

**Implications of contrarian and  
one-sided strategies for the  
fair-coin game**

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# Purpose

- **Give lower bounds of convergence**
- **for Strong Law of Large Numbers**
- **in coin-tossing games**
- **in game-theoretic probability**

# Reference

**Implications of contrarian and one-sided strategies for the fair-coin game.** *Stochastic Processes and their Applications*, doi:10.1016/j.spa.2007.11.007.  
Y.Horikoshi and A.Takemura.

# Game-theoretic probability

## Game-theoretic probability

**New framework for probability theory established in 2001 by Shafer and Vovk**

**“Probability and Finance – It’s Only a Game!”**

# Game-theoretic probability

## Game

We start with games instead of probability.

## Law of large numbers in game-theoretic probability

Game-theoretic statement: there exists a strategy forcing the law.

## Intuition

In the fair game, it is impossible to become infinitely rich.

# Coin-tossing game (0)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

⇐ Protocol

**Coin tosses are repeated indefinitely!**

# Coin-tossing game (1)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

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**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

Player

**Skeptic** predicts heads or tails  
trying to increase his  
capital to infinity

**Reality** decides heads and  
tails, trying to keep  
**Skeptic's** capital  
bounded.

## Coin-tossing game (2)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

Note:

**Reality** moves after Skeptic!

## Coin-tossing game (3)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

$M_n$  : **Skeptic's bet**

$x_n$  : **result of coin toss**

$M_n x_n$  : **return to Skeptic**

# Coin-tossing game (4)

$\mathcal{K}_0 := 1$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$

END FOR

Return (1)

**Skeptic's return** =  $M_n x_n$

**Skeptic predicts heads**

$\Rightarrow M_n > 0$

# Coin-tossing game (5)

$\mathcal{K}_0 := 1$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$

END FOR

Return (2)

**Reality** moves after **Skeptic!**

She can **ALWAYS** make

$$M_n x_n \leq 0.$$

# Coin-tossing game (6)

$\mathcal{K}_0 := 1$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$

END FOR

$\mathcal{K}_0$  : initial capital of **Skeptic**

$n$  : number of rounds

$\mathcal{K}_n$  : **Skeptic's** capital at the end of round  $n$

# Coin-tossing game (7)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

Collateral duty

**Skeptic** loses if he becomes bankrupt.

Therefore he has to choose

$$M_n \in [-\mathcal{K}_{n-1}, \mathcal{K}_{n-1}].$$

# Coin-tossing game (8)

$$\mathcal{K}_0 := 1$$

FOR  $n = 1, 2, \dots$

**Skeptic**

announces  $M_n \in \mathbb{R}$

**Reality**

announces  $x_n \in \{-1, 1\}$

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$$

END FOR

## Notations

$$s_0 := 0$$

$$s_n := x_1 + \dots + x_n$$

$$(n = 1, 2, \dots)$$

$$\bar{x}_0 := 0$$

$$\bar{x}_n := \frac{s_n}{n} \quad (n = 1, 2, \dots)$$

# Statement of game-theoretic SLLN (1)

## Strategy

Shafer and Vovk proved that there exist a **Skeptic's** strategy such that

whenever **Reality** does not observe SLLN  
(i.e.  $\bar{x}_n$  does not converge to 0),  
then **Skeptic's** capital grows to infinity.

# Statement of game-theoretic SLLN (2)

## Meaning of a strategy

**Reality** can take any of the paths

$$\{x_1 x_2 \cdots | x_i = 1 \text{ or } -1\} = \{-1, 1\}^\infty.$$

Whatever path **Reality** chooses

$$\text{either } \mathcal{K}_n \rightarrow \infty \text{ or } \bar{x}_n \rightarrow 0.$$

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**Reality** is faced with only two choices:

- Either she observes SLLN,
- Or she accepts **Skeptic** becoming infinitely rich.

**Reality** is forced the SLLN!

# Generality of game-theoretic probability

## Expectation, probability

Expectation and probability are naturally defined from the game.

## Various probability laws

SLLN for general case, Law of the Iterated Logarithm, Central Limit Theorem, Black-Scholes formula, can all be derived.

# Effectiveness of game-theoretic arguments

## Simplicity of assumptions

Same statements from fewer assumptions than the measure-theoretic formulation.

## Simplicity of proofs

With clever strategies, proofs are often much simpler than the measure-theoretic ones.

# Central Limit Theorem and Black-Scholes formula

- In the BOOK, CLT is proved by a betting strategy.

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- **Without using characteristic function**

# Central Limit Theorem and Black-Scholes formula

- In the BOOK, CLT is proved by a betting strategy.
- Without using characteristic function
- CLT and Black-Scholes are logically equivalent. (in game-theoretic probability and hence in measure-theoretic probability)

# Rate of convergence of SLLN

## SLLN

Forcing of  $|\bar{x}_n| = o(1)$

## Upper bound for convergence

Kumon and Takemura (2007) gave a strategy forcing  $|\bar{x}_n| = O(\sqrt{\log n/n})$

# Rate of convergence of SLLN

## SLLN

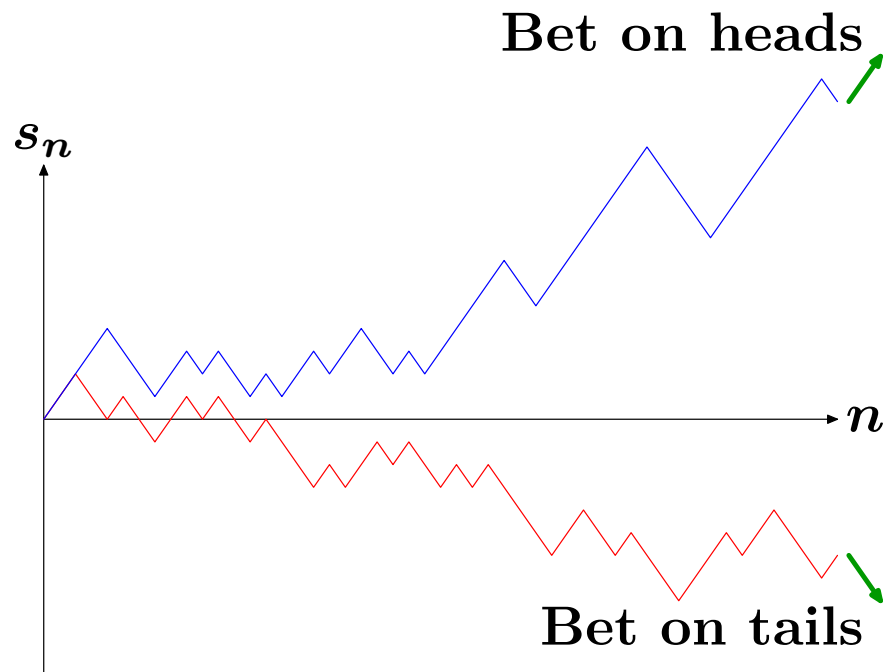
Forcing of  $|\bar{x}_n| = o(1)$

## Upper bound for convergence

Kumon and Takemura (2007) gave a strategy forcing  $|\bar{x}_n| = O(\sqrt{\log n/n})$

