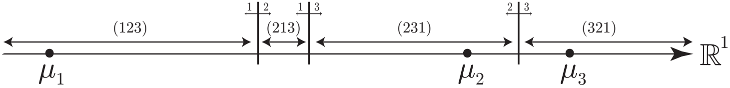
$$(i_1 \cdots i_m) : \textbf{admissible}$$
$$\overset{\text{def.}}{\iff} \ \|\boldsymbol{y} - \boldsymbol{\mu}_{i_1}\| < \cdots < \|\boldsymbol{y} - \boldsymbol{\mu}_{i_m}\|, \ \exists \boldsymbol{y} \in \mathbb{R}^n.$$

- Define the **ranking pattern**

$$\mathrm{RP}^{\mathrm{UF}}(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m) := \{\text{admissible rankings}\}$$

(of the unfolding model with $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$).

Example



- Admissible rankings:

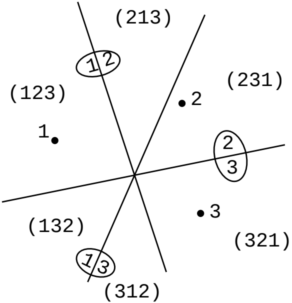
$$(123),\ (213),\ (231),\ (321).$$

- Ranking pattern:

$$\mathrm{RP}^{\mathrm{UF}}(\mu_1, \mu_2, \mu_3) \quad = \quad \{(123), (213), (231), (321)\}.$$

When the dimension is enough: $m = 3$, $n = 2$

- 3 points in $\mathbb{R}^2$:



- All rankings ($3! = 6$) appear.
- If $m - 1 \leq n$, then all rankings appear.

Number of rankings in a ranking pattern

- **Problem 1**: What is the number of admissible rankings for $m-1>n$? (common for all "generic" m points in $\mathbb{R}^n$)

- **Answer 1**: It is given by $c_0+c_1+\cdots+c_n$, where

$$(1+t)(1+2t)\cdots(1+(m-1)t)=c_0+c_1t+\cdots+c_{m-1}t^{m-1}. \quad (1)$$

- A different m-tuple $(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m)$,
  $\rightarrow$ a different $\mathrm{RP}^{\mathrm{UF}}(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m)$.

  Problem 2: How many ranking patterns are possible?

- We will discuss this later.

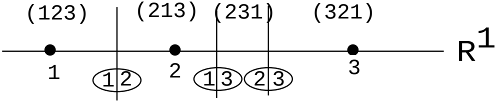
Braid slice

Answer 1 follows from the following facts:

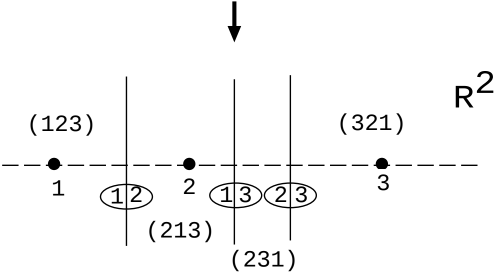
- Unfolding arrangement can be identified with an intersection of the braid arrangement with a lower-dimensional affine space.
- In the following we call the intersection a **braid slice**.
- Slicing corresponds to the truncation of the intersection lattice of the braid arrangement.
- Hence by truncating the Poincaré polynomial of the braid arrangement and using Zaslavsky's result we obtain Answer 1.

Perturbation viewpoint for braid slice

- Slicing can be intuitively understood by perturbation of points.
- Look at $m = 3$ in $\mathbb{R}^1$ again:

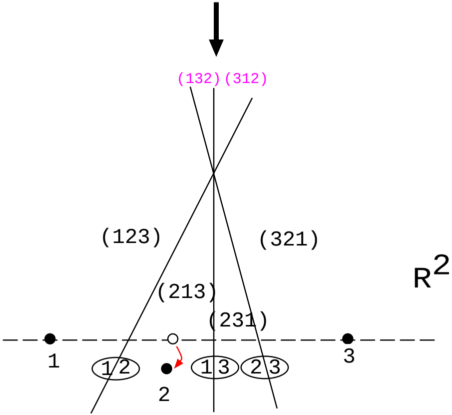


- We embed it in $\mathbb{R}^2$.



