Seiberg-Witten invariants and end-periodic Dirac operators

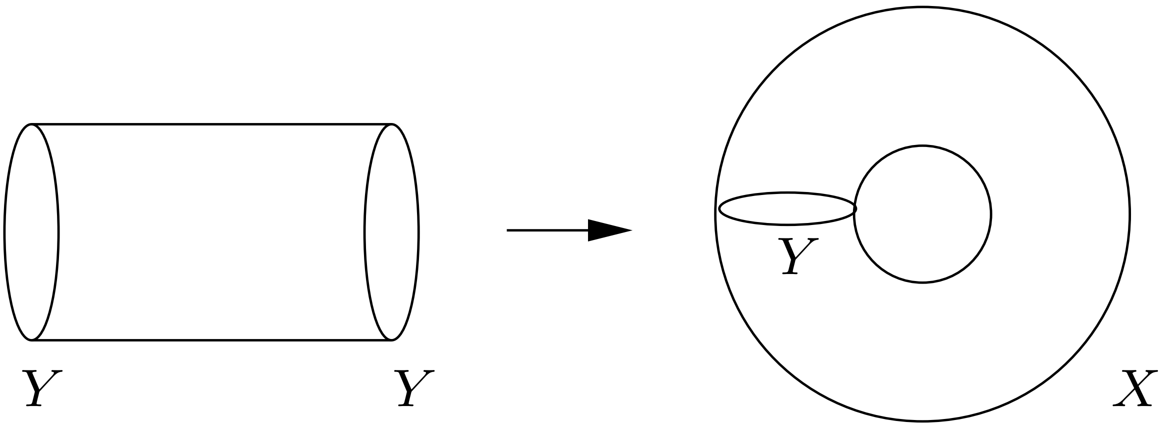
Tomasz Mrowka
Daniel Ruberman
Nikolai Saveliev

Homology $\mathbf{S^1 \times S^3}$ is a smooth oriented closed spin manifold X of dimension 4 such that

$$H_*(X) = H_*(S^1 \times S^3).$$

Example. A product $X = S^1 \times Y$, where Y is an integral homology sphere.

Example. A "furled up" homology cobordism from Y to itself:



Homology orientation of X is a choice of generator $1 \in H^1(X; \mathbf{Z})$.

Rohlin Invariant

Given an oriented spin 3-manifold Y, the **Rohlin invariant** of Y is defined as

$$\rho(Y) = \frac{1}{8}\,\text{sign}(Z) \pmod 2$$

where Z is any smooth compact spin 4-manifold with boundary $\partial Z = Y$.

Let X be a homology $S^1 \times S^3$ with a fixed homology orientation, and choose an oriented submanifold $Y \subset X$ dual to $1 \in H^1(X;\mathbf{Z})$. Define the **Rohlin invariant** of X as

$$\rho(X) = \rho(Y) \pmod 2$$

where Y has the induced spin structure. This is a well defined invariant of X.

The Rohlin invariant is tied to some difficult questions in 4-dimensional topology. Here is an example:

Question: Is there a homotopy $S^1 \times S^3$ with non-trivial Rohlin invariant?

Such a manifold, if existed, would provide a fake smooth structure on $S^1 \times S^3$.

Approach: An integer valued lift $\lambda_{\mathrm{SW}}(X)$ of the Rohlin invariant $\rho(X)$.

Seiberg–Witten Theory

Given a metric g on X and $\beta \in \Omega^1(X, i\mathbf{R})$, consider the triples

$$(A, s, \varphi) \in \Omega^1(X, i\mathbf{R}) \times \mathbf{R}_{\geq 0} \times \mathbf{C}^\infty(S^+)$$

such that

$$\begin{cases} F_A^+ - s^2 \cdot \tau(\varphi, \varphi) = d^+\beta \\ D_A^+(X, g)(\varphi) = 0, \quad \|\varphi\|_{L^2(X)} = 1 \end{cases}$$

Seiberg–Witten moduli space $\mathcal{M}(X, g, \beta)$: the gauge equivalence classes of solutions of the above system. The solutions with $s = 0$ are called reducible.

