Arrangements and statistics, Talk No.3: Other uses of hyperplane arrangement theory in statistics

Akimichi Takemura

University of Tokyo

August 5, 2009

Plan of three talks

1. Ranking and arrangements
(with H.Kamiya)
2. Mid-hyperplane arrangement and finite-field method
(with H.Kamiya, H.Terao and P.Orlik)
3. Other uses of hyperplane arrangement theory in statistics
(short survey[1] of works by other statisticians and algebraists)

[1]Totally dependent on my personal taste

Table of Contents for Talk No.3

Abstract tubes

Material in this section is largely based on a presentation of Satoshi Kuriki in Nov.2006.

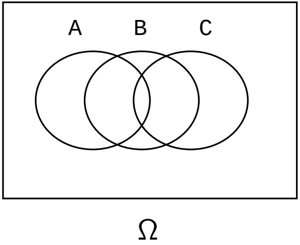
Abstract tube

- Abstract tube theory was developed by D.Q.Naiman and H.P.Wynn (1992,1997).
- It provides a nice theory on the structure of sign cancellations in inclusion-exclusion principle.
- The theory is also related to Hotelling-Weyl tube formula (variously known as Steiner formula, kinematic formula, etc).
- In fact it can be considered as a discrete version of the tube formula.

Abstract tube: simple example

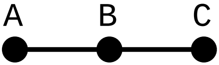
- Consider three subsets A, B, C of Ω, such that $A \cap C \subset B$.
- Then the inclusion exclusion principle for the probability $P(A \cup B \cup C)$ simplifies as

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C).$$



Abstract tube: simple example

- If there exist certain degeneracies in the way the subsets intersect each other, then the inclusion-exclusion simplifies.
- The degeneracy for this example can be depicted by a linear graph:

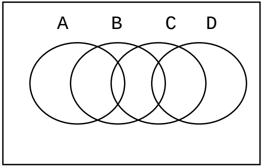


- Actually this corresponds to a simplicial complex.

$$\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}.$$

Abstract tube: another example

- Consider four sets A, B, C, D such that $A \cap C \subset B$ and $B \cap D \subset C$.



Ω

- Again it corresponds to a linear graph: 

- The inclusion-exclusion simplifies as

$$
\begin{aligned}
P(A \cup B \cup C \cup D) &= P(A) + P(B) + P(C) + P(D) \\
&\quad - P(A \cap B) - P(B \cap C) - P(C \cup D).
\end{aligned}
$$

In terms of indicator function

- In fact, the inclusion-exclusion principle applies to indicator functions of sets:

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$$

- For A, B, C with $A \cap C \subset B$,

$$1_{A \cup B \cup C} = 1_A + 1_B + 1_C - 1_{A \cap B} - 1_{B \cap C} \qquad (\forall \omega \in \Omega).$$

- Just consider how many times each ω is counted on the RHS.

Notation and definition

- Let Ω be a base space and let $A_1, \dots, A_n$ be its subsets.
- Write $V = [n] = \{1, \dots, n\}$ and let $\mathcal{T}$ be a simplicial complex of subsets of V.
- Write $\mathcal{A} = \{A_1, \dots, A_n\}$.

Notation and definition

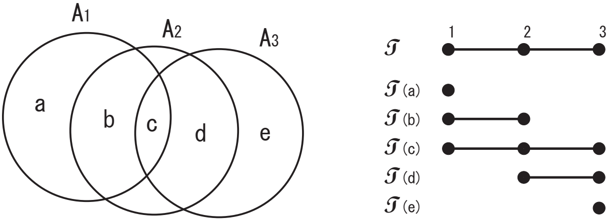
Definition 1

The pair $(\mathcal{A}, \mathcal{T})$ forms an abstract tube, if for every $\omega \in \bigcup_{i \in V} A_i$, the subcomplex

$$\mathcal{T}(\omega) = \{I \in \mathcal{T} \mid \omega \in \bigcap_{i \in I} A_i\}$$

is contractible. If $\dim(\mathcal{T}) = d - 1$, then we call d the *depth* of the tube.

