Computer Algebra based Analysis and Holonomic Gradient Method based Evaluation of MIMO Wireless Communications Systems

Costi Siriteanu, Osaka University
Akimichi Takemura, University of Tokyo
Satoshi Kuriki, Institute of Statistical Mathematics
Christoph Koutschan, Austrian Academy of Sciences

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Acronyms

- MIMO = multiple-input/multiple-output (communications system)
- ZF = zero forcing (detection method that cancels interference)
- SNR = signal-to-noise ratio (determines performance)
- H.G.M. = holonomic gradient method
- m.g.f. = moment generating function
- p.d.f. = probability density function

Outline

1. Multiple-Input/Multiple-Output (MIMO) Communications Systems
2. MIMO Zero-Forcing Detection (ZF) Analysis
3. Holonomic Gradient Method (HGM)-based ZF Evaluation
4. Computer Algebra Representation of Performance Measures
5. Other MIMO Analysis and Evaluation Applications

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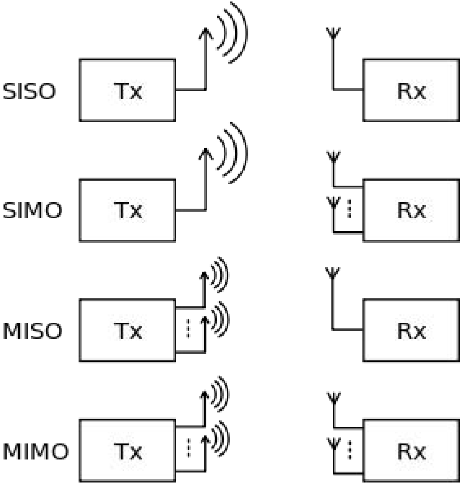
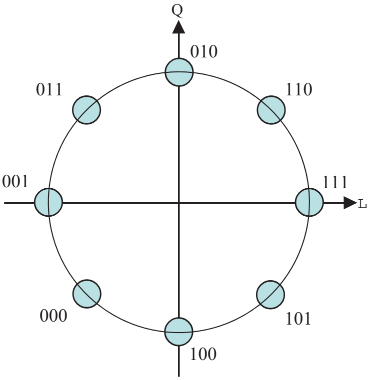
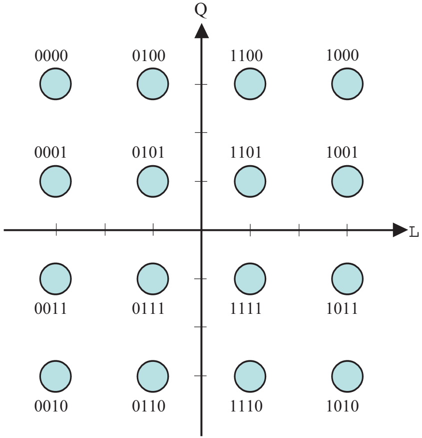
SISO-to-MIMO Evolution of Wireless Systems [1]



Signal Transmitted from Each Antenna

Phase-Shift Keying and Quadrature Amplitude Modulation [2]:





$N_\mathrm{T} \times N_\mathrm{R}$ MIMO Signal, Fading, Noise Models

If we transmit $\mathbf{x} = [x_1\ x_2\ \ldots\ x_{N_\mathrm{T}}]^\mathcal{T}$ then we receive

$$\mathbf{r} = \mathbf{H}\,\mathbf{x} + \mathbf{n} = (\mathbf{H}_\mathrm{d} + \mathbf{H}_\mathrm{r})\,\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{H} = \mathbf{H}_\mathrm{d} + \mathbf{H}_\mathrm{r}$ is complex-Gaussian $N_\mathrm{R} \times N_\mathrm{T}$ matrix, so that:

- If $[\mathbf{H}_\mathrm{d}]_{i,j} = \mathbf{0} \Rightarrow |[\mathbf{H}]_{i,j}|$ *Rayleigh*-distributed
- If $[\mathbf{H}_\mathrm{d}]_{i,j} \neq \mathbf{0} \Rightarrow |[\mathbf{H}]_{i,j}|$ *Rician*-distributed,

and $\mathbf{n}$ is complex-valued, zero-mean, Gaussian receiver noise.

Assumptions, for analysis tractability:

- Initially [3], we assumed 'Rician–Rayleigh' fading, i.e.,

$$\mathbf{H}_\mathrm{d} = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{H}_{\mathrm{d},2}) = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{0}). \tag{2}$$

- Recently [4], we allowed for $\mathbf{H}_{\mathrm{d},2} \neq \mathbf{0}$, restricted ranks of $\mathbf{H}_\mathrm{d}$, $\mathbf{H}_{\mathrm{d},2}$.

Rician K-Factor

If we write the channel matrix as

$$\mathbf{H} = \mathbf{H}_{\mathrm{d}} + \mathbf{H}_{\mathrm{r}}, \tag{3}$$

then we usually define the Rician-K factor as the power ratio

$$K = \frac{\|\mathbf{H}_{\mathrm{d}}\|^2}{\mathbb{E}\{\|\mathbf{H}_{\mathrm{r}}\|^2\}}, \tag{4}$$

whose value in practice is around $K = 5$, or, in decibels,

$$K_{\mathrm{dB}} = 10 \log_{10} 5 = 7 \text{ dB}. \tag{5}$$

Goal: evaluate MIMO performance for realistic fading type, parameters.