**We studied strategies for lower bounds.**

# Intuition behind momentum strategy (1)

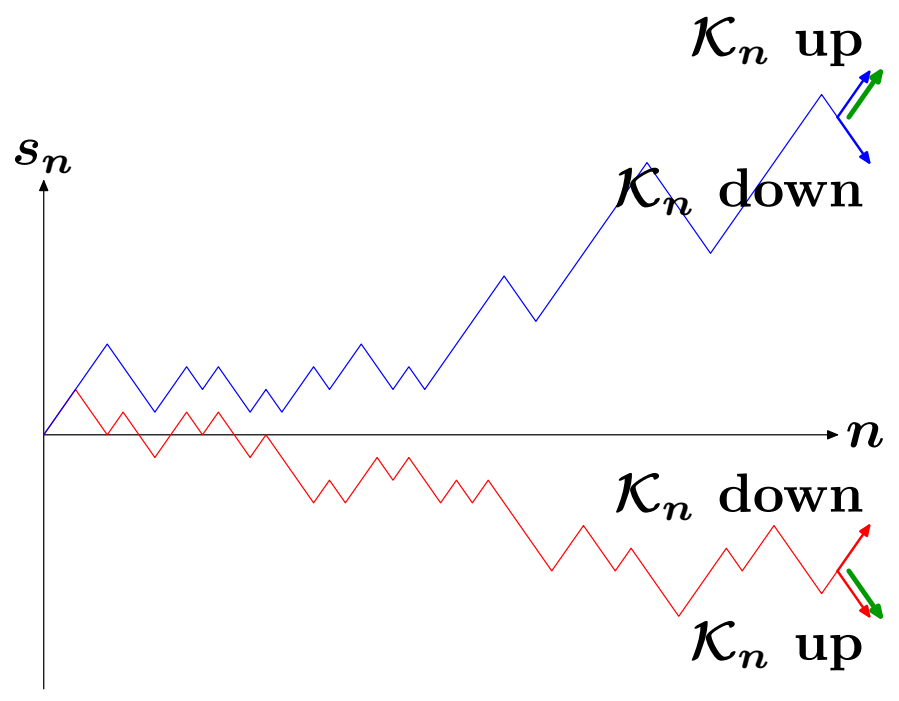


## momentum strategy

$s_{n-1} > 0 \Rightarrow$  **Bet on heads**

$s_{n-1} < 0 \Rightarrow$  **Bet on tails**

# Intuition behind momentum strategy (2)



## Intuition

$$|s_n| \nearrow \Rightarrow \mathcal{K}_n \nearrow$$

# Strategy for lower bound (1)

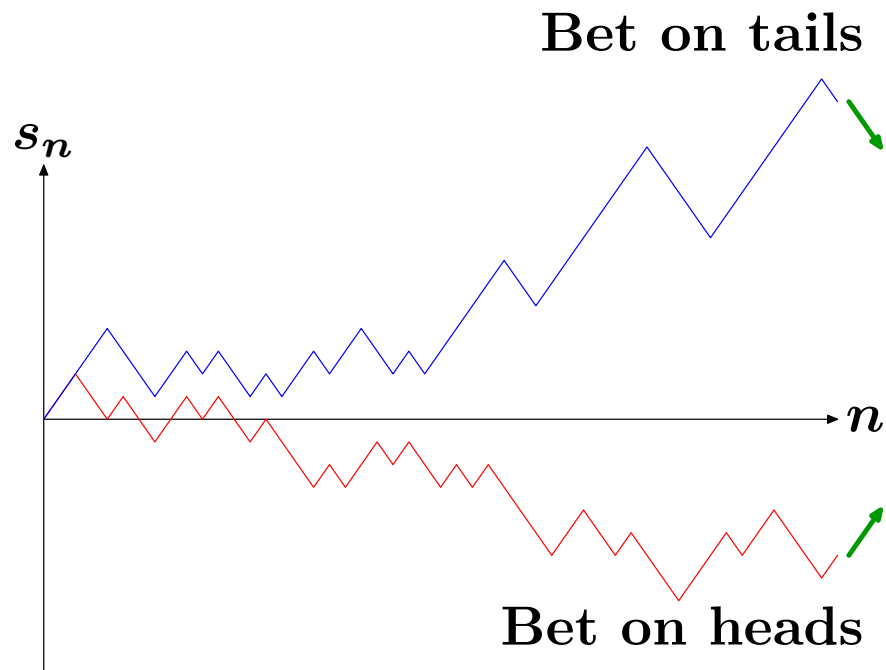
## Strategy $\mathcal{P}_c$

For  $0 < c \leq 1/2$

$$\mathcal{P}_c : M_n = -c\bar{x}_{n-1}\mathcal{K}_{n-1}$$

**This increases the capital when  $|s_n|$  is small.  
(contrarian strategy)**

# Intuition behind contrarian strategy (1)

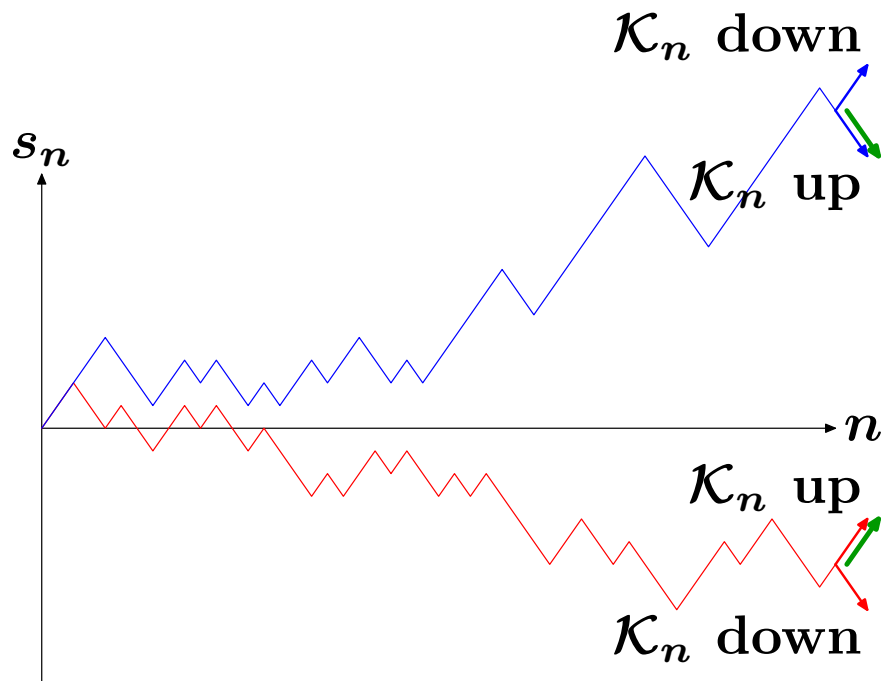


## contrarian strategy

$S_{n-1} > 0 \Rightarrow$  **Bet on tails**

$S_{n-1} < 0 \Rightarrow$  **Bet on heads**

# Intuition behind contrarian strategy (2)



contrarian strategy

$$|s_n| \searrow \Rightarrow \mathcal{K}_n \nearrow$$

# Theorem for lower bound (1)

## Theorem 1

With  $\mathcal{P}_c$

$$\limsup (1 + 2c) \sqrt{n} |\bar{x}_n| < 1$$

$$\Rightarrow \lim \mathcal{K}_n = \infty.$$

We can let  $c \downarrow 0$ . Then this theorem states that the rate of convergence is bounded from below by  $O(\sqrt{1/n})$ .

# Theorem for lower bound (2)

## Theorem 2

**Skeptic** can force that there are infinite number of  $n$ 's with

$$|s_n| \geq \sqrt{n} - 1$$

$|s_n| \geq \sqrt{n} - 1$  for only finite number of  $n$

$$\Rightarrow \mathcal{K}_n \rightarrow \infty$$

(stronger than Theorem 1.)

