

# Comparison of proof techniques in game-theoretic probability and measure-theoretic probability

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## Outline: A.Takemura

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0. Background and our contributions
1. Setup of various games and notions
2. Examples of the first part of Borel-Cantelli
3. Kolmogorov's 0-1 law
4. Martingale convergence theorem for non-negative martingales

Purpose of the talk: show that game-theoretic proofs are much more intuitive!

## 0. Background and our contributions

- My own background:
  - U.Tokyo, undergrad, Master
  - Stanford Ph.D (1982, statistics)
  - U.Tokyo since 1984
  - Main field: classical multivariate statistics
- Our group on game-theoretic probability:  
Kei Takeuchi, Masayuki Kumon, me and I  
gently push some students.

- **The BOOK: Shafer and Vovk (2001).** *Probability and Finance: It's Only a Game!*
  - I knew Glenn at Stanford “Theory of evidence”
  - Takeuchi got interested in 2002
  - I got interested in 2003
- Takeuchi wrote a book in Japanese in 2004.
- Japanese translation of The BOOK in 2006.
- By now my group wrote 7 papers (5 published).

1. Simple strategy for strong law of large numbers (bounded case)
  2. Exposition of pricing formulas
  3. SLLN for unbounded variables
  4. Bayesian strategy in game-theoretic probability
  5. Consideration of contrarian strategies
  6. Application of Bayesian strategy to continuous-time game.
  7. Multistep Bayesian strategies
- So at least in Japan, there are some disciples.

# 1. Setup of various games and notions

Setup of various games:<sup>a</sup>

- Complete information game between two players
  - **Skeptic** (statistician, investor) bets on some outcome.
  - **Reality** (nature, market) decides the outcome.;
- **Skeptic** → **Reality** → **S** → **R** → .  
They play in turn.

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<sup>a</sup>Games with explicit prices of tickets, but for simplicity without Forecaster

- One round is (**S**keptic's turn, **R**eality's turn) in this order
- $n = 1, 2, \dots$  denote rounds.
- **S**keptic's initial capital:  $\mathcal{K}_0 = 1$
- At each round, **S**keptic first announces how much he bets  $M_n \in \mathbb{R}$ .  $M_n$  can be any real number and can be arbitrarily small. Negative  $M_n$  allowed (selling).

- He has to pay some predetermined price  $p_n$  per unit bet (ticket) at round  $n$ .
- After knowing  $M_n$ , **R**eality chooses the outcome  $x_n \in X \subset \mathbb{R}$ .
- We consider various move spaces  $X$  of **R**eality
- Payoff to **S**keptic:  $M_n x_n - M_n p_n = M_n(x_n - p_n)$
- **S**keptic's capital changes as

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(x_n - p_n).$$

**In summary:**

$$\mathcal{K}_0 = 1$$

**FOR**  $n = 1, 2, \dots$

**Skeptic** announces  $M_n \in \mathbb{R}$ .

**Reality** announces  $x_n \in X$ .

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n).$$

**END FOR**

## Fair coin game

- $X = \{-1, 1\}$  and  $p_n \equiv 0$   
or equivalently
- $X = \{0, 1\}$  and  $p_n \equiv 1/2$ .

We take the second parameterization

- **R**eality can choose the sign of  $x_n - 1/2 = \pm 1/2$  as the opposite of the sign of  $M_n$ . Therefore **R**eality can always decrease **S**keptic's capital.

- **S**keptic can bet

$$M_1 = \frac{1}{2}, \quad M_2 = \frac{1}{4}, \quad M_3 = \frac{1}{8}, \dots$$

and avoid bankruptcy.

- No-win situation for Skeptic?
- But then **R**eality is “forced” to observe SLLN!

**Theorem**      There exists a Skeptic's strategy  $\mathcal{P}$ . (he can announce this strategy even before the start of the game.) If Skeptic uses  $\mathcal{P}$ , then he is never bankrupt and furthermore whenever Reality violates

$$\lim_{n \rightarrow \infty} \frac{1}{n} (x_1 + \cdots + x_n) = \frac{1}{2},$$

then

$$\lim_{n \rightarrow \infty} \mathcal{K}_n = \infty.$$

Returning to general setup:

- **Path:**  $\xi = x_1x_2 \dots$  is an infinite sequence of **R**eality's moves
- **Sample space:**  $\Xi = \{\xi\} = X^\infty$ , the set of all paths
- **Event:**  $E \subset \Xi$
- **Partial path:**  $\xi^n = x_1x_2 \dots x_n$
- **Finite event:**  $E \subset X^n$

In game-theoretic probability we do not introduce a  $\sigma$ -field on  $\Xi$ . [However do events  $E \subset \Xi$  have to be approximated by finite events?]

