Quantum ($\mathfrak{sl}_n, \wedge V_n$) link invariant
and
matrix factorizations

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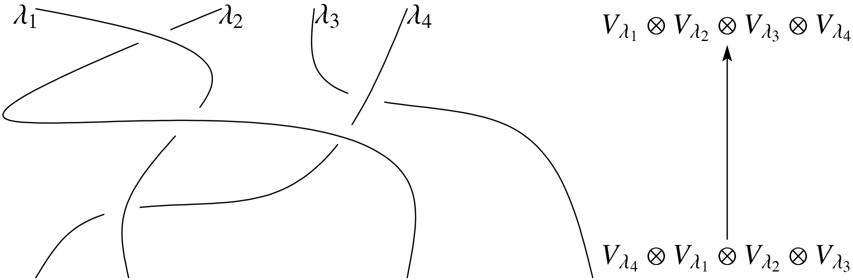
May 28 2010, Tanbara

Research Background

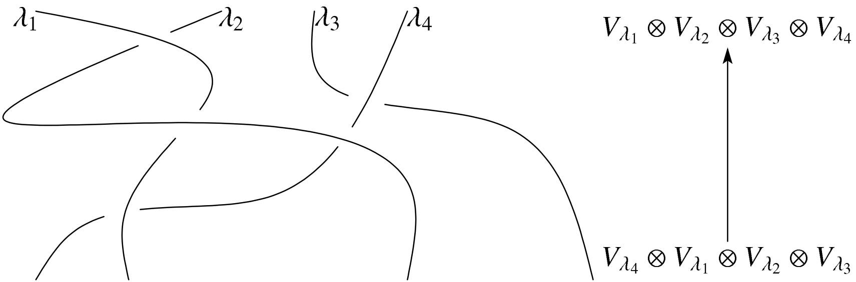
Quantum link invariant (Reshetikhin-Turaev)

Quantum link inv. $\leftsquigarrow$ $U_q(\mathfrak{g})$ and V_λ: irred. reps.



Quantum link invariant (Reshetikhin-Turaev)

Quantum link inv.	$\leftsquigarrow$	$U_q(\mathfrak{g})$ and V_λ: irred. reps.
Ex.		
Jones poly. $J_2(q)$	$\leftsquigarrow$	$U_q(\mathfrak{sl}_2)$ and V_2: 2-dim rep.
HOMFLY-PT poly. $J_n(q)$	$\leftsquigarrow$	$U_q(\mathfrak{sl}_n)$ and V_n: n-dim rep.



Research Background

HOMFLY-PT poly. $J_n(q)$ (Quantum $(\mathfrak{sl}_n, V_n)$ link invariant)

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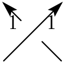
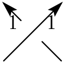
Crossing 

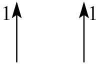
$\in \mathrm{End}(V_n^{\otimes 2})$

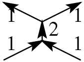
HOMFLY-PT poly. $J_n(q)$ (Quantum $(\mathfrak{sl}_n, V_n)$ link invariant)

Crossing 

$\in \mathrm{End}(V_n^{\otimes 2})$

Basis of $\mathrm{End}(V_n^{\otimes 2})$



$= \quad \mathrm{Id}_{V_n^{\otimes 2}}$ 

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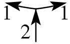
$\circ$ 

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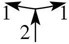
$\in \mathrm{Hom}(V_n^{\otimes 2}, \wedge^2 V_n)$ 

$\in \mathrm{Hom}(\wedge^2 V_n, V_n^{\otimes 2})$ $\Bigg)$

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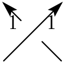
HOMFLY-PT poly. $J_n(q)$ (Quantum $(\mathfrak{sl}_n, V_n)$ link invariant)

Crossing 

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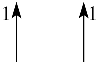

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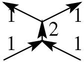

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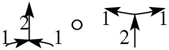
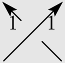
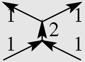