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MIMO Zero-Forcing Detection (ZF)

For $N_\mathrm{R} \times 1$ received-signal vector

$$\mathbf{r} = \mathbf{H}\,\mathbf{x} + \mathbf{n}, \quad \mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_\mathrm{R}}), \tag{6}$$

ZF determines constellation symbols closest to each element of

$$\left[\mathbf{H}^{\mathcal{H}}\,\mathbf{H}\right]^{-1}\,\mathbf{H}^{\mathcal{H}}\,\mathbf{r} = \mathbf{x} + \left[\mathbf{H}^{\mathcal{H}}\,\mathbf{H}\right]^{-1}\,\mathbf{H}^{\mathcal{H}}\,\mathbf{n}. \tag{7}$$

where we denoted the transposed complex-conjugate of $\mathbf{H}$ as $\mathbf{H}^{\mathcal{H}}$.

Then, ZF signal-to-noise ratio (SNR) for Stream 1 is

$$\gamma = \frac{1}{\left[(\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}\right]_{1,1}}, \tag{8}$$

where matrix $\mathbf{H}^{\mathcal{H}}\mathbf{H}$ is noncentral Wishart-distributed.

ZF SNR M.G.F. for Rician–Rayleigh Fading [3]

Assuming

$$\mathbf{H}_\mathrm{d} = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{H}_{\mathrm{d},2}) = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{0}), \tag{9}$$

we have shown that SNR moment generating function (m.g.f.) is [3]

$$M(s,a) = \frac{1}{(1-s)^N} \, {}_1F_1\left(N; N_\mathrm{R}; a\frac{s}{1-s}\right),$$

where

$$\begin{aligned} N &= N_\mathrm{R} - (N_\mathrm{T} - 1) = \text{number of degrees of freedom} \\ a &\propto K N_\mathrm{R} N_\mathrm{T}, \quad (e.g., a \approx 188, \text{for } 6 \times 6 \text{ system}, K = 5), \end{aligned}$$

and *confluent hypergeometric function* has infinite-series expression

$${}_1F_1(N; N_\mathrm{R}; \sigma) = \sum_{n=0}^{\infty} \frac{(N)_n}{(N_\mathrm{R})_n} \frac{\sigma^n}{n!} = \sum_{n=0}^{\infty} A_n(\sigma).$$

ZF SNR M.G.F. and P.D.F.

Thus, SNR m.g.f. can be written as

$$M(s,a) = \frac{{}_1F_1\left(N; N_\mathrm{R}; a\frac{s}{1-s}\right)}{(1-s)^N} = \sum_{n=0}^{\infty} A_n(a) \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(1-s)^{N+n-m}},$$

so that the SNR p.d.f. is given by the infinite series

$$p(t,a) = \sum_{n=0}^{\infty} A_n(a) \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m t^{(N+n-m)-1} e^{-t}}{[(N+n-m)-1]!\, {}^{N+n-m}}. \tag{10}$$

Goal is, for realistic K, i.e., large $a \propto K N_\mathrm{R} N_\mathrm{T}$:

- Compute $p(t,a)$.
- Compute MIMO performance measures.

MIMO Performance Measures

- Outage probability, for threshold SNR γ_{th}:

$$P_{\text{o}}(\gamma_{\text{th}}, a) = \text{Probability}\,\{\gamma \le \gamma_{\text{th}}\} = \int_0^{\gamma_{\text{th}}} p(t, a)\,\mathrm{d}t. \tag{11}$$

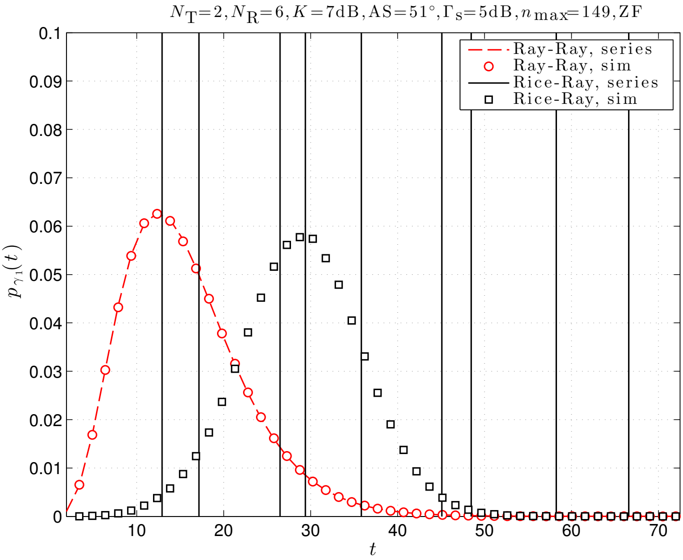
- Ergodic (i.e., average) capacity:

$$C(a) = \mathbb{E}_{\gamma}\{C(\gamma)\} = \int_0^{\infty} \log_2(1 + t)p(t, a)\,\mathrm{d}t. \tag{12}$$

Infinite-series expression of $p(t, a)$ yields infinite-series expressions for $P_{\text{o}}(\gamma_{\text{th}}, a)$ and $C(a)$.