## Strategy for lower bound 2

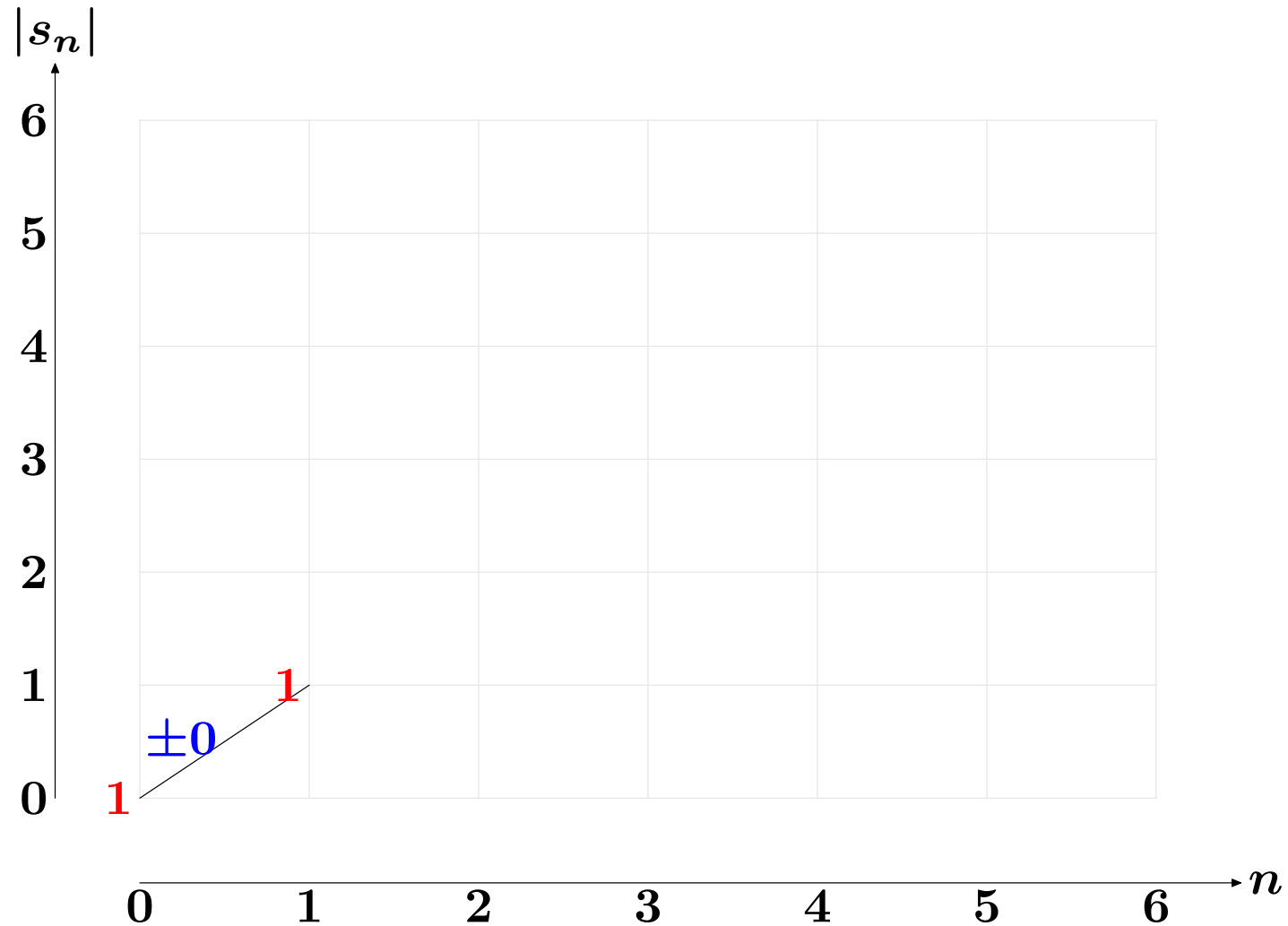
For Theorem 2 we use the strategy

$$\mathcal{P}^\epsilon : M_n = -\epsilon S_{n-1}$$

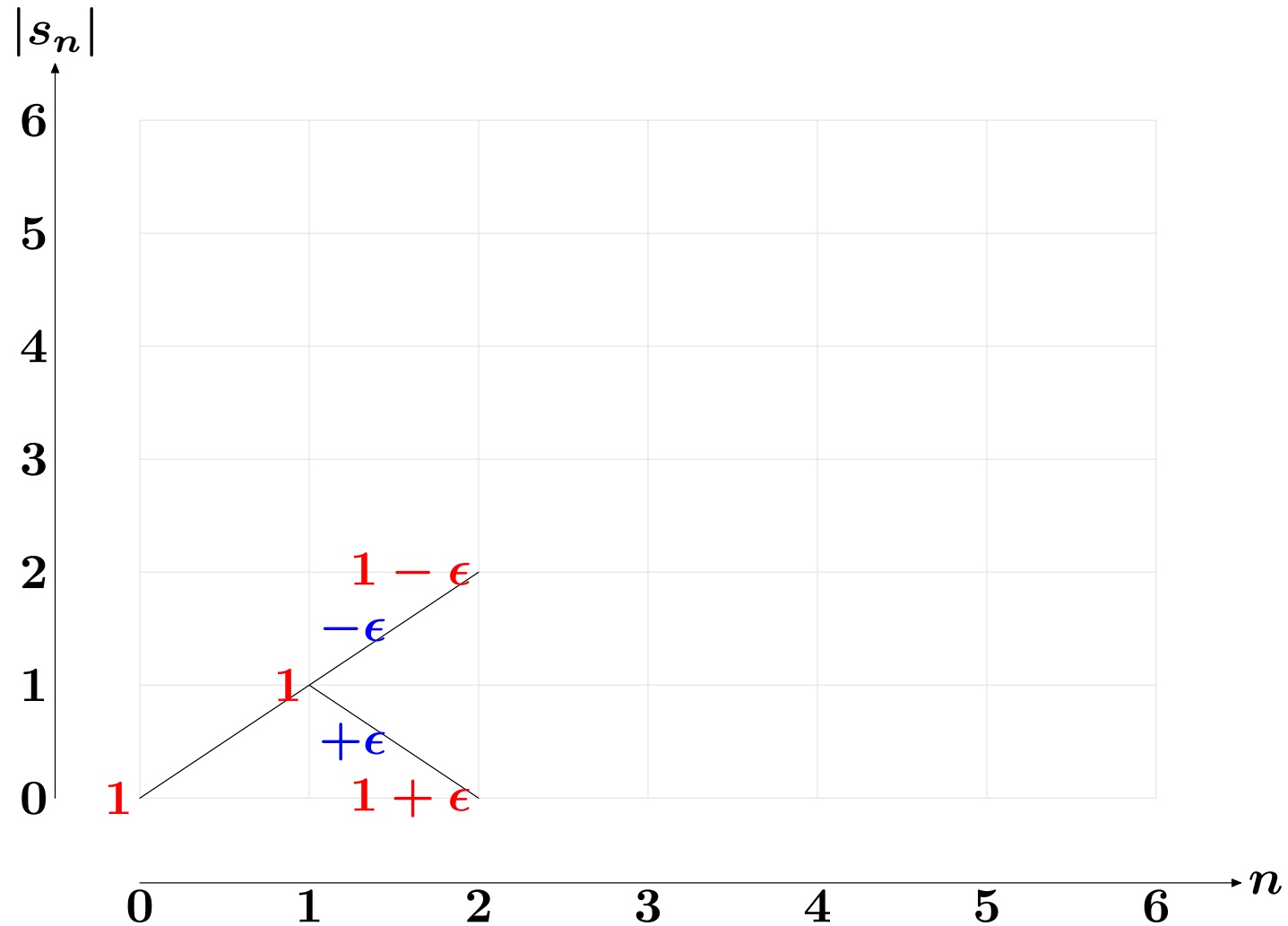
(again a contrarian strategy.)

