Properties of Ehrhart quasi-polynomials for hyperplane arrangements with integral coefficients

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Contents

Motivation for studying Ehrhart polynomials

- There are two interesting polynomials: “tube formula” and “Ehrhart polynomial”.
- Ehrhart polynomial seems to be a discrete analog of tube formula to me. (I am not aware of any explicit result showing the relation.)

- **Tube formula**: let M be a compact set in $\mathbb{R}^m$ and let $B = \{x \mid \|x\| \leq 1\}$ the unit ball. Then

$$\begin{aligned}\mathrm{Vol}(M + rB) = {} & \mathrm{Vol}(M) + r\mathrm{Vol}(\partial M) \\ & + \cdots + r^m \mathrm{Vol}(B)\chi(M)\end{aligned}$$

  - $\chi(M)$ in the last term is the Euler characteristic of M.
  - Coefficients are various “curvatures” and theoretically well understood. However often they are hard to evaluate.

- **Ehrhart polynomial:** Let P be a polytope in $\mathbb{R}^m$ with integral extreme points. Let $k \in \mathbb{Z}_>$ be a positive integer. Then

$$\#\{kP \cap \mathbb{Z}^m\} = k^m \mathrm{Vol}(P) + \frac{1}{2} \mathit{Vol}(\partial P) k^{m-1} + \cdots + 1.$$

  - The coefficient of k^{m-1} is not exactly the volume of the boundary of P. It depends on how the facets are "skewed" w.r.t. the integer lattice $\mathbb{Z}^m$. It has also a mysterious $1/2$ in front.

– The constant term 1 is again the Euler characteristic.

– It is instructive to think of refining the integer lattice k-fold and think of counting the number of points of $(1/k)\mathbb{Z}^d$ in P, i.e.

$$\#\{kP \cap \mathbb{Z}^m\} = \#\{P \cap \frac{1}{k}\mathbb{Z}^m\}.$$

– If the extreme points of P are rational, then $\#\{kP \cap \mathbb{Z}^d\}$ is a quasi-polynomial of k (i.e. with periodic coefficients).

Comparison of two polynomials

- **The coefficients of Ehrhart polynomial are not well understood. On the other hand, there are fast algorithms to compute them for a given generic polytope P.**

- **The meaning of coefficients of tube formula is well understood. But there is no easy algorithm to obtain them for a given generic compact set M.**

- **How about counting the lattice points in M and in $M + rB$ for various r and fit a polynomial?**

Introduction to hyperplane arrangement with integral coefficient

- **Consider an arrangement $\mathcal{A}$ of hyperplanes (i.e. a collection of hyperplanes) defined by linear forms with integral coefficients.**

$$\mathcal{A} = \{H_1, \ldots, H_n\},\ H_j : c_{1j}x_1 + \cdots + c_{mj}x_m = 0,$$
$$c_{ij} \in \mathbb{Z}, \qquad (m : \text{dimension},\ \text{central case})$$

- **Finite field method**: consider $\mathcal{A}$ in $\mathbb{F}_q^m$, where q is a large prime. Write $\mathcal{A}_q$.
  - Complement of $\mathcal{A}_q$: $M(\mathcal{A}_q) = \mathbb{F}_q^m \setminus \cup_i H_i$
  - For sufficiently large q, the characteristic polynomial[a] $\chi(\mathcal{A}, t)$ of $\mathcal{A}$ coincides with the cardinality of $M(\mathcal{A}_q)$.

$$\chi(\mathcal{A}, q) = |M(\mathcal{A}_q)|$$

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[a]In today's talk I am not defining the characteristic polynomial.