Problem: we cannot compute (accurately or even at all) the infinite series for $p(t, a)$, $P_{\text{o}}(\gamma_{\text{th}}, a)$, and $C(a)$ for realistic K (i.e., large a).

MATLAB Results for SNR P.D.F. for $a = 0, 188$



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${}_1F_1(N; N_\mathrm{R}; \sigma)$ from Differential Equation, with HGM

Since ${}_1F_1(N; N_\mathrm{R}; \sigma)$ satisfies

$$\sigma\, {}_1F_1^{(2)}(N; N_\mathrm{R}; \sigma) + (N_\mathrm{R} - \sigma)\, {}_1F_1^{(1)}(N; N_\mathrm{R}; \sigma) - N\, {}_1F_1(N; N_\mathrm{R}; \sigma) = 0,$$

or, equivalently,

$$\partial_\sigma \begin{pmatrix} {}_1F_1(N; N_\mathrm{R}; \sigma) \\ {}_1F_1^{(1)}(N; N_\mathrm{R}; \sigma) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{N}{\sigma} & 1 - \frac{N_\mathrm{R}}{\sigma} \end{pmatrix} \begin{pmatrix} {}_1F_1(N; N_\mathrm{R}; \sigma) \\ {}_1F_1^{(1)}(N; N_\mathrm{R}; \sigma) \end{pmatrix}. \tag{13}$$

it can be computed with the holonomic gradient method (HGM):

- Compute infinite series ${}_1F_1(N; N_\mathrm{R}; \sigma_0)$ for σ_0 small.
- Numerically solve system (13) from σ_0 to desired $\sigma \gg 1$.

Similar Approach for $p(t, a)$? Yes [5]

In [3] we expressed the SNR m.g.f. as

$$M(s, a) = \int_0^\infty e^{st} p(t, a)\, \mathrm{d}t = \frac{1}{(1-s)^N} {}_1F_1\left(N; N_\mathrm{R}; a\frac{s}{1-s}\right). \qquad (14)$$

Then, we can try to obtain $p(t, a)$ at large values of $a \propto K$ as follows:

- Deduce differential equations for $M(s, a)$ based on

$$\sigma\, {}_1F_1^{(2)}(N; N_\mathrm{R}; \sigma) + (N_\mathrm{R} - \sigma)\, {}_1F_1^{(1)}(N; N_\mathrm{R}; \sigma) - N\, {}_1F_1(N; N_\mathrm{R}; \sigma) = 0$$

- Invert Laplace to obtain differential equations for $p(t, a)$.
- Use HGM to numerically compute $p(t, a)$.

Integrate numerically to evaluate MIMO performance measures.

Differential Equation for $M(s,a)$ w.r.t. a

For $\sigma = \frac{a\,s}{1-s}$, the differential equation for ${}_1F_1(N;N_\mathrm{R};\sigma)$ is

$$\frac{a\,s}{1-s}\,{}_1F_1^{(2)}\left(N;N_\mathrm{R};\frac{a\,s}{1-s}\right)+\left(N_\mathrm{R}-\frac{a\,s}{1-s}\right){}_1F_1^{(1)}\left(N;N_\mathrm{R};\frac{a\,s}{1-s}\right)$$
$$-N\,{}_1F_1\left(N;N_\mathrm{R};\frac{a\,s}{1-s}\right)=0, \qquad (15)$$

where we can substitute ${}_1F_1^{(k)}$ from

$$\partial_a^k M(s,a)=\frac{s^k\,{}_1F_1^{(k)}\left(N;N_\mathrm{R};\frac{a\,s}{1-s}\right)}{(1-s)^{N+k}} \qquad (16)$$

to obtain

$$\left(a\,(1-s)\partial_a^2+[N_\mathrm{R}(1-s)-a\,s]\,\partial_a-N\right)\bullet M(s,a)=0. \qquad (17)$$

Differential Equation for $M(s, a)$ w.r.t. s and a

On the other hand, differentiating w.r.t. s

$$M(s,a) = \frac{1}{(1-s)^N} \, {}_1F_1\left(N; N_{\mathrm{R}}; a\frac{s}{1-s}\right) \tag{18}$$

yields

$$\partial_s M(s,a) = \tfrac{N}{(1-s)^{N+1}} \, {}_1F_1\left(N; N_{\mathrm{R}}; \tfrac{a\,s}{1-s}\right) + \tfrac{a}{(1-s)^{N+2}} \, {}_1F_1^{(1)}\left(N; N_{\mathrm{R}}; \tfrac{a\,s}{1-s}\right),$$

where again substitute ${}_1F_1^{(k)}$ from

$${}_1F_1^{(k)}\left(N; N_{\mathrm{R}}; \tfrac{a\,s}{1-s}\right) = \tfrac{(1-s)^{N+k}}{s^k} \partial_a^k M(s,a), \tag{19}$$

to obtain

$$\left[\partial_a - \tfrac{s(1-s)}{a}\partial_s + N\tfrac{s}{a}\right] \bullet M(s,a) = 0. \tag{20}$$