- Skeptic's strategy  $\mathcal{P}$ :

$$\mathcal{P} : \xi^{n-1} = x_1 x_2 \dots x_{n-1} \mapsto M_n$$

- Capital process for  $\mathcal{P}$ :

$$\mathcal{K}_n^{\mathcal{P}}(\xi) = \mathcal{K}_n^{\mathcal{P}}(\xi^n) = \mathcal{K}_0 + \sum_{i=1}^n M_i(\xi^{i-1})(x_i - p_i)$$

- **Collateral duty:**  $\mathcal{P}$  satisfies the collateral duty for Skeptic with the initial capital  $\mathcal{K}_0 = \delta > 0$  if

$$\mathcal{K}_n^{\mathcal{P}}(\xi) \geq 0, \quad \forall \xi, \forall n.$$

- **Weak forcing of an event**  $\mathcal{P}$  weakly forces an event  $E \subset \Xi$  if  $\mathcal{P}$  satisfies the collateral duty with some  $\delta > 0$  and

$$\limsup_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty, \quad \forall \xi \notin E.$$

- **Forcing of an event**  $\mathcal{P}$  forces an event  $E \subset \Xi$  if “ $\limsup_n$ ” is replaced by “ $\lim_n$ ” above.
- **“Skeptic can (weakly) force  $E$ ”**: if **S**keptic can construct a strategy  $\mathcal{P}$  as above.  
We also say “ $E$  happens almost surely”.

- **Upper probability  $\bar{P}(E)$  of an event  $E$ :**

Let  $I_E$  denote a ticket which pays 1 dollar if  $E$  occurs. The upper probability  $\bar{P}(E)$  of  $E$  is the price of the ticket  $I_E$ .

- **Definition**

$$\bar{P}(E) = \inf\{\mathcal{K}_0^{\mathcal{P}} \mid \exists \mathcal{P} \text{ s.t. } \mathcal{K}_n^{\mathcal{P}}(\xi) \geq I_E(\xi), \forall \xi \in \Xi\},$$

- **If we start with the initial capital  $\delta > \bar{P}(E)$ , then we can superreplicate the ticket  $I_E$ . So that the value of the ticket  $I_E$  is at most  $\delta$ .**

## 2. Examples of the first part of Borel-Cantelli

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Example 1  $X = \{0, 1\}$  (coin-tossing) and  
 $\sum_{n=1}^{\infty} p_n < \infty$  :

$$\mathcal{K}_0 = 1$$

**FOR**  $n = 1, 2, \dots$

**Skeptic** announces  $M_n \in \mathbb{R}$ .

**Reality** announces  $x_n \in \{0, 1\}$ .

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n).$$

**END FOR**

- Let  $E$  be the event that  $x_n = 1$  for only finite  $n$ .
- Skeptic can force  $E$ .

*Proof.* Let  $C = \sum_n p_n < \infty$ . Starting with the initial capital  $\delta = 1$ , consider the strategy  $M_n \equiv 1/C$ . The capital process is

$$\begin{aligned} \mathcal{K}_n^{\mathcal{P}}(\xi) &= 1 + \frac{1}{C} \sum_{i=1}^n (x_i - p_i) = 1 - \frac{1}{C} \sum_{i=1}^n p_i + \frac{1}{C} \sum_{i=1}^n x_i \\ &\geq \frac{1}{C} \sum_{i=1}^n x_i. \end{aligned}$$

If  $x_n = 1$  for infinitely many  $x_n$ , then  $\lim_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$ .  $\square$

## Example 2

- $X = [0, \infty)$  and the price  $p_n = \nu$  is a constant:

$$\mathcal{K}_0 = 1$$

**FOR**  $n = 1, 2, \dots$

**Skeptic** announces  $M_n \in \mathbb{R}$ .

**Reality** announces  $x_n \geq 0$ .

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - \nu).$$

**END FOR**

- Let  $E_n$  be the event  $x_n \geq n^{1+\epsilon}$ ,  $\epsilon > 0$ .
- Skeptic can force the event

$$E = \{E_n \text{ only for finite } n\}$$

*Proof.* We combine Markov inequality with Borel-Cantelli argument. Let  $C = \sum_{n=1}^{\infty} 1/n^{1+\epsilon} < \infty$ . Consider the strategy  $M_n = 1/(C\nu n^{1+\epsilon})$ . Starting with