- **Question**: the characteristic polynomial $\chi(A, t)$ can be evaluated at a non-prime q. We can also define arrangement of "hyperplanes" in $\mathbb{Z}_q^m$, $\mathbb{Z}_q = \mathbb{Z}/(q\mathbb{Z})$, by

$$H_{j,q} : c_{1j}x_1 + \cdots + c_{mj}x_m \equiv 0 \pmod{q}$$

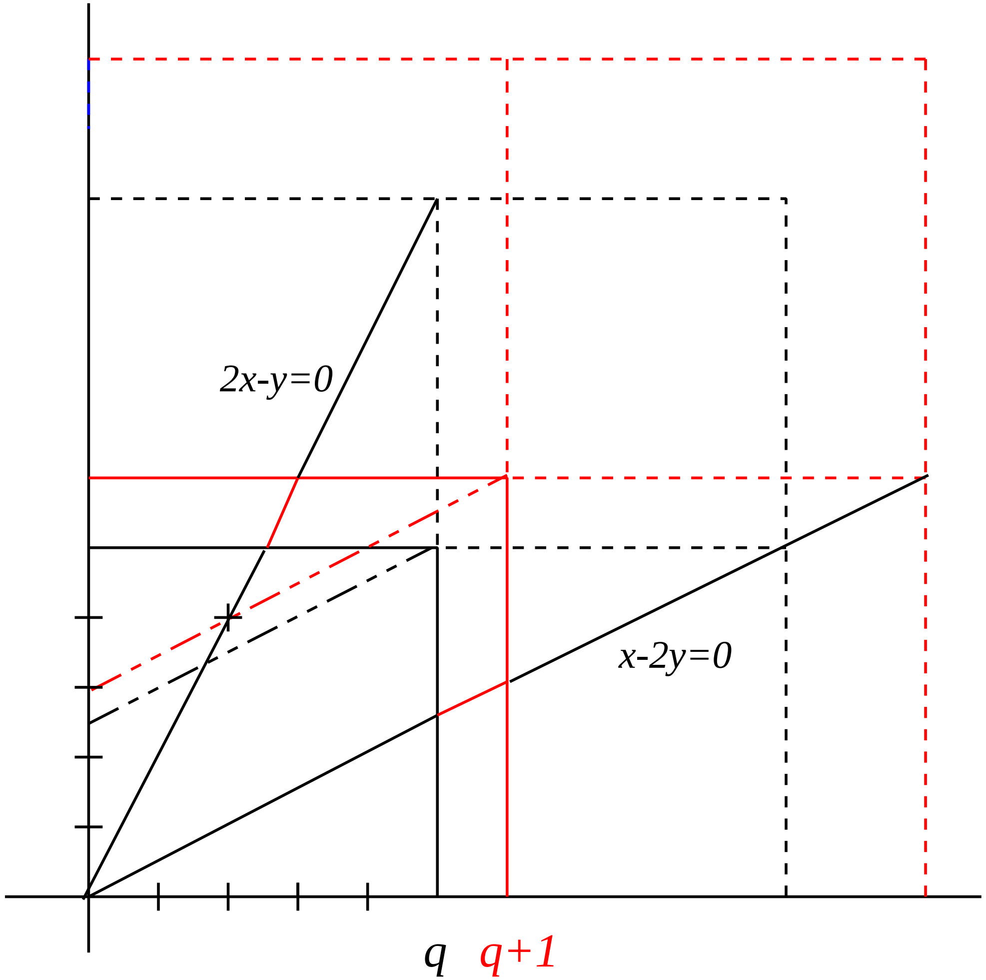
and count the number of points in the complement of $\mathcal{A}_q$. Are they the same?

$\Rightarrow$ generally NO!

- However $|M(\mathcal{A}_q)|$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$.

$\Rightarrow$ "characteristic quasi-polynomial"

- **Intuitively, the hyperplanes have more chances to meet at integer points, if q has many divisors.**



Results on characteristic quasi-polynomials

<u>References:</u>

- **Periodicity of hyperplane arrangements with integral coefficients modulo positive integers.** *Journal of Algebraic Combinatorics*, **27, 317–330 H.Kamiya, A.Takemura and H.Terao. 2008.**

- **Periodicity of non-central integral arrangements modulo positive integers. H.Kamiya, A.Takemura and H.Terao.** `arXiv:0803.2755v1`**, 2008. To appear in** *Annals of Combinatorics.*

- **Coefficient matrix $C = (c_{ij}) : m \times n$. Each column determines a hyperplane.**

- **Let $J \subseteq \{1, \ldots, n\}$ be a subset of hyperplanes and let C_J denote the submatrix of C consisting of columns $j \in J$.**

- **Let $e(J)$ denote the largest elementary divisor of C_J.**

- **Let**

$$\rho_0 = \mathrm{lcm}\{e(J) \mid J \subseteq \{1, \ldots, n\}, J \neq \emptyset\}.$$

Theorem 1 The function $|M(\mathcal{A}_q)|$ is a monic quasi-polynomial in $q \in \mathbb{Z}_{>0}$ of degree m with a period ρ_0. Furthermore the coefficients of the quasi-polynomial depend only on $\gcd\{\rho_0, q\}$.