Notation and definition



- Let $\mathcal{A} = \{A_1, A_2, A_3\}$

$$\mathcal{T} = \{\{1,2\},\{2,3\}, \text{and subsets}\}.$$

Then $(\mathcal{A}, \mathcal{T})$ forms an abstract tube.

Note on definition

- Abstract tube is not necessarily unique. Let

$$\mathcal{T} = \{\{1, 2, 3\}, \text{and subsets}\}$$

(2-simplex with vertices $\{1\}, \{2\}, \{3\}$). Then $(\mathcal{A}, \mathcal{T})$ is also an abstract tube.

- Smaller $\mathcal{T}$ gives a better inclusion-exclusion.
- However there may not exist a unique minimum $\mathcal{T}$, such that $(\mathcal{A}, \mathcal{T})$ forms an abstract tube.

Theorem by Naiman and Wynn

Theorem 2

(Naiman-Wynn): If $\{\mathcal{A}, \mathcal{T}\}$ forms an abstract tube, then

$$1_{\cup_{i=1}^n A_i}(\omega) = \sum_{F \in \mathcal{T}} (-1)^{\dim F} 1_{\cap_{i \in F} A_i}(\omega), \qquad \forall \omega \in \Omega. \tag{1}$$

- A proof is easy based on the fact
  1. Euler characteristic is 1 for a contractible simplicial complex.
  2. Euler characteristic is the alternating sum of number of faces of each dimension.

Proof of the theorem in formulas

- The Euler characteristic of a simplicial complex $\mathcal{K}$ is given by

$$\chi(\mathcal{K}) = \sum_{F \in \mathcal{K}} (-1)^{\dim F}.$$

- Both sides of (1) is zero for $\omega \notin \cup_i A_i$.
- For $\omega \in \cup_i A_i$, LHS=1. For RHS, by the definition of $\mathcal{T}(\omega)$, we have

$$1_{\bigcap_{i \in F} A_i}(\omega) = \begin{cases} 1 & (F \in \mathcal{T}(\omega)) \\ 0 & (F \notin \mathcal{T}(\omega)) \end{cases}.$$

Therefore

$$\text{RHS} = \sum_{F \in \mathcal{T}(\omega)} (-1)^{\dim F} = \chi(\mathcal{T}(\omega)) = 1.$$

Inequality version by Naiman and Wynn

- Inclusion-exclusion principle gives upper bounds and lower bounds in an alternating way.
- Using basic homology theory (corresponding to Morse's inequality), Naiman and Wynn have shown the inequality version too.

Theorem 3

(Naiman-Wynn): If $(\{A_i\}_{i\in V}, \mathcal{T})$ is an abstract tube, then

$$1_{\bigcup_{v\in V} A_v} \geq \sum_{\substack{I\in\mathcal{T}\\ |I|\leq r}} (-1)^{|I|-1} 1_{\bigcap_{i\in I} A_i}, \quad (r : \text{even})$$

$$1_{\bigcup_{v\in V} A_v} \leq \sum_{\substack{I\in\mathcal{T}\\ |I|\leq r}} (-1)^{|I|-1} 1_{\bigcap_{i\in I} A_i}, \quad (r : \text{odd}).$$

Relation to hyperplane arrangements

- FACT: A set of half-spaces in $\mathbb{R}^d$ forms an abstract tube with depth d.
- Intuitively, many half-spaces in $\mathbb{R}^d$ can intersect each other only in certain degenerate manner.

- Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, where each A_i is a half-space not containing the origin:
$$A_i = \{x \in \mathbb{R}^d \mid \langle x, u_i \rangle \geq 1\} \quad (u_i \neq 0).$$
- We are considering the case that $A_1, \ldots, A_n$ do not cover the whole $\mathbb{R}^d$. So without loss of generality we can assume that the origin does not belong to $\cup_i A_i$.