$\delta = 1$ , the capital process is

$$\begin{aligned}\mathcal{K}_n^{\mathcal{P}}(\xi) &= 1 + \sum_{i=1}^n \frac{1}{C\nu i^{1+\epsilon}} (x_i - \nu) \\ &= 1 - \sum_{i=1}^n \frac{1}{C i^{1+\epsilon}} + \frac{1}{C\nu} \sum_{i=1}^n \frac{x_i}{i^{1+\epsilon}} \\ &\geq \frac{1}{C\nu} \sum_{i=1}^n \frac{x_i}{i^{1+\epsilon}}.\end{aligned}$$

If  $x_n \geq n^{1+\epsilon}$  for infinitely many  $x_n$ , then  
 $\lim_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$ . □

## General game-theoretic statement:

Let  $p_n$  be the price for the event  $E_n$ . If  $\sum \bar{P}(E_n) < \infty$  then  $E_n$  happens only for finite  $n$  almost surely.

Suppose that for each event  $E_n$  there is a unit ticket  $I_{E_n}$  which pays you 1 dollar when  $E_n$  happens. Assume that the sum of the prices for all the tickets is finite  $\sum_n p_n < \infty$ . Then you can buy all the tickets with a finite amount of money. Now if  $E_n$  happens for infinitely many  $n$ , then you become infinitely rich!

**Such a simple argument!**

## Review of the measure-theoretic proof

- $\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$ . (O.K.)
- Since  $\{D_n = \bigcup_{m=n}^{\infty} E_m\}$  is a decreasing sequence of events

$$P(\limsup_n E_n) = \lim_n P(\bigcup_{m=n}^{\infty} E_m)$$

$\Leftarrow$  uses the continuity of probability measure  
(why do we need continuity?)

- $P(\bigcup_{m=n}^{\infty} E_m) \leq \sum_{m=n}^{\infty} P(E_m) \rightarrow 0 \quad (n \rightarrow \infty)$  (O.K.)

## Kolmogorov's 0-1 law

- $E \subset \Xi$  is a tail event if

$$x_1 \dots x_N x_{N+1} \dots \in E \Leftrightarrow \forall N \ * \dots \ * \ x_N x_{N+1} \dots \in E.$$

- Suppose that  $\bar{P}(E) < 1$ . Actually we have to define  $\bar{P}(E)$  carefully because  $E$  is a subset of the set of infinite sequences  $X^\infty$ .
- Define  $\bar{P}(E) < 1$  as follows. There exist  $\delta < 1$  and a strategy  $\mathcal{P}$  satisfying the collateral duty with initial  $\delta$  such that

$$\liminf_n \mathcal{K}_n^{\mathcal{P}}(\xi) \geq 1 \quad \forall \xi \in E.$$

- In words, if  $E$  happens then starting with  $\delta < 1$  you can wait and there is a time point  $n$  such that  $\mathcal{K}_n^{\mathcal{P}}(\xi) \geq 1 - \epsilon$ , where  $\epsilon$  is arbitrary small.
- Multiplying everything by  $1/\delta$ ,  $\bar{P}(E) < 1$  means the following: There exists  $\epsilon > 0$  such that starting with  $\delta = 1$ , there will be a time point where your capital is at least  $1 + \epsilon$ .

Now we have the following game-theoretic 0-1 law.

Let  $E$  be a tail event. If  $\bar{P}(E) < 1$  then  $\bar{P}(E) = 0$ .

*Proof.* (In words). Suppose that  $E$  happens. You start with the initial capital of  $\delta = 1$ . Wait until your capital becomes  $1 + \epsilon$ . Then save  $\epsilon$ . Now start all over again. Because  $E$  is a tail event, your situation is “the same” as the beginning of the game. Therefore there will be a time point that you get another  $\epsilon$ . This repeats infinite number of times and your capital becomes infinite. □

( $\bar{P}(E) = 0$  if you can get arbitrarily rich when  $E$  happens.)

This material on Kolmogorov's 0-1 law has just been written up in the following manuscript:

**“The game-theoretic martingales behind the zero-one laws”** by Akimichi Takemura, Vladimir Vovk, and Glenn Shafer.  
(<http://arXiv.org/abs/0803.3679>)

**Measure-theoretic statement:** Suppose that  $X_1, X_2, \dots$  are independent random variables. If  $E$  is a tail event, then  $P(E) = 0$  or  $1$ .