Differential Equation for $M(s,a)$ w.r.t. s

Rewriting (20) as

$$\partial_a M(s,a) = \left[\frac{s\,(1-s)}{a}\partial_s - N\frac{s}{a}\right] \bullet M(s,a),$$

differentiating again w.r.t. a, and substituting into

$$\Big(a\,(1-s)\partial_a^2 + [N_\mathrm{R}(1-s) - a\,s]\,\partial_a - N\Big) \bullet M(s,a) = 0,$$

yields

$$\Big(s(1-s)^2\partial_s^2 - [2(N+1)s(1-s) - (1-s)N_\mathrm{R} + a\,s]\,\partial_s$$
$$+N\,[(N+1)s - N_\mathrm{R} - a]\Big) \bullet M(s,a) = 0. \qquad (21)$$

Goal: invert Laplace to get differential equation for $p(t,a)$ w.r.t. t.

Switching Order of s and ∂_s

Property (from [6, Th. 6.1.2 (Liebniz Formula), p. 282])

$$s^l \partial_s^k = \sum_{m=0}^{\min(l,k)} \frac{(-1)^m (l-m+1)_m (k-m+1)_m}{m!} \partial_s^{k-m} s^{l-m} \tag{22}$$

yields the following particular rules

$$\begin{aligned} s\partial_s &= \partial_s s - 1, \\ s\partial_s^2 &= \partial_s^2 s - 2\partial_s, \\ s^2\partial_s &= \partial_s s^2 - 2s, \\ s^2\partial_s^2 &= \partial_s^2 s^2 - 4\partial_s s + 2, \\ s^3\partial_s^2 &= \partial_s^2 s^3 - 6\partial_s s^2 + 6s, \end{aligned}$$

which produce the following annihilator of $M(s,a)$ w.r.t. s:

$$\begin{aligned} &\partial_s^2 s^3 - 2\partial_s^2 s^2 + \partial_s^2 s + (2N-4)\partial_s s^2 + (6 - 2N - N_R - a)\,\partial_s s \\ &+ (N_R - 2)\partial_s + (N-1)(N-2)s + (N-1)(2 - N_R - a). \end{aligned} \tag{23}$$

Taking the Inverse-Laplace Transform

It can be shown that $\int_0^\infty e^{st}\left[t^k p^{(l)}(t,a)\right]\mathrm{d}t$ is given by

$$(-1)^l \partial_s^k [s^l M(s,a)] + \sum_{m=k+1}^{l} (-1)^m p^{(l-m)}(0+,a)\,\frac{(m-1)!}{(m-k-1)!} s^{m-k-1}.$$

Applied to the terms in (23), it yields the following Laplace pairs:

$$\begin{aligned} \partial_s^2 s^3 M(s,a) + 2!\,p(0+,a) \quad &\leftrightarrow \quad -t^2 p^{(3)}(t,a) \\ &\dots \\ (N-1)(2-N_\mathrm{R}-a)M(s,a) \quad &\leftrightarrow \quad (N-1)(2-N_\mathrm{R}-a)\,p(t,a). \end{aligned}$$

We have found that the left-hand-side terms sum to 0, $\forall N \geq 1$, so that the right-hand-side terms also sum to 0.

Differential Equation for $p(t, a)$ w.r.t. t

This yields the following differential equation w.r.t. t for $p(t, a)$:

$$\begin{aligned} p^{(3)}(t,a) \quad = \quad & \frac{(N_\mathrm{R}-2)t+(N-1)(2-N_\mathrm{R}-a)}{t^2}p(t,a) \\ & -\frac{t^2+(6-2N-N_\mathrm{R}-a)t+(N-1)(N-2)}{t^2}p^{(1)}(t,a) \\ & -\frac{2t^2-(2N-4)t}{t^2}p^{(2)}(t,a). \end{aligned}$$

If we define the function vector

$$\mathbf{p}(t,a) = \Big(p(t,a) \quad p^{(1)}(t,a) \quad p^{(2)}(t,a)\Big)^{\mathcal{T}},$$

then the above can be written as the system of differential equations

$$\partial_t \mathbf{p}(t,a) = \mathbf{P}(t,a)\,\mathbf{p}(t,a).$$

Differential Equation for $p(t, a)$ w.r.t. a

From (20) we deduce

$$a\partial_a M(s, a) = \left[s\partial_s - s^2\partial_s - Ns\right] \bullet M(s, a). \tag{24}$$