Relation to hyperplane arrangements

- Define a closed polyhedron P containing the origin by

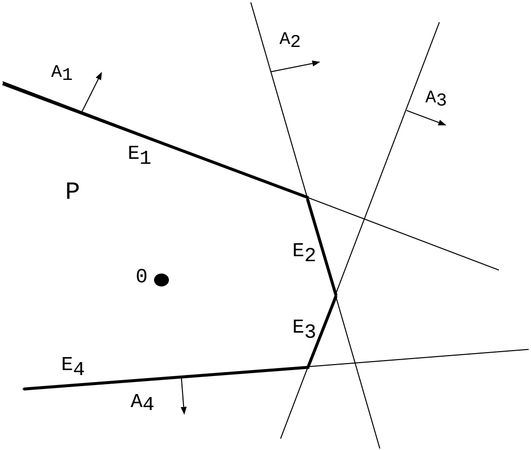
$$P = \bigcap_{i=1}^{n} \overline{A_i^c} = \overline{(\cup_{i=1}^{n} A_i)^c}.$$

- Let ∂A_i denote the boundary of A_i and define

$$E_i = P \cap \partial A_i$$

  which is the common boundary between P and A_i.
- We can assume that each E_i is a facet of P, because we can omit other A_i's from $\mathcal{A}$.

Relation to hyperplane arrangements



Relation to hyperplane arrangements

- FACT: Define

$$\mathcal{T} = \Big\{ F \subset \{1, \dots, n\} \mid \bigcap_{i \in F} E_i \text{ is a face of } P \Big\}.$$

Then $(\mathcal{A}, \mathcal{T})$ forms an abstract tube.

Voting theory

This section was prepared by Hidehiko Kamiya. We mainly present Terao's result (2007).

Voting theory

- A society of m **individuals**.
- ℓ **policy options**.
- Every individual has his/her **ranking** of the ℓ policy options.
- A **social welfare function (SWF)** is an aggregation rule (or a voting system) which assigns a ranking (interpreted as the **social ranking**) to any profile of the m individuals' rankings.

Voting theory

Two requirements on SWF

- **Pareto property**: If every individual ranks option i above option j, then the society ranks i above j.
- **Pairwise independence**: Social ranking over any two options $\{i, j\}$ depends only on individuals' rankings over these two options $\{i, j\}$.

Voting paradox

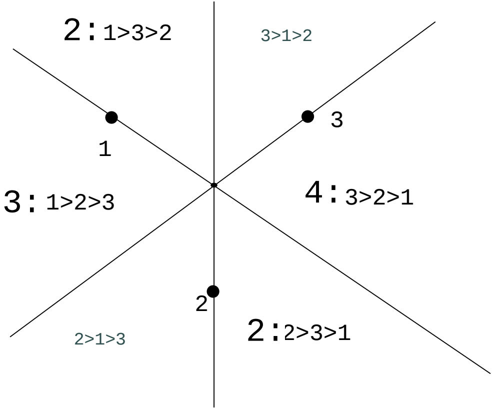
- The **simple majority rule** satisfies these two requirements, but can cause a **voting cycle**:
- If $a : 1 \succ 2 \succ 3,\ b : 2 \succ 3 \succ 1,\ c : 3 \succ 1 \succ 2$ (individual a ranks option 1 best, 2 second best and 3 worst, etc.), then the simple majority rule yields

$$1 \succ 2 \succ 3 \succ 1$$

  and cannot give a social ranking.
- An example from Saari (Notices of the AMS, April 2008).

$$\begin{aligned} &3 \text{ voters} : 1 \succ 2 \succ 3, \quad 2 \text{ voters} : 2 \succ 3 \succ 1 \\ &2 \text{ voters} : 1 \succ 3 \succ 2, \quad 4 \text{ voters} : 3 \succ 2 \succ 1 \end{aligned}$$

Voting paradox



Voting paradox

$$3 \text{ voters} : 1 \succ 2 \succ 3, \quad 2 \text{ voters} : 2 \succ 3 \succ 1$$
$$2 \text{ voters} : 1 \succ 3 \succ 2, \quad 4 \text{ voters} : 3 \succ 2 \succ 1$$

- **Positional voting rule** with voting vector
  - vote-for-one: (1,0,0) 1 wins with $1 \succ 3 \succ 2$ ranking.
  - vote-for-two: (1,1,0) 2 wins with $2 \succ 3 \succ 1$ ranking.
  - Borda count: (2,1,0) 3 wins with $3 \succ 2 \succ 1$ ranking.
- Different weights lead to different results.
- Furthermore positional voting rule does not satisfy pairwise indepedence.
- Then, does there exits an SWF (voting system) satisfying the two requirements?