However we need a condition for collateral duty.

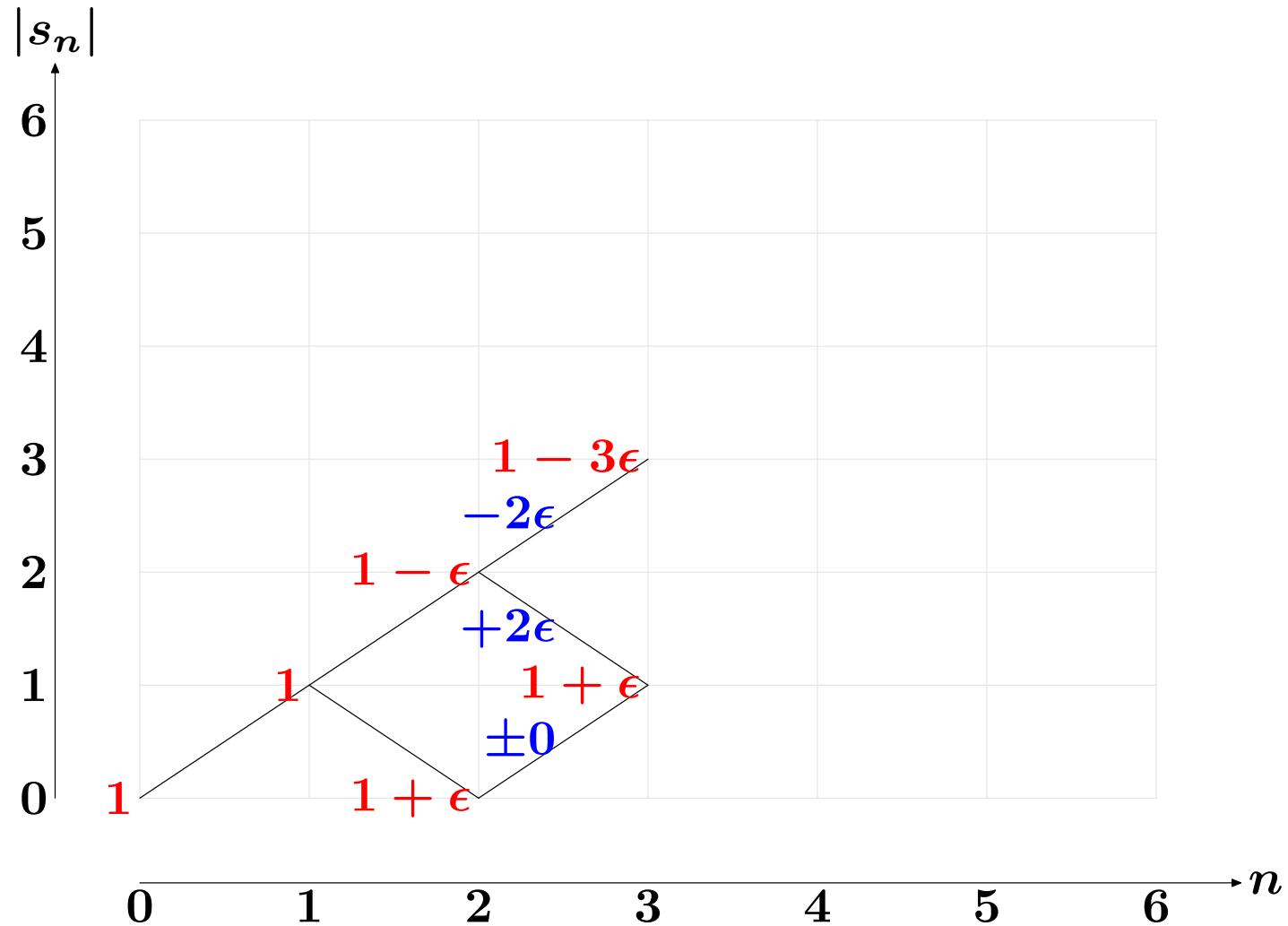
# Capital process of $\mathcal{P}^\epsilon : M_n = -\epsilon S_{n-1}$