An example

- Let

$$C = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

- Corresponding hyperplanes in $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ is $\mathcal{A} = \{H_1, H_2, H_3\}$:

$$H_1 : x - y = 0, \quad H_2 : x + y = 0, \quad H_3 : -2x + y = 0.$$

- $\rho_0 = 6.$

- **Characteristic quasi-polynomial:**

$$|M(\mathcal{A}_q)| = \begin{cases} q^2 - 3q + 2 & \text{when } \gcd\{6, q\} = 1, \\ q^2 - 3q + 3 & \text{when } \gcd\{6, q\} = 2, \\ q^2 - 3q + 4 & \text{when } \gcd\{6, q\} = 3, \\ q^2 - 3q + 5 & \text{when } \gcd\{6, q\} = 6. \end{cases}$$

- **Relation to the characteristic polynomial (already stated by Athanasiadis).**

  Theorem 2 Let ρ be a period of the quasi-polynomial $|M(\mathcal{A}_q)|$ and q be a positive integer relatively prime to ρ. Then $|M(\mathcal{A}_q)| = \chi(\mathcal{A}, q)$.

- **This theorem shows that we can apply the "finite field method" with a composite q relatively prime to ρ for obtaining the characteristic polynomial of $\mathcal{A}$.**

Periodicity of intersection posets

- **The intersection posets of $\mathcal{A}_q$ are also periodic.**
- **Periodicity of $|M(\mathcal{A}_q)|$ and that of the intersection poset are not equivalent.**
- **Our example:**
  $H_1 : x - y = 0, H_2 : x + y = 0, H_3 : -2x + y = 0.$

- “Hyperplanes” (all modulo q)

$$H_{1,q} = \{(0,0),(1,1),\ldots,(q-1,q-1)\}$$
$$H_{2,q} = \{(0,0),(1,q-1),\ldots,(q-1,1)\}$$
$$H_{3,q} = \{(0,0),(1,2),(2,4),\ldots,(q-1,q-2)\}$$

- **Intersections for $q \geq 4$,**

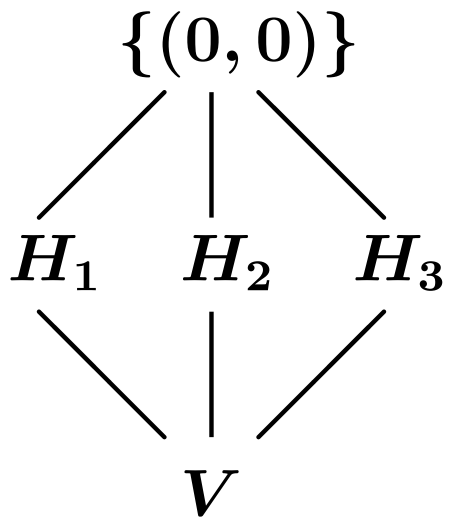
$$H_{\{1,2\},q} = \begin{cases} \{(0,0)\}, & q : \text{odd}, \\ \{(0,0), (\frac{q}{2}, \frac{q}{2})\}, & q : \text{even}, \end{cases}$$

$$H_{\{2,3\},q} = \begin{cases} \{(0,0)\}, & 3 \not| \; q, \\ \{(0,0), (\frac{q}{3}, \frac{2q}{3}), (\frac{2q}{3}, \frac{q}{3})\}, & 3 \mid q, \end{cases}$$