Arrow's impossibility theorem

- Kenneth Arrow proved a shocking result: *Suppose* $\ell \geq 3$. *Then, every SWF satisfying the above two requirements is* **dictatorial**, *i.e, the social ranking is equal to a particular individual's ranking.*

  Arrow, K. J., *Social Choice and Individual Values*, Cowles Commission Monograph 12, John Wiley & Sons, New York, 1951.

- Elegant proof of Arrow's impossibility theorem by the theory of hyperplane arrangements.

  Terao, H.: Chambers of arrangements of hyperplanes and Arrow's impossibility theorem, *Advances in Math.* **214** (2007), 366–378.

Terao's work on Arrow's impossibility theorem: Setup

- $\mathcal{A}$: a nonempty central arrangement in $\mathbb{R}^\ell$.
- $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$: the set of chambers of $\mathcal{A}$.
- Each $H = \ker(\alpha) \in \mathcal{A}$ defines half-spaces

$$
\begin{aligned}
H^+ &:= \{x \in \mathbb{R}^\ell : \alpha(x) > 0\}, \\
H^- &:= \{x \in \mathbb{R}^\ell : \alpha(x) < 0\}.
\end{aligned}
$$

- $B := \{+, -\}$.
- For $H \in \mathcal{A}$, define $\epsilon_H^+, \epsilon_H^- : \mathbf{Ch} \to B$ by

$$
\epsilon_H^+(C) := \begin{cases} + & \text{if } C \subseteq H^+, \\ - & \text{if } C \subseteq H^-, \end{cases}
$$

$$
\epsilon_H^-(C) := \begin{cases} - & \text{if } C \subseteq H^+, \\ + & \text{if } C \subseteq H^-. \end{cases}
$$

Terao's work on Arrow's impossibility theorem: Setup

- Let

$$\begin{aligned}\mathbf{Ch}^m &:= \underbrace{\mathbf{Ch}\times\cdots\times\mathbf{Ch}}_{m \text{ times}},\\ B^m &:= \underbrace{B\times\cdots\times B}_{m \text{ times}}.\end{aligned}$$

- Then $\epsilon_H^{\pm} : \mathbf{Ch} \to B$ induce $\epsilon_H^{\pm} : \mathbf{Ch}^m \to B^m$:

$$\begin{aligned}\epsilon_H^{\sigma}(C_1,\ldots,C_m) := (\epsilon_H^{\sigma}(C_1),\ldots,\epsilon_H^{\sigma}(C_m)),\\ \sigma \in B = \{+,-\}.\end{aligned}$$

Terao's work: Admissible map

- A map

$$\Phi : \mathbf{Ch}^m \to \mathbf{Ch}$$

is called an **admissible map** if there exists a family of maps

$$\varphi_H^\sigma : B^m \to B \quad (H \in \mathcal{A},\ \sigma \in B = \{+, -\})$$

satisfying the following two conditions.

Terao's work: Admissible map

(1) $\varphi_H^\sigma(\tau, \dots, \tau) = \tau, \ \forall \sigma, \tau \in B$,

(2) the diagram

$$\begin{array}{ccc} \mathbf{Ch}^m & \xrightarrow{\Phi} & \mathbf{Ch} \\ \epsilon_H^\sigma \Big\downarrow & & \Big\downarrow \epsilon_H^\sigma \\ B^m & \xrightarrow{\varphi_H^\sigma} & B \end{array}$$

commutes for each $H \in \mathcal{A}$ and $\sigma \in B$.