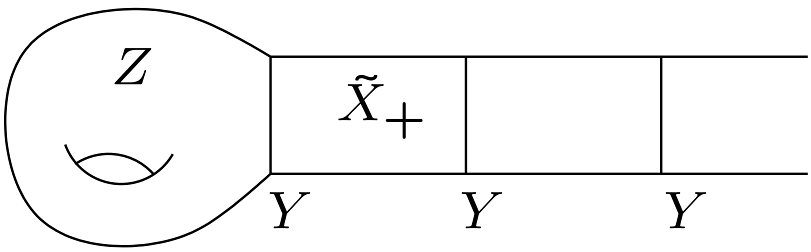
Theorem 1. For generic (g, β), the moduli space $\mathcal{M}(X, g, \beta)$ is a compact oriented 0-dimensional manifold with no reducibles.

Denote by $\#\,\mathcal{M}(X, g, \beta)$ the signed count of points in this moduli space.

Correction Term

Let $\tilde{X} \to X$ be the $\mathbf{Z}$–fold covering corresponding to $1 \in H^1(X;\mathbf{Z})$ and $\tilde{X}_+$ its "positive half".

End-periodic manifold is a smooth manifold $Z_+ = Z \cup \tilde{X}_+$, where Z is a compact smooth spin 4-manifold with $\partial Z = -\partial \tilde{X}_+$.



Product case: $X = S^1 \times Y$ gives rise to $Z_+ = Z \cup ([0,+\infty) \times Y)$. The index theory was studied by Atiyah, Patodi and Singer.

General case: the basics of index theory on Z_+ were established by Taubes. We develop this theory far enough to prove the following two theorems.

Theorem 2. For generic (g, β), the operator $D^+(Z_+, g) + \beta : L^2_1 \to L^2$ is Fredholm, and

$$w(X, g, \beta) = \text{ind}(D^+(Z_+, g) + \beta) + \text{sign}(Z)/8$$

is independent of the choice of Z and the way g and β are extended over $Z \subset Z_+$.

Theorem 3. The quantity

$$\lambda_{\text{SW}}(X) \;=\; \#\mathcal{M}(X, g, \beta) - w(X, g, \beta)$$

is an invariant of X (with a choice of orientation and homology orientation). Moreover,

$$\lambda_{\text{SW}}(X) = \rho(X) \pmod 2.$$

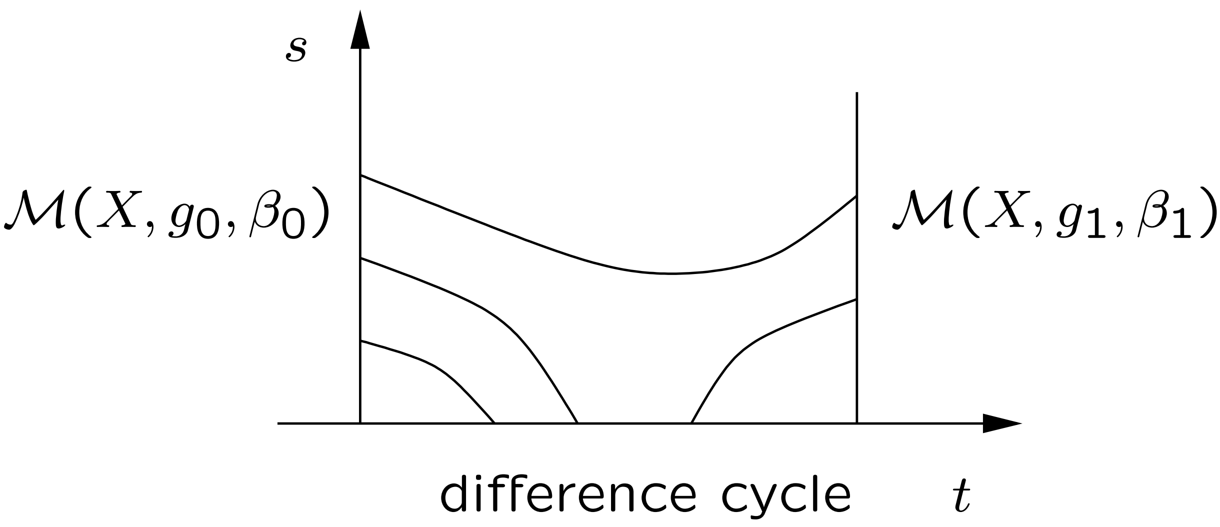
Product case: Weimin Chen and Yuhan Lim.