Perturbation viewpoint for braid slice

- Perturb the point 2.



Unfolding model as a braid slice in a matrix form

- We confirm the intuitive idea by writing relevant quantities.
- All of our discussions are in

$$H_0 : x_1 + \cdots + x_m = 0.$$

- Let $\mathcal{B}_m$ be the **braid arrangement** in H_0:

$$\mathcal{B}_m := \{H_{ij} : 1 \le i < j \le m\}, \qquad H_{ij} : x_i = x_j \text{ in } H_0.$$

- For any generic $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m \in \mathbb{R}^n$,

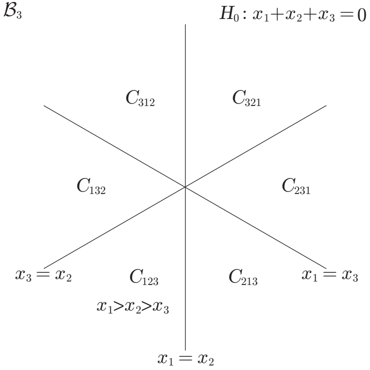
$$\mathrm{RP}^{\mathrm{UF}}(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m)$$

  can be obtained by slicing the braid arrangement $\mathcal{B}_m$ by an n-dimensional affine subspace K of H_0.

Unfolding model as a braid slice in a matrix form

- $\mathcal{B}_m$ has $m!$ chambers

$$C_{i_1 \cdots i_m} : \; x_{i_1} > \cdots > x_{i_m}.$$



Unfolding model as a braid slice in a matrix form

- Wlog, we assume $\sum_{i=1}^m \boldsymbol{\mu}_i = \mathbf{0}$ and $(1/m)\sum_{i=1}^m \|\boldsymbol{\mu}_i\|^2 = 1$.
- For an $m \times n$ matrix and a vector

$$\boldsymbol{W} := \begin{pmatrix} \boldsymbol{\mu}_1^T \\ \vdots \\ \boldsymbol{\mu}_m^T \end{pmatrix}, \quad \boldsymbol{u} := -\frac{1}{2}\begin{pmatrix} \|\boldsymbol{\mu}_1\|^2 - 1 \\ \vdots \\ \|\boldsymbol{\mu}_m\|^2 - 1 \end{pmatrix},$$

define an affine subspace

$$K := \boldsymbol{u} + \mathrm{col}\,\boldsymbol{W} \subset H_0 \qquad (\dim K = n, \dim H_0 = m-1).$$

- Then we have

$$(i_1 \cdots i_m) \text{ is admissible} \iff K \text{ intersects with } C_{i_1 \cdots i_m}.$$

Unfolding model as a braid slice in a matrix form

- The proof is quite simple:

$$\begin{aligned}
&\|\boldsymbol{y}-\boldsymbol{\mu}_{i_1}\| < \cdots < \|\boldsymbol{y}-\boldsymbol{\mu}_{i_m}\|,\ \exists \boldsymbol{y}\in\mathbb{R}^n\\
\iff\ & \boldsymbol{\mu}_{i_1}^T\boldsymbol{y} - \frac{1}{2}(\|\boldsymbol{\mu}_{i_1}\|^2-1)\\
&\qquad > \cdots > \boldsymbol{\mu}_{i_m}^T\boldsymbol{y} - \frac{1}{2}(\|\boldsymbol{\mu}_{i_m}\|^2-1),\ \exists \boldsymbol{y}\in\mathbb{R}^n\\
\iff\ & \boldsymbol{W}\boldsymbol{y}+\boldsymbol{u}\in C_{i_1\cdots i_m},\ \exists \boldsymbol{y}\in\mathbb{R}^n\\
\iff\ & K\cap C_{i_1\cdots i_m}\neq\emptyset.
\end{aligned}$$