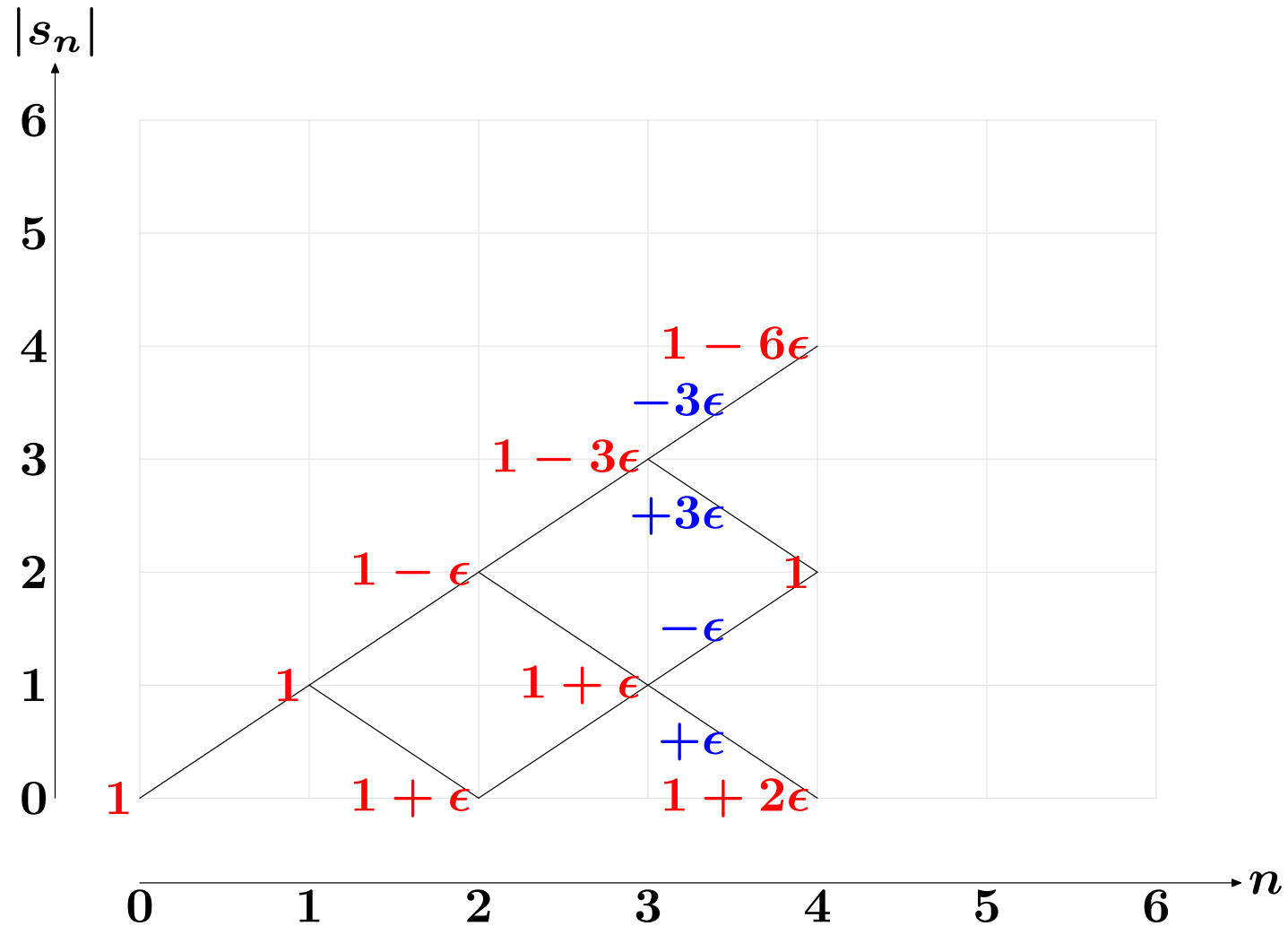
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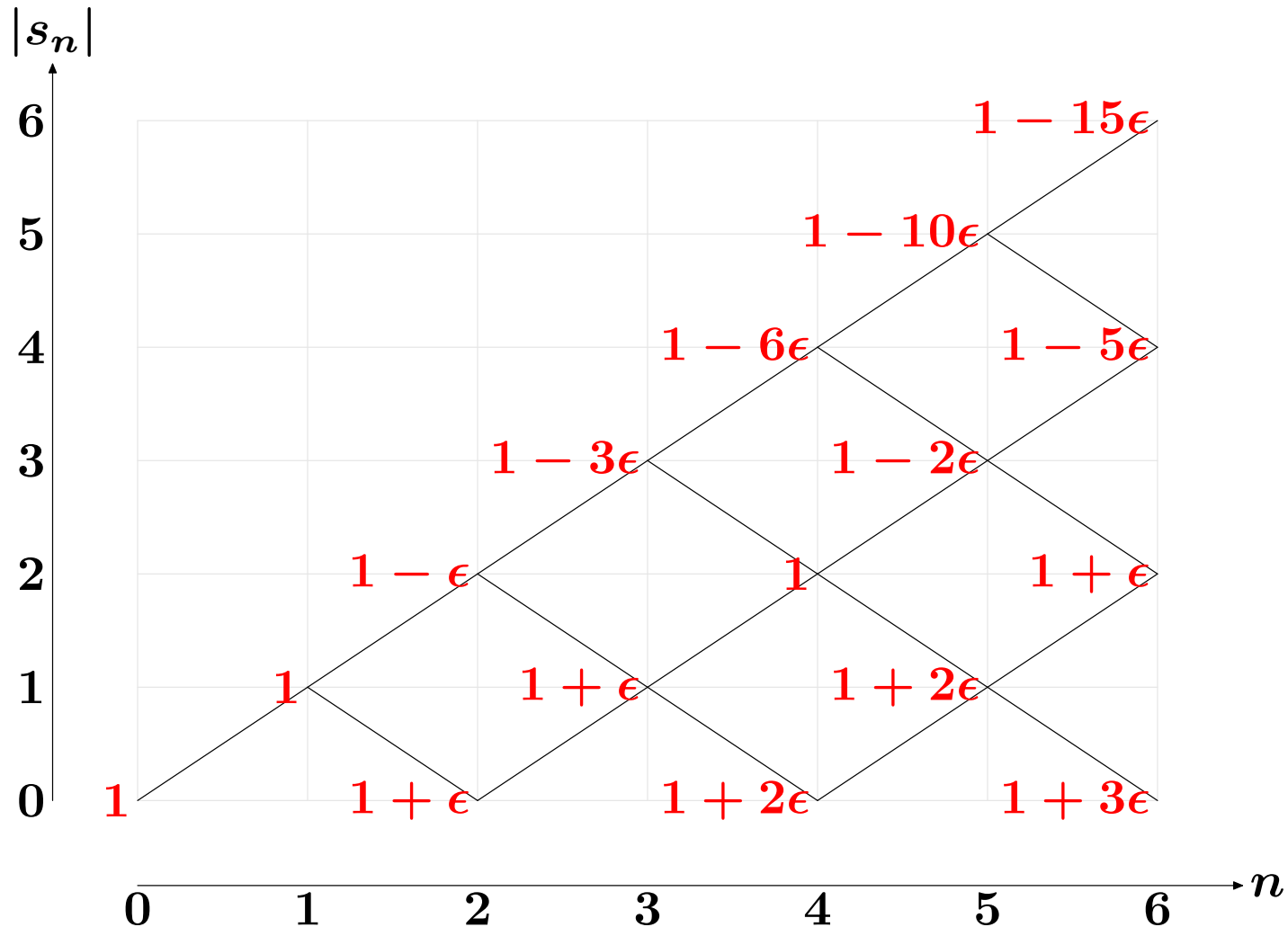
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Monotone increasing at the same level of  $|s_n|$ .

## Explicit form of $\mathcal{K}_n$

Generally,

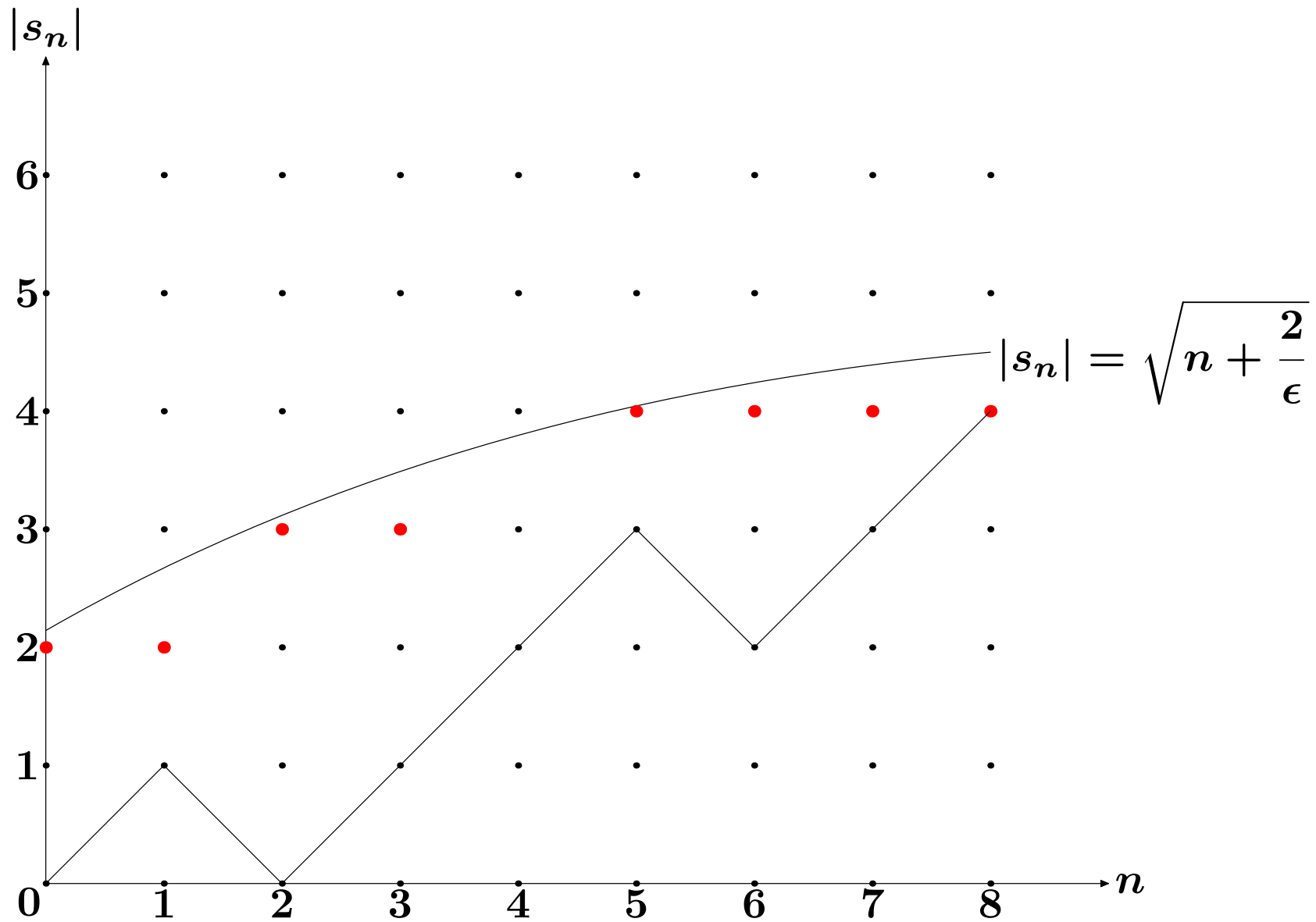
$$\mathcal{K}_n = 1 + \frac{\epsilon}{2}(n - s_n^2).$$