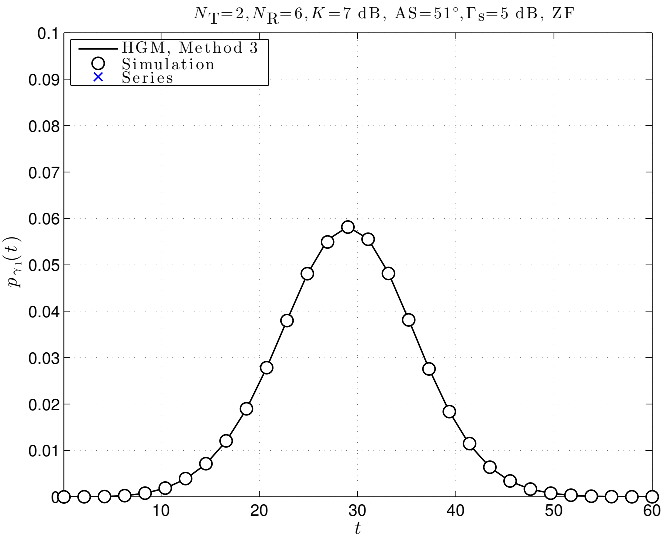
Switching s and ∂_s order, and Laplace-transforming yields

$$a\partial_a p(t, a) = -p(t, a) + (N - t - 2)\, p^{(1)}(t, a) - t\, p^{(2)}(t, a).$$

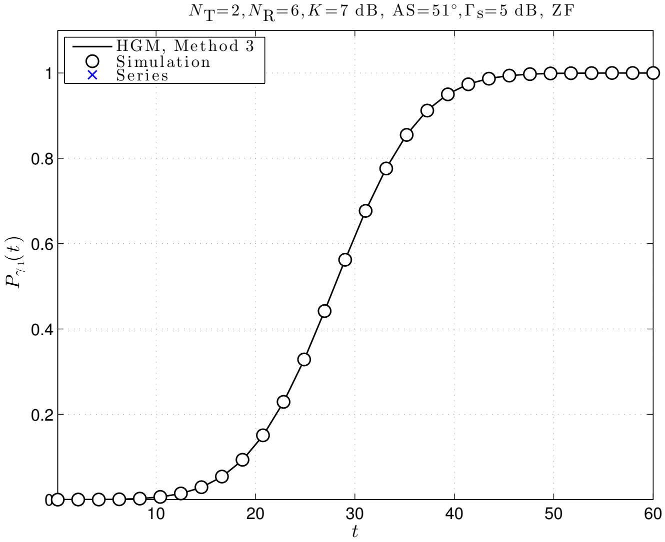
Differentiating it twice w.r.t. t yields the system of differential equations

$$\partial_a \mathbf{p}(t, a) = \frac{1}{a}\, \mathbf{Q}(t, a)\, \mathbf{p}(t, a). \tag{25}$$

MATLAB Results for $p(t, a)$ [5]



MATLAB Results for $\int_0^t p(y, a)\mathrm{d}y$ [5]



MIMO ZF Performance Measures

Recall:

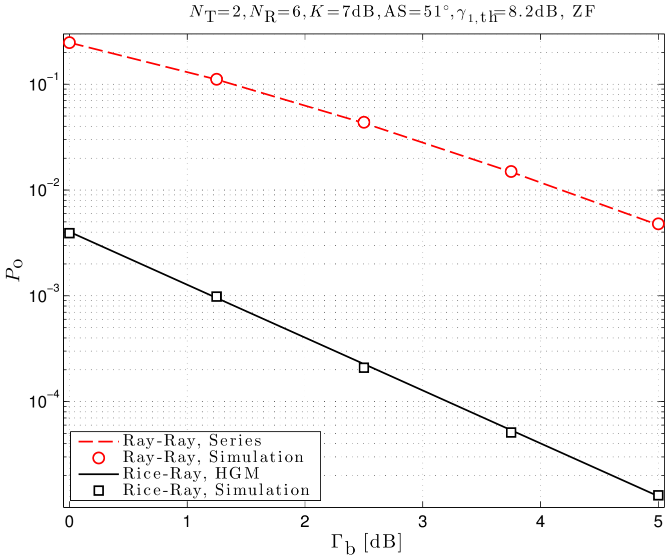
- Outage probability, for threshold SNR $\gamma_{1,\text{th}}$:

$$P_\text{o}(\gamma_{1,\text{th}}, a) = \text{Probability}\,\{\gamma_1 \leq \gamma_{1,\text{th}}\} = \int_0^{\gamma_{1,\text{th}}} p(t,a)\,\mathrm{d}t. \tag{26}$$

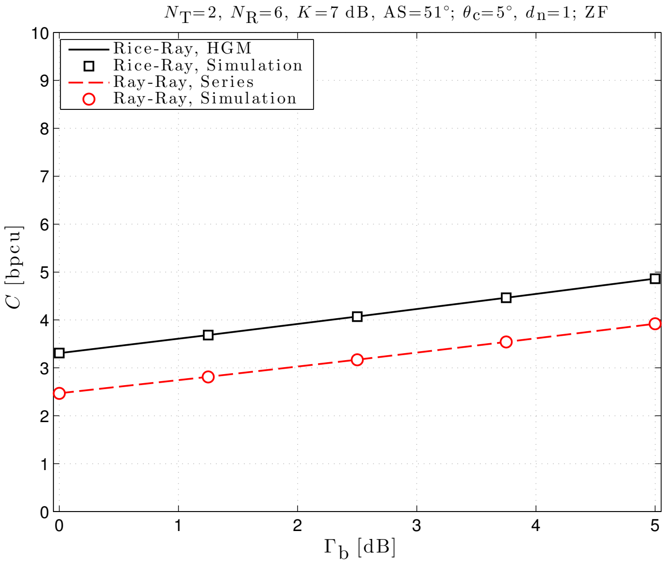
- Ergodic (i.e., average) capacity:

$$C(a) = \mathbb{E}_{\gamma_1}\{C(\gamma_1)\} = \int_0^{\infty} \log_2(1+t)p(t,a)\,\mathrm{d}t. \tag{27}$$