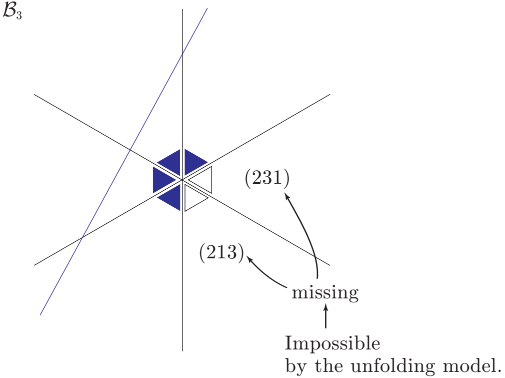
- Hence for any $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m\in\mathbb{R}^n$,

$$\mathrm{RP}^{\mathrm{UF}}(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m)=\{(i_1\cdots i_m) : K\cap C_{i_1\cdots i_m}\neq\emptyset\}. \qquad (2)$$

- We call the RHS (the ranking pattern of) the **braid slice by** K.

The problem of unrealizable braid slice

- Not all braid slices can be realized by the unfolding model:



- $\boldsymbol{u}$ in the definition of K is restricted, i.e., $\boldsymbol{u}$ depends on $\boldsymbol{W}$ in a nonlinear manner.

Two difficult problems for general $m > n + 1$

- Characterization of braid slices by all generic n-dimensional affine subspaces K.
- Identifying those slices which are realizable by unfolding model
- Both problems are currently intractable for general $m > n + 1$.

- For the one-dimensional case $n = 1$, we can analyze *realizable* slices by "mid-hyperplane arrangement"[1].
- For the case of codimension one $m - 2 = n$ (or $\dim K = \dim H_0 - 1$), we can analyze both problems by "all-subset arrangement" (restricted to H_0).

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[1] Non-realizable ones are not solved yet.

Number of ranking patterns for one-dimensional case

Mid-hyperplane arrangement

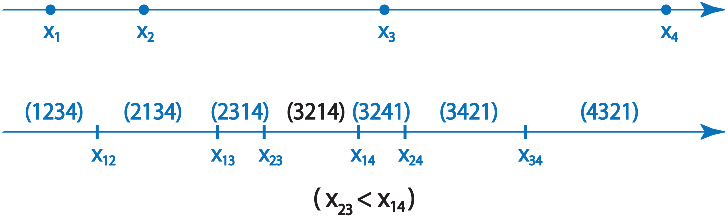
- Objects: $x_1, \dots, x_m \in \mathbb{R}^1$.

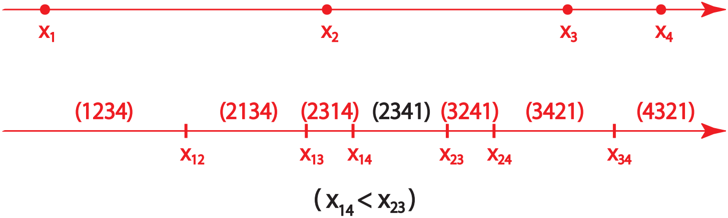
  (We write x_i instead of μ_i for mid-hyperplane arrangement.)
- The hyperplanes H_{ij} are the midpoints

$$x_{ij} := \frac{x_i + x_j}{2}, \quad 1 \le i < j \le m,$$

  of $x_1, \dots, x_m$.
- For $m = 3$, if we order $x_1 < x_2 < x_3$, there is only one ranking pattern.
- For $m = 4$, if we order $x_1 < x_2 < x_3 < x_4$, there are two possible patterns.

Number of ranking patterns for one-dimensional case





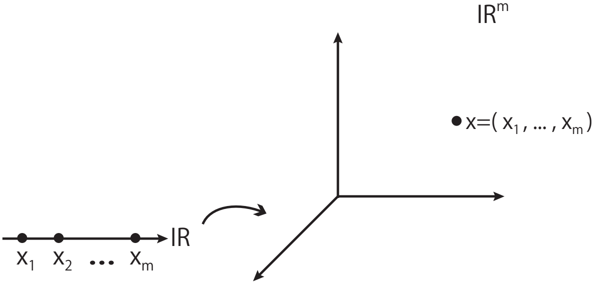
(Actually these are flips of each other, but we do not identify these two in this talk.)

