On the twisted Alexander polynomial for metabelian $\mathrm{SL}_2(\mathbb{C})$–representations with the adjoint action

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Basic notation and Notion

The twisted Alexander polynomial

The twisted Alexander polynomial	$=$	A refinement of $\Delta_K(t)$ (the Alexander polynomial) with $\rho : \pi_1 \to \mathrm{GL}(V)$

Notation

$E_K := S^3 \setminus N(K)$ a knot exterior,

$$\begin{array}{ccccc} Ad \circ \rho : \pi_1(E_K) & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{C}) & \xrightarrow{Ad} & Aut(\mathfrak{sl}_2(\mathbb{C})) \\ \gamma & \mapsto & \rho(\gamma) & \mapsto & Ad_{\rho(\gamma)} : v \mapsto \rho(\gamma) v \rho(\gamma)^{-1} \end{array}$$

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The adjoint action *Ad* gives a connection with the character variety $Hom(\pi_1(E_K), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$.

Metabelian representations

Definition of metabelian reps.

- $\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ is **metabelian**

$$\iff \quad \rho([\pi_1(E_K), \pi_1(E_K)]) \subset \mathrm{SL}_2(\mathbb{C}) \text{ abelian.}$$

Remark

- $\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ is **abelian**

$$\iff \quad \rho(\pi_1(E_K)) \subset \mathrm{SL}_2(\mathbb{C}) \text{ abelian,}$$
$$\iff \quad \rho([\pi_1(E_K), \pi_1(E_K)]) = \{\mathbf{1}\}.$$

Background

$\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ **metabelian**

- ρ : **reducible**

$$\rho : [\pi_1(E_K), \pi_1(E_K)] \to \left\{ \pm \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \middle| \omega \in \mathbb{C} \right\} \subset \mathrm{SL}_2(\mathbb{C})$$

$\Delta_K(t)$ appears in the twisted Alexander.
$\left(\longrightarrow \right.$ Hyperbolic torsion at "bifurcation points" in $Hom(\pi_1(E_K, \mathrm{SL}_2(\mathbb{C})))//\mathrm{SL}_2(\mathbb{C})$. $\left. \right)$

- ρ : **irreducible**

$$\rho : [\pi_1(E_K), \pi_1(E_K)] \to \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C} \setminus \{0\} \right\} \subset \mathrm{SL}_2(\mathbb{C})$$

Does $\Delta_K(t)$ appear in the twisted Alexander?

Main Theorem

Theorem

Suppose that

$\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ *s.t.* $\left\{ \begin{array}{l} \text{an } \textbf{irred. metabelian} \text{ and;} \\ \text{“}\textbf{longitude–regular}\text{”.} \end{array} \right.$

(the twisted Alexander poly.)

Then $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) \doteq (t-1)\Delta_K(-t)P(t),$

where $\Delta_K(t)$ *is the Alexander polynomial of* K.

The details of $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$

Homomorphisms

- $\rho : \pi_1(E_K) \to \mathrm{SL}_2(\mathbb{C})$ is **metabelian**

  $$\iff \quad \rho([\pi_1(E_K), \pi_1(E_K)])(\subset \mathrm{SL}_2(\mathbb{C})) \text{ is abelian,}$$

- Suppose that $\rho([\pi_1(E_K), \pi_1(E_K)]) \neq \{\mathbf{1}\}$.
- $\alpha : \pi_1(E_K) \to \pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)] \simeq H_1(E_K) = \langle t \rangle$
  s.t. $\alpha(\mu) = t$

the twisted Alexander poly.

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) = \frac{\det\left(\alpha \otimes Ad \circ \rho\left(\frac{\partial r_i}{\partial g_j}\right)_{i,j\neq 1}\right)}{\det\left(\alpha \otimes Ad \circ \rho\,(g_1 - 1)\right)}$$

from a presentation $\pi_1(E_K) = \langle g_1, g_2, \ldots, g_k \,|\, r_i, \ldots, r_{k-1}\rangle$.

Main tools

- We need a “good” presentation of $\pi_1(E_K)$ for metabelian reps.
- X-S. Lin introduced a suitable presentation of $\pi_1(E_K)$ by using a free Seifert surface of K.
  a Seifert surface S is free
  $\Leftrightarrow S^3 = S \times [-1, 1] \cup \overline{S^3 \setminus S \times [-1, 1]}$
  : a Heegaard decompo.