MATLAB Results for Outage Probability [5]



MATLAB Results for Ergodic Capacity [5]



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Christoph Koutschan's Mathematica Tool [7]

Properties of the `HolonomicFunctions` package:

- Deals with multivariate holonomic functions and sequences.
- Computes annihilating ideal, i.e., set of linear differential equations, linear recurrences, q-difference equations, and mixed linear equations that a given function satisfies.
- Executes closure properties (addition, multiplication, substitutions) for such functions.
- Implements creative telescoping for the summation and integration of multivariate holonomic functions.
- Subtasks:
  - Computations in Ore algebras (noncommutative polynomial arithmetic with mixed difference-differential operators)
  - Noncommutative Gröbner bases
  - Solving coupled linear systems of differential or difference equations.

Using Christoph Koutschan's Mathematica Tool [7]

For our m.g.f. expression

$$M(s,a) = \frac{1}{(1-s)^N}\,{}_1F_1\left(N; N_\mathrm{R}; a\frac{s}{1-s}\right), \tag{28}$$

the `HolonomicFunctions` command

$$\texttt{Annihilator}\left(\frac{{}_1F_1\left(N;N_R;\frac{as}{1-s}\right)}{(1-s)^N}, \{\texttt{Der}(s), \texttt{Der}(a)\}\right),$$

yields the annihilators:

- $a\partial_a + \partial_s\left(s^2 - s\right) + Ns$,
- $\partial_a\left(as + sN_R - N_R\right) + \partial_a^2\left(as - a\right) + Ns$.

Differential Equation w.r.t. *a* for P.D.F.

From

$$M(s,a) = \frac{1}{(1-s)^N} \, {}_1F_1\left(N; N_{\mathrm{R}}; a\frac{s}{1-s}\right), \qquad (29)$$

a few `HolonomicFunctions` commands produced the following differential equation, whose by-hand derivation had been intractable:

$$\begin{aligned} & a^2\partial_a^3 p(t,a) + (2a^2 + 2aN_{\mathrm{R}})\partial_a^2 p(t,a) \\ & + \left(a^2 + aN + 2aN_{\mathrm{R}} + {N_{\mathrm{R}}}^2 - N_{\mathrm{R}} - at\right)\partial_a p(t,a) \\ & + \left(aN - N + NN_{\mathrm{R}} - (N_{\mathrm{R}} - 1)t\right)p(t,a) = 0. \end{aligned} \qquad (30)$$

Differential Equations for Performance Measures

Creative telescoping [6]: deducing differential equations satisfied by the integral of a function from the differential equation satisfied by the function. `HolonomicFunctions` yielded for $P_o(a)$, $C(a)$:

$$a^2\partial_a^4 P_o(a) =$$
$$-(-N_R + a + aN + N_R + NN_R)\partial_a P_o(a)$$
$$-(-a + 3a + +a^2 + aN + N_R + 2aN_R + {N_R}^2)\partial_a^2 P_o(a)$$
$$-2a(1 + a + N_R)\partial_a^3 P_o(a).$$

$$a^2\partial_a^5 C(a) =$$
$$-2a(N_R + a + 2)\partial_a^4 C(a)$$
$$-(2 + a + 7a + {N_R}^2 + a^2 + 3N_R + Na + 2N_R a)\partial_a^3 C(a)$$
$$-(3 + N_R + 3a + N + 3N_R + NN_R + Na + 1)\partial_a^2 C(a)$$
$$-(N + 1)\partial_a C(a).$$

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Our More Recent ZF Work for Full-Rician Fading [4]

Partitioning again (corresponding to intended/interfering streams) as

$$\mathbf{H}_{\mathrm{d}} = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{H}_{\mathrm{d},2}), \tag{31}$$

allowing for $\mathbf{H}_{\mathrm{d},2} \neq \mathbf{0}$, but assuming, for tractability, that

$$\mathrm{rank}(\mathbf{H}_{\mathrm{d}}) = \mathrm{rank}(\mathbf{H}_{\mathrm{d},2}) = 1, \tag{32}$$

we derived the ZF SNR m.g.f. for Stream 1 as

$$M_{\gamma_1}(s) = \frac{e^{-y}}{(1-s)^N} \sum_{n=0}^{\infty} \frac{y^n}{n!} {}_1F_1\left(N; n + N_{\mathrm{R}}; \frac{s}{1-s} x\right),$$

where x, y depend on noncentrality $\mathbf{H}_{\mathrm{d}}$.