Number of ranking patterns for one-dimensional case

- We may only consider the case

$$x_1 < \cdots < x_m.$$

- Identify $x_1, \ldots, x_m \in \mathbb{R}$ with $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$:



Order of midpoints

$$\text{Ranking pattern} \quad \Leftrightarrow \quad \text{Order of midpoints } x_{ij}.$$

Example: $m = 4$

- Ranking pattern:

$$\{(1234), (2134), (2314), (3214), \\ (3241), (3421), (4321)\}$$

$$\updownarrow$$

- Order of x_{ij}'s:

$$x_{12} < x_{13} < x_{23} < x_{14} < x_{24} < x_{34}.$$

Counting orders of midpoints

- Problem 2:
  What is the number of ranking patterns?
  → Find the number of possible orders of x_{ij}'s.
- For each pair x_{ij}, x_{kl}, we have

$$\begin{aligned} x_{ij} < x_{kl} \quad &\iff x_i + x_j < x_k + x_l \\ &\iff \mathbf{x} = (x_1, \ldots, x_m) \text{ is on one side} \\ &\ \text{of the hyperplane } H_{ijkl} \text{ in } \mathbb{R}^m, \end{aligned}$$

where

$$H_{ijkl} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i + x_j = x_k + x_l\}.$$

Counting orders of midpoints

- What we want to know boils down to:

  Problem 2': Into how many regions is the chamber

  $$C_0 : x_1 < \cdots < x_m$$

  of $\mathcal{B}_m$ divided by H_{ijkl}'s?

- This argument leads us to define

  $$\mathcal{M}_m := \mathcal{B}_m \cup \{H_{ijkl} \mid (i,j,k,l) \in I_4\}$$

  (the **mid-hyperplane arrangement**), where $I_4 := \{(i,j,k,l) \mid i,j,k,l$ are all distinct, $1 \le i < j \le m,\ i < k < l \le m\}$.

Answer to Problem 2 in one dimension

Theorem 1 (KOTT(2006))

The number of ranking patterns when $x_1, \dots, x_m$ are restricted so that $x_1 < \cdots < x_m$ (and that x_{ij}'s are all distinct) is

$$r(m) := \frac{\#\{\textit{chambers of } \mathcal{M}_m\}}{m!}.$$

- $r(m)$ for $m \leq 10$ (by finite field method):

$$r(3) = 1,\ r(4) = 2,\ r(5) = 12,\ r(6) = 168,\ r(7) = 4680,$$
$$r(8) = 229386,\ r(9) = 18330206,\ r(10) = 2241662282$$

Asymptotics for $r(m)$

Theorem 2 (Kamiya, Takemura, Tokushige (2010))

For all $m \geq 4$, we have

$$2\left(\frac{3}{4}\right)^{m-4}\{(m-3)!\}^2 \leq r(m) < \frac{2}{m!}\left\{\frac{em(m-1)^2}{8}\right\}^{m-2}.$$

For all sufficiently large m,

$$(0.10m^2)^m < r(m) < (0.93m^2)^m.$$

- Let $\ell(m)$ and $u(m)$: two bounds in the first statement of the theorem.
- Let

$$a(m) := \frac{(m-2)\{(m-2)^{m-3}-1\}\cdot(m-4)!}{m-3}.$$

Asymptotics for $r(m)$

- Let $f(m)$ denote an upper bound for $r(m)$ by Thrall.

$$f(m) := \frac{\{\frac{m(m-1)}{2}\}! \prod_{i=1}^{m-2} i!}{\prod_{i=1}^{m-1}(2i-1)!}.$$

Table 1: $r(m)$, $a(m)$, $\ell(m)$, $u(m)$, $f(m)$, for $4 \le m \le 10$.