Morphism description of crossing



$= q^{n-1}$


$- q^{n}$


Categorification of quantum link invariant

Jones poly. $J_2(q)$	$\overset{\text{Euler char.}}{\leftsquigarrow}$	Khovanov homology
HOMFLY-PT poly. $J_n(q)$	$\overset{\text{Euler char.}}{\leftsquigarrow}$	Khovanov-Rozansky homology

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Khovanov-Rozansky homology

: Matrix Factorization (MF)

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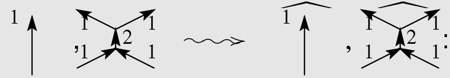
Khovanov-Rozansky homology



: Matrix Factorization (MF)

: Cpx. of MF

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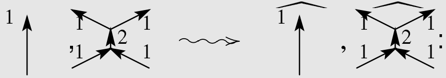

: Matrix Factorization (MF)

: Cpx. of MF $\rightsquigarrow$ Link homology : link inv.

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Khovanov-Rozansky homology

: Matrix Factorization (MF)

: Cpx. of MF $\rightsquigarrow$ Link homology : link inv.

$\rightsquigarrow$ Polynomial link inv. $P_n(q,t,s)$ where $s^2 = 1$

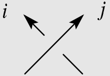
$(P_n(q,-1,1) = J_n(q))$

Research Background

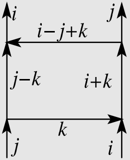
Quantum ($\mathfrak{sl}_n$,$\wedge V_n$) link invariant (Murakami-Ohtsuki-Yamada)

$\wedge V_n = \{V_n, \wedge^2 V_n, ..., \wedge^{n-1} V_n\} \cdots$ Fund. reps. of $U_q(\mathfrak{sl}_n)$.

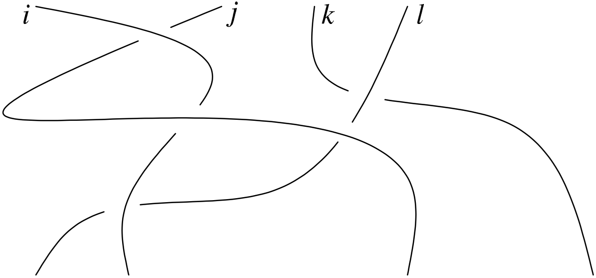
$[i, j]$-crossing $\cdots$

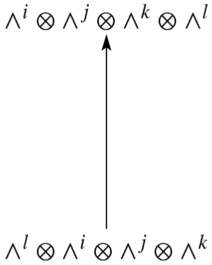


$$= \sum_{k=0}^{j} q^{a+k}$$



$$1 \le i, j \le n-1$$





Plan of talk

1. MF for morphism between fund. reps.
   - $\mathbb{Z}$-graded matrix factorization
   - MF and morphism
2. Complex for $[1,k]$-crossing
   - Definition
   - Main Theorem 1
3. New link invariant
   - Approximate $[i,j]$-crossing
   - Main theorem 2
   - New polynomial invariant
   - Main theorem 3 (Cor of Main theorem 2)

$\mathbb{Z}$-graded matrix factorization

Koszul matrix factorization

$R \cdots$ graded polynomial ring,
$M \cdots$ free R-module,
$a, b \cdots$ graded homog. polynomial.