Idea of proof

Choose a (generic) path (g_t, β_t), $0 \leq t \leq 1$, between two generic pairs of metrics and perturbations. Then the parameterized moduli space

$$\bigcup_{t \in [0,1]} \{t\} \times \mathcal{M}(X, g_t, \beta_t)$$

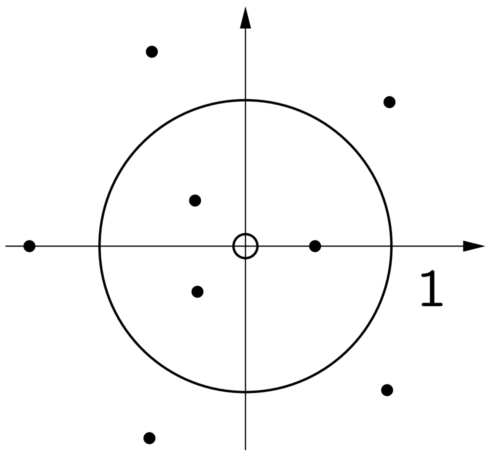
is a 1-dimensional manifold with boundary:



A version of Fourier transform associates with $D^+(Z_+, g)$ the holomorphic family

$$D_z^+(X, g) = D^+(X, g) - \log z \cdot df,$$

where $f : X \to S^1$ is such that $[df] = 1 \in H^1(X; \mathbf{Z})$.

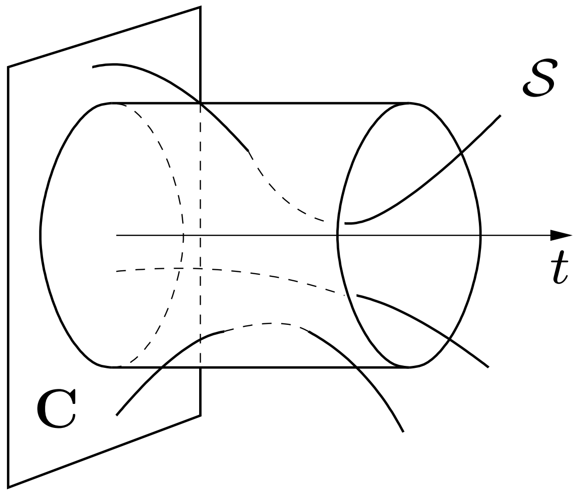


Spectral points

Fredholmness means no spectral points on the circle $|z| = 1$.

Then ind $(D^+(Z_+, g)+\beta)$ changes along (g_t, β_t) by the spectral flow of the family

$$D_z^+(X, g_t) + \beta_t$$



The well definedness of $\lambda_{\mathsf{SW}}(X, g)$ follows by matching this with the Seiberg-Witten difference cycle.

The Rohlin invariant part is the hardest because it requires Fredholmness of $D^+(Z_+, g)$ with $\beta = 0$, by perturbing metric g alone.

Product case

If $X = S^1 \times Y$ then $D^+(X,g) = d/d\theta + D$ with D the self-adjoint Dirac operator on Y.