m	$r(m)$	$a(m)$	$\ell(m)$	$u(m)$	$f(m)$
4	2	2	2	12	2
5	12	12	6	334	12
6	168	168	41	18,744	286
7	4,680	4,680	486	1.82×10^6	33,592
8	229,386	223,920	9,113	2.76×10^8	23,178,480
9	18,330,206	16,470,720	246,038	6.06×10^{10}	108,995,910,720
10	2,241,662,282	1,725,655,680	9.05×10^6	1.81×10^{13}	3,973,186,258,569,120

Number of ranking patterns for the case of codimension 1

- We consider the case $n = m - 2$.
- We first solve the problem of enumerating braid slices by introducing "all-subset arrangement" restricted to H_0.
- This step is independent of unfolding model.
- Then we show that almost all braid slices are realizable by unfolding model.

- We assume that $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m$ are generic. Then $K = \boldsymbol{u} + \operatorname{col} \boldsymbol{W}$ is a hyperplane in H_0:

$$\dim K = \dim H_0 - 1.$$

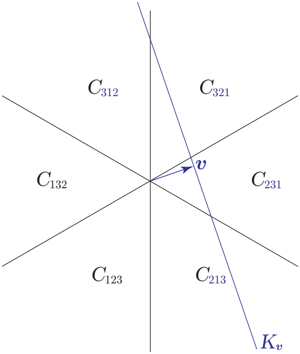
- An arbitrary affine hyperplane in H_0 can be indexed by its normal vector $\boldsymbol{v} \in H_0$:

$$K_{\boldsymbol{v}} := \{\boldsymbol{x} \in H_0 : \boldsymbol{v}^T \boldsymbol{x} = \|\boldsymbol{v}\|^2\}, \quad \boldsymbol{v} \neq \boldsymbol{0}.$$

Number of ranking patterns for the case of codimension 1

- Write the braid slice by $K_{\boldsymbol{v}}$ as $\mathrm{RP}(\boldsymbol{v})$:

$$\mathrm{RP}(\boldsymbol{v}) := \{(i_1 \cdots i_m) : K_{\boldsymbol{v}} \cap C_{i_1 \cdots i_m} \neq \emptyset\}.$$



$\mathrm{RP}(\boldsymbol{v}) = \{ (312), (321), (231), (213) \}$

Number of ranking patterns for the case of codimension 1

Basic lemma

Suppose $\boldsymbol{v} = (v_1, \ldots, v_m)^T \in H_0$ satisfies $\sum_{i \in I} v_i \neq 0$ for all $I \subset \{1, \ldots, m\}$, $1 \leq \#I \leq m-1$. Then for $(i_1 \cdots i_m)$ we have

$$\begin{aligned} & K_{\boldsymbol{v}} \cap C_{i_1 \cdots i_m} = \emptyset \\ & \quad \iff v_{i_1} < 0,\ v_{i_1} + v_{i_2} < 0, \ldots, v_{i_1} + \cdots + v_{i_{m-1}} < 0, \\ & K_{\boldsymbol{v}} \cap C_{i_1 \cdots i_m} \neq \emptyset \text{ is bounded} \\ & \quad \iff v_{i_1} > 0,\ v_{i_1} + v_{i_2} > 0, \ldots, v_{i_1} + \cdots + v_{i_{m-1}} > 0, \\ & K_{\boldsymbol{v}} \cap C_{i_1 \cdots i_m} \text{ is unbounded} \\ & \iff v_{i_1} + v_{i_2} + \cdots + v_{i_k} > 0 \text{ for some } k, \text{ and} \\ & \qquad v_{i_1} + v_{i_2} + \cdots + v_{i_\ell} < 0 \text{ for some } \ell. \end{aligned}$$

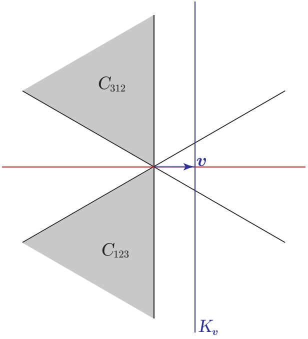
Number of ranking patterns for the case of codimension 1