Our More Recent ZF Work for Full-Rician Fading [4]

We have analyzed MIMO ZF for full-Rician fading. Partitioning as

$$\mathbf{H}_{\mathrm{d}} = (\mathbf{h}_{\mathrm{d},1} \quad \mathbf{H}_{\mathrm{d},2}), \tag{33}$$

and assuming, for tractability, that

$$\mathrm{rank}(\mathbf{H}_{\mathrm{d}}) = \mathrm{rank}(\mathbf{H}_{\mathrm{d},2}) = 2, \tag{34}$$

we derived the ZF SNR m.g.f. for Stream 1 as

$$\begin{aligned} M_{\gamma_1}(s) &= \frac{e^{-(x_{12}+x_{22})}}{(1-s)^N} \sum_{n_{12}=0}^{\infty} \sum_{n_{22}=0}^{\infty} \frac{x_{12}^{n_{12}} x_{22}^{n_{22}}}{n_{12}! n_{22}!} \\ &\quad \times {}_2F_2(N, N_{\mathrm{R}} - 1 + n_{22}; N_{\mathrm{R}} - 1, N_{\mathrm{R}} + n_{12} + n_{22}; \delta_{11}(s)), \end{aligned}$$

where $\delta_{11}(s) = \frac{s}{1-s} \|\mathbf{h}_{\mathrm{d},1}\|^2$, and x_{12}, x_{22} depend on noncentrality $\mathbf{H}_{\mathrm{d}}$.

SIMO Optimum Combining for Rician Fading [8]

For SIMO system, when the received signal vector is

$$\mathbf{r} = \mathbf{h}x + \mathbf{n}, \tag{35}$$

optimum combining means

$$r = \frac{\mathbf{h}^{\mathcal{H}}}{\|\mathbf{h}\|}\mathbf{r} = \|\mathbf{h}\|x + \frac{\mathbf{h}^{\mathcal{H}}}{\|\mathbf{h}\|}\mathbf{n}, \tag{36}$$

and yields symbol-detection SNR $\propto \|\mathbf{h}\|^2 = \sum_{i=1}^{N_\mathrm{R}} |h_i|^2$.

Satellite channel is described by shadowed-Rician fading.

Then, p.d.f. of $\|\mathbf{h}\|^2$ is known in terms of ${}_1F_1(\cdot;\cdot;\cdot)$ [8, Eq. (6)].

MIMO Space–Time Coding for General Fading [9]

Output SNR $Y = \sum_{i=1}^{n} X_i^2$ m.g.f. is $M_Y(s) = \prod_{i=1}^{n} M_{X_i^2}(s)$, where per-branch SNR m.g.f. for various fading types is:

Channel model	Fading Parameters	MGF
Nakagami-m	$\Omega_j, m_j \geq 0$	$M^{N}_{X_j^2}(s) = \frac{1}{\left(1 - s\frac{\Omega_j}{m_j}\right)^{m_j}}$
Rice	$\Omega_j, K_j \geq 0$	$M^{R}_{X_j^2}(s) = \frac{1}{1 - s\frac{\Omega_i}{K_j+1}} \exp\left(\frac{\frac{K_j \Omega_j}{K_j+1} s}{1 - s\frac{\Omega_j}{K_j+1}}\right)$
Hoyt	$\Omega_j \geq 0, 0 \leq q \leq 1$	$M^{H}_{X_j^2}(s) = \frac{1}{\sqrt{\left(1 - \frac{2\Omega_j q_j^2}{1+q_j^2} s\right)\left(1 - \frac{2\Omega_j}{1+q_j^2} s\right)}}$
Beckmann	$\Omega_j > 0, \mu_{j,1}, \mu_{j,2}, \sigma_{j,1}, \sigma_{j,2}$	$M^{B}_{X_j^2}(s) = \frac{\exp\left(\frac{\mu_{j,1}^2 \Omega_j s}{1-2\sigma_{j,1}^2 \Omega_j s} + \frac{\mu_{j,2}^2 \Omega_j s}{1-2\sigma_{j,2}^2 \Omega_j s}\right)}{\sqrt{\left(1-2\sigma_{j,1}^2 \Omega_j s\right)\left(1-2\sigma_{j,2}^2 \Omega_j s\right)}}$
Shadowed Rice	$\Omega_j, m_j, \lambda_j \geq 0$	$M^{SR}_{X_j^2}(s) = \frac{\left(1-2\lambda_j s\right)^{m_j - 1}}{\left(1 - \left(2\lambda_j + \frac{\Omega_j}{m_j}\right) s\right)^{m_j}}$

MIMO Space–Time Coding for General Fading [9]

MIMO space–time coding output SNR $Y = \sum_{i=1}^{n} X_i^2$ m.g.f. is

$$M_Y(s) = \prod_{i=1}^{n} \frac{\exp\left(\delta_i \frac{a_i s}{1-a_i s}\right)(1-b_i s)^{k_i}}{(1-a_i s)^{p_i}}, \tag{37}$$

which can be written as infinite series (β controls convergence)

$$M_Y(s) = A \sum_{r=0}^{\infty} \frac{c_r}{1-s\beta^{\rho+r}}, \tag{38}$$

so that p.d.f. of SNR is

$$f_Y(y) = A \sum_{r=0}^{\infty} c_r \frac{y^{\rho+r-1} e^{-y/\beta}}{\beta^{\rho+r}\Gamma(\rho+r)}. \tag{39}$$

MIMO ML Detection for Rician Fading [10]