Theorem (Atiyah-Patodi-Singer)

$$\operatorname{ind} D^+(Z_+, g) = \int_Z \widehat{A}(Z,g) - \frac{1}{2}\eta(Y,g),$$

where

$$\eta(Y,g) = \sum_{0 \neq \lambda \in \mathrm{Spec}(D)} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$$

evaluated at $s = 0$.

Theorem (Yuhan Lim)

$$\lambda_{\mathrm{SW}}(S^1 \times Y) = -\lambda(Y),$$

the **Casson invariant** of Y, obtained by counting irreducible representations $\pi_1(Y) \to \mathrm{SU}(2)$.

Mapping torus case

Let Y be a homology sphere and X the mapping torus of $\tau : Y \to Y$ of **finite order**. Then $\tilde{X} = \mathbf{R} \times Y$ as in the product case.

Theorem 4. Let $Y = \Sigma(a_1, \ldots, a_n)$ and X the mapping torus of $\tau : Y \to Y$ induced by complex conjugation on the link so that $Y/\tau = S^3$ with branch set a Montesinos knot k. Then

$$\lambda_{\mathsf{SW}}(X) = -\frac{1}{8}\,\mathrm{sign}(k),$$

also known as the **equivariant Casson** $\lambda^\tau(Y)$ (Collin–Saveliev).

Conjecture. For any mapping torus X of finite order orientation preserving diffeomorphism $\tau : Y \to Y$, one has

$$\lambda_{\mathsf{SW}}(X) = -\lambda^\tau(Y).$$

Furuta–Ohta invariant

Conjecture. If X is a $\mathbf{Z}[\mathbf{Z}]$–homology $S^1 \times S^3$ then (cf. Witten's conjecture)

$$\lambda_{\mathsf{SW}}(X) = -\lambda_{\mathsf{FO}}(X),$$

the **Furuta–Ohta invariant** obtained by counting irreducible representations $\pi_1(X) \to \mathsf{SU}(2)$. Note that $\lambda_{\mathsf{FO}}(X) = \lambda^{\tau}(Y)$ for the finite order mapping tori.

If true, this conjecture would give a negative answer to the question about homotopy $S^1 \times S^3$.

End-periodic index theorem

(work in progress)

Assume there is $Y \subset X$ dual to $1 \in H^1(X;\mathbf{Z})$ such that

(1) X is isometric to $N(Y) = [-\varepsilon, \varepsilon] \times Y$ near Y, and

(2) df is supported in $N(Y)$

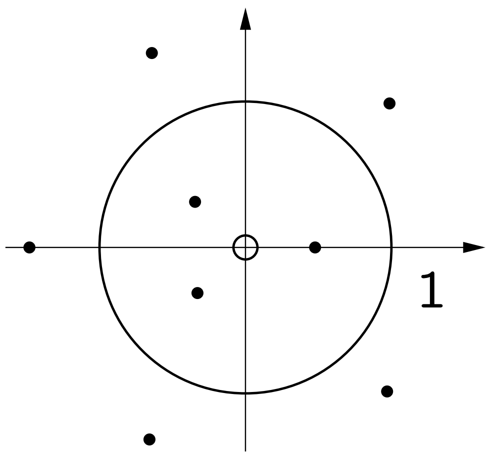
(if not, the formulas will be more complicated).

Then

$$\operatorname{ind} D^+(Z_+, g) = \int_Z \hat{A}(Z, g) - \frac{1}{2}\eta(X, g),$$

where

$$\eta(X,g) = \sum_{\text{ker } D_z^+ \neq 0} \text{sign}(\log|z|),$$



properly regularized:

$\eta(X,g) =$

$$\frac{1}{\pi i}\int_0^\infty \oint_{|z|=1} \text{Tr}\left(df \cdot D_z^+ e^{-tD_z^- D_z^+}\right) \frac{dz}{z}\, dt.$$

In the product case, $z = e^\lambda \in \mathbf{R}$, and we get back the η-invariant of Atiyah-Patodi-Singer.