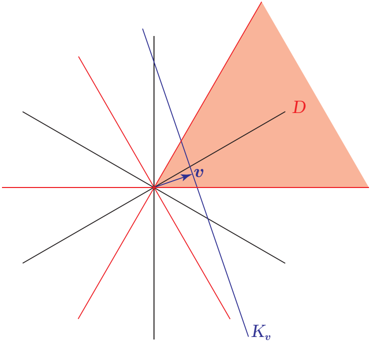
- Define the **all-subset arrangement** $\mathcal{A}_m$ in H_0 by

$$\mathcal{A}_m := \{H_I : I \subset \{1,\dots,m\},\ 1 \le \#I \le m-1\},$$

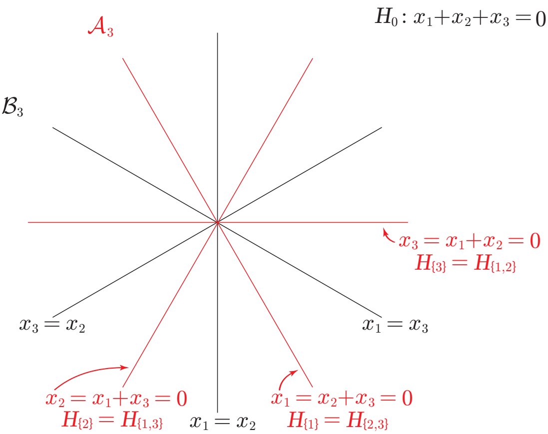
where

$$H_I : \sum_{i \in I} x_i = 0 \quad \text{in } H_0.$$





Number of ranking patterns for the case of codimension 1



Number of ranking patterns for the case of codimension 1

- Then we have a bijection

$$\{\text{chambers of } \mathcal{A}_m\} \leftrightarrow \{\text{braid slices}\}$$

- By counting the number of chambers of $\mathcal{A}_m$, we obtain the number of braid slices. (We can use the finite field method.)
- Let

$$\begin{aligned}
\mathcal{V}_1 &= \{\boldsymbol{v} = (v_1, \ldots, v_m)^T \in H_0 : \\
&\qquad \boldsymbol{v} \text{ has exactly one positive entry} \\
&\qquad\quad \text{or exactly one negative entry}\}, \\
\mathcal{V}_2 &:= \{\boldsymbol{v} = (v_1, \ldots, v_m)^T \in H_0 : \\
&\qquad \boldsymbol{v} \text{ has at least two positive entries} \\
&\qquad\quad \text{\& at least two negative entries}\}.
\end{aligned}$$

Realizability for the case of codimension 1

Theorem 3

For $v \in \mathcal{V}_2$*, both* $\mathrm{RP}(v)$ *and* $\mathrm{RP}(-v)$ *can be realized. For* $v \in \mathcal{V}_1$*, exactly one of* $\mathrm{RP}(v)$ *and* $\mathrm{RP}(-v)$ *can be realized.*

- Let $q(m)$ be the number of ranking patterns of unfolding models of codimension one:

$$\begin{aligned} q(m) \quad &:= \quad \#\{\mathrm{RP}^{\mathrm{UF}}(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m) : \\ &\qquad\qquad \text{generic } \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in \mathbb{R}^{m-2}\}. \end{aligned}$$

Then finally we have the following result.

Theorem 4

$$q(m) = (\textit{number of chambers of } \mathcal{A}_m) - m$$

Realizability for the case of codimension 1

- By the finite field method, for $m \leq 8$ we have

$$q(3) = 3,\ q(4) = 28,\ q(5) = 365,$$
$$q(6) = 11286,\ q(7) = 1066037,\ q(8) = 347326344.$$