Terao's work: Decomposability

- $\mathcal{A}$ is **decomposable** if there exist nonempty $\mathcal{A}_1, \mathcal{A}_2$ such that

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \text{ (disjoint)}$$

  and

$$r(\mathcal{A}) = r(\mathcal{A}_1) + r(\mathcal{A}_2)$$

- Here

$$r(\mathcal{A}) := \text{codim}_{\mathbb{R}^\ell} \bigcap_{H \in \mathcal{A}} H \quad \text{(the rank of } \mathcal{A}\text{).}$$

- $\mathcal{A}$ is said to be **indecomposable** if it is not decomposable.

Terao's result

- Terao proves: When $|\mathcal{A}| \geq 3$ and $m \geq 2$,

$$\begin{aligned} &\mathcal{A} \text{ is indecomposable} \\ &\iff \text{Every admissible map } \Phi \text{ is} \\ &\qquad \text{a projection to a component} \\ &\qquad \text{(say, the } h\text{-th component).} \end{aligned}$$

- Note that: When Φ is the projection to the h-th component, (Φ is admissible and) the corresponding φ_H^σ ($H \in \mathcal{A}$, $\sigma \in B$) are the projections to the h-th component.

A: Braid arrangement

- Now consider the case where $\mathcal{A}$ is the braid arrangement in $\mathbb{R}^\ell$ ($\ell \geq 3$).
- In this case, $\mathcal{A}$ is indecomposable, and thus we obtain the following result: *When $\mathcal{A}$ is a braid arrangement with $\ell \geq 3$, every admissible map Φ is a projection.*

A: Braid arrangement

- This result can be seen to be equivalent to Arrow's impossibility theorem:

$$
\begin{array}{rcl}
\text{Chamber} & \leftrightarrow & \text{Ranking} \\
\Phi & \leftrightarrow & \text{SWF} \\
\text{Projection} & \leftrightarrow & \text{Dictatorship} \\
\text{Condition (1)} & \leftrightarrow & \text{Pareto property} \\
\text{Condition (2)} & \leftrightarrow & \text{Pairwise independence.}
\end{array}
$$

Voting theory and hyperplane arrangement: Other topics

A. Caplin, B. Nalebuff, On 64%-majority rule, *Econometrica* **56** (1988) 787–814.
A. Caplin, B. Nalebuff, Aggregation and social choice: a mean voter theorem, *Econometrica* **59** (1991) 1–23.

- With the restriction on the individual rankings by the unfolding model (together with some restriction on the distribution of individuals), Caplin and Nalefuff established the existence of a super-majority winner.
- If the status quo is based on the preference of an average voter, then it can not be overtuled by any alternative under the requirement that the alternative has to be supported by more than 64% of the voters.

Voting theory and hyperplane arrangement: Other topics

D. Lepelley, A. Louichi, H. Smaoui, On Ehrhart polynomials and probability calculations in voting theory, *Social choice and Welfare* **30** (2008) 363-383.

- Three options $1, 2, 3$.
- $3! = 6$ rankings:

$$1 \succ 2 \succ 3, \quad 1 \succ 3 \succ 2, \quad 2 \succ 1 \succ 3,$$
$$2 \succ 3 \succ 1, \quad 3 \succ 1 \succ 2, \quad 3 \succ 2 \succ 1.$$

- Let n_{123} be

$$\text{the number of individuals with } 1 \succ 2 \succ 3,$$

etc.

Voting theory and hyperplane arrangement: Other topics

- The event "Option 1 is the Condorcet winner" corresponds to

$$
\begin{aligned}
1 \succ 2 : n_{123} + n_{132} + n_{312} &> n_{213} + n_{231} + n_{321}, \\
1 \succ 3 : n_{123} + n_{132} + n_{213} &> n_{231} + n_{312} + n_{321}.
\end{aligned}
$$

- The probability of such an event can be obtained by counting the number of integer solutions to a system of integral linear inequalities (i.e., the number of lattice points in a rational polytope).
  $\rightarrow$ The Ehrhart theory.
- The Ehrhart theory was rediscovered in social choice literature in 2000!