*Proof.* • Approximate  $E$  by  $E_n \in \sigma(X_1, \dots, X_n)$ .

- Because  $E$  is a tail event,  $E$  is independent of  $E_n$  and

$$P(E \cap E_n) = P(E) \times P(E_n)$$

- Taking the limit we have  $P(E) = P(E)^2$ . Then  $P(E) = 0$  or  $1$ .

□

**Unfortunately this proof is so artificial.**

# Martingale convergence theorem for non-negative martingales

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- If there exists a Skeptic's strategy  $\mathcal{P}$  satisfying the collateral duty with initial  $\delta$ , then its capital process  $\mathcal{K}_n^{\mathcal{P}}$  is called a game-theoretic non-negative martingale.

Then we have the following statement (Lemma 4.5 of Shafer and Vovk (2005)).

A non-negative martingale  $\mathcal{K}_n^{\mathcal{P}}$  converges to a non-negative finite value almost surely.

From Williams book (“Probability with Martingales”):

### 11.1. The picture that says it all

The top part of Figure 11.1 shows a sample path  $n \mapsto X_n(\omega)$  for a process  $X$  where  $X_n - X_{n-1}$  represents your winnings per unit stake on game  $n$ . The lower part of the picture illustrates your total-winnings process  $Y := C \bullet X$  under the previsible strategy  $C$  described as follows:

Pick two numbers  $a$  and  $b$  with  $a < b$ .