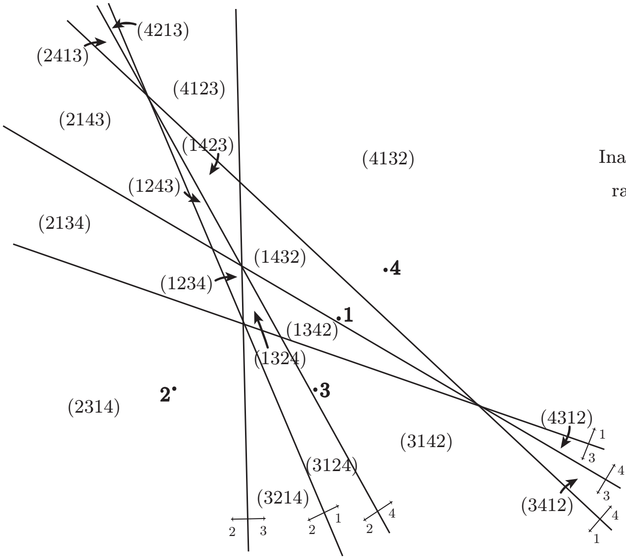
- Note that the symmetric group S_m acts on rankings by permutation of the labels of objects.
- Recently, the number of orbits ("inequivalent rankings") $q'(m)$, $m \leq 8$, have been obtained in "Arrangements stable under the Coxeter groups" as

$$q'(3) = 1,\ q'(4) = 3,\ q'(5) = 11,$$
$$q'(6) = 55,\ q'(7) = 575,\ q'(8) = 16639.$$

- In the following 3 slides, we show 3 inequivalent ranking patterns for $m = 4$.

Inequivalent ranking patters for $m = 4$

D$_1$



Inadmissible

rankings:

(2341)

(2431)

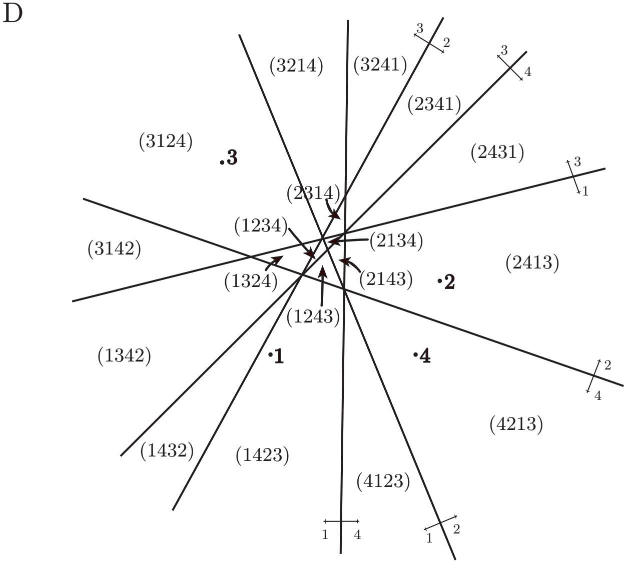
(3241)

(3421)

(4231)

(4321)

Inequivalent ranking patters for $m = 4$

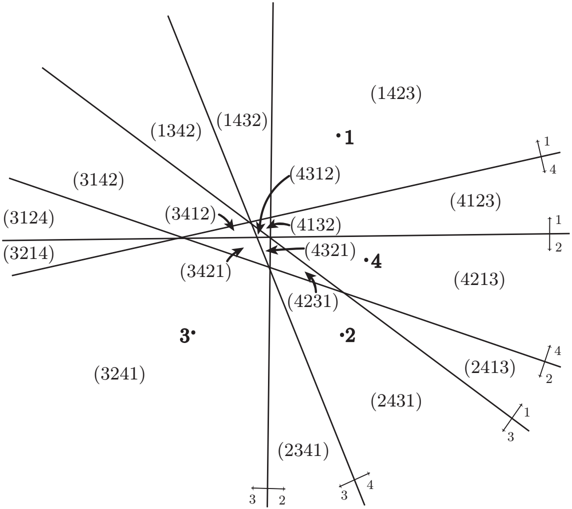


Inadmissible rankings:

(3412)
(3421)
(4312)
(4321)
(4132)
(4231)

Inequivalent ranking patters for $m = 4$

-D



Inadmissible rankings:

(2143)

(2134)

(1243)

(1234)

(1324)

(2314)

How did the all-subset arrangement come up?