$\mathbb{Z}$-graded matrix factorization

Koszul matrix factorization

$R \cdots$ graded polynomial ring,
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Koszul matrix factorization $K(a;b)_M$ is 2-cyclic chain

$$K(a;b)_M \quad = \quad \cdots \xrightarrow{b} M \xrightarrow{a} M \xrightarrow{b} M \xrightarrow{a} M \xrightarrow{b} \cdots$$

$(ab = ba \neq 0)$

Category of MF

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- (outer) tensor product.

$$\overline{M} \boxtimes \overline{N}$$

- Krull-Schmidt prop.

$$\overline{M} \oplus \overline{L} \simeq \overline{N} \oplus \overline{L} \Longrightarrow \overline{M} \simeq \overline{N}$$

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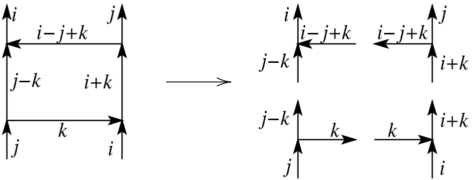
Behavior of matrix factorization

Graded matrix factorization $\sim$ Graded (super) vector space

Construction of MF for morphism between fund. reps.



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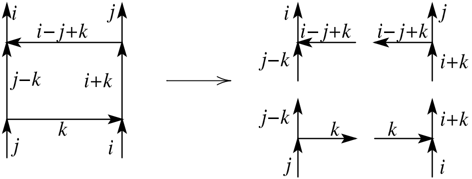
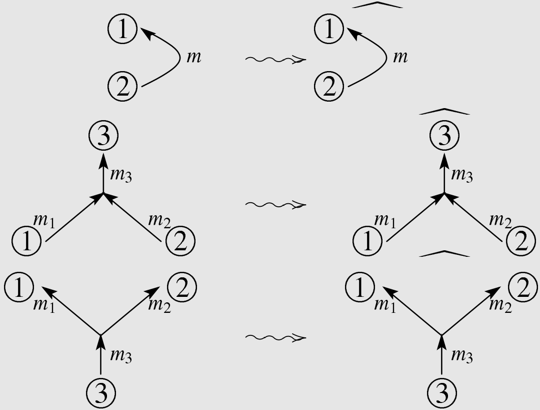
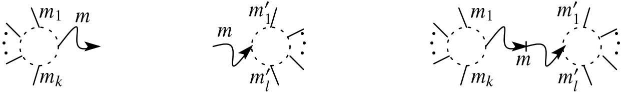


MF for identity between fund. rep. and essential morphism



Gluing MFs for morphism

Consider two morphisms with same coloring and opposite orientation legs.
Define gluing MFs for these diag. at end of legs.



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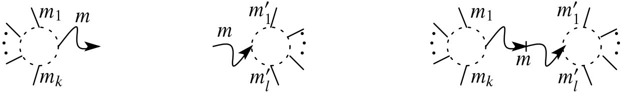
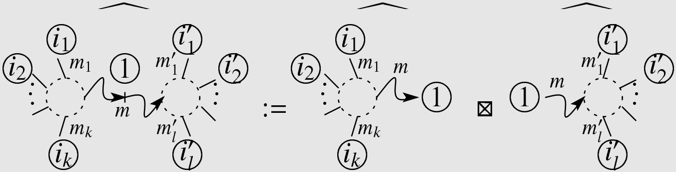


Def (Gluing MFs for morphisms)



Gluing MF for morphism

Consider morphism with neighboring legs such that same coloring and opposite orientation.
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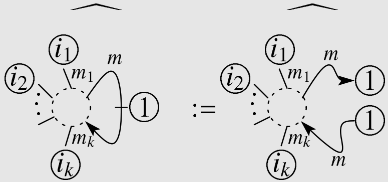

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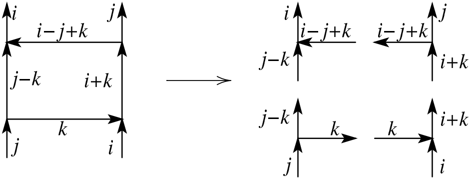