REPEAT

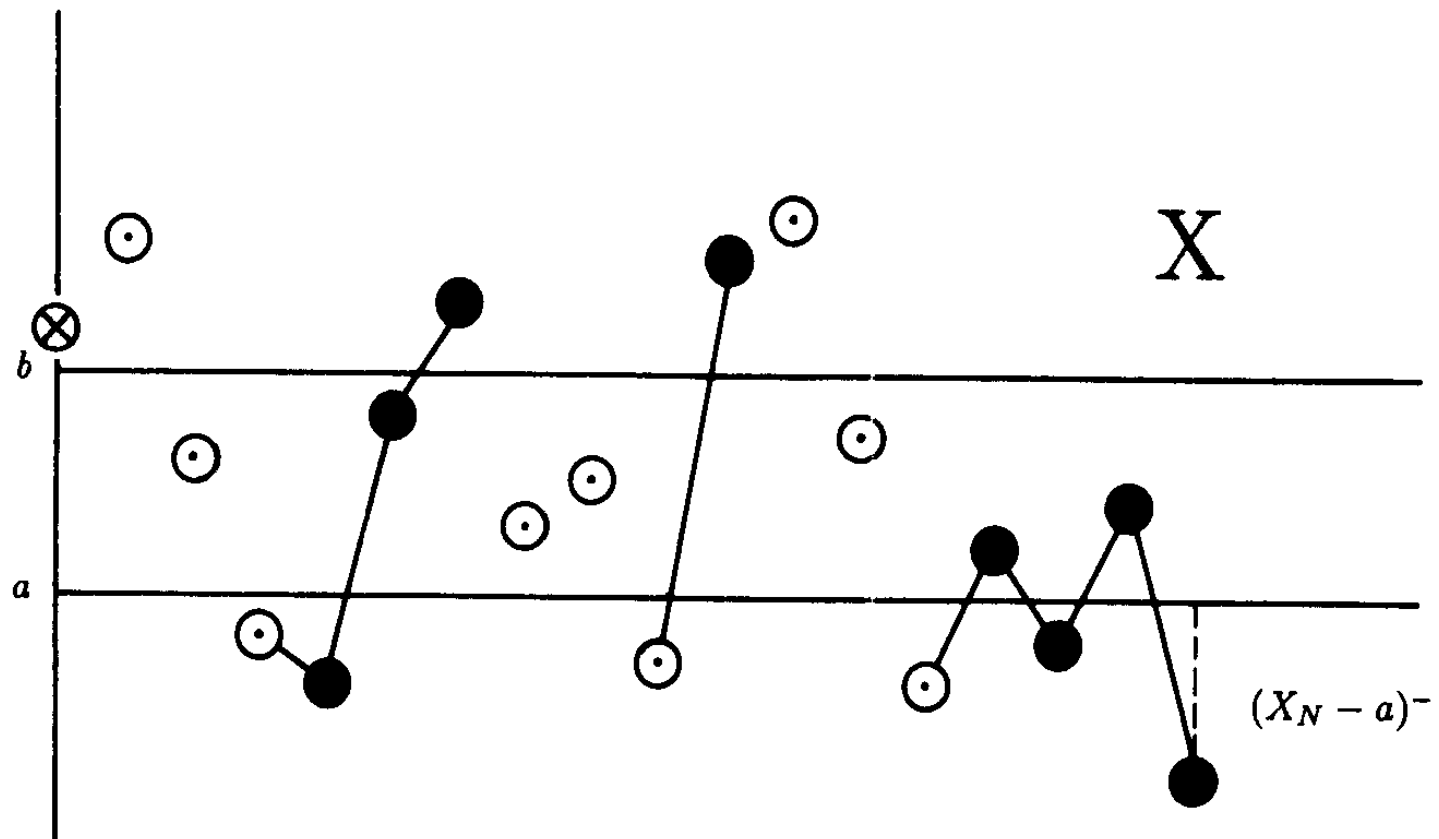
    Wait until  $X$  gets below  $a$

    Play unit stakes until  $X$  gets above  $b$  and stop playing

UNTIL FALSE (that is, forever!).

Black blobs signify where  $C = 1$ ; and open circles signify where  $C = 0$ . Recall that  $C$  is not defined at time 0.

Figure 11.1





*Proof.* • Let  $E$  denote the set of paths such that  $\mathcal{K}_n^{\mathcal{P}}$  converges to a finite value. We need to construct a strategy  $\mathcal{Q}$  such that  $\limsup_n \mathcal{K}_n^{\mathcal{Q}} = \infty$  for each  $\xi \notin E$ .

- Use  $\mathcal{P}$  itself with the initial capital of  $\delta = 1/2$ . If  $\limsup_n \mathcal{K}_n^{\mathcal{P}} = \infty$ , we do not need any other strategy.
- Divide the remaining initial capital  $1/2$  as

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

- Enumerate pairs of positive rational numbers  $(a_i, b_i)$ ,  $0 < a_i < b_i$ ,  $i = 1, 2, \dots$ . Note that the pairs of rational numbers are countable.

- Assign the initial capital  $1/2^{i+1}$  to the account and strategy  $\mathcal{Q}^{(i)}$  based on the Doob's upcrossing lemma for  $(a_i, b_i)$ .
- The strategy associated  $\mathcal{Q}^{(i)}$  is watching the capital process  $\mathcal{K}_n^{\mathcal{P}}$ . If  $\mathcal{K}_n^{\mathcal{P}}$  comes below  $a_i$ , then  $\mathcal{Q}^{(i)}$  tells us to start betting as  $\mathcal{P}$  until  $\mathcal{K}_n^{\mathcal{P}}$  exceeds  $b_i$ .
- Form the convex combination

$$\mathcal{Q} = \frac{1}{2}\mathcal{P} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \mathcal{Q}^{(i)}$$

- Then for  $\xi \notin E$ ,  $\limsup_n \mathcal{K}_n^{\mathcal{Q}}(\xi) = \infty$ .

□

Now we look at measure-theoretic proof of the same statement:

“A non-negative martingale converges to a non-negative finite value almost surely”.

I take a look at Chapter 11 of Williams (1991).

*Proof.* • Let  $X$  be a martingale. Let  $U_N[a, b]$  be the number of upcrossings of  $[a, b]$  by time  $N$ . Then

$$(b - a)E(U_N[a, b]) \leq E[(X_N - a)^-], \quad (1)$$

where  $x^- = \max(-x, 0)$ . Proof of this is very instructive but very counterintuitive and hard to explain to students.

- Let  $X$  be a non-negative martingale, then

$$P(U_\infty[a, b] = \infty) = 0. \quad (2)$$

This follows easily from (1)

- Use (2) for enumeration of pairs  $\{(a_i, b_i)\}$  of positive rational numbers. Countable sum of 0 probability is 0. This proves the theorem.



- **Again the game-theoretic proof is much more intuitive. You can explain it by words and pictures.**
- **On the other hand, for measure-theoretic proof, you definitely need monotone convergence theorem and other machinery of measure theory.**
- **You need to say “almost surely” so many times in measure-theoretic proof.**

## Summary and Discussion:

- I presented some examples, where game-theoretic arguments are much easier.
- Game-theoretic statement is “pathwise”
- What is the role of measure theory? Why do you need measurability? For some statements, we can use outer measure (like Borel-Cantelli).