Non-regular fractional factorial designs

Non-regular fractional factorial designs

- Relation of the material in this section to arrangements is very tentative.
- Fractional factorial designs are important in application.
- Theory of regular fractional factorial designs is well developed.
- A regular fractional design corresponds to a single affine subspace of a vector space over $\mathcal{F}_2$. (equivalent to a linear code in coding theory)
- Other "non-regular" fractional factorial designs have not been studied very well.

References for this section

- Aoki, S. and Takemura, A., Some characterizations of affinely full-dimensional factorial designs. *Journal of Statistical Planning and Inference*, **139**, No.10, (2009). 3525–3532.
- Pistone, G., Riccomagno, E. and Wynn, H. P., *Algebraic Statistics, Computational Commutative Algebra in Statistics*. Chapman & Hall. (2000).
- Wu, C. F. J. and Hamada, M., *Experiments: Planning, Analysis, and Parameter Design Optimization*. Wiley. (2000).

Introduction to regular fractional factorial designs

- For simplicity, we only consider the case that each "factor" has two levels, $+1$ and -1.
- Consider some factors influencing quality of products in a production process, such as temperature, pressure, types of ingredients, etc.
- We conduct experiments by setting each factor either to a "high level" $(+1)$ or to a "low level" (-1).
- Suppose that there are 6 factors and denote them as A, B, C, D, E, F.
- We need $64 = 2^6$ "runs", if we do experiments for every combination of levels of 6 factors.
- If an experiment is expensive, we want to cut down the number of runs.

Introduction to regular fractional factorial designs

- Suppose that we want to cut down the number of experiments to $16 = 64/4$.
- We can use the following rule:

$$I = ABE, \quad I = ACDF.$$

- $I = ABE$ means that we get 1 when we multiply levels ± 1 of A,B,E, e.g. $A = 1, B = -1, E = -1$.
- Equivalently the level of E is determined by $E = AB$ (in view of $1 = A^2 = \cdots = F^2$).
- Similarly $I = ACDF$ can be rewritten as $F = ACD$.

Introduction to regular fractional factorial designs

Combination of levels for 16 runs

A	B	C	D		E	F
+	+	+	+		+	+
+	+	+	−		+	−
+	+	−	+		+	−
+	+	−	−		+	+
+	−	+	+		−	+
+	−	+	−		−	−
+	−	−	+		−	−
+	−	−	−		−	+
−	+	+	+		−	−
−	+	+	−		−	+
−	+	−	+		−	+
−	+	−	−		−	−
−	−	+	+		+	−
−	−	+	−		+	+
−	−	−	+		+	+
−	−	−	−		+	−

Rewriting to additive form

- We can code the high level as 0 and the low level as 1:

$$+ \to 0, \qquad - \to 1$$

- Then the multiplication on the previous page becomes mod 2 addition ("exclusive or").
- Let $x_1, \dots, x_6$ denote the "bits" (i.e. 0 or 1) of 6 levels.
- $E = AB, F = ACD$ is rewritten as

$$\begin{aligned} x_5 &\equiv x_1 + x_2 \pmod 2, \\ x_6 &\equiv x_1 + x_3 + x_4 \pmod 2, \end{aligned}$$

which is the parity check.

Rewriting to additive form

The following 16 points are on a 4-dimensional linear subspace of $\mathbb{F}_2^6$.

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	0	1
0	0	1	1	0	0
0	1	0	0	1	0
0	1	0	1	1	1
0	1	1	0	1	1
0	1	1	1	1	0
1	0	0	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	0	1	0	0
1	1	1	0	0	0
1	1	1	1	0	1

Back to the multiplicative form

- Multiplicative notation has merits.
- Consider A, B, C, D, E, F as "indeterminates" and consider a polynomial ring $k[A, B, C, D, E, F]$, where k is some field.
- Let

$$I = \langle A^2 - 1, B^2 - 1, C^2 - 1, D^2 - 1, E^2 - 1, F^2 - 1, \\ ABE - 1, ACDF - 1 \rangle$$

  denote an ideal of $k[A, B, C, D, E, F]$.
- Pistone et al. call this the "design ideal".
- The 16 runs are points of the variety $V(I)$.