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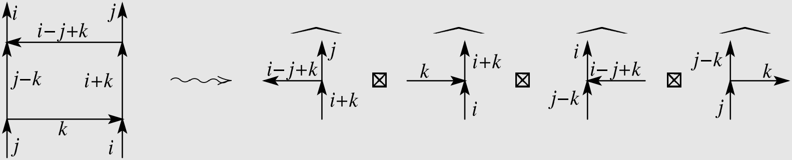
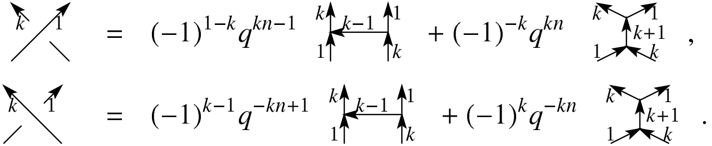
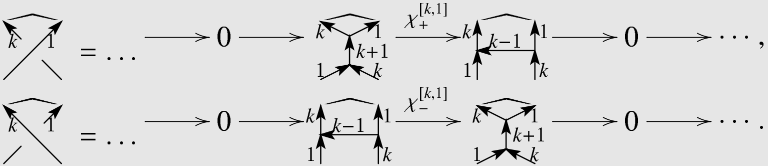


Complex for $[1, k]$-crossing

For $[1, k]$-crossing, quantum link inv. forms as follows;



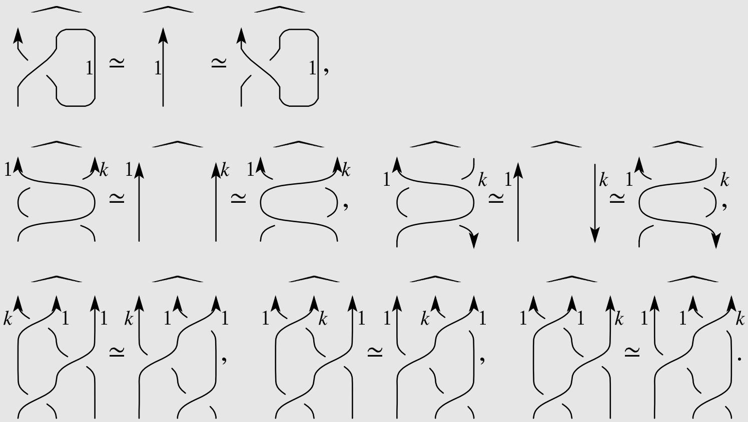
Def [Y] (Complex for $[1, k]$-crossing)

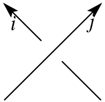


Boundary maps $\chi_+^{[k,1]}, \chi_-^{[k,1]}$ are explicitly forms.

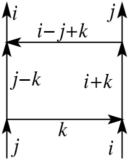
Complex for $[1, k]$-crossing

Main theorem 1 [Y](Invariance of Reidemeister move), in $k = 1$ [KR]



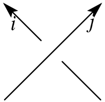


$$= \sum_{k=0}^{j} (-1)^{a-k} q^{b+k}$$



for $i \geq j$

Approximate $[i, j]$-crossing



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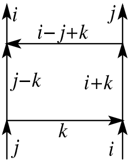
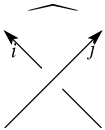
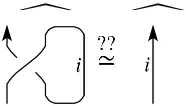


for $i \geq j$

It's both difficult to define cpx. of $[i, j]$-crossing and to proof invariance under Reidemeister moves if we define cpx.

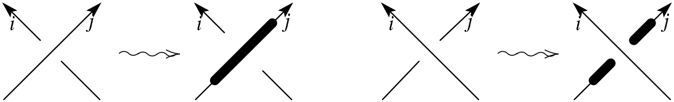


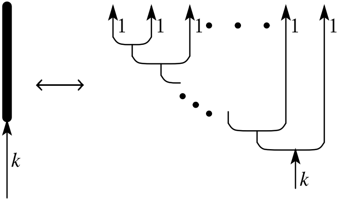
$=??$



Approximate $[i, j]$-crossing

Consider approximate $[i, j]$-crossing as follows.