Therefore

$$\mathcal{K}_n \geq 0 \Leftrightarrow |s_n| \leq \sqrt{n + \frac{2}{\epsilon}}.$$

and for collateral duty, **Skeptic** has to stop betting at the first  $n$  such that

$$|s_n| > \sqrt{n + 1 + \frac{2}{\epsilon}} - 1.$$



**Stop when  $(n, s_n)$  comes to a red point.**

# $s_n$ can not be bounded

**The capital process of contrarian strategy**

$$\mathcal{K}_n = 1 + \frac{\epsilon}{2}(n - s_n^2).$$

**implies that Skeptic becomes infinitely rich if  $|s_n|$  stays bounded.**

# Lemma

## Lemma 1

**Skeptic** can force that there are infinitely many  $n$  with

$$s_n = 0.$$

**Sketch of proof:** Suppose that  $s_n$  stays non-negative. We know that  $s_n$  can not be bounded. But then a one-sided strategy makes Skeptic infinitely rich.

## Proof of Theorem 2

**Proof** (We show pictures later)

We divide the initial capital of 1 (dollar) into two accounts  $A_1, A_2$  with the initial capital of  $1/2$  each. We employ the strategy of Lemma 1 for  $A_1$  for forcing  $s_n = 0$  infinitely often.

For the account  $A_2$  we divide its initial capital of  $1/2$  into two further accounts  $A_2^1, A_2^2$  (initial capital of  $1/4$  each).

Let  $L$  be a large number and let  $n_0$  be the first time after  $L$  such that  $s_n = 0$ .

We keep the initial capital of  $A_2^2$  for later use.

In the account  $A_2^1$ , we consider  $n_0$  to be the initial round (i.e. reset the clock) and employ the strategy  $\mathcal{P}^{\epsilon_0}$ , where  $\epsilon_0 := \frac{\epsilon}{2}(n_0 - 1)$ .

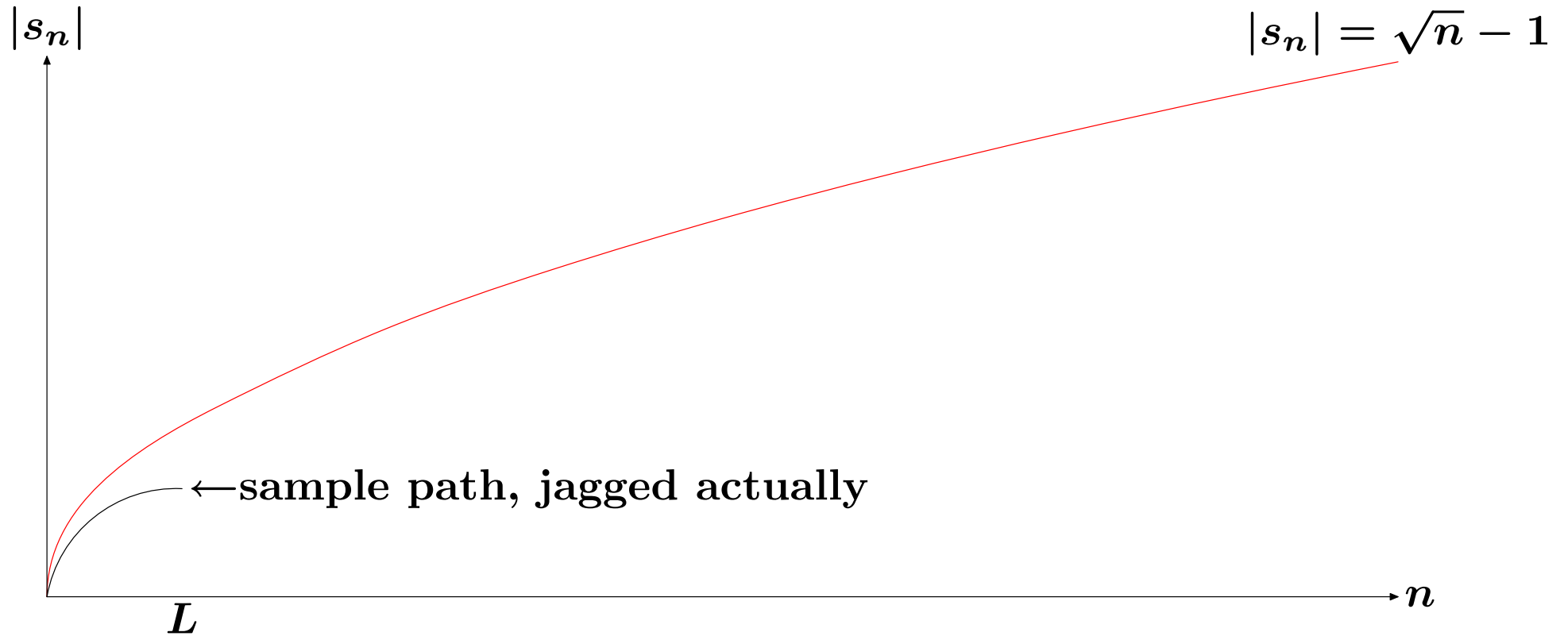
Then we can keep betting for the account  $A_2^1$  until

$$|s_n| > \sqrt{(n - n_0) + 1 + \frac{2}{\epsilon}} - 1 = \sqrt{n} - 1. \quad (1)$$

If (1) does not happen after  $n_0$ , then the amount of account  $A_2^1$  becomes arbitrarily large.

If (1) happens at some  $n > n_0$ , we stop betting and wait until  $n$  such that  $s_n = 0$ . Let this  $n$  be  $n_1$ . Recall that we have saved the capital of  $1/4$  in the account  $A_2^2$ . We move  $1/8$  out of  $1/4$  into the account  $A_2^1$ . Then we can repeat the procedure!