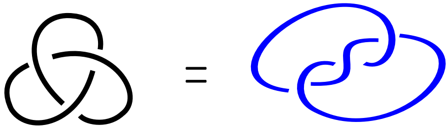

Figure: a free Seifert surface of the trefoil knot

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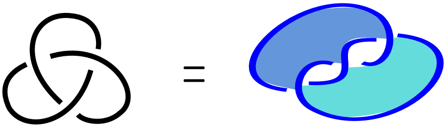

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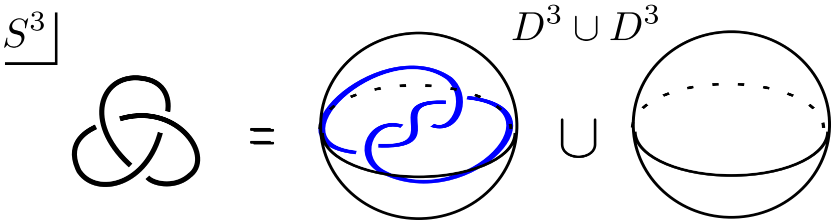

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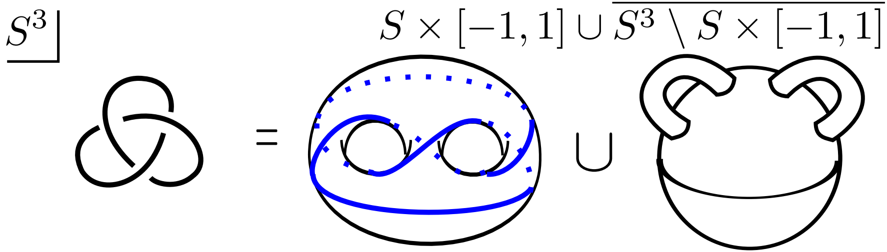


Figure: a free Seifert surface of the trefoil knot

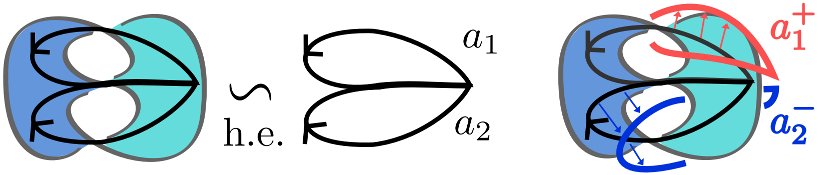
Lin's presentation

By using a free Seifert surface S with genus $2g$,

$$\pi_1(E_K) = \langle \mu, x_1, \ldots, x_{2g} \mid \mu a_i^+ \mu^{-1} = a_i^- \ (i = 1, \ldots 2g) \rangle$$

where

- x_i is a closed loop corresponding to 1–handle in $S^3 \setminus S \times [-1, 1]$,
- $a_i^\pm$ is a word in $x_1, \ldots, x_{2g}$, corresponding to closed loops in the spine of S.



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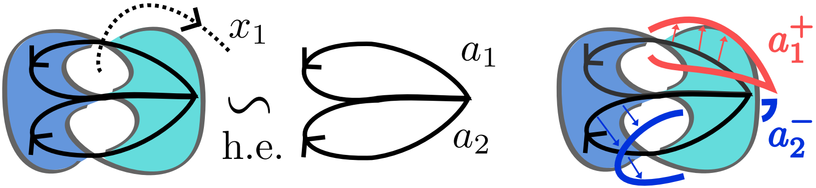

Figure: a free Seifert surface of the trefoil knot

Remark & Examples of Lin's presentation

Remark

The generators $x_1, \dots, x_{2g}$ are null–homologous,
i.e., $x_i \in [\pi_1(E_K), \pi_1(E_K)]$.

- K = trefoil knot
$$\pi_1(E_K) = \left\langle \mu, x_1, x_2 \;\middle|\; \begin{array}{l} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \\ \mu x_2^{-1} x_1 \mu^{-1} = x_2^{-1} \end{array} \right\rangle$$
- K = figure eight knot
$$\pi_1(E_K) = \left\langle \mu, x_1, x_2 \;\middle|\; \begin{array}{l} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \\ \mu x_2 x_1 \mu^{-1} = x_2 \end{array} \right\rangle$$

Explicit form of metabelian reps.

X-S. Lin, F. Nagasato

- The correspondence

$$\mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_i \mapsto \begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i^{-1} \end{pmatrix}$$

  gives a metabelian rep.
- They gives all representatives of conj. classes of metabelian reps.
- $\sharp$(conj. classes of irred. metabelian reps) $= \dfrac{|\Delta_K(-1)| - 1}{2}$.

Sketch of proof

$$Ad \circ \rho : \pi_1(E_K) \to Aut(\mathfrak{sl}_2(\mathbb{C}))$$
$$= Aut(\mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$$

$$\mu \mapsto \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

$$x_i \mapsto \begin{pmatrix} \xi^2 & & \\ & 1 & \\ & & \xi^{-2} \end{pmatrix}$$

The subspace $\mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is invariant space.