Merits of the multiplicative form

- Consider d factors. Let

$$\mathcal{D} = \{-1, 1\}^d = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid x_1^2 = \cdots = x_d^2 = 1\}$$

- A fractional factorial design $\mathcal{F}$ is any subset of $\mathcal{D}$.
- Suppose that you have a budget constraint to conduct only certain number of runs (say 13) of the experiment. How do you choose $\mathcal{F}$ (say 13 points from $\mathcal{D}$) ?
- Clearly you can not choose a regular design, because the run size of a regular design is a power of 2.
- Let $f(x_1, \ldots, x_d)$ be a square-free polynomial in $x_1, \ldots, x_d$.
- With multiplicative notation we can specify $\mathcal{F} = V(I)$, where

$$I = \langle x_1^2 - 1, \ldots, x_d^2 - 1, f(x_1, \ldots, x_d) - 1 \rangle.$$

Merits of the multiplicative form

- Pistone et al. show that properties of $\mathcal{F}$ can be studied by computing Gröbner basis of I.
- In Aoki and Takemura (2009) we investigated some properties of design $\mathcal{F}$ such that $\mathcal{F}$ is a subset of no regular fractional designs.
- Such a design is very opposite to regular designs, but some good designs are found in the proposed class.
  (good: maximizing "statistical information")
- More recently, we are looking at cases where $\mathcal{F}$ is a union of some regular fractional designs (finally related to subspace arrangements).
- We find that, depending on the run size, some good designs are union of a few regular fractional factorial designs.

Concluding remarks

- We presented abstract tube theory of Naiman and Wynn.
- We presented voting theory and a result by Prof. Terao.
- We presented some results on non-regular fractional factorial experimental designs.

Bibliography I

- Aoki, S. and Takemura, A., Some characterizations of affinely full-dimensional factorial designs. *Journal of Statistical Planning and Inference*, **139**, No.10, (2009). 3525–3532.
- Arrow, K. J., *Social Choice and Individual Values*, Cowles Commission Monograph 12, John Wiley & Sons, New York. (1951).
- Caplin, A. and Nalebuff B., On 64%-majority rule, *Econometrica* **56** (1988) 787–814.
- Caplin, A. and Nalebuff B., Aggregation and social choice: a mean voter theorem, *Econometrica* **59** (1991) 1–23.
- Klaus Dohmen, *Improved Bonferroni Inequalities via Abstract Tubes, Inequalities and Identities of Inclusion-Exclusion Type*, Lecture Notes in Mathematics, 1826, Springer. (2003)

Bibliography II

- Satoshi Kuriki, On tube formula and abstract tube theory. (in Japanese) A presentation in a workshop on developments of algebraic statistics at RIMS, Kyoto, Nov.8, 2006.
- Lepelley, D. Louichi, A. and Smaoui, H., On Ehrhart polynomials and probability calculations in voting theory, *Social choice and Welfare*, **30**, (2008). 363–383.
- Naiman, D.Q. and Wynn, H.P., Inclusion-exclusion-Bonferroni identities and inequalities for discrete tube-like problems via Euler characteristics, *Ann. Statist.*, **20**, (1992). 43–76.
- Naiman, D.Q. and Wynn, H.P., Abstract tubes, improved inclusion exclusion identities and inequalities and importance sampling, *Ann. Statist.*, **25**, (1997), 1954–1983.

Bibliography III

- Pistone, G., Riccomagno, E. and Wynn, H. P., *Algebraic Statistics, Computational Commutative Algebra in Statistics*. Chapman & Hall. (2000).
- Saari, D. G., Mathematics and voting, *Notices of the AMS*, **55**, No.4, (2008). 448–455.
- Terao, H., Chambers of arrangements of hyperplanes and Arrow's impossibility theorem, *Advances in Math.* **214** (2007), 366–378.
- Wu, C. F. J. and Hamada, M., *Experiments: Planning, Analysis, and Parameter Design Optimization*. Wiley. (2000).