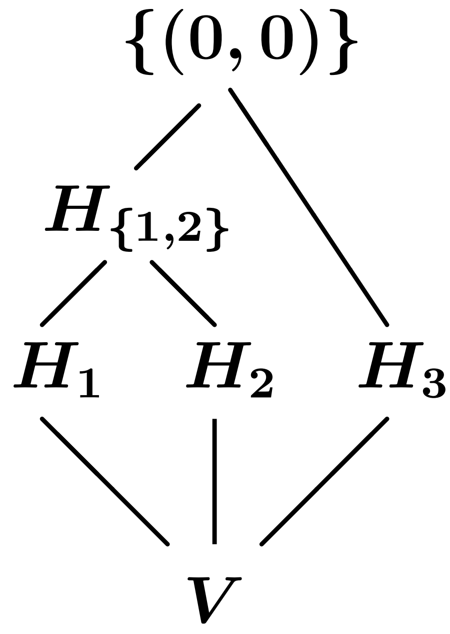
$$H_{\{1,3\},q} = H_{\{1,2,3\},q} = \{(0,0)\}$$

(all modulo q)

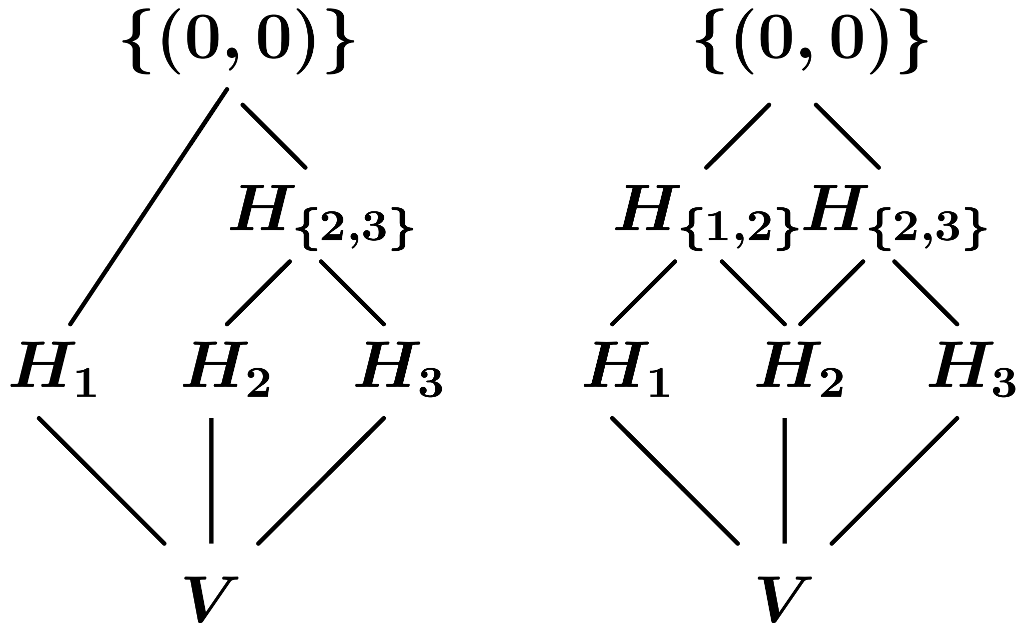
- **Hasse diagrams of intersection lattices for $q \geq 4$**



$\gcd\{6, q\} = 1$



$\gcd\{6, q\} = 2$



$\gcd\{6, q\} = 3$ $\qquad$ $\gcd\{6, q\} = 6$

Theorem 3 **The intersection lattices $L(\mathcal{A}_q)$ are periodic for all sufficiently large q with a period ρ_0.**

- **NOTE: $|M(\mathcal{A}_q)|$ is periodic for all $q > 0$. On the other hand $L(\mathcal{A}_q)$ are periodic from some q on.**

Summary

- **I have stated a personal motivation for studying Ehrhart polynomials in view of my research on tube formulas.**

- **I have stated that the number of lattice points in the complement of hyperplane arrangement modulo q possesses a particular periodicity property.**

- **The characteristic polynomials for hyperplane arrangements are actually "valuations" in the sense of Klain and Rota's book. Valuations may be useful for studying various related fields from a unified viewpoint.**