MIMO maximum-likelihood (ML) receiver performance is determined by $\lambda_{\min}(\mathbf{H}^{\mathcal{H}}\mathbf{H})$. For Wishart matrix with rank-1 noncentrality [10, Eq. (16)]

$$p_{\lambda_{\min}}(\lambda) \propto \Phi_3(b, c; w, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b)_k}{(c)_{k+m}} \frac{w^k z^m}{k!m!}. \qquad (40)$$

Because confluent hypergeometric function Φ_3 is difficult to evaluate, it was written as a finite series [10, Eq. (12)] in terms of the generalized Marcum-Q function, which can be expressed accurately, i.e.,

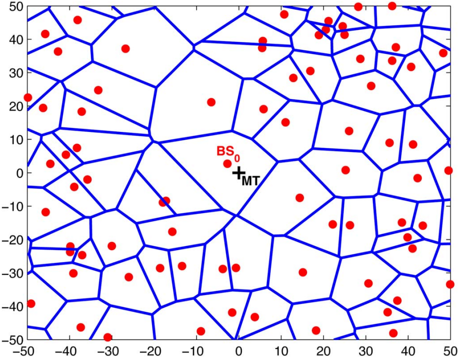
$$Q_m(a, b) = \int_b^{\infty} \frac{x^m}{a^{m-1}} \exp\left(-\frac{a^2 + x^2}{2}\right) I_{m-1}(ax)\mathrm{d}x, \qquad (41)$$

where I_m is the mth order modified Bessel function of the first kind.

MIMO ML under Interference [11]

Study MIMO *network* performance by assuming Poisson point process base-station locations (e.g., with $\lambda = 0.008$) and characterizing other-cell interference using stochastic geometry.

ML performance measures expressed in terms of hypergeometric functions, integrated from 0 to ∞. Authors notice difficulties at large argument values and employ approximations [11, Eqs. (19), (20)].



MIMO Beamforming for Rician Fading [12]

- Assuming knowledge of $\mathbf{H}$ the transmitter one can transmit symbol x into a certain 'direction' $\mathbf{w}_\mathrm{T}$, i.e., *beamforming*, so that received signal model is:

$$\mathbf{r} = \mathbf{H}\,\mathbf{w}_\mathrm{T} x + \mathbf{n} = (\mathbf{H}_\mathrm{d} + \mathbf{H}_\mathrm{r})\,\mathbf{w}_\mathrm{T} x + \mathbf{n}. \tag{42}$$

- Optimum receive processing: combine $\mathbf{r}$ with $\mathbf{w}_\mathrm{R} \propto \mathbf{H}\,\mathbf{w}_\mathrm{T}$; then, detection SNR is proportional with

$$(\mathbf{H}\,\mathbf{w}_\mathrm{T})^\mathcal{H}(\mathbf{H}\,\mathbf{w}_\mathrm{T}) = \mathbf{w}_\mathrm{T}^\mathcal{H}\mathbf{H}^\mathcal{H}\mathbf{H}\,\mathbf{w}_\mathrm{T} \tag{43}$$

- Thus, if $\mathbf{H}^\mathcal{H}\mathbf{H} = \mathbf{U\Lambda U}^\mathcal{H} = \sum \lambda_i \mathbf{u}_i\mathbf{u}_i^\mathcal{H}$ and $\lambda_1 = \lambda_\mathrm{max} = \max\{\lambda_i\}$, then the optimum beamformer is $\mathbf{w}_\mathrm{T} \propto \mathbf{u}_1$.
- Finally, SNR $\propto \lambda_\mathrm{max}$, i.e., we need its distribution. It was found for complex-valued noncentral Wishart in terms of hypergeometric function in [12].

Dominant-Eigenvalue Statistics, for Real Case [13]

C.d.f. of $\lambda_{\max}$ of real-valued zero-mean Wishart matrix is known in terms of [13, Eq. (4)]

$$I(a, b; x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} P(b, t) \mathrm{d}t, \tag{44}$$

where

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} \mathrm{d}t. \tag{45}$$

The latter be computed recursively with

$$P(a+n, x) = P(a, x) - e^{-x} \sum_{k=0}^{n-1} \frac{x^{a+k}}{\Gamma(a+k+1)}. \tag{46}$$

C.d.f. computation requires 70 seconds for 500×500 matrices.

Dominant-Eigenvalue Statistics, for Complex Case [13]

For larger matrices[1], use the Tracy–Widom (TW) limiting distribution, i.e., the c.d.f. of $\lambda_{\max}$

$$F_2(x) = \exp\left\{ -\int_x^{\infty} (y-x) q^2(y) \mathrm{d}y \right\}, \tag{47}$$

where $q(y)$ satisfies Painlevé II differential equation

$$q''(y) = yq(y) + 2q^3(y), \tag{48}$$

which is solved numerically in [13] for c.d.f. evaluation.

[1] Massive MIMO envisions thousands of antennas.

Outline

1. Multiple-Input/Multiple-Output (MIMO) Communications Systems
2. MIMO Zero-Forcing Detection (ZF) Analysis
3. Holonomic Gradient Method (HGM)-based ZF Evaluation
4. Computer Algebra Representation of Performance Measures
5. Other MIMO Analysis and Evaluation Applications

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