# Picture for explanation

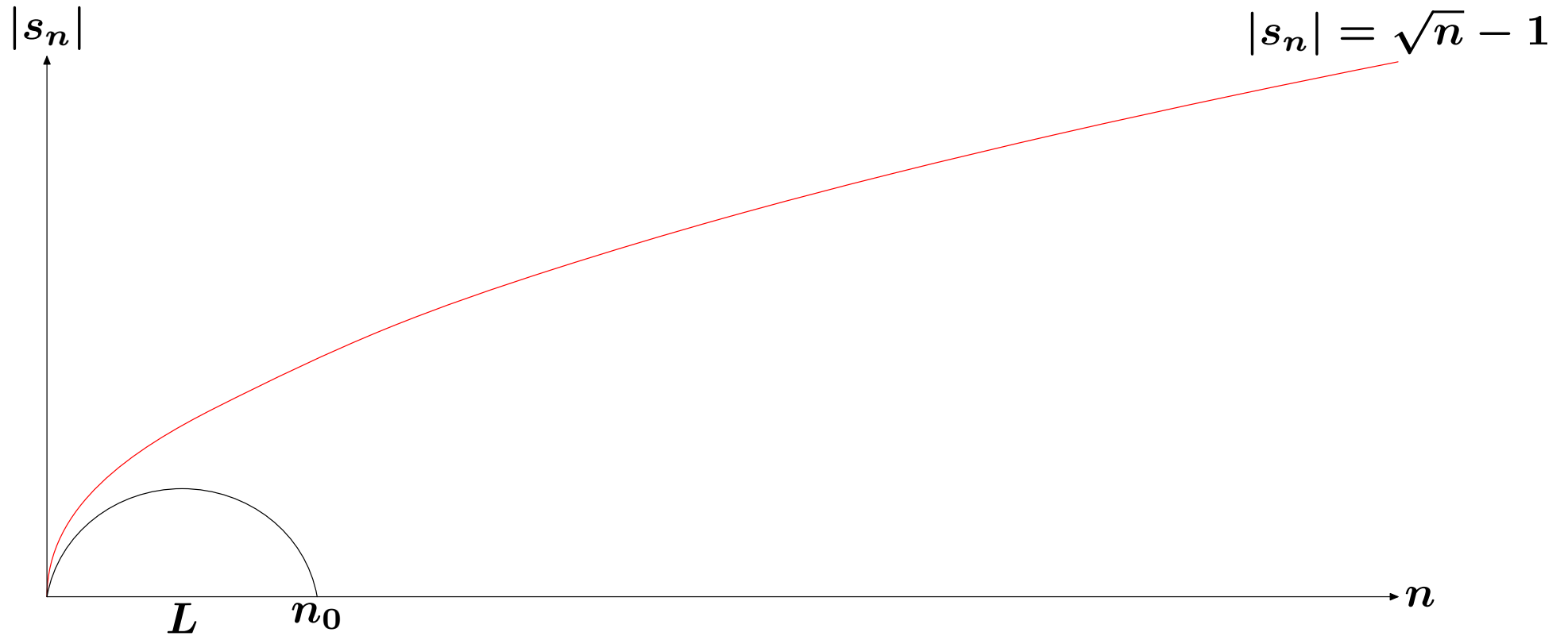


$A_2^1$

$A_2^2$

No bet at the beginning

# Picture for explanation



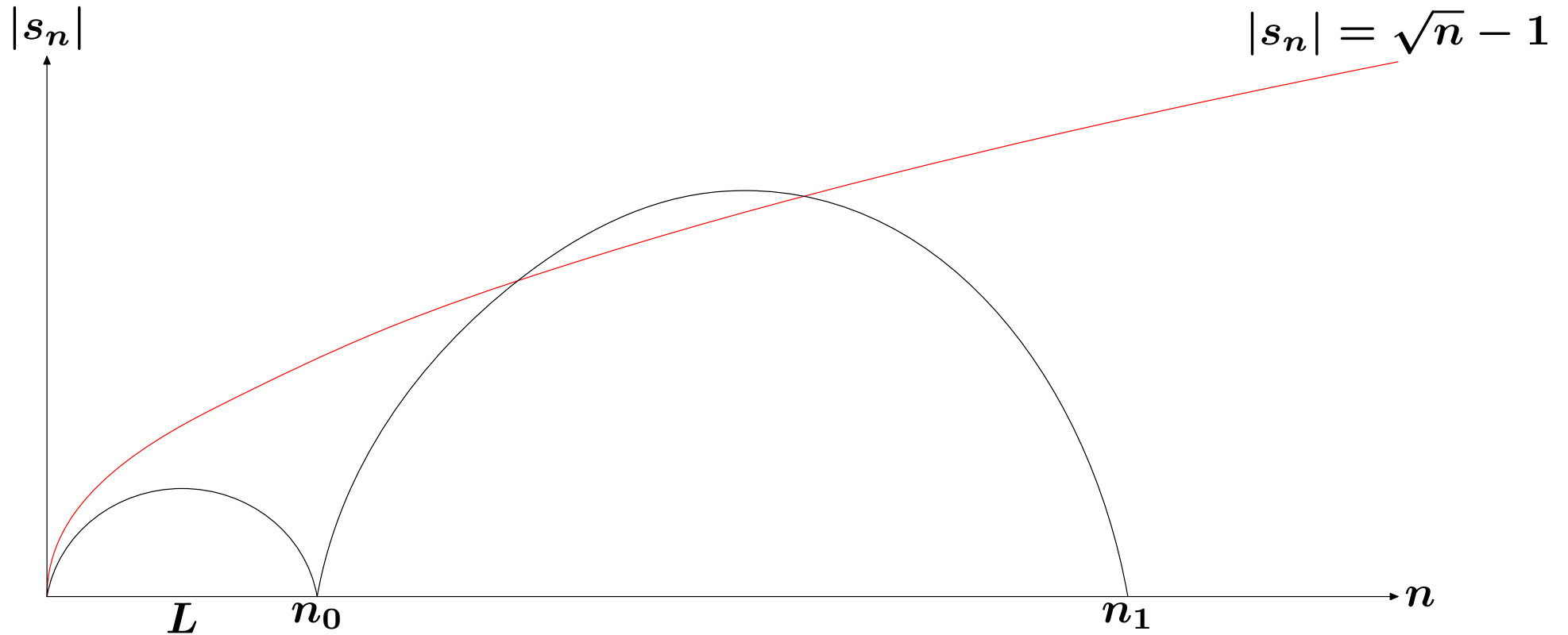
$A_2^1$

$A_2^2$

We start when  $s_n = 0$



# Picture for explanation

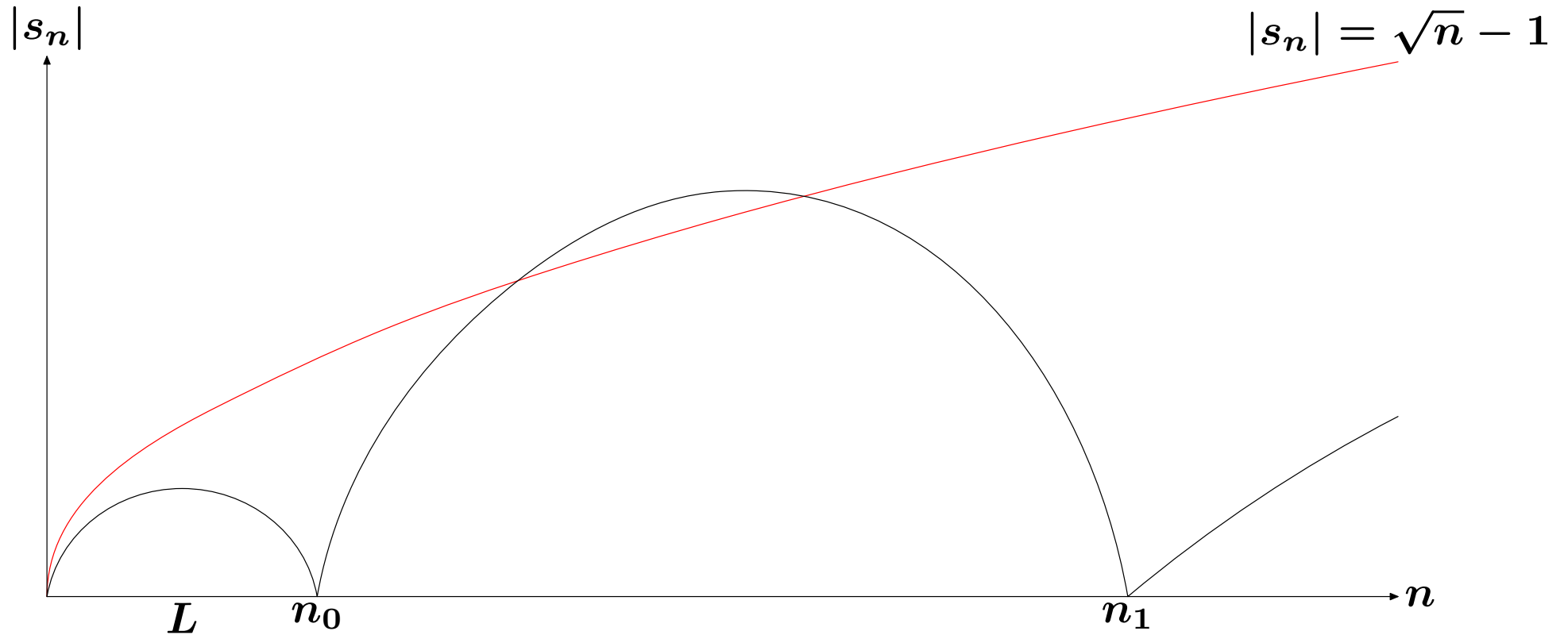


$A_2^1$

$A_2^2$

We halve  $A_2^2$  and wait until  $s_n = 0$ .

# Picture for explanation



$A_2^1$

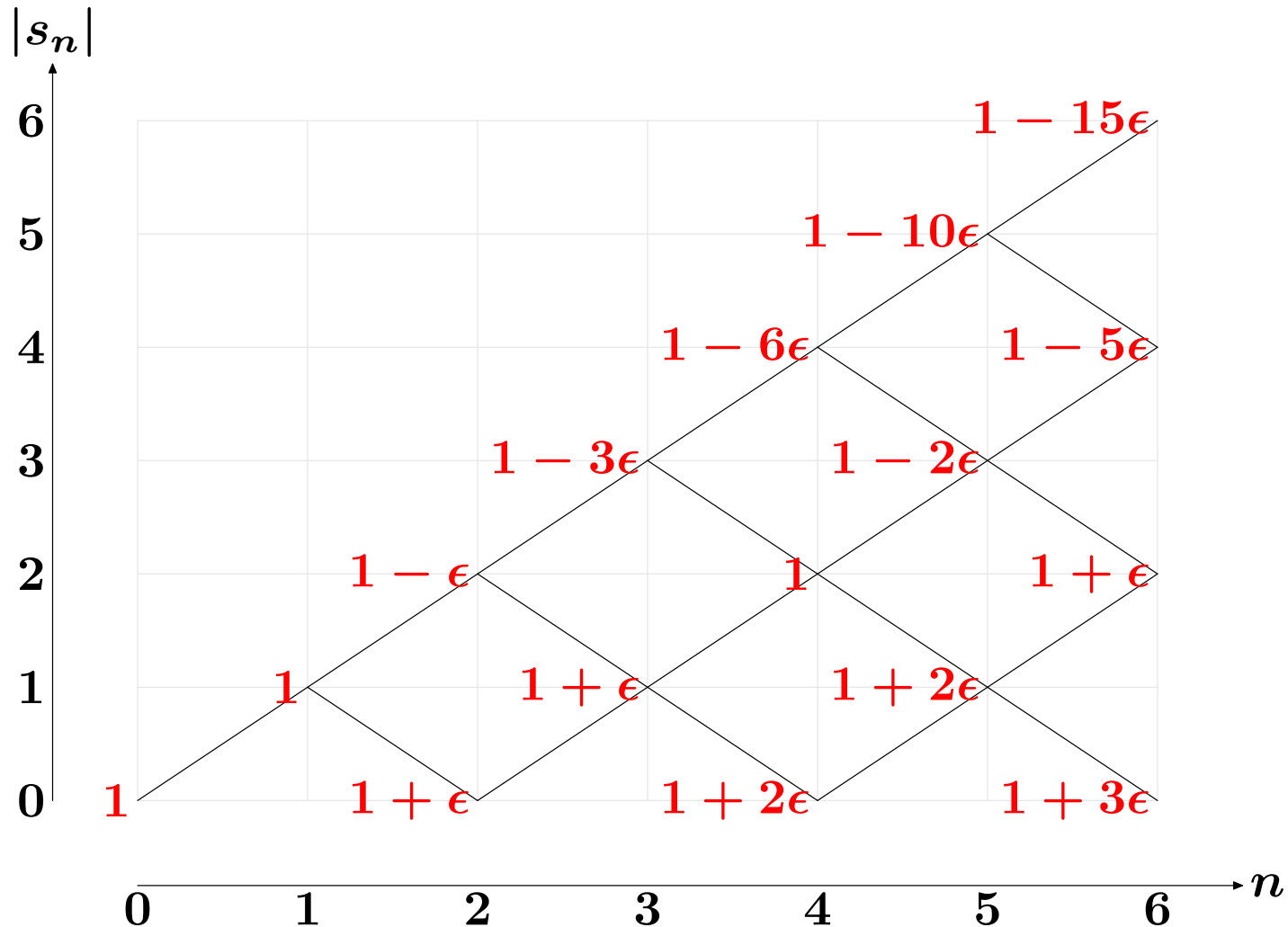


$A_2^2$



If  $s_n$  does not approach  $\sqrt{n} - 1$  then  $A_2^1$  goes up.

# Capital process of $\mathcal{P}^\epsilon : M_n = -\epsilon S_{n-1}$



Monotone increasing at the same level of  $|s_n|$ .

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- **contrarian strategy is symmetric w.r.t. the sign.**

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  - It can only force properties of  $|s_n|$ .
  - i.e. **Reality** can keep  $s_n$  always positive.
- **We can construct a strategy for preventing this.**
  - Need to force the symmetry between  $s_n$  and  $-s_n$ .
  - Then we can show that our theorem holds for  $s_n$  instead of  $|s_n|$ .

# Summary and concluding remarks

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  - The idea of contrarian strategy was essential.
- We also showed result on  $s_n$  instead of  $|s_n|$ .

# Summary and concluding remarks

- **Game-theoretic probability is a very new field. It offers new ways of looking at probability from the viewpoint of betting.**
- **Recently rapid advances have been made (Vladimir Vovk taking up an approach by our group) on how to treat continuous-times trading games.**