- Let $\mathcal{A}$ be an essential central arrangement in $\mathbb{R}^m$.
- Let $L_1(\mathcal{A})$ denote the set of lines (one-dimensional elements) of the intersection lattice of $\mathcal{A}$.
- For each $\ell \in L_1(\mathcal{A})$, let H_ℓ be the linear hyperplane having the normal vector ℓ.
- We call $\{H_\ell : \ell \in L_1(\mathcal{A})\}$, the arrangement defined by the one-dimensional elements of the intersection lattice of $\mathcal{A}$.
- It can be shown that the all-subset arrangement is the arrangement defined by the one-dimensional elements of the braid arrangement.
- It can also be shown that the chambers of $\{H_\ell : \ell \in L_1(\mathcal{A})\}$ are in one-to-one relation to the pattern of slices of $\mathcal{A}$ by generic affine hyperplanes.

Relation to arrangements generated by generic points

- The arrangement considered by Koizumi, Numata and Takemura:

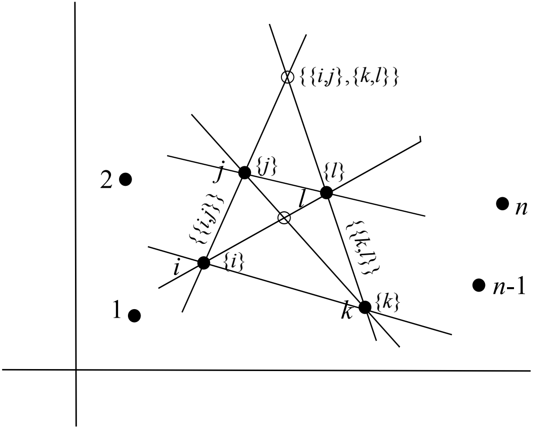


Figure 1: Arrangement for dimension two

Relation to arrangements generated by generic points

- Consider n generic points in $\mathbb{R}^m$.
- Each m-tuple of points defines (spans) an affine hyperplane in $\mathbb{R}^m$. The set of these hyperplanes defines an arrangement, which we call the arrangement generated by these points.
- It is known (from Michael Falk (1994)) that these arrangements are "linearly equivalent" to discriminantal arrangements.
- In Koizumi, Numata and Takemura we have studied the structure of intersection posets of the arrangements generated by generic points and evaluated characteristic polynomials up to dimension 6.

Relation to arrangements generated by generic points

- Consider the central version, i.e. arrangements generated by generic vectors.
- Take n generic vectors $v_1, \ldots, v_n$ (emanating from the origin) in $\mathbb{R}^m$.
- Each $(m-1)$-tuple of vectors spans a hyperplane in $\mathbb{R}^m$. The set of these hyperplanes defines an arrangement $\mathcal{A}$, which we call the arrangement generated by these vectors.
- If $m \leq n$, then this arrangement is central and essential.
- On the other hand, consider linear hyperplanes $H_1, \ldots, H_n$ where H_i has the normal v_i, $i = 1, \ldots, n$.

Relation to arrangements generated by generic points

- $\mathcal{B} = \{H_1, \dots, H_n\}$ is an arrangement of generic linear hyperplanes.
- By this correspondence, the hyperplane spanned by $m-1$ vectors $v_{i_1}, \dots, v_{i_{m-1}}$ corresponds to $H_{i_1} \cap \cdots \cap H_{v_{m-1}}$, which is exactly the one-dimensional element of $\mathcal{B}$.
- We see that an arrangement defined by one-dimensional elements of intersection lattice of another arrangement appears in various studies.

Summary

- We gave an introduction to ranking models and arrangements, as in Sapporo meeting in 2009.
- We gave an asymptotic evaluation of the number of ranking patterns for one-dimensional unfolding model.
- We explained the results for the case of codimension 1.
- We also pointed out the connection to arrangements generated by generic points.