Decomposition of $Ad \circ \rho$

$$Ad \circ \rho = \rho_2 \oplus \rho_1$$

where ρ_1 is 1–dim. rep. and ρ_2 is 2–dim. rep.

Observation about $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) = \Delta_{E_K}^{\alpha\otimes\rho_2}(t)\cdot\Delta_{E_K}^{\alpha\otimes\rho_1}(t)$$

Wada ⟶ ⟵ Milnor

$$= Q(t)\cdot\frac{\Delta_K(-t)}{(-t-1)}$$

longitude–regular
& inv. of conj.
for R–torsion

$$= (t-1)(t+1)P(t)\cdot\frac{\Delta_K(-t)}{-t-1}$$

$$= -(t-1)\cdot P(t)\cdot\Delta_K(-t).$$

Examples

- trefoil knot $\quad \Delta_K(t) = t^2 - t + 1$ $(\frac{|\Delta_K(-1)|-1}{2} = 1)$.

$$\pi_1(E_K) = \left\langle \mu, x_1, x_2 \;\middle|\; \begin{array}{l} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \\ \mu x_2^{-1} x_1 \mu^{-1} = x_2^{-1} \end{array} \right\rangle$$

$$\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x_1) = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^{-2} \end{pmatrix}$$

where $\zeta_3 = e^{\frac{2\pi\sqrt{-1}}{3}}$

$$\Rightarrow \quad \Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = (t-1)\Delta_K(-t)$$

- figure eight knot

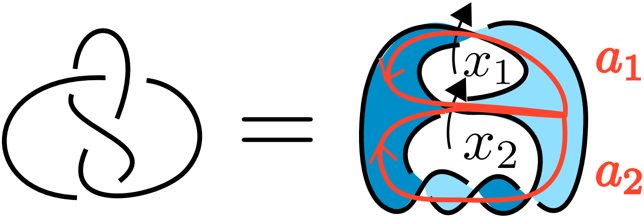

Figure: a free Seifert surface of the figure eight knot

- figure eight knot $\quad \Delta_K(t) = t^2 - 3t + 1 \; (\frac{|\Delta_K(-1)|-1}{2} = 2)$.

$$\pi_1(E_K) = \left\langle \mu, x_1, x_2 \;\middle|\; \begin{array}{l} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \\ \mu x_2 x_1 \mu^{-1} = x_2 \end{array} \right\rangle$$

  - $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x_1) = \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \zeta_5^2 & 0 \\ 0 & \zeta_5^{-2} \end{pmatrix}$
  - $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x_1) = \begin{pmatrix} \zeta_5^2 & 0 \\ 0 & \zeta_5^{-2} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \zeta_5^4 & 0 \\ 0 & \zeta_5^{-4} \end{pmatrix}$

  where $\zeta_5 = e^{\frac{2\pi\sqrt{-1}}{5}}$

$$\Rightarrow \quad \Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = (t-1)\Delta_K(-t)$$

(both reps. have the same result.)

- 5_2 knot

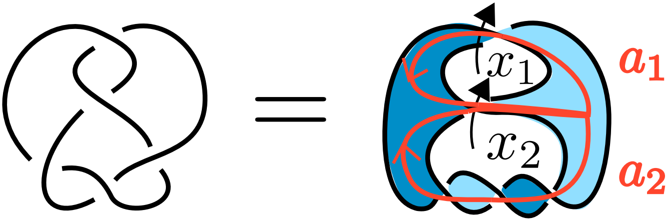

Figure: a free Seifert surface of 5_2 knot

- 5_2 knot $\quad \Delta_K(t) = 2t^2 - 3t + 2$ ($\frac{|\Delta_K(-1)|-1}{2} = 3$).

$$\pi_1(E_K) = \left\langle \mu, x_1, x_2 \;\middle|\; \begin{array}{l} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \\ \mu x_2^{-2} x_1 \mu^{-1} = x_2^{-2} \end{array} \right\rangle$$

  - $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x_1) = \begin{pmatrix} \zeta_7 & 0 \\ 0 & \zeta_7^{-1} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^{-2} \end{pmatrix}$
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  where $\zeta_7 = e^{\frac{2\pi\sqrt{-1}}{7}}$

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$$\Rightarrow \quad \Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) = (t-1)(2+e^{6\pi\sqrt{-1}/7}+e^{-6\pi\sqrt{-1}/7})\Delta_K(-t)$$

- $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x_1) = \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^{-2} \end{pmatrix}, \rho(x_2) = \begin{pmatrix} \zeta_7^4 & 0 \\ 0 & \zeta_7^{-4} \end{pmatrix}$

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$$\Rightarrow \quad \Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) = (t-1)(2+e^{4\pi\sqrt{-1}/7}+e^{-4\pi\sqrt{-1}/7})\Delta_K(-t)$$