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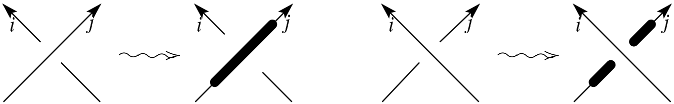
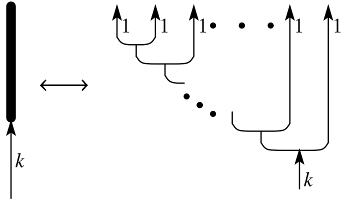
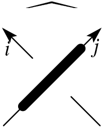


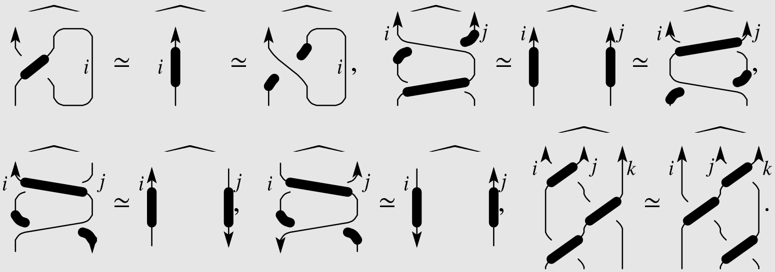


Approximate $[i, j]$-crossing consists of $[i, 1]$-crossings only..
Therefore, we can obtain cpx of approximate $[i, j]$-crossing.

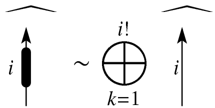


Theorem (Properties of cpx. of approximate $[i, j]$-crossing)

Main theorem 2 [Y]



This claim is not enough to construct link homology. We need to pick out cpx. of original $[i, j]$-crossing. Unfortunately, I could not construct link homology in my thesis.



Polynomial link invariant

$D \cdots$ Colored link diagram

We obtain new link invariant by the following procedure.

Polynomial link invariant

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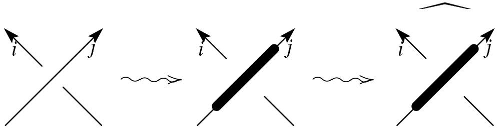

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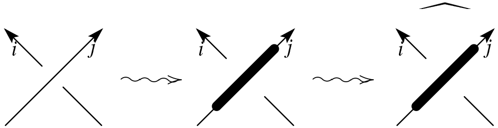

$\rightsquigarrow$ $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-homology $H^{i,j,k}(D)$ (Not link invariant)

Polynomial link invariant

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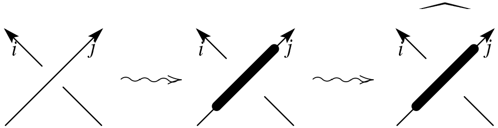

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$\rightsquigarrow$ Poincaré poly. (Not link invariant)

$$\overline{P}(D) = \sum_{i,j,k} t^i q^j s^k \dim_{\mathbb{Q}} H^{i,j,k}(D) \in \mathbb{Z}[q,t,s]/\langle s^2 - 1 \rangle$$

Polynomial link invariant

$D \cdots$ Colored link diagram

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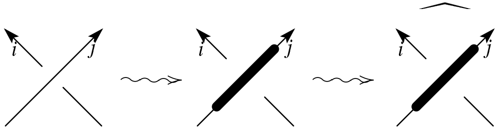

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$\rightsquigarrow$ Normalized Poincaré poly. (Link invariant)

$$P(D) := \overline{P}(D) \prod_{i=1}^{n-1} \frac{1}{\left([i]_q!\right)^{\mathrm{Cr}_i(D)}}.$$

$$\mathrm{Cr}_i(D) := \#\{\ [*, i]\text{-crossing in } D\ \}.$$

Polynomial link invariant

Main theorem 3 (Cor of Main theorem 2)

If diagram D translates D' each other by Reidemeister moves, these evaluations by P are the same

$$P(D) = P(D').$$

That is, we have following equations: